Finite-dimensional solutions of coupled harmonic oscillator quantum systems

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Abstract. This paper describes a quantum system consisting of a one-dimensional chain of M identical harmonic oscillators coupled by means of springs. We approach this as a Wigner Quantum System, not imposing the canonical commutation relations, but using instead weaker relations following from the compatibility of Hamilton's equations and the Heisenberg equations. We show that a class of solutions can be obtained using generators of the Lie superalgebra $gl(1|M)$, and study the properties and spectra of the physical operators in a class of unitary representations of $gl(1|M)$.

1 Introduction and description of the problem

Consider a quantum system consisting of a linear chain of M identical harmonic oscillators (with mass m and natural frequency ω) coupled by some nearest neighbour interaction, the coupling being represented by springs obeying Hooke's law. The Hamiltonian of such a system is given by:

$$\hat{H} = \sum_{k=1}^{M} \Big(\frac{\hat{p}_k^2}{2m} + \frac{m\omega^2}{2}\hat{q}_k^2 + \frac{cm}{2}(\hat{q}_k - \hat{q}_{k+1})^2 \Big). \tag{1}$$

The position and momentum operator for the kth oscillator are given by $\hat{q}_k$ and $\hat{p}_k$; more precisely $\hat{q}_k$ measures the displacement of the kth mass point with respect to its equilibrium position. The last term in (1) represents the nearest neighbour coupling by means of "springs", with coupling strength c ($c > 0$). We shall also assume periodic boundary conditions, i.e.

$$\hat{q}_{M+1} \equiv \hat{q}_1. \tag{2}$$

Such quantum systems are relevant in quantum information theory, in quantum optics (photonic crystals), or for describing phonons in a crystal [1, 2].

In the standard approach for the quantum system governed by (1), one assumes the canonical commutation relations (CCR's)

$$[\hat{q}_k, \hat{q}_l] = 0, \qquad [\hat{p}_k, \hat{p}_l] = 0, \qquad [\hat{q}_k, \hat{p}_l] = i\hbar\delta_{kl}. \tag{3}$$

Here, we start from a more general quantization procedure. This is based upon the compatibility of Hamilton's equations with the Heisenberg equations. Such systems are called Wigner Quantum Systems (WQS's) [3]. For the Hamiltonian (1), the equivalence of Hamilton's equations $\dot{\hat{q}}_k = \frac{\partial \hat{H}}{\partial \hat{p}_k}$, $\dot{\hat{p}}_k = -\frac{\partial \hat{H}}{\partial \hat{q}_k}$ and the Heisenberg equations $\dot{\hat{p}}_k = \frac{i}{\hbar}[\hat{H}, \hat{p}_k]$, $\dot{\hat{q}}_k = \frac{i}{\hbar}[\hat{H}, \hat{q}_k]$ leads to the following compatibility conditions:

$$[\hat{H}, \hat{q}_k] = -\frac{i\hbar}{m}\hat{p}_k, \tag{4}$$

$$[\hat{H}, \hat{p}_k] = -i\hbar cm\,\hat{q}_{k-1} + i\hbar m(\omega^2 + 2c)\,\hat{q}_k - i\hbar cm\,\hat{q}_{k+1}, \tag{5}$$

where $k = 1, 2, \ldots, M$, and – extending (2) – $\hat{q}_0$ stands for $\hat{q}_M$, or more generally

$$\hat{q}_k = \hat{q}_{\,k \bmod M}, \quad \hat{p}_k = \hat{p}_{\,k \bmod M} \tag{6}$$

whenever k is out of the range $\{1, 2, \ldots, M\}$. The task is to find operator solutions for $\hat{q}_k$ and $\hat{p}_k$ such that the compatibility conditions (4)-(5), together with (1), are satisfied. Furthermore, since $\hat{q}_k$ and $\hat{p}_k$ correspond to physical observables, the operators should be Hermitian:

$$\hat{q}_k^\dagger = \hat{q}_k, \quad \hat{p}_k^\dagger = \hat{p}_k \qquad (k = 1, 2, \ldots, M). \tag{7}$$

Using finite Fourier transforms of $\hat{q}_k$ and $\hat{p}_k$,

$$\hat{Q}_r = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \mathrm{e}^{-\frac{2\pi irk}{M}} \hat{q}_k, \quad \hat{P}_r = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} \mathrm{e}^{\frac{2\pi irk}{M}} \hat{p}_k, \tag{8}$$

with $\hat{Q}_r^\dagger = \hat{Q}_{M-r}$ and $\hat{P}_r^\dagger = \hat{P}_{M-r}$ (using the same convention as in (6)), the Hamiltonian becomes, just as in the canonical case [1, 2]

$$\hat{H} = \sum_{r=1}^{M} \Big(\frac{1}{2m} \hat{P}_r \hat{P}_r^\dagger + \frac{m\omega_r^2}{2} \hat{Q}_r \hat{Q}_r^\dagger \Big), \tag{9}$$

where, for $r = 1, 2, \ldots, M$, the quantities ω_r are positive numbers with

$$\omega_r^2 = \omega^2 + 2c - 2c\cos(\frac{2\pi r}{M}) = \omega^2 + 4c\sin^2(\frac{\pi r}{M}), \tag{10}$$

and clearly $\omega_{M-r} = \omega_r$.

As a final step it is convenient to introduce linear combinations of the unknown operators $\hat{Q}_r$ and $\hat{P}_r$, say a_r^+ and a_r^- ($r = 1, 2, \ldots, M$), by

$$a_r^- = \sqrt{\frac{m\omega_r}{2\hbar}}\hat{Q}_r + \frac{i}{\sqrt{2m\omega_r\hbar}}\hat{P}_r^\dagger, \quad a_r^+ = \sqrt{\frac{m\omega_r}{2\hbar}}\hat{Q}_r^\dagger - \frac{i}{\sqrt{2m\omega_r\hbar}}\hat{P}_r, \qquad (11)$$

with $(a_r^\pm)^\dagger = a_r^\mp$. It is important to remember that, since the CCR's do not hold, the operators $a_r^\pm$ do *not* satisfy the usual boson relations. Now the Hamiltonian (9) becomes:

$$\hat{H} = \sum_{r=1}^{M} \frac{\hbar\omega_r}{2}\{a_r^-, a_r^+\} = \sum_{r=1}^{M} \frac{\hbar\omega_r}{2}(a_r^- a_r^+ + a_r^+ a_r^-), \qquad (12)$$

and the conditions (4)-(5) are equivalent to:

$$[\hat{H}, a_r^\pm] = \pm\hbar\omega_r a_r^\pm, \quad (r = 1, 2 \ldots, M). \qquad (13)$$

Thus in the approach of (1) as a Wigner Quantum System, the problem is reduced to finding $2M$ operators $a_r^\pm$ ($r = 1, \ldots, M$), acting in some Hilbert space, such that $(a_r^\pm)^\dagger = a_r^\mp$ and

$$[\sum_{j=1}^{M} \omega_j(a_j^- a_j^+ + a_j^+ a_j^-), a_r^\pm] = \pm 2\omega_r a_r^\pm, \quad (r = 1, 2 \ldots, M). \qquad (14)$$

2 Lie superalgebraic solution of the problem

The main result is that the algebraic relations (14) have a solution in terms of generators of the Lie superalgebra $gl(1|M)$. For this Lie superalgebra, consider the standard basis elements e_{jk}, with $j, k = 0, 1, \ldots, M$ (e_{k0} and e_{0k}, $k = 1, \ldots, M$, being the *odd* elements having degree $\deg(e_{k0}) = \deg(e_{0k}) = 1$) and Lie superalgebra bracket

$$[\![e_{ij}, e_{kl}]\!] = \delta_{jk} e_{il} - (-1)^{\deg(e_{ij})\deg(e_{kl})}\delta_{il} e_{kj}. \qquad (15)$$

Then the compatibility conditions (14) have a solution for the operators $a_k^\pm$ in terms of $gl(1|M)$ generators:

$$a_k^- = \sqrt{\frac{2\beta_k}{\omega_k}}\, e_{k0}, \quad a_k^+ = \sqrt{\frac{2\beta_k}{\omega_k}}\, e_{0k}, \quad (k = 1, \ldots, M). \qquad (16)$$

Herein, we have introduced new constants β_k in terms of the quantities ω_r:

$$\beta_k = -\omega_k + \frac{1}{M-1}\sum_{j=1}^{M}\omega_j, \tag{17}$$

with $\beta_{M-k} = \beta_k$ and

$$\beta \equiv \sum_{j=1}^{M}\beta_j = \sum_{j=1}^{M}\omega_j. \tag{18}$$

In order to satisfy the Hermiticity conditions $(a_r^{\pm})^\dagger = a_r^{\mp}$, being equivalent with the star condition $(e_{0k})^\dagger = e_{k0}$ for the Lie superalgebra elements (corresponding to the "compact form" $u(1|M)$ of $gl(1|M)$, for which finite-dimensional unitary representations exist), all constants β_k in (16) should be positive. This is true only if the coupling constant c is in a certain interval $]0,c_0[$, where c_0 is a critical value depending upon M. It can be shown that $c \le \frac{\omega^2}{2(M-2)}$ is a sufficient condition for all $\beta_k > 0$ [4].

3 A class of $gl(1|M)$ representations as state spaces

We consider a class of unitary representations $W(p)$ of $gl(1|M)$ as state spaces of the system. These are finite-dimensional representations, labelled by a number p, with either $p \in \{0,1,2,\ldots,M-1\}$ [atypical case] or else p real with $p > M-1$ [typical case]. The basis vectors of $W(p)$ are given by:

$$w(\theta) \equiv w(\theta_1,\theta_2,\ldots,\theta_M), \qquad \theta_i \in \{0,1\},\ \text{and}\ |\theta| = \theta_1 + \cdots + \theta_M \le p. \tag{19}$$

Clearly, for $p > M-1$, the dimension of $W(p)$ is given by 2^M.

The action of the $gl(1|M)$ generators on the basis vectors of $W(p)$ is now given by:

$$\begin{aligned}
e_{00}w(\theta) &= (p-|\theta|)\,w(\theta);\\
e_{kk}w(\theta) &= \theta_k\,w(\theta);\\
e_{k0}w(\theta) &= (1-\theta_k)(-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-|\theta|}\,w(\theta_1,\ldots,\theta_k+1,\ldots,\theta_M);\\
e_{0k}w(\theta) &= \theta_k(-1)^{\theta_1+\cdots+\theta_{k-1}}\sqrt{p-|\theta|+1}\,w(\theta_1,\ldots,\theta_k-1,\ldots,\theta_M),
\end{aligned}$$

where $1 \le k \le M$. The representation $W(p)$ is unitary for the star condition $e_{jk}^\dagger = e_{kj}$ with respect to the inner product $\langle w(\theta)|w(\theta')\rangle = \delta_{\theta,\theta'}$.

The main purpose is now the study of the spectrum of the physical operators $\hat{H}$, $\hat{q}_r$ and $\hat{p}_r$, for which we shall present the results only in the typical case ($p > M-1$). Under the solution (16), the Hamiltonian $\hat{H}$ takes the form

$$\hat{H} = \hbar(\beta\, e_{00} + \sum_{k=1}^{M} \beta_k\, e_{kk}). \tag{20}$$

It follows that the vectors $w(\theta)$ are eigenvectors for $\hat{H}$, $\hat{H}\, w(\theta) = \hbar E_\theta\, w(\theta)$, with eigenvalues

$$E_\theta = \beta(p - |\theta|) + \sum_{k=1}^{M} \theta_k \beta_k = \beta(p - \frac{M-2}{M-1}|\theta|) - \sum_{k=1}^{M} \theta_k \omega_k. \tag{21}$$

In this expression, $\theta = (\theta_1, \ldots, \theta_M)$, with each $\theta_k \in \{0,1\}$, and $|\theta| = \sum_{k=1}^{M} \theta_k$. Some analysis, using the symmetries of β_k, shows that (for $0 < c < c_0$) the multiplicity of E_θ is given by $2^{\sum_{k=1}^{M}(\theta_k - \theta_{M-k})^2/2}$. Whereas there are $M+1$ (equidistant) energy levels for $c = 0$, the number of energy levels for $0 < c < c_0$ is $2 \cdot 3^{(M-1)/2}$ (M odd) or $4 \cdot 3^{(M-2)/2}$ (M even).

The determination of the spectrum of position and momentum operators in $W(p)$ requires an extensive analysis. The result is as follows: the operator $\hat{q}_r$ has $2M$ distinct eigenvalues given by $\pm x_K = \pm\sqrt{\frac{\hbar}{mM}(p-K)\gamma}$, where $0 \leq K \leq M-1$ and $\gamma = \sum_{k=1}^{M} \beta_k/\omega_k^2$. The multiplicity of the eigenvalue $\pm x_K$ is $\binom{M-1}{K}$.

The eigenvectors of the position operator $\hat{q}_r$ enable us to determine position probability distributions for the stationary states $w(\theta)$ and to describe the probability distributions of the other oscillator positions when one oscillator is in a fixed (eigenvalue) position [4].

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