

**Efficient computation
of updated lower expectations for
imprecise continuous-time hidden
Markov chains**



**Thomas
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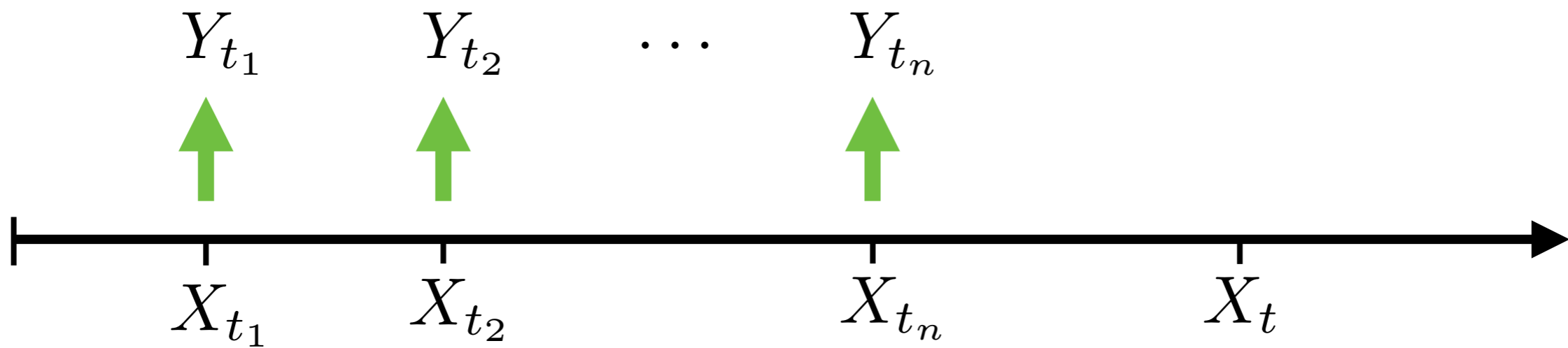
**Arno
Siebes**

Imprecise continuous-time Markov chain



Imprecise continuous-time Markov chain

hidden

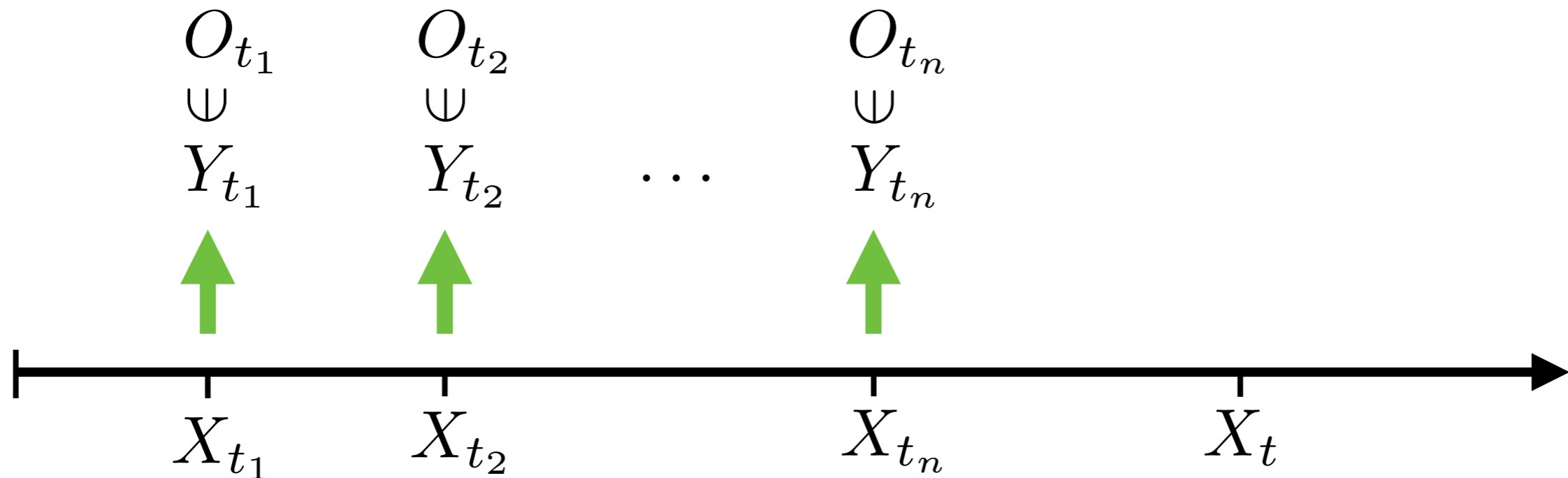


Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} \in O_{t_1}, \dots, Y_{t_n} \in O_{t_n})$$



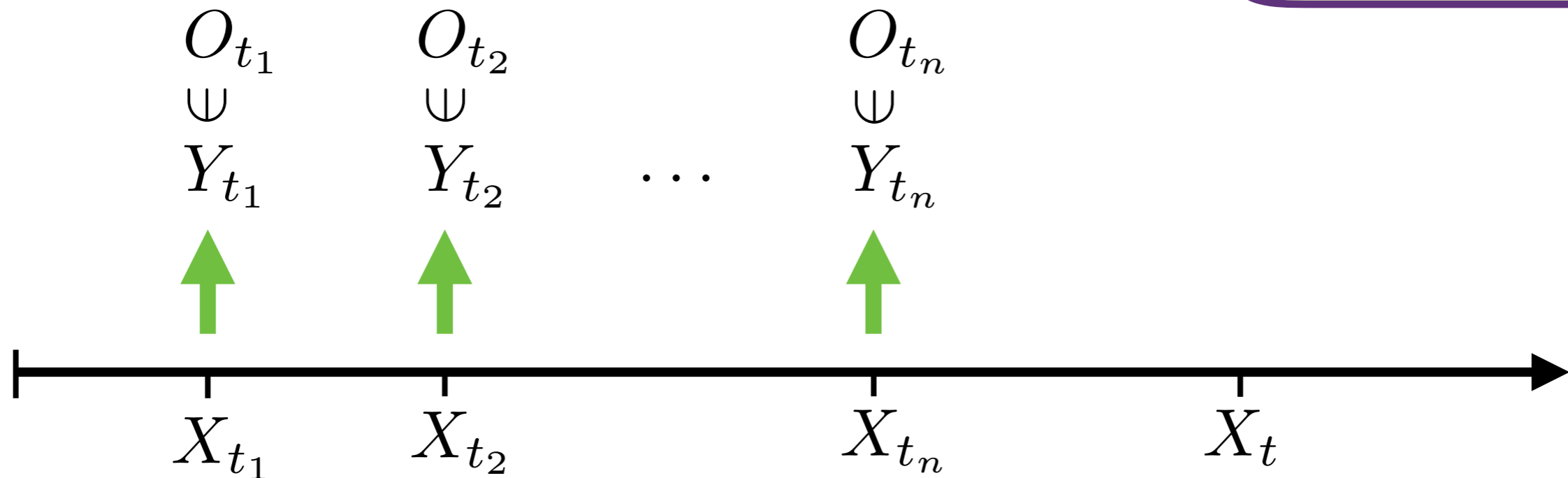
Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} \in O_{t_1}, \dots, Y_{t_n} \in O_{t_n})$$

efficient algorithms



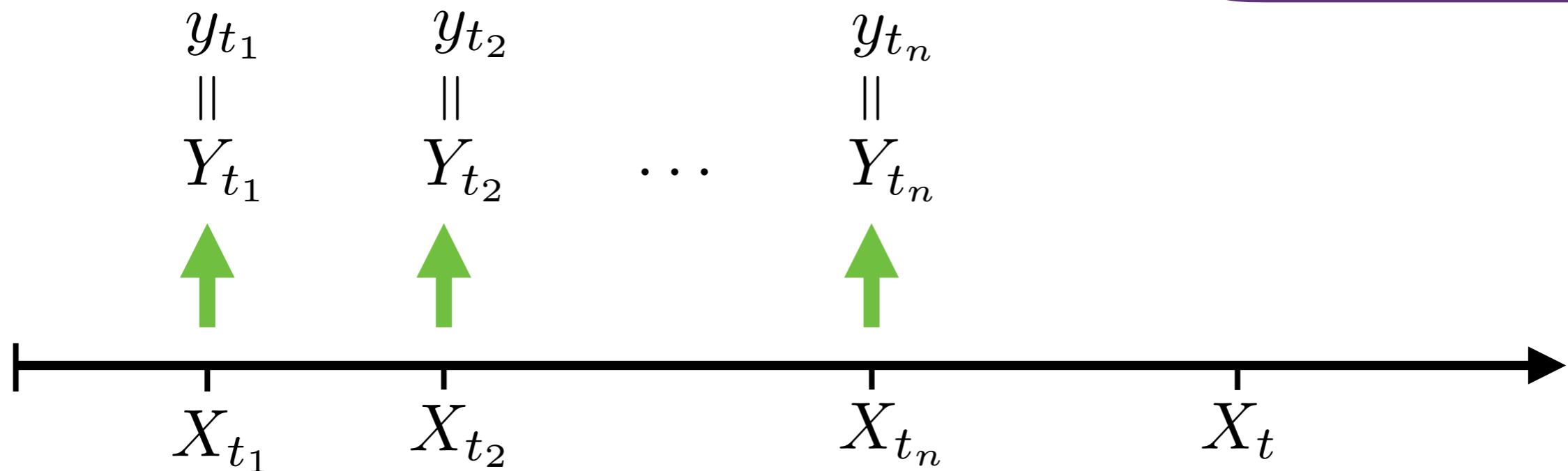
Imprecise continuous-time Markov chain

hidden

updated lower expectations

$$\underline{E}(f(X_t) | Y_{t_1} = y_{t_1}, \dots, Y_{t_n} = y_{t_n})$$

point observations



Want to know more?



See you at the poster...

Efficient Computation of Updated Lower Expectations for Imprecise Continuous-Time Hidden Markov Chains

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Abstract We consider the problem of performing inference with *imprecise continuous-time hidden Markov chains*, that is, *imprecise continuous-time hidden Markov chains* that are augmented with random *output variables* whose distribution depends on the hidden state of the chain. The prefix 'imprecise' refers to the fact that we do not consider a classical continuous-time Markov chain, but replace it with a robust extension that allows us to represent various types of model uncertainty, using the theory of *imprecise probabilities*. The inference problem amounts to computing lower expectations of functions on the state-space of the chain, given observations of the output variables. We develop and investigate this problem with very few assumptions on the output variables; in particular, they can be chosen to be either discrete or continuous random variables. Our main result is a polynomial runtime algorithm to compute the lower expectation of functions on the state-space at any given time-point, given a collection of observations of the output variables.

"Precise" Continuous-Time Markov Chains

State-space X (e.g., $X = \{\text{healthy, sick}\}$)

Continuous-time Markov chain P specifies r.v. X_t at each time $t \in \mathbb{R}_{\geq 0}$

For any finite number of time-points, e.g. $0 < t < r < s$, P induces a Bayesian network:

Satisfies Markov property: $P(X_t | X_0, X_1, \dots, X_r) = P(X_t | X_r)$

Imprecise Continuous-Time Markov Chains

Now a set \mathcal{P} of distributions.

Each $P \in \mathcal{P}$ specifies r.v. X_t at each time $t \in \mathbb{R}_{\geq 0}$

For any finite number of time-points, e.g. $0 < t < r < s$, \mathcal{P} induces a credal network:

Satisfies imprecise Markov property: $\underline{P}(X_t | X_0, X_1, \dots, X_r) = \underline{P}(X_t | X_r)$

Imprecise CT Hidden Markov Chains

States X_t cannot be directly observed. Instead we observe Y_t , which "correlates" with X_t (e.g., symptoms of a disease).

For simplicity, we use a *precise, homogeneous output model*:

$$\underline{P}(Y_t | X_t) = P(Y_t | X_t) = P(Y | X), t \in \mathbb{R}_{\geq 0}$$

We are interested in inferences about the states given observations. For example, given some $O \subseteq Y$, we want to know $\underline{E}[f(X_t) | Y_t \in O]$.

Outputs with Positive (Upper) Probability

If the observation $(Y_t \in O)$ has positive probability, we use Bayes' rule:

$$E_p[f(X_t) | Y_t \in O] = \sum_{x \in X} f(x) \frac{P(X_t = x, Y_t \in O)}{P(Y_t \in O)}$$

For the imprecise model, we use *regular extension*:

$$\underline{E}[f(X_t) | Y_t \in O] = \inf\{E_p[f(X_t) | Y_t \in O] : P \in \mathcal{P}, P(Y_t \in O) > 0\}$$

whenever $\bar{P}(Y_t \in O) > 0$.

This lower expectation satisfies a *generalised Bayes' rule*:

$$\underline{E}[f(X_t) | Y_t \in O] = \max\{\mu \in \mathbb{R} : \underline{E}[P(Y_t \in O | X_t)(f(X_t) - \mu)] \geq 0\}$$

Continuous Outputs

If Y_t is continuous, then usually $P(Y_t = y) = 0$ for all $P \in \mathcal{P}$. Assume a (conditional) probability density function $\phi: Y \times X \rightarrow \mathbb{R}$:

$$P(Y_t \in O | X_t = x) = \int_O \phi(y|x) dy$$

Take a sequence $(O_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow \infty} O_n = \{y\}$. Then define

$$E_p[f(X_t) | Y_t = y] = \lim_{n \rightarrow \infty} E_p[f(X_t) | Y_t \in O_n]$$

This limit exists under suitable assumptions; if $E_p[\phi(y | X_t)] > 0$:

$$E_p[f(X_t) | Y_t = y] = \frac{E_p[f(X_t)\phi(y | X_t)]}{E_p[\phi(y | X_t)]}$$

Continuous Outputs, Imprecise Case

For the imprecise case, when $\underline{E}[\phi(y | X_t)] > 0$ we define

$$\underline{E}[f(X_t) | Y_t = y] = \inf\{E_p[f(X_t) | Y_t = y] : P \in \mathcal{P}\}$$

This lower expectation satisfies a limit interpretation

$$\underline{E}[f(X_t) | Y_t = y] = \lim_{n \rightarrow \infty} \underline{E}[f(X_t) | Y_t \in O_n]$$

and a *generalised Bayes' rule for (finite) mixtures of densities*:

$$\underline{E}[f(X_t) | Y_t = y] = \max\{\mu \in \mathbb{R} : \underline{E}[\phi(y | X_t)(f(X_t) - \mu)] \geq 0\}$$

Solving the Generalised Bayes' Rule(s)

In both cases, we have a generalised Bayes' rule:

$$\underline{E}[f(X_t) | Y_t \in O] = \max\{\mu \in \mathbb{R} : \underline{E}[P(Y_t \in O | X_t)(f(X_t) - \mu)] \geq 0\}$$

$$\underline{E}[f(X_t) | Y_t = y] = \max\{\mu \in \mathbb{R} : \underline{E}[\phi(y | X_t)(f(X_t) - \mu)] \geq 0\}$$

See the paper for a polynomial runtime algorithm to solve these.