# $3nj$-coefficients of $su(1,1)$ as connection coefficients between orthogonal polynomials in $n$ variables

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**Abstract**

In the tensor product of $n+1$ positive discrete series representations of $su(1,1)$, a coupled basis vector can be described by a certain binary coupling tree. To every such binary coupling tree, polynomials $R_l^{(k)}(x)$ and $\mathcal{R}_l^{(k)}(x)$ are associated. These polynomials are $n$-variable Jacobi and continuous Hahn polynomials, and are orthogonal with respect to a weight function. The connection coefficients expressing such a polynomial associated with a given binary coupling tree in terms of those polynomials associated with another binary coupling tree are proportional to $3nj$-coefficients of $su(1,1)$.

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# I Introduction

The $3nj$-coefficients of $su(2)$ (or $so(3)$) play a dominant role in quantum theory of angular momentum [1, 2, 3, 4, 5] and its applications in nuclear, atomic and molecular physics. For $3j$- and $6j$-coefficients of $su(2)$ there exist expressions in terms of hypergeometric series [5, 6, 7], explaining the close relation with orthogonal polynomials such as Hahn and Racah polynomials [7]. For example, the $6j$-coefficient of $su(2)$ is expressed in terms of a terminating balanced ${}_4F_3$ series of unit argument. The parameters of the ${}_4F_3(1)$ series are written in terms of the six representation labels (angular momenta) of the $6j$-coefficient. By the nature of these representation labels (integer or half-integer positive numbers), the parameters of the ${}_4F_3(1)$ series are integers [6, 7]. When identifying the $6j$-coefficient with a Racah polynomial $R_m(\lambda(x);\alpha,\beta,\gamma,\delta)$, it is not easy to decide which parameters correspond to the degree $m$, which to the variable $x$, and which to the parameters $\alpha,\beta,\gamma,\delta$ of the polynomial. In a way, this identification becomes easier when dealing with positive discrete series representations of $su(1,1)$ rather than with $su(2)$ representations. This is for us one of the main reasons to consider couplings of such $su(1,1)$ representations, their $3nj$-coefficients, and the connections with orthogonal polynomials in this paper.

The Lie algebra $su(1,1)$, or $so(2,1)$, plays itself an important role in physical models. It has been extensively used as spectrum generating algebra in many simple quantum systems, such as the nonrelativistic Coulomb problem, the isotropic harmonic oscillator, Schrödinger's relativistic equation, and the Dirac-Coulomb problem (Ref. [9], and references therein). In certain boson models [10, 11], the relevant representations are the positive discrete series representations $\mathcal{D}^+(k)$. To fix the notation, let $J_0$, $J_\pm$ be the generators of $su(1,1)$ subject to

$$[J_0,J_\pm]=\pm J_\pm, \qquad [J_+,J_-]=-2J_0, \tag{1.1}$$

with the conditions $J_0^\dagger=J_0$ and $J_\pm^\dagger=J_\mp$. The positive discrete series representations [7] $\mathcal{D}^+(k)$ $(k>0)$ have a basis $|k,n\rangle$, with $n=0,1,2,\ldots$, and the action of the generators is given by :

$$\begin{aligned}
J_0|k,n\rangle &= (n+k)|k,n\rangle,\\
J_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}|k,n+1\rangle,\\
J_-|k,n\rangle &= \sqrt{n(2k+n-1)}|k,n-1\rangle.
\end{aligned} \tag{1.2}$$

The tensor product of two positive discrete series representations[†] $(k_1)=\mathcal{D}^+(k_1)$ and $(k_2)=$

[†]In the rest of the paper, we always mean "positive discrete series representations" whenever we say "representations" of $su(1,1)$

$\mathcal{D}^+(k_2)$ (or the *coupling of two representations* $(k_1)$ and $(k_2)$) decomposes as follows [7] :

$$(k_1) \otimes (k_2) = \bigoplus_{j=0}^{\infty} (k_1 + k_2 + j). \tag{1.3}$$

The "coupled basis vectors" are written in terms of the uncoupled ones by means of the $su(1,1)$ Clebsch-Gordan coefficients :

$$|(k_1 k_2)k, n\rangle = \sum_{n_1, n_2} C^{k_1, k_2, k}_{n_1, n_2, n} \; |k_1, n_1\rangle \otimes |k_2, n_2\rangle. \tag{1.4}$$

Herein, $k = k_1 + k_2 + j$ for some integer $j \geq 0$, and the sum is such that $n_1 + n_2 = j + n$. Explicit expressions for the Clebsch-Gordan coefficients are given, e.g. in Ref. [7, 12, 13], in terms of a ${}_3F_2(1)$ series.

In this paper we shall be dealing with the coupling or tensor product of $n+1$ such representations, labeled by positive integers $k_1, k_2, \ldots, k_{n+1}$. For the description of coupled basis vectors in such a tensor product, the notion of binary coupling trees is essential (see [4, Topic 12] and [14, 15]). Binary couplings find their origin in the recoupling theory of $n+1$ angular momenta [4, Topic 12] (thus in the context of tensor products of $(n+1)$ $su(2)$ representations), but apply also to the "recoupling theory" of $n+1$ positive discrete series representations of $su(1,1)$ [14, 15]. Binary coupling trees describe the coupling scheme (the way of taking tensor products), i.e. the sequential pairwise coupling [4, Topic 12]. A $3nj$-coefficient of $su(1,1)$ is then the coefficient of a unitary transformation which connects two basis vectors corresponding to different binary coupling schemes of $n+1$ representations [4, 6, 14, 15]. Thus a $3nj$-coefficient is characterized by two binary trees $T_1$ and $T_2$ (with labeled leaves and nodes), and since it is the transformation coefficient or overlap coefficient, it is usually denoted by $\langle T_1, T_2 \rangle$. For the reader not familiar with binary coupling trees, their meaning will become transparent from the examples given in this paper.

The motivation for the work presented here stems from interpretations of identities involving orthogonal polynomials and $su(1,1)$ Racah coefficients (or $su(1,1)$ $6j$-coefficients). One such identity appears already in the seminal work of Granovskiĭ and Zhedanov [16, Eq. (9)] : this is a convolution identity involving products of Jacobi polynomials and Racah coefficients. The identity can be interpreted as a connection coefficient identity between orthogonal polynomials in two variables, with $su(1,1)$ Racah coefficients as connection coefficients (see [12] and [13]). It was later extended to the case of continuous Hahn polynomials [13, Theorem 3.13] : here the connection coefficients are the same (namely $su(1,1)$ Racah coefficients), but the orthogonal polynomials in two variables are certain products of two continuous Hahn polynomials.

In the present paper orthogonal polynomials in $n$ variables will be associated with the tensor product of $n+1$ representations of $su(1,1)$. A different class of orthogonal polynomials arises depending upon the (internal) binary coupling tree. Then, the connection coefficients relating two such orthogonal polynomials associated with a different binary coupling tree are given by $3nj$-coefficients (associated with the coupling trees). For the case of three couplings Jacobi polynomials and continuous Hahn polynomials appear. Also for the $n$-variable case, these two families are present : the orthogonal polynomials we are dealing with are either products of certain Jacobi polynomials, or products of continuous Hahn polynomials.

In general, orthogonal polynomials in several variables give rise to certain difficulties that are not present in the one variable situation. For example, orthogonal polynomials in $n$ variables are no longer uniquely defined by the area $\Omega$ and the weight function on the area. This is closely related to the fact that there is no obvious natural order for polynomials in several variables.

The space of all polynomials in the variables $x_1, \ldots, x_n$ with real coefficients is denoted $\mathbb{R}[x_1, \ldots, x_n]$ or $\Pi^n$ for short. The degree of a polynomial $P \in \Pi^n$ is the highest degree of any of its monomials. Let $\langle \cdot, \cdot \rangle$ be an inner product defined on $\Pi^n$, then $P$ is an orthogonal polynomial if $\langle P, Q \rangle = 0$ for all polynomials $Q$ with $\deg Q < \deg P$. This definition does not require that $P$ is orthogonal with other (orthogonal) polynomials of the same degree.

In our case the inner product will be defined in terms of some (classical) weight function $W$ on some (classical) area $\Omega$ in $\mathbb{R}^n$: $\langle P, Q \rangle = \int_\Omega P(x)Q(x)W(x)\,dx$.

The outline of the rest of this article is as follows. In section II we confine ourselves to the two variable case. The basic identities are convolution identities involving $su(1,1)$ Racah coefficients and Jacobi polynomials [16, 12] or continuous Hahn polynomials [13]. The new aspect here is that we show how to deduce these identities from the Biedenharn-Elliott identity [4]. More particularly, we show how the Biedenharn-Elliott identity yields a new convolution identity involving the Wilson and Racah polynomials. The products of Wilson polynomials on both sides of the identity are shown to be orthogonal in $\mathbb{R}^2$ with respect to some weight function. Using limiting relations between the Wilson, continuous Hahn and Jacobi polynomials, two other (known) convolution identities are (re)derived.

Section III generalizes the result involving Jacobi polynomials to $n$ variables. With the coupling of $n+1$ representations in $su(1,1)$ we associate a set of orthogonal polynomials on the simplex with respect to the classical weight function $x_1^{\alpha_1}\cdots x_n^{\alpha_n}(1-x_1-\cdots-x_n)^{\alpha_{n+1}}$ $(\alpha_i>-1)$. For $n=2$ these polynomials are due to Proriol [17]. For arbitrary $n$, but associated to a special coupling tree, these polynomials appear in [18]. The more general case was also considered by Rosengren [19] in the context of multilinear Hankel forms.

Section IV does the same for the result involving continuous Hahn polynomials. In this case the area is the complete space $\mathbb{R}^n$, and the weight function is in terms of Gamma functions.

In section V we prove that the sets of polynomials defined in the previous sections form bases for $\Pi^n$. Moreover, we show that the generalized recoupling coefficients of $su(1,1)$, which are essentially sums of products of Racah coefficients, are the connection coefficients between the different bases.

In the rest of this paper we will use the following abbreviations : $k$ always stands for the $(n+1)$-tuple $(k_1,\ldots,k_{n+1})\in\mathbb{R}_+^{n+1}$, and $l$ is an $n$-tuple $(l_1,\ldots,l_n)\in\mathbb{N}^n$. Moreover, we use $|k|=k_1+\cdots+k_{n+1}$ and $|l|=l_1+\cdots+l_n$. The Kronecker delta symbol $\delta_{l,l'}$ then stands for the product $\delta_{l_1,l'_1}\ldots,\delta_{l_n,l'_n}$, etc.

## II A convolution identity involving Wilson polynomials

In general recoupling theory of $su(2)$, the Biedenharn-Elliott identity is well known [4, 1, 6]. In the case of positive discrete series representations of $su(1,1)$, the identity is essentially the same. It is easily derived by considering the overlap or transformation coefficient [4, p. 456-457], [14, 15]

$$\langle(((k_1,k_2)k_{12},k_3)k_{13},k_4)k_{14}\,|\,(k_1,(k_2,(k_3,k_4)k_{34})k_{24})k_{14}\rangle$$

and computing this in two different ways : either directly, yielding a product of two recoupling coefficients, or in three steps, introducing a summation variable and thus yielding a sum of a product of three recoupling coefficients. Explicitly we get :

$$\sum_{k_{23}} U_{k_3,k_{13},k_{23}}^{k_1,k_2,k_{12}} U_{k_4,k_{14},k_{24}}^{k_1,k_{23},k_{13}} U_{k_4,k_{24},k_{34}}^{k_2,k_3,k_{23}} = U_{k_4,k_{14},k_{34}}^{k_{12},k_3,k_{13}} U_{k_{34},k_{14},k_{24}}^{k_1,k_2,k_{12}}, \tag{2.1}$$

where $k_{23}$ is restricted to some range, and $U$ stands for a Racah coefficient (or recoupling coefficient, or $6j$-coefficient) of $su(1,1)$. All $k_i$'s or $k_{ij}$'s refer to $su(1,1)$ representation labels and are positive real numbers. An explicit form for the recoupling coefficients in terms of terminating balanced ${}_4F_3(1)$ series is known [7, 12, 13] :

$$U^{k_1,k_2,k_{12}}_{k_3,k_0,k_{23}} = \binom{j+j_{12}}{j_{23}} \frac{(2k_2)_{j_{12}}(2k_3)_j(2k_1+2k_2+2k_3+j+j_{12}-1)_{j_{23}}}{(2k_3, 2k_2+2k_3+j_{23}-1)_{j_{23}}(2k_2+2k_3+2j_{23})_{j'}}$$
$$\times \sqrt{\frac{j'!(2k_1, 2k_{23}, 2k_1+2k_{23}+j'-1)_{j'}\; j_{23}!(2k_2, 2k_3, 2k_2+2k_3+j_{23}-1)_{j_{23}}}{j!(2k_{12}, 2k_3, 2k_{12}+2k_3+j-1)_j\; j_{12}!(2k_1, 2k_2, 2k_1+2k_2+j_{12}-1)_{j_{12}}}} \qquad (2.2)$$
$$\times\, {}_4F_3\left(\begin{matrix} 2k_1+2k_2+j_{12}-1, 2k_2+2k_3+j_{23}-1, -j_{12}, -j_{23} \\ 2k_2, 2k_1+2k_2+2k_3+j+j_{12}-1, -j-j_{12} \end{matrix}; 1\right),$$

with the following restrictions and definitions :

$$k_{12} = k_1+k_2+j_{12}, \qquad k_{23} = k_2+k_3+j_{23},$$
$$k_0 = k_{12}+k_3+j = k_1+k_{23}+j', \qquad j_{12}, j, j_{23}, j' \in \mathbb{N}, \text{ and } j_{12}+j = j_{23}+j'.$$

In (2.2), we follow the classical notation for Pochhammer symbols $(a)_n$ and for hypergeometric series ${}_pF_q$ [20, 21, 22].

Both the Racah and Wilson polynomials are defined in terms of terminating balanced ${}_4F_3(1)$ series. We will now perform an appropriate renaming of the nine free parameters and the summation variable in (2.1) to derive a convolution identity involving Wilson and Racah polynomials.

Wilson polynomials, denoted $W_m(x^2; a, b, c, d)$, are defined as follows :

$$W_m(x^2; a, b, c, d) =$$
$$(a+b)_m(a+c)_m(a+d)_m\; {}_4F_3\left(\begin{matrix} -m, m+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}; 1\right), \qquad (2.3)$$

while Racah polynomials, denoted $R_m(\lambda(x); \alpha, \beta, \gamma, \delta)$, are defined by :

$$R_m(\lambda(x); \alpha, \beta, \gamma, \delta) = {}_4F_3\left(\begin{matrix} -m, m+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}; 1\right), \qquad (2.4)$$

where $\lambda(x) = x(x+\gamma+\delta+1)$ and one of the denominator parameters equals $-M$ with $M \in \mathbb{N}$ and $0 \le m \le M$.

Here (and in the rest of the paper), we use the notation of [8] for Wilson and Racah polynomials (and for all other one variable orthogonal polynomials).

**Theorem 1** *The Wilson polynomials satisfy the following convolution identity :*

$$\sum_{l=0}^{m+j} \binom{j+m}{l} \frac{(2k_2)_m(2k_3)_j(2k_1+2k_2+2k_3+j+m-1)_l}{(2k_3)_l(2k_2+2k_3+l-1)_l(2k_2+2k_3+2l)_{j+m-l}}$$

$$
\begin{aligned}
&\times R_l(\lambda(m); 2k_2-1, 2k_3-1, -j-m-1, 2k_1+2k_2+j+m-1)\\
&\times W_{m+j-l}((x_1-t)^2; k_1+it, k_2+k_3+l-is+it, k_1-it, k_2+k_3+l+is-it)\\
&\times W_l((x_1+x_2-t)^2; k_2-ix_1+it, k_3-is+it, k_2+ix_1-it, k_3+is-it) \qquad (2.5)\\
= \ & W_m((x_1-t)^2; k_1+it, k_2-i(x_1+x_2)+it, k_1-it, k_2+i(x_1+x_2)-it)\\
&\times W_j((x_1+x_2-t)^2; k_1+k_2+m+it, k_3-is+it, k_1+k_2+m-it, k_3+is-it),
\end{aligned}
$$

*where* $j, m \in \mathbb{N}$, $k_1, k_2, k_3, x_1, x_2, x_3, t \in \mathbb{R}$ *and* $s = x_1+x_2+x_3$.

**Proof**: In the case of $su(1,1)$, the summation range for $k_{23}$ in (2.1) is from $k_2+k_3$ to $\min(k_{13}-k_1, k_{24}-k_4)$. The summation variable $k_{23}$ thus takes *real* values, starting with $k_2+k_3$ and increasing in steps of one. Substituting (2.2) in (2.1) yields an identity between terminating balanced ${}_4F_3(1)$ series :

$$
\begin{aligned}
&\sum_{k_{23}=k_2+k_3}^{\min(k_{13}-k_1,k_{24}-k_4)} f \times {}_4F_3\left({k_1+k_2+k_{12}-1, k_2+k_3+k_{23}-1, k_1+k_2-k_{12}, k_2+k_3-k_{23} \atop 2k_2, k_{13}+k_1+k_2+k_3-1, k_1+k_2+k_3-k_{13}}; 1\right)\\
&\quad\times {}_4F_3\left({k_1+k_{23}+k_{13}-1, k_{23}+k_4+k_{24}-1, k_1+k_{23}-k_{13}, k_{23}+k_4-k_{24} \atop k_1+k_{23}+k_4-k_{14}, 2k_{23}, k_{14}+k_1+k_{23}+k_4-1}; 1\right)\\
&\quad\times {}_4F_3\left({k_2+k_3+k_{23}-1, k_2+k_3-k_{23}, k_3+k_4+k_{34}-1, k_3+k_4-k_{34} \atop k_2+k_3+k_4-k_{24}, 2k_3, k_{24}+k_2+k_3+k_4-1}; 1\right)\\
&= {}_4F_3\left({k_3+k_4+k_{34}-1, k_{12}+k_3+k_{13}-1, k_3+k_4-k_{34}, k_{12}+k_3-k_{13} \atop k_{12}+k_3+k_4-k_{14}, 2k_3, k_{14}+k_{12}+k_3+k_4-1}; 1\right)\\
&\quad\times {}_4F_3\left({k_1+k_2+k_{12}-1, k_1+k_2-k_{12}, k_2+k_{34}+k_{24}-1, k_2+k_{34}-k_{24} \atop 2k_2, k_1+k_2+k_{34}-k_{14}, k_{14}+k_1+k_2+k_{34}-1}; 1\right),
\end{aligned}
$$

where $f$ is a numerical factor that is easily calculated from (2.2). Renaming the following positive integer differences as

$$
m = k_{12}-k_1-k_2, \ j = k_{13}-k_{12}-k_3 \text{ and } l = k_{23}-k_2-k_3,
$$

and performing appropriate Bailey transformations [23, Theorem 3.3.3] on the balanced ${}_4F_3(1)$'s, yields that (when $k_{13}-k_1 \le k_{24}-k_4$) :

$$
\begin{aligned}
&\sum_{l=0}^{j+m} f' \times {}_4F_3\left({-l, l+2k_2+2k_3-1, -m, m+2k_1+2k_2-1 \atop 2k_2, 2k_1+2k_2+2k_3+j+m-1, -j-m}; 1\right)\\
&\quad\times {}_4F_3\left({-m-j+l, m+j+l+2k_1+2k_2+2k_3-1, k_1+k_{14}+k_{24}-1, k_1+k_{14}-k_{24} \atop 2k_1, k_1+k_2+k_3+l+k_4+k_{14}-1, k_1+k_2+k_3+l+k_{14}-k_4}; 1\right)\\
&\quad\times {}_4F_3\left({-l, l+2k_2+2k_3-1, k_2+k_{24}+k_{34}-1, k_2+k_{24}-k_{34} \atop 2k_2, k_2+k_3+k_{24}-k_4, k_{24}+k_2+k_3+k_4-1}; 1\right)
\end{aligned}
$$

$$= {}_4F_3\left(\begin{matrix}-j, j+2m+2k_1+2k_2+2k_3-1, m+k_1+k_2+k_{14}-k_{34}, m+k_1+k_2+k_{14}+k_{34}-1\\ 2m+2k_1+2k_2, m+k_1+k_2+k_3+k_4+k_{14}-1, m+k_1+k_2+k_3+k_{14}-k_4\end{matrix};1\right)$$
$$\times {}_4F_3\left(\begin{matrix}-m, m+2k_1+2k_2-1, k_1+k_{14}+k_{24}-1, k_1+k_{14}-k_{24}\\ 2k_1, k_1+k_2+k_{14}-k_{34}, k_1+k_2+k_{14}+k_{34}-1\end{matrix};1\right),$$

where, once again, $f'$ is a numerical factor that is easily calculated.

This last identity can be written in terms of Wilson and Racah polynomials by putting

$$ix_1 = k_{14} - k_{24},\ ix_2 = k_{24} - k_{34},\ ix_3 = k_{34} - k_4 \text{ and } it = k_4 + i(x_1 + x_2 + x_3) - 1/2.$$

Note that all the renamings are invertible. Determination of the factor $f'$ now yields the desired result. Since (2.5) is a rational identity in the parameters $k_i$, $x_i$ and $t$, it is valid for all values of these parameters. $\square$

From Theorem 1 we can easily rederive a convolution identity for continuous Hahn polynomials. The continuous Hahn polynomials, denoted $p_m(x; a, b, c, d)$, are defined as [8]

$$p_m(x;a,b,c,d) = i^m \frac{(a+c)_m(a+d)_m}{m!}\,{}_3F_2\left(\begin{matrix}-m, m+a+b+c+d-1, a+ix\\ a+c, a+d\end{matrix};1\right); \tag{2.6}$$

for their orthogonality (when $\Re(a,b,c,d) > 0$, $\bar{c} = a$ and $\bar{d} = b$), see [8] or (4.3). Using the limit transition [8] :

$$\lim_{t\to-\infty} \frac{W_m((x-t)^2; a+it, b+it, c-it, d-it)}{(2t)^m m!} = p_m(x;a,b,c,d), \tag{2.7}$$

in (2.5), one finds the following corollary (see also [13, Theorem 3.13]).

**Corollary 2** *The continuous Hahn polynomials satisfy the following convolution identity :*

$$\begin{aligned}
&\sum_{l=0}^{m+j} \binom{j+m}{m} \frac{(2k_2)_m(2k_3)_j(2k_1+2k_2+2k_3+j+m-1)_l}{(2k_3)_l(2k_2+2k_3+l-1)_l(2k_2+2k_3+2l)_{j+m-l}}\\
&\qquad \times R_l(\lambda(m); 2k_2-1, 2k_3-1, -j-m-1, 2k_1+2k_2+j+m-1)\\
&\qquad \times p_{m+j-l}(x_1; k_1, k_2+k_3+l-is, k_1, k_2+k_3+l+is)\\
&\qquad \times p_l(x_2; k_2, k_3-i(s-x_1), k_2, k_3+i(s-x_1))\\
&= p_m(x_1; k_1, k_2-i(x_1+x_2), k_1, k_2+i(x_1+x_2))\\
&\qquad \times p_j(x_1+x_2; k_1+k_2+m, k_3-is, k_1+k_2+m, k_3+is),
\end{aligned} \tag{2.8}$$

*where $j, m \in \mathbb{N}$, $k_1$, $k_2$, $k_3$, $x_1$, $x_2$, $x_3 \in \mathbb{R}$ and $s = x_1 + x_2 + x_3$.* $\square$

The classical Jacobi polynomials are defined by :

$$P_m^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_m}{m!}\,{}_2F_1\left(\begin{matrix}-m, m+\alpha+\beta+1\\ \alpha+1\end{matrix};\frac{1-x}{2}\right); \tag{2.9}$$

for $\alpha, \beta > -1$, they are orthogonal over the interval $[-1,1]$ for the weight function $(1-x)^\alpha(1+x)^\beta$. Replacing $x_i$ by $sx_i$ in (2.8) and letting $s$ tend to infinity yields Corollary 3 (see also [13, Corollary 3.15], or [16, 12]) :

**Corollary 3** *The Jacobi polynomials satisfy the following convolution identity :*

$$\begin{aligned}
&\sum_{l=0}^{m+j}\binom{j+m}{m}\frac{(2k_2)_m(2k_3)_j(2k_1+2k_2+2k_3+j+m-1)_l}{(2k_3)_l(2k_2+2k_3+l-1)_l(2k_2+2k_3+2l)_{j+m-l}} \\
&\qquad \times R_l(\lambda(m); 2k_2-1, 2k_3-1, -j-m-1, 2k_1+2k_2+j+m-1) \\
&\qquad \times P_{m+j-l}^{(2k_1-1,2k_2+2k_3+2l-1)}(1-2x_1)(1-x_1)^l P_l^{(2k_2-1,2k_3-1)}\left(\frac{1-x_1-2x_2}{1-x_1}\right) \\
&= (x_1+x_2)^m P_m^{(2k_1-1,2k_2-1)}\left(\frac{x_2-x_1}{x_1+x_2}\right) P_j^{(2k_1+2k_2+2m-1,2k_3-1)}(1-2x_1-2x_2).
\end{aligned}$$

□

Both Corollary 2 and 3 can be written in a more symmetric (and unified) way using a different scaling for the continuous Hahn and Jacobi polynomials. Defining the polynomial $\mathcal{S}_m^{k_1,k_2}(x_1,x_2)$ as

$$\begin{aligned}
\mathcal{S}_m^{k_1,k_2}(x_1,x_2) &= \sqrt{\frac{m!(2m+2k_1+2k_2-1)\Gamma(m+2k_1+2k_2-1)}{\Gamma(m+2k_1)\Gamma(m+2k_2)}} \\
&\quad \times p_m(x_1; k_1, k_2-i(x_1+x_2), k_1, k_2+i(x_1+x_2)),
\end{aligned} \tag{2.10}$$

we have the following identity [13, Eq. (3.15)] :

$$\begin{aligned}
&\mathcal{S}_{k_{12}-k_1-k_2}^{k_1,k_2}(x_1,x_2)\mathcal{S}_{k_0-k_{12}-k_3}^{k_{12},k_3}(x_1+x_2,x_3) = \\
&\qquad \sum_{k_{23}=k_2+k_3}^{k_0-k_1} U_{k_3,k_0,k_{23}}^{k_1,k_2,k_{12}}\mathcal{S}_{k_{23}-k_2-k_3}^{k_2,k_3}(x_2,x_3)\mathcal{S}_{k_0-k_1-k_{23}}^{k_1,k_{23}}(x_1,x_2+x_3).
\end{aligned} \tag{2.11}$$

Formula (2.11) is easily remembered by considering two ways in which three $su(1,1)$ representations can be coupled, as shown in Figure 1. Notice how the left side of (2.11) follows from the tree on the left side of this figure. With each non-leaf node (i.e. with each intermediate or final coupling) one associates an $\mathcal{S}$-polynomial. The first (resp. second) variable of this $\mathcal{S}$-polynomial is the sum of all the variables associated with the leaves in the left (resp. right) subtree of the considered node. The upper parameters are determined by the value of the representation labels of the left and right child (in that order). The (positive integer) lower parameter is the difference between the value of the coupled representation label and the consisting labels. The $\mathcal{S}$-polynomials on the right side of (2.11) are formed in the same way but working with the tree on the right side of the figure. The recoupling coefficient appearing in (2.11) is that associated with a recoupling of three representations as shown in Figure 1.

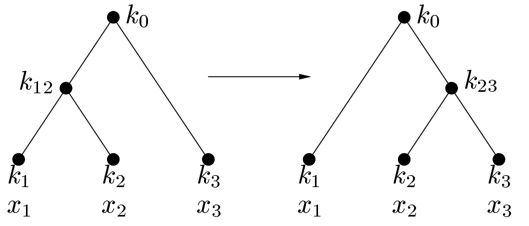


Figure 1: Two possible ways of coupling three representations

The $\mathcal{S}$-polynomials have the following property :

$$\mathcal{S}_m^{k_1,k_2}(x_1,x_2) = (-1)^m \mathcal{S}_m^{k_2,k_1}(x_2,x_1), \tag{2.12}$$

which is a direct consequence of Whipple's transformation for terminating ${}_3F_2(1)$ series [23, p. 142].

If we define [16, 12]

$$S_m^{k_1,k_2}(x_1,x_2) = (-1)^m \sqrt{\frac{m!}{(2k_1,2k_2,2k_1+2k_2+m-1)_m}}(x_1+x_2)^m P_m^{(2k_1-1,2k_2-1)}\left(\frac{x_2-x_1}{x_2+x_1}\right), \tag{2.13}$$

then relations (2.11) and (2.12) are still valid (after replacing $\mathcal{S}$ by $S$). This follows from the fact that

$$\lim_{u\to\infty} \frac{\mathcal{S}_m^{k_1,k_2}(ux_1,ux_2)}{u^m} = \sqrt{\frac{\Gamma(2m+2k_1+2k_2)}{\Gamma(2k_1)\Gamma(2k_2)}} S_m^{k_1,k_2}(x_1,x_2). \tag{2.14}$$

It is known [13] that the products of continuous Hahn polynomials in both the left and right side of (2.8) (or (2.11)) are orthogonal on $\mathbb{R}^2$ for the weight function

$$\Gamma(k_1 \pm ix_1)\Gamma(k_2 \pm ix_2)\Gamma(k_3 \pm i(s-x_1-x_2)), \tag{2.15}$$

where, for conciseness of notation, the product $\Gamma(k_j+ix)\Gamma(k_j-ix)$ is written as $\Gamma(k_j \pm ix)$. This convention is adopted throughout this paper. Moreover, we will write $\Gamma(\pm k_j \mp ix)$ instead of $\Gamma(k_j - ix)\Gamma(-k_j + ix)$, etc. Observe that (2.11), with $x_3 = s - x_1 - x_2$, can thus be interpreted as a connection coefficient formula between orthogonal polynomials in two variables, and the $su(1,1)$ Racah coefficients are the connection coefficients. Similarly, replacing $\mathcal{S}$ by $S$ in (2.11), one obtains again a connection coefficient formula. This time, the orthogonal polynomials are orthogonal on the simplex determined by $x_1, x_2 > 0$, $x_1 + x_2 < s$, and the weight function is $x_1^{2k_1-1} x_2^{2k_2-1} (s - x_1 - x_2)^{2k_3-1}$.

In this section we have shown how the Biedenharn-Elliott identity implies two connection coefficient formulas for orthogonal polynomials in two variables, one constructed with continuous Hahn polynomials, and one constructed with Jacobi polynomials. In the following sections this will be generalized to orthogonal polynomials in $n$ variables. Finally, observe that also the products of Wilson polynomials in Theorem 1 are related to orthogonal polynomials in two variables, see Appendix A.

# III Orthogonal polynomials related to Jacobi polynomials

When considering orthogonal polynomials in $n$ variables, one of the classical areas is the simplex $T_s^n$ :

$$T_s^n = \{x \in \mathbb{R}^n \mid 0 < x_j \text{ and } x_1 + \cdots + x_n < s\}. \tag{3.1}$$

Herein, $s$ denotes some positive constant, and in almost all cases $s$ is taken to be equal to 1. The classical weight function in this case is :

$$x_1^{\kappa_1 - 1/2} \cdots x_n^{\kappa_n - 1/2} (s - |x|)^{\kappa_{n+1} - 1/2}, \tag{3.2}$$

where each $\kappa_i > -1/2$. In [24, Proposition 2.3.8] an explicit orthonormal basis is given associated with the weight function (3.2) on the simplex (3.1). Such a basis is not unique. In fact, with every binary coupling tree on $n+1$ leaves, a different basis can be constructed. In this section, an $n$-variable orthonormal polynomial will be constructed out of a product of $n$ $S$-polynomials (2.13), and associated with a binary coupling of $n+1$ representations of $su(1,1)$. We will show that this polynomial is orthogonal on the simplex $T_s^n$ for the classical weight function. The outline of this proof is as follows : by a change of variables, we will transform the simplex into the cube on $\mathbb{R}^n$. This transformation will map the integral over the simplex into an integral over the cube, where one part can be interpreted as the multiple Jacobi weight function :

$$\prod_{i=1}^{n} (1 - x_i)^{a_i} (1 + x_i)^{b_i}, \tag{3.3}$$

while the other parts are the corresponding Jacobi polynomials.

**Theorem 4** *With every coupling of $(n+1)$ $su(1,1)$ representations labeled by $k_1, \ldots, k_{n+1}$, i.e. with every binary coupling tree with $n$ internal nodes, we associate a set of polynomials $R_l^{(k)}(x)$ in $n$ variables orthogonal on the simplex $T_s^n$ for the weight function*

$$w^{(k)}(x) = x_1^{2k_1-1} \cdots x_n^{2k_n-1}(s - |x|)^{2k_{n+1}-1}, \tag{3.4}$$

*where each $k_i > 0$. Explicitly the orthogonality reads :*

$$\int_{T_s^n} R_l^{(k)}(x) R_{l'}^{(k)}(x) w^{(k)}(x)\, dx = \delta_{l,l'} \frac{s^{2|k|+2|l|-1}}{\Gamma(2|k|+2|l|)} \prod_{i=1}^{n+1} \Gamma(2k_i). \tag{3.5}$$

*Note that in (3.5) we also have orthogonality between polynomials of the same degree, which is more than the definition of orthogonality requires.*

**Remark 5** In principal the notation of the polynomial should contain a reference to the binary coupling tree it corresponds to. For the moment, we can assume that the binary coupling tree is fixed, and we do not mention it in the notation of $R_l^{(k)}(x)$. When we want to emphasize the dependence of $R_l^{(k)}(x)$ on the given binary coupling tree $T$, we shall write $R_{l,T}^{(k)}(x)$. The meaning of the subscript $l$ is related to the labeling of the internal nodes, and will soon become apparent. ∎

The association of a polynomial with a binary coupling tree is an extension of the method described after Eq. (2.11). For a given binary coupling tree, the polynomial $R_l^{(k)}(x)$ consists of a product of $S$-polynomials, each of these associated with a non-leaf node of the tree. Let us first describe an example.

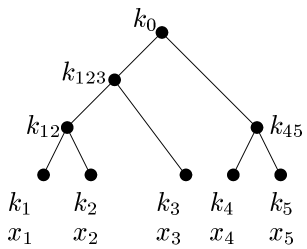


Figure 2: Example binary coupling tree

**Example 6** With the binary coupling tree shown in Figure 2 we associate the following polynomial :

$$\begin{aligned}
&R(x_1,x_2,x_3,x_4,x_5)\\
&= S^{k_1,k_2}_{k_{12}-k_1-k_2}(x_1,x_2)S^{k_{12},k_3}_{k_{123}-k_{12}-k_3}(x_1+x_2,x_3)S^{k_4,k_5}_{k_{45}-k_4-k_5}(x_4,x_5)\\
&\quad\times S^{k_{123},k_{45}}_{k_0-k_{123}-k_{45}}(x_1+x_2+x_3,x_4+x_5)\\
&= C\times(x_1+x_2)^{k_{12}-k_1-k_2}P^{(2k_1-1,2k_2-1)}_{k_{12}-k_1-k_2}\left(\frac{x_2-x_1}{x_2+x_1}\right)\\
&\quad\times(x_1+x_2+x_3)^{k_{123}-k_{12}-k_3}P^{(2k_{12}-1,2k_3-1)}_{k_{123}-k_{12}-k_3}\left(\frac{x_3-x_1-x_2}{x_3+x_1+x_2}\right)\\
&\quad\times(x_4+x_5)^{k_{45}-k_4-k_5}P^{(2k_4-1,2k_5-1)}_{k_{45}-k_4-k_5}\left(\frac{x_5-x_4}{x_5+x_4}\right)\\
&\quad\times(x_1+x_2+x_3+x_4+x_5)^{k_0-k_{123}-k_{45}}P^{(2k_{123}-1,2k_{45}-1)}_{k_0-k_{123}-k_{45}}\left(\frac{x_4+x_5-x_1-x_2-x_3}{x_4+x_5+x_1+x_2+x_3}\right),
\end{aligned}$$

herein, $C$ is some numerical factor, that can be determined from (2.13). ∎

So the $S$-polynomial, associated to a non-leaf node of the tree, has : as (upper) parameters the representation labels of left and right child of the node; as degree (the sub-index) the difference between the representation label of the node and those of the children (this is a nonnegative integer); as left (resp. right) argument the sum of all the variables associated with the leaves in the left (resp. right) subtree of the considered node.

Such a polynomial $R(x_1,\ldots,x_{n+1})$, defined as a product of $S$-polynomials in this way, is homogeneous in the variables $x_1,\ldots,x_{n+1}$. So we can choose the constraint :

$$x_1+x_2+\cdots+x_{n+1}=s, \tag{3.6}$$

where $s$ is some arbitrary, positive constant. Note that this constraint is compatible with the definition of the weight function (3.4). The resulting polynomial will be denoted by $R^{(k)}_l(x)$. The subscript $l$ in $R^{(k)}_l(x)$ stands for the sequence of degrees of the $S$-polynomials, in a chosen order (see later for this choice).

In order to prove (3.5), we shall : transform variables from $(x_1,\ldots,x_n)$ to $(v_1,\ldots,v_n)$; determine the Jacobian and integration area for this transformation; determine the transformed weight function; and finally deduce the orthogonality. All of this can be done quite explicitly for any given binary coupling tree, and will be presented in the following subsections.

### Change of variables

We will change from variables $x$ to variables $v$. The $v_i$ are the arguments of the Jacobi polynomials appearing in the product expansion of $R_l^{(k)}(x)$, see for instance Example 6.

Since each $S$-polynomial (and thus each Jacobi polynomial) in the product is associated with a specific non-leaf node of the tree, we can associate a variable $v_i$ with each non-leaf node. The exact order is irrelevant but, for simplicity, we choose *postorder* [25, Section 2.3.1]. See Figure 3 for an illustration.

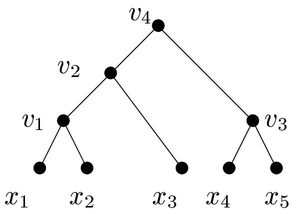


Figure 3: Association of the variables $v_i$ with non-leaf nodes

**Example 7** The arguments of the Jacobi polynomials in Example 6 are :

$$\begin{cases} v_1 = (x_2 - x_1)/(x_2 + x_1) \\ v_2 = (x_3 - x_1 - x_2)/(x_3 + x_1 + x_2) \\ v_3 = (x_5 - x_4)/(x_5 + x_4) \\ v_4 = (x_4 + x_5 - x_1 - x_2 - x_3)/(x_4 + x_5 + x_1 + x_2 + x_3). \end{cases} \tag{3.7}$$

The set of equations (3.7), together with the constraint (3.6), has a unique solution :

$$\begin{cases} x_1 = s(1 - v_1)(1 - v_2)(1 - v_4)/8 \\ x_2 = s(1 + v_1)(1 - v_2)(1 - v_4)/8 \\ x_3 = s(1 + v_2)(1 - v_4)/4 \\ x_4 = s(1 - v_3)(1 + v_4)/4 \\ x_5 = s(1 + v_3)(1 + v_4)/4. \end{cases} \tag{3.8}$$

Notice how the solution (3.8) of (3.7) can easily be deduced from the tree in Figure 3 : each $x_j$ consists of a factor $s$ and factors $(1 \pm v_i)/2$. There is a factor $(1 + v_i)/2$ if $x_j$ is in the right subtree of $v_i$, and there is a factor $(1 - v_i)/2$ if $x_j$ is in the left subtree of $v_i$. ∎

This observation can be generalized to an arbitrary tree. Then we have :

$$\begin{cases} v_i = \left( \displaystyle\sum_{\substack{\text{right leaves} \\ \text{of } v_i}} x_j - \sum_{\substack{\text{left leaves} \\ \text{of } v_i}} x_j \right) \Big/ \displaystyle\sum_{\substack{\text{all leaves} \\ \text{of } v_i}} x_j, \quad \text{for } i = 1, \ldots, n \\ x_1 + x_2 + \cdots + x_{n+1} = s. \end{cases} \tag{3.9}$$

**Lemma 8** *The system (3.9) of $n+1$ (linear) equations for the variables $x_j$ has the unique solution*

$$x_j = s \prod_{\substack{\text{right ancestors} \\ \text{of } x_j}} \frac{1+v_i}{2} \prod_{\substack{\text{left ancestors} \\ \text{of } x_j}} \frac{1-v_i}{2}, \textit{ for } j = 1, \ldots, n+1, \tag{3.10}$$

*where we call $v_i$ a right (resp. left) ancestor of $x_j$ if $x_j$ is in the right (resp. left) subtree of $v_i$.*

**Proof**: By induction on $n$. It is easily verified that this statement is true when $n = 1$. If $n > 1$ and the left subtree of $v_n$ has $0 \leq n_1 < n$ internal nodes, then the last two equations of (3.9) read

$$\begin{cases} v_n = \left( \displaystyle\sum_{\substack{\text{right leaves} \\ \text{of } v_n}} x_j - \sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j \right) / \sum_{j=1}^{n+1} x_j \\ x_1 + x_2 + \cdots + x_{n+1} = s. \end{cases} \tag{3.11}$$

These equations have the following unique solution for $\sum_{\substack{\text{right leaves} \\ \text{of } v_n}} x_j$ and $\sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j$ (compare this with the case $n = 1$) :

$$\begin{cases} \displaystyle\sum_{\substack{\text{right leaves} \\ \text{of } v_n}} x_j = s \frac{1+v_n}{2} \\ \displaystyle\sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j = s \frac{1-v_n}{2}. \end{cases} \tag{3.12}$$

Using this solution for $\sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j = \sum_{\substack{\text{leaves} \\ \text{of } v_{n_1}}} x_j$ with the first $n_1$ equations of (3.9), which involve only variables $x$ that are in the left subtree of $v_n$ yields (by induction) the desired form of the unique solution for the variables $x$ in the left subtree. The same applies to the variables $x$ in the right subtree. This proves formula (3.10). We can even say more :

$$\sum_{\substack{\text{leaves} \\ \text{of } v_i}} x_j = s \prod_{\substack{\text{right ancestors} \\ \text{of } v_i}} \frac{1+v_m}{2} \prod_{\substack{\text{left ancestors} \\ \text{of } v_i}} \frac{1-v_m}{2}, \tag{3.13}$$

and

$$\sum_{\substack{\text{left leaves} \\ \text{of } v_i}} x_j = s \frac{1-v_i}{2} \prod_{\substack{\text{right ancestors} \\ \text{of } v_i}} \frac{1+v_m}{2} \prod_{\substack{\text{left ancestors} \\ \text{of } v_i}} \frac{1-v_m}{2}. \tag{3.14}$$

□

### The Jacobian of the transformation

**Example 9** The Jacobian $\frac{\partial x_j}{\partial v_i}$ of (the first four equations of) (3.8) is the following matrix :

$$\begin{pmatrix} -s(1-v_2)(1-v_4)/8 & -s(1-v_1)(1-v_4)/8 & 0 & -s(1-v_1)(1-v_2)/8 \\ s(1-v_2)(1-v_4)/8 & -s(1+v_1)(1-v_4)/8 & 0 & -s(1+v_1)(1-v_2)/8 \\ 0 & s(1-v_4)/4 & 0 & -s(1+v_2)/4 \\ 0 & 0 & -s(1+v_4)/4 & s(1-v_3)/4 \end{pmatrix}. \tag{3.15}$$

At first sight, this matrix is quite arbitrary. However, by taking linear combinations of rows and by swapping rows, it can be transformed into a simple upper triangular matrix. From (3.8) (or directly from (3.14)) we see that

$$\begin{cases} \sum\limits_{\substack{\text{left leaves} \\ \text{of } v_1}} x_j = x_1 = s(1-v_1)(1-v_2)(1-v_4)/8 \\ \sum\limits_{\substack{\text{left leaves} \\ \text{of } v_2}} x_j = x_1 + x_2 = s(1-v_2)(1-v_4)/4 \\ \sum\limits_{\substack{\text{left leaves} \\ \text{of } v_3}} x_j = x_4 = s(1-v_3)(1+v_4)/4 \\ \sum\limits_{\substack{\text{left leaves} \\ \text{of } v_4}} x_j = x_1 + x_2 + x_3 = s(1-v_4)/2. \end{cases}$$

So, performing the row combinations $R_2 \leftarrow R_2 + R_1$, $R_3 \leftarrow R_3 + R_2$ and swapping the rows $R_3$ and $R_4$ of the Jacobian results in the following upper triangular matrix :

$$\begin{pmatrix} -s(1-v_2)(1-v_4)/8 & -s(1-v_1)(1-v_4)/8 & 0 & -s(1-v_1)(1-v_2)/8 \\ 0 & -s(1-v_4)/4 & 0 & -s(1-v_2)/4 \\ 0 & 0 & -s(1+v_4)/4 & s(1-v_3)/4 \\ 0 & 0 & 0 & -s/2 \end{pmatrix}. \tag{3.16}$$

The determinant of the Jacobian is thus

$$-s^4(1-v_2)(1-v_4)^2(1+v_4)/256. \tag{3.17}$$

Notice that the determinant only contains factors $s/2$ and $(1 \pm v_i)/2$. The power of each of these factors can easily be read from the tree in Figure 3. There are 4 factors $s/2$, and the tree has 4 internal nodes. There are 2 factors $(1-v_4)/2$ and there are 3 leaves in the left subtree of $v_4$. There is 1 factor $(1+v_4)/2$ and there are 2 leaves in the right subtree of $v_4$. The same applies to $v_1$, $v_2$ and $v_3$. ∎

Again, these observations can be generalized to an arbitrary tree.

**Lemma 10** *The absolute value of the determinant of the Jacobian, denoted $J$, of the transformation (3.10) (with $j = 1, \dots, n$) equals*

$$J = \left| \det \frac{\partial x_j}{\partial v_i} \right| = \left(\frac{s}{2}\right)^n \prod_{i=1}^{n} \left(\frac{1+v_i}{2}\right)^{nr_i - 1} \left(\frac{1-v_i}{2}\right)^{nl_i - 1}, \tag{3.18}$$

*where $nr_i$ (resp. $nl_i$) is the number of leaves in the right (resp. left) subtree of $v_i$.*

**Proof**: We will prove (3.18) by transforming the matrix $\frac{\partial x_j}{\partial v_i}$ into an upper triangular matrix by taking linear combinations of rows and by swapping rows. These manipulations do not change (up to a sign factor) the determinant of this matrix.

From (3.14) and the fact that we choose postorder, it is easily seen that $\sum_{\substack{\text{left leaves} \\ \text{of } v_i}} x_j$ depends on $v_i$ but not on $v_m$ when $m < i$. Therefore, we wish to perform row operations such that the $m$-th row becomes

$$\frac{\partial \left( \sum_{\substack{\text{left leaves} \\ \text{of } v_m}} x_j \right)}{\partial v_i}. \tag{3.19}$$

We use induction on $n$ to show that is possible. By the induction hypothesis, we can create the desired linear combinations in the left and right subtree. Note that the linear combinations in the right subtree do not depend on the variables $v_1, \dots, v_{nl_n - 1}$ (i.e. the variables $v$ in the left subtree). In this process the row corresponding to the variable associated with the rightmost leaf of the left subtree is not used. For clarity, assume that this variable is $x_{nl_n}$. The Jacobian now has the following form :

$$\begin{array}{c} \\ \\ \\ \\ \\ x_{nl_n} \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} v_1 & v_2 & v_3 & \cdots & v_{nl_n-1} & v_{nl_n} & v_{nl_n+1} & \cdots & v_{n-1} & v_n \end{array} \\
\begin{pmatrix}
* & \star & \star & \cdots & * & 0 & 0 & \cdots & 0 & * \\
0 & * & \star & \cdots & * & 0 & 0 & \cdots & 0 & * \\
\vdots & & & & \vdots & \vdots & \vdots & & \vdots & \\
0 & 0 & 0 & \cdots & * & 0 & 0 & \cdots & 0 & * \\
\star & \star & \star & \cdots & * & 0 & 0 & \cdots & 0 & * \\
0 & 0 & 0 & \cdots & 0 & * & \star & \cdots & * & * \\
0 & 0 & 0 & \cdots & 0 & 0 & * & \cdots & * & * \\
\vdots & \vdots & \vdots & & \vdots & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & * & *
\end{pmatrix}
\end{array} \tag{3.20}$$

Herein, $*$ represents a non-zero value and $\star$ an arbitrary value.

We will now use the row $\partial x_{nl_n}/\partial v_i$ to create the sum of the variables corresponding to the leaves in the left subtree of the root, i.e. the row

$$\frac{\partial\left(\sum\limits_{\substack{\text{left leaves}\\ \text{of } v_n}} x_j\right)}{\partial v_i} = \frac{\partial\left(\sum\limits_{\substack{\text{left leaves}\\ \text{of } v_{nl_n-1}}} x_j\right)}{\partial v_i} + \frac{\partial\left(\sum\limits_{\substack{\text{right leaves}\\ \text{of } v_{nl_n-1}}} x_j\right)}{\partial v_i}. \tag{3.21}$$

By induction, the first term in the rhs is already a row of the matrix. If $x_{nl_n}$ is the only leaf in the right subtree of $v_{nl_n-1}$, then we are finished : simply add the row corresponding to the first term to the row corresponding to $x_{nl_n}$. If there is more than one leaf in the right subtree of $v_{nl_n-1}$, we call the root of this right subtree $v_m$ and we have :

$$\frac{\partial\left(\sum\limits_{\substack{\text{right leaves}\\ \text{of } v_{nl_n-1}}} x_j\right)}{\partial v_i} = \frac{\partial\left(\sum\limits_{\substack{\text{left leaves}\\ \text{of } v_m}} x_j\right)}{\partial v_i} + \frac{\partial\left(\sum\limits_{\substack{\text{right leaves}\\ \text{of } v_m}} x_j\right)}{\partial v_i}. \tag{3.22}$$

The first term of the rhs of this equation is a row of the matrix (by induction). If $x_{nl_n}$ is the only right leaf of $v_m$, we are finished; add the two rows corresponding to the first terms in (3.21) and (3.22) to the row corresponding to $x_{nl_n}$. It is clear that we can continue in this way to create the desired row.

By swapping the rows in this matrix, we have created an upper triangular matrix with the same determinant (up to a sign) as the original matrix.

From (3.14) it is clear that the elements of this upper triangular matrix consist only of factors $s/2$, $(1 \pm v_i)/2$ (and $-1$). Each element on the diagonal of the upper triangular matrix has a single factor $-s/2$. There is a single factor $(1+v_i)/2$ for each variable $v_m$ that is in the right subtree of $v_i$. There are $nr_i - 1$ (i.e. the number of internal nodes in the right subtree of $v_i$) such variables, so there are $nr_i - 1$ entries on the diagonal that have a factor $(1+v_i)/2$. In the same way one sees that there are $nl_i - 1$ entries on the diagonal that have a factor $(1-v_i)/2$. Multiplying the elements on the diagonal yields formula (3.18). $\square$

### Transformation of the area and weight function

The cube $\square^n$ in $\mathbb{R}^n$ is defined as :

$$\square^n = \{v \in \mathbb{R}^n \,|\, -1 < v_j < 1\}. \tag{3.23}$$

**Lemma 11** *When $x$ and $v$ are connected through (3.9) and (3.10), then $x \in T_s^n \iff v \in \Box^n$.*

Note that, although the simplex $T_n^s$ depends on $s$, the cube $\Box^n$ does not. An explicit proof of this lemma is given in Appendix B.

Next, we consider the transformation of the weight function (3.4), first for an example, and then for an arbitrary binary coupling tree.

**Example 12** For the tree of Figure 2 the weight function is

$$w^{(k)}(x_1,x_2,x_3,x_4) = x_1^{2k_1-1}x_2^{2k_2-1}x_3^{2k_3-1}x_4^{2k_4-1}(s-x_1-x_2-x_3-x_4)^{2k_5-1}. \tag{3.24}$$

Rewriting this weight function in terms of the variables $v$, using (3.8), gives :

$$\begin{aligned}\tilde{w}^{(k)}(v_1,v_2,v_3,v_4) &= s^{2k_1+2k_2+2k_3+2k_4+2k_5-5} \\ &\times \left(\frac{1+v_1}{2}\right)^{2k_2-1}\left(\frac{1+v_2}{2}\right)^{2k_3-1}\left(\frac{1+v_3}{2}\right)^{2k_5-1}\left(\frac{1+v_4}{2}\right)^{2k_4+2k_5-2} \\ &\times \left(\frac{1-v_1}{2}\right)^{2k_1-1}\left(\frac{1-v_2}{2}\right)^{2k_1+2k_2-2}\left(\frac{1-v_3}{2}\right)^{2k_4-1}\left(\frac{1-v_4}{2}\right)^{2k_1+2k_2+2k_3-3}.\end{aligned}$$

∎

**Lemma 13** *The transformation of the weight function (3.4) under the substitution (3.10) becomes :*

$$\tilde{w}^{(k)}(v) = s^{2|k|-n-1}\prod_{i=1}^{n}\left(\frac{1+v_i}{2}\right)^{\sum_{\substack{\text{right leaves}\\ \text{of } v_i}} 2k_j - nr_i}\left(\frac{1-v_i}{2}\right)^{\sum_{\substack{\text{left leaves}\\ \text{of } v_i}} 2k_j - nl_i}. \tag{3.25}$$

**Proof**: Since the weight function is essentially equal to $\prod_{j=1}^{n+1} x_j^{2k_j-1}$, it becomes a product of factors $s$, and $(1\pm v_i)/2$. Each $x_j$ has a single factor $s$; the power of $s$ is thus $\sum_{j=1}^{n+1}(2k_j-1) = 2|k|-n-1$. Furthermore, each $x_j$ which is in the right (resp. left) subtree of $v_i$ has a factor $(1+v_i)/2$ (resp. $(1-v_i)/2$). □

## Verifying the orthogonality

If two $su(1,1)$ representations $k_1$ and $k_2$ are coupled to $k_{12}$, then the difference $k_{12}-k_1-k_2$ is a nonnegative integer, see (1.3). So, with each internal node, say $v_i$, of a binary coupling tree we can associate a nonnegative integer, say $l_i$. Furthermore, we can associate a real positive value with each node of the tree; for the leaves this value is simply $k_j$, and for an internal node $v_i$ the value equals the value of the left child plus the value of the right child plus $l_i$, or explicitly :

$$\sum_{\substack{\text{leaves} \\ \text{of } v_i}} k_j + \sum_{\substack{\text{nodes in} \\ \text{subtree of } v_i}} l_j + l_i. \tag{3.26}$$

**Example 14** In Figure 4 the value of each node is indicated in the tree. ▮

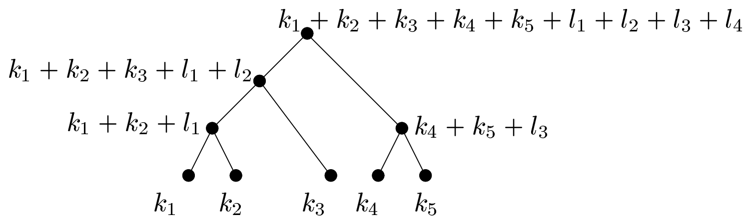


Figure 4: Value of the nodes in the tree

For a given binary coupling tree, the knowledge of $k = (k_1, k_2, \ldots, k_{n+1})$ and $l = (l_1, \ldots, l_n)$ completely determines the polynomial $R_l^{(k)}(x)$. Now we are in a position to prove Theorem 4.

As already mentioned, we will transform the integral over $T_s^n$ into an integral over $\square^n$ by changing the variables from $x$ to $v$. We will get a product of $n$ integrals over the interval $(-1, 1)$, each involving two Jacobi polynomials and a factor acting as weight function. Using the orthogonality of the Jacobi polynomials over this interval with respect to this weight function will yield the desired result. Explicitly, the orthogonality of the Jacobi polynomials reads

$$\int_{-1}^{1} P_m^{(a,b)}(t) P_n^{(a,b)}(t)(1-t)^a(1+t)^b \, dt = \delta_{m,n} h_m^{(a,b)}, \qquad a, b > -1, \tag{3.27}$$

with

$$h_m^{(a,b)} = \frac{2^{a+b+1}\Gamma(m+a+1)\Gamma(m+b+1)}{(2m+a+b+1)m!\Gamma(m+a+b+1)}.$$

After transformation, using (3.13), the polynomial $R_l^{(k)}(x)$ becomes :

$$\tilde{R}_l^{(k)}(v) = C \prod_{i=1}^{n} \left( s \prod_{\substack{\text{left ancestors} \\ \text{of } v_i}} \frac{1 - v_m}{2} \prod_{\substack{\text{right ancestors} \\ \text{of } v_i}} \frac{1 + v_m}{2} \right)^{l_i} P_{l_i}^{(a_i, b_i)}(v_i), \tag{3.28}$$

with,

$$C = (-1)^{|l|} \prod_{i=1}^{n} \sqrt{\frac{l_i!}{(a_i + 1, b_i + 1, a_i + b_i + l_i + 1)_{l_i}}}. \tag{3.29}$$

Herein, $a_i$ equals 2 times the value of the node corresponding to the left child of $v_i$ minus one, or explicitly

$$a_i = \sum_{\substack{\text{left leaves}\\ \text{of } v_i}} 2k_j + \sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} 2l_j - 1. \tag{3.30}$$

Similarly, one finds

$$b_i = \sum_{\substack{\text{right leaves}\\ \text{of } v_i}} 2k_j + \sum_{\substack{\text{nodes in right}\\ \text{subtree of } v_i}} 2l_j - 1. \tag{3.31}$$

Now we turn our attention to the transformed integrand of (3.5). Apart from the products

$$P_{l_i}^{(a_i,b_i)}(v_i)P_{l'_i}^{(a'_i,b'_i)}(v_i), \tag{3.32}$$

it consists of factors $s$ and $(1 \pm v_i)/2$. Let us determine the power of $(1 - v_i)/2$. There are three contributing parts :

- the determinant of the Jacobian, yielding a term $nl_i - 1$, see (3.18)
- the transformed weight function, yielding a term $\sum_{\substack{\text{left leaves}\\ \text{of } v_i}} 2k_j - nl_i$, see (3.25)
- the transformed polynomials, yielding a term $\sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} l_j + \sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} l'_j$, see (3.28).

It is thus clear that the power $p_i^-$ of $(1 - v_i)/2$ is

$$p_i^- = \sum_{\substack{\text{left leaves}\\ \text{of } v_i}} 2k_j + \sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} l_j + \sum_{\substack{\text{nodes in left}\\ \text{subtree of } v_i}} l'_j - 1. \tag{3.33}$$

In the same way one finds that the power $p_i^+$ of $(1 + v_i)/2$ is

$$p_i^+ = \sum_{\substack{\text{right leaves}\\ \text{of } v_i}} 2k_j + \sum_{\substack{\text{nodes in right}\\ \text{subtree of } v_i}} l_j + \sum_{\substack{\text{nodes in right}\\ \text{subtree of } v_i}} l'_j - 1. \tag{3.34}$$

**Example 15** In our example the transformed integral, denoted $I$ is, up to a constant factor, equal to the following product of four integrals :

$$\int_{-1}^{1} P_{l_1}^{(2k_1-1,2k_2-1)}(v_1)P_{l'_1}^{(2k_1-1,2k_2-1)}(v_1)\left(\frac{1-v_1}{2}\right)^{2k_1-1}\left(\frac{1+v_1}{2}\right)^{2k_2-1} dv_1$$
$$\times \int_{-1}^{1} P_{l_2}^{(2k_1+2k_2+2l_1-1,2k_3-1)}(v_2)P_{l'_2}^{(2k_1+2k_2+2l'_1-1,2k_3-1)}(v_2)$$

$$
\begin{aligned}
&\left(\frac{1-v_2}{2}\right)^{2k_1+2k_2+l_1+l'_1-1}\left(\frac{1+v_2}{2}\right)^{2k_3-1} dv_2\\
\times &\int_{-1}^{1} P_{l_3}^{(2k_4-1,2k_5-1)}(v_3)P_{l'_3}^{(2k_4-1,2k_5-1)}(v_3)\left(\frac{1-v_3}{2}\right)^{2k_4-1}\left(\frac{1+v_3}{2}\right)^{2k_5-1} dv_3\\
\times &\int_{-1}^{1} P_{l_4}^{(2k_1+2k_2+2k_3+2l_1+2l_2-1,2k_4+2k_5+2l_3-1)}(v_4)P_{l'_4}^{(2k_1+2k_2+2k_3+2l'_1+2l'_2-1,2k_4+2k_5+2l'_3-1)}(v_4)\\
&\left(\frac{1-v_4}{2}\right)^{2k_1+2k_2+2k_3+l_1+l'_1+l_2+l'_2-1}\left(\frac{1+v_4}{2}\right)^{2k_4+2k_5+l_3+l'_3-1} dv_4.
\end{aligned}
$$

Now, using the orthogonality (3.27) of the Jacobi polynomials, the first of these integrals is zero, except when $l_1 = l'_1$. Assuming that $l_1 = l'_1$, we see that the second integral is zero, except when $l_2 = l'_2$. The third integral immediately implies that $I$ is zero if $l_3 \neq l'_3$. Assuming that $l_i = l'_i$, for $i = 1, \ldots, 3$, we see that $I$ is zero, except when $l_4 = l'_4$. We thus see that $I = \delta_{l,l'} h_l^{(k)}$, where $h$ is some numerical constant. ∎

In general, the transformation yields :

$$
\begin{aligned}
I &= \int_{T_s^n} R_l^{(k)}(x)R_{l'}^{(k)}(x)w^{(k)}(x)\,dx\\
&= D\int_{\square}\prod_{i=1}^{n} P_{l_i}^{(a_i,b_i)}(v_i)P_{l'_i}^{(a'_i,b'_i)}(v_i)\left(\frac{1-v_i}{2}\right)^{p_i^-}\left(\frac{1+v_i}{2}\right)^{p_i^+} dv\\
&= D\prod_{i=1}^{n}\int_{-1}^{1} P_{l_i}^{(a_i,b_i)}(v_i)P_{l'_i}^{(a'_i,b'_i)}(v_i)\left(\frac{1-v_i}{2}\right)^{p_i^-}\left(\frac{1+v_i}{2}\right)^{p_i^+} dv_i.
\end{aligned}
\tag{3.35}
$$

Herein, the constant $D$ equals :

$$
D = Cs^{|l|}\times C's^{|l'|}\times s^{2|k|-n-1}\times\left(\frac{s}{2}\right)^n = \frac{CC's^{|l|+|l'|+2|k|-1}}{2^n}. \tag{3.36}
$$

The first two factors come from (3.28), the third factor comes from the transformed weight function (3.25), and the last factor originates from the Jacobian (3.18) of the transformation.

Notice that $p_i^-$ equals $a_i$ when $l_j = l'_j$ for the nodes in the left subtree of $v_i$. An analogous argument applies to $p_i^+$ and $b_i$. Suppose one computes the product of integrals (3.35) in the indicated order. At the moment that one is dealing with the integral involving $v_i$, the situation is so that $I$ is zero if there is a $j < i$ so that $l_j \neq l'_j$. Therefore, we can at that moment assume that $a_i = p_i^-$ and $b_i = p_i^+$, and apply orthogonality (3.27) of the Jacobi polynomials, implying that $l_i$ should equal $l'_i$ in order to have a non-zero value of the integral $I$.

From (3.35) and (3.27) we have that

$$\begin{aligned}
&\int_{T_s^n} R_l^{(k)}(x) R_{l'}^{(k)}(x) w^{(k)}(x)\,dx \\
&= D \prod_{i=1}^{n} \int_{-1}^{1} P_{l_i}^{(a_i,b_i)}(v_i) P_{l'_i}^{(a'_i,b'_i)}(v_i) \left(\frac{1-v_i}{2}\right)^{p_i^-} \left(\frac{1+v_i}{2}\right)^{p_i^+} dv_i \\
&= \delta_{l,l'} D \prod_{i=1}^{n} \frac{2\Gamma(l_i+a_i+1)\Gamma(l_i+b_i+1)}{(2l_i+a_i+b_i+1) l_i! \Gamma(l_i+a_i+b_i+1)} \\
&= \delta_{l,l'} s^{2|l|+2|k|-1} \prod_{i=1}^{n} \frac{\Gamma(l_i+a_i+1)}{(a_i+1)_{l_i}} \frac{\Gamma(l_i+b_i+1)}{(b_i+1)_{l_i}} \\
&\quad \times \frac{1}{(2l_i+a_i+b_i+1)\Gamma(l_i+a_i+b_i+1)(l_i+a_i+b_i+1)_{l_i}} \\
&= \delta_{l,l'} s^{2|l|+2|k|-1} \prod_{i=1}^{n} \frac{\Gamma(a_i+1)\Gamma(b_i+1)}{\Gamma(2l_i+a_i+b_i+2)} \\
&= \delta_{l,l'} \frac{s^{2|l|+2|k|-1}}{\Gamma(2|k|+2|l|)} \prod_{i=1}^{n+1} \Gamma(2k_i). \qquad (3.37)
\end{aligned}$$

The last equation follows from the previous one by induction on $n$, using the result on the left and right subtree.

We thus have proved that with every binary coupling on $n$ internal nodes there is an associated polynomial $R_l^{(k)}(x)$ in $n$ variables that is orthogonal on the simplex $T_s^n$ for the weight function $w^{(k)}(x)$.

As final comment, for $n = 2$ the polynomials constructed here are due to Proriol [17]. The polynomials $R_l^{(k)}(x)$ associated with the special binary coupling tree in the shape of a *spine* [15] were already constructed in [18]. The polynomials $R_l^{(k)}(x)$ associated with a general binary tree were also studied by Rosengren [19] in the context of multilinear Hankel forms. In his approach, orthogonality of these polynomials follows quite easily. His methods, however, are less accessible for a mathematical physicist. Furthermore, the relation to classical $3nj$-coefficients (see Section V) is more explicit here.

## IV Orthogonal polynomials related to continuous Hahn polynomials

The notation in this section is as before, i.e. $k$ stands for $(k_1, \ldots, k_{n+1})$ and $l$ for $(l_1, \ldots, l_n)$. But we will also need a notation for parts of the components, so $\mathbf{k}_m = (k_1, \ldots, k_m)$, $\mathbf{k}^m =$

$(k_m, \ldots, k_{n+1})$ (and similarly for $\mathbf{l}_m$ and $\mathbf{l}^m$, and $\mathbf{l}_m^j = (l_m, \ldots, l_j)$ $(m \leq j)$). As usual, $|\mathbf{l}_m^j| = l_m + \cdots + l_j$.

In the way that a binary coupling tree defines $R_l^{(k)}(x)$ as a product of $S$-polynomials (Jacobi polynomials), we define new polynomials $\mathcal{R}_l^{(k)}(x)$ as the same product of $\mathcal{S}$-polynomials (continuous Hahn polynomials). In this section, we will prove the following theorem.

**Theorem 16** *With every coupling of $(n+1)$ $su(1,1)$ representations $k_1, \ldots, k_{n+1}$, i.e. with every binary coupling tree with $n$ internal nodes, we associate a set of polynomials $\mathcal{R}_l^{(k)}(x)$ in $n$ variables. This set is orthogonal on $\mathbb{R}^n$ for the weight function*

$$w^{(k)}(x) = \Gamma(k_1 \pm ix_1) \cdots \Gamma(k_n \pm ix_n)\Gamma(k_{n+1} \pm i(s - |x|)). \tag{4.1}$$

*Explicitly, the orthogonality reads :*

$$\int_{\mathbb{R}^n} \mathcal{R}_l^{(k)}(x)\mathcal{R}_{l'}^{(k)}(x)w^{(k)}(x)\,dx = \delta_{l,l'}(2\pi)^n\Gamma(|k| + |l| \pm is). \tag{4.2}$$

Recall that the $\mathcal{S}$-polynomials are defined by (2.10). The $\mathcal{S}$-polynomial in (2.10) is of degree $m$ in the variables $x_1$ and $x_2$, but it is not homogeneous in these variables, see the proof of Lemma 19. This implies that the product $\mathcal{R}_l^{(k)}(x)$ is *not* homogeneous in the variables $x_i$, however we still put $x_1 + \cdots + x_{n+1} = s$. Note that this is consistent with the definition of the weight function (4.1).

**Example 17** With the binary coupling tree shown in Figure 2 we associate the following polynomial (before the replacement of $x_5$ by $s - x_1 - x_2 - x_3 - x_4$) :

$$\begin{aligned}
&\mathcal{R}(x_1, x_2, x_3, x_4, x_5) \\
&= \mathcal{S}_{l_1}^{k_1,k_2}(x_1, x_2)\mathcal{S}_{l_2}^{k_{12},k_3}(x_1 + x_2, x_3)\mathcal{S}_{l_3}^{k_4,k_5}(x_4, x_5)\mathcal{S}_{l_4}^{k_{123},k_{45}}(x_1 + x_2 + x_3, x_4 + x_5) \\
&= C \times p_{l_1}(x_1; k_1, k_2 - i(x_1 + x_2), k_1, k_2 + i(x_1 + x_2)) \\
&\quad \times p_{l_2}(x_1 + x_2; k_1 + k_2 + l_1, k_3 - i(x_1 + x_2 + x_3), k_1 + k_2 + l_1, k_3 + i(x_1 + x_2 + x_3)) \\
&\quad \times p_{l_3}(x_4; k_4, k_5 - i(x_4 + x_5), k_4, k_5 + i(x_4 + x_5)) \\
&\quad \times p_{l_4}(x_1 + x_2 + x_3; k_1 + k_2 + k_3 + l_1 + l_2, k_4 + k_5 + l_3 - i(x_1 + x_2 + x_3 + x_4 + x_5), \\
&\qquad\qquad k_1 + k_2 + k_3 + l_1 + l_2, k_4 + k_5 + l_3 + i(x_1 + x_2 + x_3 + x_4 + x_5)),
\end{aligned}$$

where $C$ is some numerical factor that can be computed from (2.10). The subscripts $l_i$ are the same as before. ∎

It is well known that the continuous Hahn polynomials, appearing in the definition of the $\mathcal{S}$-polynomials, are orthogonal on $\mathbb{R}$ for the weight function $\Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)$ when $\Re(a,b,c,d)>0$, $c=\overline{a}$ and $d=\overline{b}$ :

$$\int_{\mathbb{R}} \Gamma(a+it)\Gamma(b+it)\Gamma(c-it)\Gamma(d-it)p_m(t;a,b,c,d)p_j(t;a,b,c,d)\,dt =$$
$$2\pi\delta_{m,j}\frac{\Gamma(j+a+c)\Gamma(j+b+d)\Gamma(j+a+d)\Gamma(j+b+c)}{j!(2j+a+b+c+d-1)\Gamma(j+a+b+c+d-1)}. \tag{4.3}$$

Using this orthogonality relation, (4.2) is easily established in the case $n=1$.

To investigate the general case, we introduce new variables $u$, but use a different approach than the one in the previous section. We do not define the variables $u$ as the arguments of the continuous Hahn polynomials in the product, but we introduce essentially only one new variable, namely $x_1+\cdots+x_{nl_n}=y$ (and thus $x_{nl_n+1}+\cdots+x_{n+1}=s-y$). Here, we assumed for simplicity that the variables associated with the leaves in the left subtree are $x_1,\ldots,x_{nl_n}$. This will enable us to use induction on $n$. Again, it is constructive to consider first an example.

**Example 18** For the tree in Figure 2, we define the variables $u_i$ as follows :

$$\begin{cases} u_1 = x_1 \\ u_2 = x_2 \\ u_3 = x_1+x_2+x_3 \\ u_4 = x_4. \end{cases}$$

We thus leave all the variables $x$ unchanged, except the rightmost variable of the left subtree (in this case thus $x_3$). We choose the sum of all the variables in the left subtree equal to a new variable (in this case $u_3$). Integrating last to this variable allows this variable to be viewed as a constant in the integrands of the other integrals, enabling us to use induction.

When integrating last to $u_3$, the integral becomes :

$$\int_{\mathbb{R}^4} \mathcal{R}_l^{(k)}(x)\mathcal{R}_{l'}^{(k)}(x)w^{(k)}(x)\,dx =$$
$$\sqrt{\frac{l_4!(2|l|+2|k|-1)\Gamma(l_4+2|k|+2|\mathbf{l}_3|-1)}{\Gamma(l_4+2|\mathbf{k}_3|+2|\mathbf{l}_2|)\Gamma(l_4+2|\mathbf{k}^4|+2l_3)}}\sqrt{\frac{l_4'!(2|l'|+2|k|-1)\Gamma(l_4'|+2|k|+2|\mathbf{l}_3'|-1)}{\Gamma(l_4'+2|\mathbf{k}_3|+2|\mathbf{l}_2'|)\Gamma(l_4'+2|\mathbf{k}^4|+2l_3')}}$$
$$\times\int_{\mathbb{R}} p_{l_4}(u_3;|\mathbf{k}_3|+|\mathbf{l}_2|,|\mathbf{k}^4|+l_3-is,|\mathbf{k}_3|+|\mathbf{l}_2|,|\mathbf{k}^4|+l_3+is)$$
$$\times\, p_{l_4'}(u_3;|\mathbf{k}_3|+|\mathbf{l}_2'|,|\mathbf{k}^4|+l_3'-is,|\mathbf{k}_3|+|\mathbf{l}_2'|,|\mathbf{k}^4|+l_3'+is)\ I_{\text{left}}\ I_{\text{right}}\,du_3,$$

with

$$
\begin{aligned}
I_{\text{left}} = &\sqrt{\frac{l_1!(2l_1+2|\mathbf{k}_2|-1)\Gamma(l_1+2|\mathbf{k}_2|-1)}{\Gamma(l_1+2k_1)\Gamma(l_1+2k_2)}}\sqrt{\frac{l_1'!(2l_1'+2|\mathbf{k}_2|-1)\Gamma(l_1'+2|\mathbf{k}_2|-1)}{\Gamma(l_1'+2k_1)\Gamma(l_1'+2k_2)}} \\
&\times \sqrt{\frac{l_2!(2|\mathbf{l}_2|+2|\mathbf{k}_3|-1)\Gamma(l_2+2|\mathbf{k}_3|+2l_1-1)}{\Gamma(l_2+2|\mathbf{k}_2|+2l_1)\Gamma(l_2+2k_3)}}\sqrt{\frac{l_2'!(2|\mathbf{l}_2'|+2|\mathbf{k}_3|-1)\Gamma(l_2'+2|\mathbf{k}_3|+2l_1'-1)}{\Gamma(l_2'+2|\mathbf{k}_2|+2l_1')\Gamma(l_2'+2k_3)}} \\
&\times \int_{\mathbb{R}^2} p_{l_1}(u_1;k_1,k_2-i|\mathbf{u}_2|,k_1,k_2+i|\mathbf{u}_2|)p_{l_1'}(u_1;k_1,k_2-i|\mathbf{u}_2|,k_1,k_2+i|\mathbf{u}_2|) \\
&\quad \times p_{l_2}(|\mathbf{u}_2|;|\mathbf{k}_2|+l_1,k_3-iu_3,|\mathbf{k}_2|+l_1,k_3+iu_3)p_{l_2'}(|\mathbf{u}_2|;|\mathbf{k}_2|+l_1',k_3-iu_3,|\mathbf{k}_2|+l_1',k_3+iu_3) \\
&\quad \times \Gamma(k_1\pm iu_1)\Gamma(k_2\pm iu_2)\Gamma(k_3\pm i(u_3-|\mathbf{u}_2|))\,du_1du_2.
\end{aligned}
$$

The integral $I_{\text{left}}$ thus corresponds to the orthogonality relation (4.2) restricted to the left subtree of the tree in Figure 2 and with $s$ replaced by $u_3$. By induction, the value of this integral is :

$$
I_{\text{left}} = \delta_{l_1,l_1'}\delta_{l_2,l_2'}(2\pi)^2\Gamma(|\mathbf{k}_3|+|\mathbf{l}_2|\pm iu_3).
$$

The integral $I_{\text{right}}$ equals

$$
\begin{aligned}
I_{\text{right}} = &\sqrt{\frac{l_3!(2l_3+2|\mathbf{k}^4|-1)\Gamma(l_3+2|\mathbf{k}^4|-1)}{\Gamma(l_3+2k_4)\Gamma(l_3+2k_5)}}\sqrt{\frac{l_3'!(2l_3'+2|\mathbf{k}^4|-1)\Gamma(l_3'+|\mathbf{k}^4|-1)}{\Gamma(l_3'+2k_4)\Gamma(l_3'+2k_5)}} \\
&\times \int_{\mathbb{R}} p_{l_3}(u_4;k_4,k_5-i(s-u_3),k_4,k_5+i(s-u_3)) \\
&\quad \times p_{l_3'}(u_4;k_4,k_5-i(s-u_3),k_4,k_5+i(s-u_3)) \\
&\quad \times \Gamma(k_4\pm iu_4)\Gamma(k_5\pm i(s-u_3-u_4))\,du_4.
\end{aligned} \tag{4.4}
$$

The integral $I_{\text{right}}$ corresponds to the orthogonality relation (4.2) restricted to the right subtree with $s$ replaced by $s-u_3$. The value of this integral is thus :

$$
I_{\text{right}} = \delta_{l_3,l_3'}2\pi\Gamma(|\mathbf{k}^4|+l_3\pm i(s-u_3)).
$$

Using the values of $I_{\text{left}}$ and $I_{\text{right}}$ enables us to use the orthogonality of the continuous Hahn polynomials when integrating over $u_3$, yielding the following :

$$
\int_{\mathbb{R}^4}\mathcal{R}_l^{(k)}(x)\mathcal{R}_{l'}^{(k)}(x)w^{(k)}(x)\,dx = \delta_{l,l'}(2\pi)^4\Gamma(|k|+|l|\pm is).
$$

∎

In the general case, assume that the variables associated with the leaves of the tree are (from left to right) $x_1,\dots,x_{n+1}$. Define the variables $u$ as follows :

$$
u_j = x_j,\ j\in\{1,\dots,n\}\setminus nl_n \text{ and } u_{nl_n} = x_1+\cdots+x_{nl_n}. \tag{4.5}
$$

It is clear that the absolute value of the Jacobian of this transformation is 1 and that $u \in \mathbb{R}^n \iff x \in \mathbb{R}^n$.

Continuous Hahn polynomials associated with internal nodes of the left subtree only involve variables $x_1, \dots, x_{nl_n}$, as do the Gamma functions of the weight function associated with the leaves of the left subtree. Replacing $x_{nl_n}$ by $u_{nl_n} - u_1 - \cdots - u_{nl_n-1}$ (see (4.5)) yields an integral over $\mathbb{R}^{nl_n-1}$ involving the variables $u_1, \dots, u_{nl_n-1}$, and with $u_{nl_n}$ playing the role of $s$. The way in which this integral is constructed allows induction. Denoting this integral by $I_{\text{left}}$, we have :

$$I_{\text{left}} = \delta_{\mathbf{l}_{nl_n-1}, \mathbf{l}'_{nl_n-1}} (2\pi)^{nl_n-1} \Gamma(|\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}| \pm i u_{nl_n}). \tag{4.6}$$

On the other hand, continuous Hahn polynomials associated with internal nodes in the right subtree involve variables $x_{nl_n+1}, \dots, x_n$, but may also involve variables $x_1, \dots, x_{nl_n}$. When the latter is the case, it is always the difference $-(x_1 + \cdots + x_{nl_n}) = -u_{nl_n}$ which occurs. So essentially, $s$ is replaced by $s - u_{nl_n}$; this integral involving the variables $u_{nl_n+1}, \dots, u_n$ can also be calculated by induction. Denoting this integral over $\mathbb{R}^{nr_n-1}$ by $I_{\text{right}}$ we have :

$$I_{\text{right}} = \delta_{\mathbf{l}^{n-1}_{nl_n}, \mathbf{l}'^{n-1}_{nl_n}} (2\pi)^{nr_n-1} \Gamma(|\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| \pm i(s - u_{nl_n})). \tag{4.7}$$

The integral $I$, i.e. the left side of (4.2), then reduces to :

$$\begin{aligned}
I = {} & \sqrt{\frac{l_n!(2|l| + 2|k| - 1)\Gamma(l_n + 2|k| + 2|\mathbf{l}_{n-1}| - 1)}{\Gamma(l_n + 2|\mathbf{k}_{nl_n}| + 2|\mathbf{l}_{nl_n-1}|)\Gamma(l_n + 2|\mathbf{k}^{nl_n+1}| + 2|\mathbf{l}^{n-1}_{nl_n}|)}} \\
& \times \sqrt{\frac{l'_n!(2|l'| + 2|k| - 1)\Gamma(l'_n + 2|k| + 2|\mathbf{l}'_{n-1}| - 1)}{\Gamma(l'_n + 2|\mathbf{k}_{nl_n}| + 2|\mathbf{l}'_{nl_n-1}|)\Gamma(l'_n + 2|\mathbf{k}^{nl_n+1}| + 2|\mathbf{l}'^{n-1}_{nl_n}|)}} \\
& \times \int_{\mathbb{R}} p_{l_n}(u_{nl_n}; |\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| - is, |\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| + is) \\
& \qquad \times p_{l'_n}(u_{nl_n}; |\mathbf{k}_{nl_n}| + |\mathbf{l}'_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}'^{n-1}_{nl_n}| - is, |\mathbf{k}_{nl_n}| + |\mathbf{l}'_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}'^{n-1}_{nl_n}| + is) \\
& \qquad \times I_{\text{left}} \times I_{\text{right}} \, du_{nl_n} \\
= {} & \delta_{\mathbf{l}_{n-1}, \mathbf{l}'_{n-1}} (2\pi)^{n-1} \sqrt{\frac{l_n!(2|l| + 2|k| - 1)\Gamma(l_n + 2|k| + 2|\mathbf{l}_{n-1}| - 1)}{\Gamma(l_n + 2|\mathbf{k}_{nl_n}| + 2|\mathbf{l}_{nl_n-1}|)\Gamma(l_n + 2|\mathbf{k}^{nl_n+1}| + 2|\mathbf{l}^{n-1}_{nl_n}|)}} \\
& \times \sqrt{\frac{l'_n!(2|l'| + 2|k| - 1)\Gamma(l'_n + 2|k| + 2|\mathbf{l}'_{n-1}| - 1)}{\Gamma(l'_n + 2|\mathbf{k}_{nl_n}| + 2|\mathbf{l}'_{nl_n-1}|)\Gamma(l'_n + 2|\mathbf{k}^{nl_n+1}| + 2|\mathbf{l}'^{n-1}_{nl_n}|)}} \\
& \times \int_{\mathbb{R}} p_{l_n}(u_{nl_n}; |\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| - is, |\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| + is) \\
& \qquad \times p_{l'_n}(u_{nl_n}; |\mathbf{k}_{nl_n}| + |\mathbf{l}'_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}'^{n-1}_{nl_n}| - is, |\mathbf{k}_{nl_n}| + |\mathbf{l}'_{nl_n-1}|, |\mathbf{k}^{nl_n+1}| + |\mathbf{l}'^{n-1}_{nl_n}| + is) \\
& \qquad \times \Gamma(|\mathbf{k}_{nl_n}| + |\mathbf{l}_{nl_n-1}| \pm i u_{nl_n}) \Gamma(|\mathbf{k}^{nl_n+1}| + |\mathbf{l}^{n-1}_{nl_n}| \pm i(s - u_{nl_n})) \, du_{nl_n}
\end{aligned}$$

$$= \quad \delta_{l,l'}(2\pi)^n\Gamma(|k|+|l|\pm is).$$

Once again, we have used the orthogonality (4.3) of continuous Hahn polynomials. This completes the proof of Theorem 16.

# V Connection coefficients between different bases of orthogonal polynomials

We will show that the set of polynomials associated with a *fixed* binary coupling tree and *fixed* leaf values $k_j$, but varying values $l_i$ form a basis for $\Pi^n$.

**Lemma 19** *For any binary coupling tree $T$ the degree of the polynomial $R_l^{(k)}(x)$ (or $\mathcal{R}_l^{(k)}(x)$) associated with $T$ is $|l|$.*

**Proof**: In the case of Jacobi polynomials, i.e. the case of $S$-polynomials, the result immediately follows from the fact that $S_m^{k_1,k_2}(x_1,x_2)$ is homogeneous of degree $m$ in the variables $x_1$ and $x_2$.

Now consider the case of continuous Hahn polynomials. Although $p_m(t;a,b,c,d)$ is a polynomial of degree $m$ in $t$, we have to be careful because, in the case of $\mathcal{S}$-polynomials, the variables $x_i$ also appear in the parameters $b$ and $d$. The polynomials associated with an internal node not on the path from the leaf $x_{n+1}$ to the root are, up to a constant factor, $p_m(x;a,b-i(x+y),a,b+i(x+y))$ ($x$ and $y$ stand for a sum of variables $x_j$). Using the definition of the continuous Hahn polynomials we have :

$$\begin{aligned}
&p_m(x;a,b-i(x+y),a,b+i(x+y)) \\
&\quad= i^m\frac{(2a)_m(a+b+i(x+y))_m}{m!}\,{}_3F_2\left(\begin{matrix}-m,m+2a+2b-1,a+ix\\2a,a+b+i(x+y)\end{matrix};1\right) \\
&\quad= i^m\frac{(2a)_m(a+b+i(x+y))_m}{m!}\sum_{j=0}^{m}\frac{(-m)_j(m+2a+2b-1)_j(a+ix)_j}{(2a)_j(a+b+i(x+y))_j j!} \\
&\quad= i^m\frac{(2a)_m}{m!}\sum_{j=0}^{m}\frac{(-m)_j(m+2a+2b-1)_j(a+ix)_j(a+b+i(x+y)+j)_{m-j}}{(2a)_j j!}.
\end{aligned}$$

From this last equation one sees that the degree of this polynomial is at most $m$. It is easy to see that the coefficient of $y^m$ is $\frac{i^{2m}(2a)_m}{m!}\neq 0$, since in our case $a>0$. The same can be done for a polynomial associated with an internal node on the path from $x_{n+1}$ to the root. $\square$

**Theorem 20** *The polynomials $R_l^{(k)}(x)$ and $\mathcal{R}_l^{(k)}(x)$ associated with a fixed binary coupling tree $T$ on $n$ internal nodes form a basis for $\Pi^n$.*

**Proof**: Theorems 4 and 16 imply that the sets of polynomials associated with $T$ are linearly independent.

The number of polynomials associated with $T$ that have degree $m$ equals the number of compositions $J(m,n)$ of $m$ into $n$ parts, i.e. the number of ways that one can write $m$ as a sum of $n$ nonnegative integers whereby the order of the summands is important. There are thus $\sum_{k=0}^{m} J(k,n)$ polynomials of degree at most $m$ associated with $T$. It is not difficult to see that this is exactly the dimension of $\Pi_m^n$, the set of polynomials in $n$ variables with degree at most $m$. □

We recall the two properties (2.11) and (2.12) of the $\mathcal{S}$-polynomials. We can use these two properties to determine the connection coefficients between the different bases. This is stated in the following theorem.

**Theorem 21** *Consider a binary coupling tree, $T_1$, with fixed values $k_j$ and $l_i$. Consider another binary coupling tree $T_2$ with the same fixed values $k_j$ but varying values $l_i'$, such that $|l| = |l'|$. Then the polynomials $R_{l,T_1}^{(k)}(x)$ (resp. $\mathcal{R}_{l,T_1}^{(k)}(x)$) can be written as a linear combination of polynomials $R_{l',T_2}^{(k)}(x)$ (resp. $\mathcal{R}_{l',T_2}^{(k)}$) :*

$$R_{l,T_1}^{(k)}(x) = \sum_{|l'|=|l|} C_{l'} R_{l',T_2}^{(k)}(x); \qquad \mathcal{R}_{l,T_1}^{(k)}(x) = \sum_{|l'|=|l|} C_{l'} \mathcal{R}_{l',T_2}^{(k)}(x). \tag{5.1}$$

*The connection coefficient $C_{l'}$ is equal to the $3nj$-coefficient $\langle T_1(l), T_2(l')\rangle$ (which is zero anyway if $|l| \neq |l'|$).*

**Proof**: This follows from Theorem 20 and the two basic properties of $S$- and $\mathcal{S}$-polynomials. These basic properties are given in (2.11) and (2.12). Observe that (2.11) simply expresses

$$\mathcal{R}_{(l_1,l_2),T_1}^{(k_1,k_2,k_3)}(x_1,x_2) = \sum_{|l'|=|l|} U_{k_3,|k|+|l|,k_2+k_3+l_1'}^{k_1,k_2,k_1+k_2+l_1} \mathcal{R}_{(l_1',l_2'),T_2}^{(k_1,k_2,k_3)}(x_1,x_2), \tag{5.2}$$

where $T_1$ (resp. $T_2$) is the tree in the left hand side (resp. right hand side) of Figure 1 with $k_{12} = k_1 + k_2 + l_1$ and $k_0 = |k| + |l|$ (resp. with $k_{23} = k_2 + k_3 + l_1'$ and $k_0 = |k| + |l'|$). By definition the Racah coefficient (or $6j$-coefficient) can be expressed as the overlap coefficient of two binary coupling trees, i.e.

$$U^{k_1,k_2,k_1+k_2+l_1}_{k_3,|k|+|l|,k_2+k_3+l'_1} = \langle T_1(l), T_2(l')\rangle.$$

In the general case, let $T_1$ (with fixed values $k_j$ and $l_i$) and $T_2$ (with the same fixed values $k_j$) be given. The expansion of $\mathcal{R}^{(k)}_{l,T_1}(x)$ in terms of $\mathcal{S}$-polynomials is then fixed. In order to express $\mathcal{R}^{(k)}_{l,T_1}(x)$ in terms of polynomials associated with the second tree $T_2$, one can use (2.11) and (2.12) a number of times. Eq. (2.11) corresponds to an elementary tree operation (the flop operation of [14]), depicted in Figure 1. Eq. (2.12) corresponds to an exchange operation [14] on trees. So, to express $\mathcal{R}^{(k)}_{l,T_1}(x)$ in terms of polynomials $\mathcal{R}^{(k)}_{l',T_2}(x)$, one has to perform sufficiently many elementary tree operations on $T_1$ until one ends up with a tree of shape $T_2$. Each such operation corresponds to an application of (2.11), introducing a Racah coefficient and a summation index, or to an application of (2.12), introducing only a phase factor. As a consequence, the coefficient $C_{l'}$ in (5.1) stands for a certain sum over products of Racah coefficients. But this sum over products of Racah coefficients is just the $3nj$-coefficient defined by the left and right binary coupling trees, since the "method of trees" [14, 15] yields that the expansion of a $3nj$-coefficient in terms of Racah coefficients is obtained exactly by such elementary tree operations. □

Observe there are some alternative ways of expressing the previous results. For example, for two $n$-variable Jacobi polynomials corresponding to the same binary coupling tree, their inner product is given by (3.5). For two $n$-variable Jacobi polynomials with different binary coupling tree, the inner product is essentially given by a $3nj$-coefficient :

$$\int_{T^n_s} R^{(k)}_{l,T_1}(x) R^{(k)}_{l',T_2}(x) w^{(k)}(x) dx = \langle T_1(l), T_2(l')\rangle \frac{s^{2|k|+2|l|-1}}{\Gamma(2|k|+2|l|)} \prod_{i=1}^{n+1} \Gamma(2k_i), \tag{5.3}$$

where $w^{(k)}(x)$ is the classical weight function (3.4).

In the same way, we have for $n$-variable continuous Hahn polynomials corresponding to different binary coupling trees that

$$\int_{\mathbb{R}^n} \mathcal{R}^{(k)}_{l,T_1}(x) \mathcal{R}^{(k)}_{l',T_2}(x) w^{(k)}(x)\, dx = \langle T_1(l), T_2(l')\rangle (2\pi)^n \Gamma(|k|+|l| \pm is). \tag{5.4}$$

where in this case $w^{(k)}(x)$ is given by (4.1).

# Appendix A

In this appendix we show that the products of Wilson polynomials in (2.5) also satisfy an orthogonality relation on $\mathbb{R}^2$; thus also (2.5) can be interpreted as a connection coefficient formula.

When $\Re(a,b,c,d) > 0$ and non-real parameters occur in conjugate pairs, Wilson polynomials are orthogonal on $\mathbb{R}_+$ for the weight function $|\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)/\Gamma(2ix)|^2$ :

$$\begin{aligned}
&\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 W_m(x^2;a,b,c,d)W_j(x^2;a,b,c,d)\,dx \\
&= \;2\pi\frac{\Gamma(j+a+b)\Gamma(j+a+c)\Gamma(j+a+d)\Gamma(j+b+c)\Gamma(j+b+d)\Gamma(j+c+d)}{\Gamma(2j+a+b+c+d)} \\
&\quad \times (j+a+b+c+d-1)_j j!\delta_{m,j}, \qquad \text{(A.1)}
\end{aligned}$$

see e.g. Ref. [8].

**Theorem 22** *The products of Wilson polynomials in both the left and right side of equation (2.5) are orthogonal on* $\mathbb{R}^2$ *for the weight function*

$$\begin{aligned}
&\Gamma(k_1 \pm ix_1)\Gamma(k_2 \pm ix_2)\Gamma(k_3 \pm i(s-x_1-x_2)) \\
&\quad\times \frac{\Gamma(k_1 \pm i(x_1-2t))\Gamma(k_2 \pm i(2x_1+x_2-2t))\Gamma(k_3 \pm i(s-2t+x_1+x_2))}{\Gamma(\pm 2ix_1 \mp 2it)\Gamma(\pm 2ix_1 \pm 2ix_2 \mp 2it)}, \qquad \text{(A.2)}
\end{aligned}$$

*if* $k_1, k_2, k_3 > 0$.

Following the convention mentioned earlier, each factor in (A.2) stands for the product of two Gamma functions.

**Proof**: Consider the polynomial in the right side of (2.5). The parameters of the Wilson polynomials occur in conjugate pairs and the real parts of the parameters are positive. This allows us to use the orthogonality relation (A.1).

Denoting the weight function (A.2) by $w(x_1,x_2)$, we want to determine the value of

$$\begin{aligned}
&\iint_{\mathbb{R}^2} w(x_1,x_2)W_m((x_1-t)^2;k_1\pm it,k_2\pm i(x_1+x_2-t)) \\
&\quad\times W_{m'}((x_1-t)^2;k_1\pm it,k_2\pm i(x_1+x_2-t)) \\
&\quad\times W_j((x_1+x_2-t)^2;k_1+k_2+m\pm it,k_3\pm i(s-t)) \\
&\quad\times W_{j'}((x_1+x_2-t)^2;k_1+k_2+m'\pm it,k_3\pm i(s-t))\,dx_1dx_2.
\end{aligned}$$

The notation is obvious : each entry of the form $k_1 \pm it$ stands for two parameters of the Wilson polynomial. In order to compute the integral, introduce two new variables, namely the arguments of the Wilson polynomials : $u_1 = x_1 - t$ and $u_2 = x_1 + x_2 - t$. Changing variables and integrating first with respect to $u_1$ and then with respect to $u_2$ gives a constant times $\delta_{m,m'}\delta_{j,j'}$ for the above integral, if we use the facts that $\Gamma(\overline{z}) = \overline{\Gamma(z)}$ and that the integrands are even functions.

Similar arguments yield the desired result for the Wilson polynomials on the left side of (2.5). $\square$

# Appendix B

In this appendix we prove Lemma 11. First, we show that if $x \in T_s^n$, then $-1 < v_i < 1$ for each $v_i$ given by (3.9). The position of the variable $v_i$ in the binary coupling tree plays a role in its expression. For a variable $v_i$ not on the path from $x_{n+1}$ to the root of the tree, the equation expressing $v_i$ in terms of $x$ is

$$v_i = \left( \sum_{\substack{\text{right leaves}\\ \text{of } v_i}} x_j - \sum_{\substack{\text{left leaves}\\ \text{of } v_i}} x_j \right) \Big/ \sum_{\substack{\text{all leaves}\\ \text{of } v_i}} x_j,$$

which is of the form $f(y,z) = \frac{y-z}{y+z}$ with $0 < y$, $0 < z$ (and $y + z < s$). It is easy to see that $-1 < f(y,z) < 1$ if $y, z > 0$. On the other hand, if $v_i$ is on the path from $x_{n+1}$ to the root, we have, after substitution of $x_{n+1}$ by $s - |x|$ that

$$v_i = \left( s - \sum_{\substack{\text{non-leaves}\\ \text{of } v_i}} x_j - 2 \sum_{\substack{\text{left leaves}\\ \text{of } v_i}} x_j \right) \Big/ \left( s - \sum_{\substack{\text{non-leaves}\\ \text{of } v_i}} x_j \right),$$

which is of the form $g(y,z) = \frac{s-y-2z}{s-y}$ with $0 < y$, $0 < z$ and $y + z < s$. A simple examination shows that $g(y,z)$ reaches its maximum $+1$ in this area when $z = 0$, and its minimum $-1$ when $y + z = s$. Thus $-1 < g(y,z) < 1$ for $y, z > 0$ with $y + z < s$. So we have that $-1 < v_i < 1$ for each $1 \le i < n$. For the variable $v_n$ a completely analogous reasoning can be given.

Before proving $v \in \square^n \Rightarrow x \in T_s^n$ in general, consider an example.

**Example 23** Let $v \in \square^4$ and consider equations (3.7) with $x_5$ replaced by $s - x_1 - x_2 - x_3 - x_4$ :

$$\begin{cases} v_1 = (x_2 - x_1)/(x_2 + x_1) \\ v_2 = (x_3 - x_1 - x_2)/(x_3 + x_1 + x_2) \\ v_3 = (s - x_1 - x_2 - x_3 - 2x_4)/(s - x_1 - x_2 - x_3) \\ v_4 = (s - 2x_1 - 2x_2 - 2x_3)/s. \end{cases} \tag{B.1}$$

Since $-1 < v_4 < 1$, the fourth equation of (B.1) implies

$$-1 < v_4 < 1 \Rightarrow 0 < x_1 + x_2 + x_3 < s. \tag{B.2}$$

The denominator of the second equation of (B.1) is thus positive, and

$$-1 < v_2 \Rightarrow x_3 > 0 \text{ and } v_2 < 1 \Rightarrow x_1 + x_2 > 0, \tag{B.3}$$

hence

$$-1 < v_1 \Rightarrow x_2 > 0 \text{ and } v_1 < 1 \Rightarrow x_1 > 0. \tag{B.4}$$

From (B.2) it also follows that the denominator of the third equation of (B.1) is positive, implying :

$$-1 < v_3 \Rightarrow x_1 + x_2 + x_3 + x_4 < s \text{ and } v_3 < 1 \Rightarrow x_4 > 0. \tag{B.5}$$

So we clearly have : $v \in \square^4 \Rightarrow x \in T_s^4$. ∎

In general, let $v \in \square^n$. We know that

$$v_n = \left( s - 2 \sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j \right) / s,$$

with $s > 0$, and thus :

$$-1 < v_n < 1 \Rightarrow 0 < \sum_{\substack{\text{left leaves} \\ \text{of } v_n}} x_j < s. \tag{B.6}$$

Now consider the variable associated with the left child of $v_n$, that is $v_{nl_n-1}$ :

$$v_{nl_n-1} = \left( \sum_{\substack{\text{right leaves} \\ \text{of } v_{nl_n-1}}} x_j - \sum_{\substack{\text{left leaves} \\ \text{of } v_{nl_n-1}}} x_j \right) / \sum_{\substack{\text{all leaves} \\ \text{of } v_{nl_n-1}}} x_j.$$

From (B.6) we know that the denominator of the previous formula is positive, and we have :

$$-1 < v_{nl_n-1} \Rightarrow \sum_{\substack{\text{right leaves} \\ \text{of } v_{nl_n-1}}} x_j > 0, \tag{B.7}$$

and

$$v_{nl_n-1} < 1 \Rightarrow \sum_{\substack{\text{left leaves} \\ \text{of } v_{nl_n-1}}} x_j > 0. \tag{B.8}$$

Continuing in this way for all internal nodes of the left subtree (in *in-order* [25]) yields that $x_j > 0$ for every variable in the left subtree, since

$$-1 < \frac{y-z}{y+z} < 1 \Rightarrow y > 0 \text{ and } z > 0,$$

if $y + z > 0$.

Consider the variable associated with the right child of $v_n$, i.e. $v_{n-1}$. For $v_{n-1}$ we have the following expression :

$$v_{n-1} = \left( s - \sum_{\substack{\text{non-leaves} \\ \text{of } v_{n-1}}} x_j - 2 \sum_{\substack{\text{left leaves} \\ \text{of } v_{n-1}}} x_j \right) \Big/ \left( s - \sum_{\substack{\text{non-leaves} \\ \text{of } v_{n-1}}} x_j \right).$$

The "non-leaves of $v_{n-1}$" are exactly the "left leaves of $v_n$", so from (B.6) we have once again that the denominator of the previous formula is positive. We thus have :

$$-1 < v_{n-1} \Rightarrow \sum_{\substack{\text{non-leaves} \\ \text{of } v_{n-1}}} x_j + \sum_{\substack{\text{left leaves} \\ \text{of } v_{n-1}}} x_j < s, \tag{B.9}$$

and

$$v_{n-1} < 1 \Rightarrow \sum_{\substack{\text{left leaves} \\ \text{of } v_{n-1}}} x_j > 0. \tag{B.10}$$

Starting from this last equation, we can perform the same actions as in the left subtree to conclude that $x_j > 0$ for all leaves in the left subtree of $v_{n-1}$. Continuing in this way we get that $x_j > 0$, for $j = 1, \ldots, n$ and $|x| < s$, since

$$-1 < \frac{s-y-2z}{s-y} < 1 \Rightarrow y + z < s \text{ and } z > 0,$$

if $s - y > 0$. This completes the proof that $v \in \Box^n \Rightarrow x \in T_s^n$.

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