Hitting Times and Probabilities for Imprecise Markov Chains

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Markov Chains

Stochastically evolving dynamical system with uncertain state X_n

- Time $n \in \mathbb{N}_0$ (*discrete time* model)
- Finite state space $\mathscr{X}$

A stochastic process P is called a *Markov chain* if

$$P(X_{n+1} = x_{n+1} \,|\, X_{0:n} = x_{0:n}) = P(X_{n+1} = x_{n+1} \,|\, X_n = x_n)$$

A Markov chain is called *homogeneous* if, moreover,

$$P(X_{n+1} = y \,|\, X_n = x) = P(X_1 = y \,|\, X_0 = x)$$

Markov Chains and Transition Matrices

A *transition matrix* T is an $|\mathscr{X}| \times |\mathscr{X}|$ matrix that is row-stochastic:

- $\sum_{y \in \mathscr{X}} T(x,y) = 1$ and $T(x,y) \geq 0$

Such a T determines a homogeneous Markov chain P for which

$$P(X_{n+1} = y \,|\, X_n = x) = T(x,y) \quad \text{for all } x, y \in \mathscr{X} \text{ and } n \in \mathbb{N}_0 \,.$$

What if we don't know T?

Or: what if Markov assumption is unwarranted?

$\Rightarrow$ Instead use an *imprecise* Markov chain

Imprecise Markov Chains

Parameterised by a *set* $\mathscr{T}$ of transition matrices.

- $\mathscr{T}$ must satisfy some technical closure properties.

Inferences are the *lower* and *upper expectations* of quantities of interest.

These depend on the type of imprecise Markov chain!

Imprecise Markov Chains

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For the set $\mathscr{P}^{\mathrm{H}}_{\mathscr{T}}$ of homogeneous Markov chains with transition matrix T in $\mathscr{T}$,

$$\underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot\,|\,\cdot] = \inf_{P\in\mathscr{P}^{\mathrm{H}}_{\mathscr{T}}} \mathbb{E}_P[\cdot\,|\,\cdot] \quad \text{and} \quad \overline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot\,|\,\cdot] = \sup_{P\in\mathscr{P}^{\mathrm{H}}_{\mathscr{T}}} \mathbb{E}_P[\cdot\,|\,\cdot]$$

What other types are there?

Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathscr{T}$.

$$\underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot \mid \cdot]$$

Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathscr{T}$.

- *Game-theoretic probability model* with local uncertainty models described by $\mathscr{T}$.

$$\underline{\mathbb{E}}^{\mathrm{V}}_{\mathscr{T}}[\cdot \mid \cdot] \qquad \leq \qquad \underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot \mid \cdot]$$

Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathscr{T}$.

- Set of *general* stochastic processes "compatible" with $\mathscr{T}$.
  Always some $T \in \mathscr{T}$ such that

$$P(X_{n+1} = x_{n+1} \,|\, X_{0:n} = x_{0:n}) = T(x_n, x_{n+1}),$$

  but can be different T for each $x_{0:n}$.
  Called an imprecise Markov chain under *epistemic irrelevance*.
- *Game-theoretic probability model* with local uncertainty models described by $\mathscr{T}$.

$$\underline{\mathbb{E}}^{\mathrm{V}}_{\mathscr{T}}[\cdot \,|\, \cdot] \leq \underline{\mathbb{E}}^{\mathrm{I}}_{\mathscr{T}}[\cdot \,|\, \cdot] \leq \qquad \underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot \,|\, \cdot]$$

Types of Imprecise Markov Chains

- Set of homogeneous Markov chains with transition matrix $T \in \mathscr{T}$.
- Set of Markov chains such that for all $n \in \mathbb{N}_0$ there is some $T \in \mathscr{T}$ for which

  $$P(X_{n+1} = x_{n+1} \,|\, X_n = x_n) = T(x_n, x_{n+1}).$$

  Called a *Markov set chain*, or an imprecise Markov chain under *strong independence*.
- Set of *general* stochastic processes "compatible" with $\mathscr{T}$.
  Always some $T \in \mathscr{T}$ such that

  $$P(X_{n+1} = x_{n+1} \,|\, X_{0:n} = x_{0:n}) = T(x_n, x_{n+1}),$$

  but can be different T for each $x_{0:n}$.
  Called an imprecise Markov chain under *epistemic irrelevance*.
- *Game-theoretic probability model* with local uncertainty models described by $\mathscr{T}$.

$$\underline{\mathbb{E}}^{\mathrm{V}}_{\mathscr{T}}[\cdot \,|\, \cdot] \leq \underline{\mathbb{E}}^{\mathrm{I}}_{\mathscr{T}}[\cdot \,|\, \cdot] \leq \underline{\mathbb{E}}^{\mathrm{M}}_{\mathscr{T}}[\cdot \,|\, \cdot] \leq \underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[\cdot \,|\, \cdot]$$

Lower and Upper Expected Hitting Times

Given a fixed set $A \subset \mathscr{X}$ of states:

How long will it take before the system visits an element of A?

What is $\mathbb{E}_P[H_A \,|\, X_0]$, where H_A is the number of steps before A is visited?

What can we say about this for the various types of imprecise Markov chains?

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Theorem

$$\underline{\mathbb{E}}^{\mathrm{V}}_{\mathscr{T}}[H_A \,|\, X_0] = \underline{\mathbb{E}}^{\mathrm{I}}_{\mathscr{T}}[H_A \,|\, X_0] = \underline{\mathbb{E}}^{\mathrm{M}}_{\mathscr{T}}[H_A \,|\, X_0] = \underline{\mathbb{E}}^{\mathrm{H}}_{\mathscr{T}}[H_A \,|\, X_0]$$

(and similarly for the upper expected hitting time)