Computational methods for imprecise continuous-time birth-death processes: a preliminary study of flipping times

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The Precise Case

Consider a continuous time and finite-state Markov process with state space $X$. At any time $t \in [0, \infty)$, the stochastic matrix of the process $P_t$ is derived from a transition rate matrix $Q$. For $i,j \in X$, the element at the $i$ row and $j$ column of $Q$ is denoted by $Q(i,j)$. For the matrix $Q$, the following properties hold

(P1) $Q(i,j) \geq 0$ for all $i,j \in X$ such that $i \neq j$
(P2) $\sum_{j \in X} Q(i,j) = 0$, $i \in X$

A matrix $Q$ is said to be bounded if $Q(i,j) > -\infty$ for all $i,j \in X$ or, equivalently, if $\|Q\| < \infty$. Our results hold for various types of norms, but the one we consider is the infinite norm defined by $\|Q\| := \max_{i \in X} \sum_{j \in X} |Q(i,j)|$.

When $Q$ is bounded, then $Q_t$ satisfies the Kolmogorov backward equation

$$dQ_t = Q_t dt$$

(1)

If we let $f_t(i) = E_t(X_0 = i)$, with $f$ a real-valued function on the finite state space $X$ and $t \in X$ an initial state, then we can rewrite Equation (1) as

$$dQ_t = f_t dt$$

(2)

Combined with the boundary condition $f_0 = f$, the unique solution of Equation (2) is $Q_t = e^{tf}$. Instead of considering a time-invariant $Q$, we can also let $Q_t$ be a function of the time $t$. In that case, Equation (2) can be rewritten as

$$dQ_t = f_t dt$$

(3)

which, in general, has no analytical solution.

A "messy" case

Consider the state space $X := \{0, 1, 2, 3\}$, the following set of bounded matrices

$$\left\{ \begin{pmatrix} -p_i & p_i & 0 & 0 \\ q_j & -q_j & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -\tau \end{pmatrix} : a_i \in [-a, a] \text{ and } b_j \in [-b, b] \right\}$$

and a function $f$ of the form $[c_1, c_2, f_1, f_2]$. In this case, we cannot efficiently identify $Q_{0,t}$, because for $Q, Q' \in Q$, we have that $Q + Q' = Q + Q'$ for all $k \in N$.

Is there a simple way to check when two different matrices $Q, Q'$ yield the same expected value, without calculating $Qk$ and $Q'k$ for all $k$?

The Imprecise Case

Set of matrices Instead of a single transition matrix $Q$, we consider a set of such matrices, denoted by $Q$. We assume that each matrix in $Q$ is bounded and satisfies (P1) and (P2). Let $X$ be the set of all rate matrices, then for any set $Q \subseteq X$ of rate matrices, we let

$$Q := \{Q(i,j) : Q \in Q\}$$

for all $i \in X$,

and we say that $Q$ has separately specified rows if

$$Q \in Q\{\forall i \in X\} \{Q(i,) \in Q_i\}$$

We further assume that $Q$ is the convex hull of a finite number of extreme transition rate matrices.

Our Approach At any time $t \in [0, \infty)$, the only assumption we make about $Q_t$ is that it is an element of $Q$. Every such possible choice of non-stationary transition rate matrices will yield $\tau$ finite numbers of possible solutions. $\tau$ goal is to calculate exact lower and upper bounds for the set of all these solutions $f_t$ as denoted by $f_t$ and $f_t$. In the recent work of [5] and with respect to the lower bound, $f_t$ is the solution to

$$dQ_t = \min_{Q \in Q} Q_t dt$$

with boundary condition $f_t = f$.

Since $Q$ is the convex hull of a finite number of extreme transition rate matrices and that the solution to (4) is continuous, there must be time points $0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots$ such that, for all $t \in \mathbb{N}$, $\{t_0, t_1, t_2, \ldots\}$, the minimum in (4) is obtained by the same extreme transition rate matrix $Q_0 \in Q$. We call these time points $t_0$. Equation (4) is then piecewise linear and has the following solution

$$f_t = e^{tf}Q_0$$

(5)

Calculating Lower Expectations We need to find the flipping times $t_0$ and the corresponding extreme transition rate matrices $Q_0$, when calculating the lower expectation of a given $f$ on $X$. It can be proved that for any pair of matrices $Q, Q' \in Q$, we have that

$$\text{if } Q < Q', \text{ then } Q_0 \neq Q'$$

where $Q_t < Q'_t$ if $Q(i,j) \leq Q'(i,j)$ for all $i \in X$ and $Q' \neq Q$. In this way, we can eliminate matrices $Q'$, which do not yield the minimum expected value of $f$ up to some $t > 0$. Since $Q$ has separately specified rows, then the matrix $Q_0$ is the one that minimises $Q'$ at each row separately. Hence, $Q_0$ belongs to the set $Q = \{Q : Q(i,j) \leq Q'(i,j), i \in X \text{ and } Q \in Q \}$.

In practice, $Q_0$ might not be a singleton and in this case, for any two matrices $Q, Q' \in Q$, we have that $Q < Q'$: it can be further proved that

$$Q < Q' \Rightarrow Q' < Q' \Rightarrow Q'$$

(6)

From (6), we understand that if $Q_0$ is not a singleton, for any matrix $Q \in Q$, we calculate $Q' < Q'$ and we compare them, in order to eliminate more matrices. If still the result set is a singleton, from the remaining ones we calculate $Q' < Q'$ and so on, till we are left with one matrix, which will be the matrix $Q_0$.

Having found $Q_0$, then, due to (5), $f_t = e^{tf}Q_0$ and $Q_0 = [0, t]$. The only thing left to find is the flipping time $t_f$. If there exists $Q_0$, such that $Q < Q_0$ and for which we obtain the minimum expected value for some $t > 0$, then due to continuity, the derivative of the system evaluated at $t = t_f$ should be equal for both $Q_0$ and $Q_\in$. Therefore, from (3) combined with the boundary condition $f_t$, we have that

$$Q_0 e^{tf}Q_0 = Q_0 e^{tf}Q_0$$

(7)

We solve (7) with respect to $t_f$, for each row separately. At each row $i$, for $Q_0$, we test all possible extreme matrices from $Q_0$. Among the solutions of $t_i$, the smallest positive real one is the first flipping time and the corresponding matrix—if it is unique—is the matrix $Q_0$. If we cannot uniquely identify $Q_0$ in this way, we follow a procedure that is similar to the one that we used to identify $Q_0$. By continuing in this way, we can find all the flipping times and their corresponding transition rate matrices.

Imprecise Birth-Death Process

We focus on the case where every state has an interval-valued birth and/or death rate. The transition rate matrices have the following form

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -\mu_i & \lambda_i & \cdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_j - \mu_j & 0 \end{pmatrix}$$

where, for all $i \in \{0, \ldots, L - 1\}$ and $j \in \{1, \ldots, L\}$, $\lambda_i \in [\lambda_i, \bar{\lambda}_i]$ and $\mu_j \in [\mu_j, \bar{\mu}_j]$ and $L \in \mathbb{N}$. In this way, we have a set of matrices $Q$ with finite numbers of extreme points, separately specified rows and which avoids the special case above.

Numerical Results

We calculate the lower expected probability of state 1, $E(X_t = 1 | X_0 = 1)$, of an imprecise birth-death chain with state space $X := \{0, 1, 2, 3\}$ for $t$ approaching infinity. The set of transition rate matrices $Q$ is derived from the intervals $\lambda_i \in [1, 3]$ and $\mu_j \in [2, 5]$. For $i \in \{0, \ldots, L - 1\}$ and $j \in \{1, \ldots, L\}$ and the input function is $f = [0, 1, 0, 0]$. Following the procedure described before, we start by finding a matrix $Q_0$, such that $Q_0 < Q' \forall Q' \in Q$. Due to the values of $f$, there are multiple $Q_0$ which minimise $Q'$. These matrices have the following form:

$$Q' = \begin{pmatrix} -1 & 0 & 0 \\ 5 & -8 & 3 \\ 0 & 2 & -2 \end{pmatrix}$$

where $\lambda_1 \in [1, 3]$ and $\mu_3 \in [2, 5]$. Let $Q_0$ be the set containing all the matrices of the above form.

Continuing with the procedure, we check whether there is a matrix $Q_0$ in $Q'$, such that $Q' < Q'_0 \forall Q' \in Q' \subseteq Q' \setminus \{Q_0\}$. Indeed, there is such a matrix and therefore we have that

$$Q_0 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 5 & -8 & 3 & 0 \\ 0 & 2 & -2 & -2 \end{pmatrix}$$

for which the flipping time is $t_f = 0.6403991$ and $Q_0$ is

$$Q_0 = \begin{pmatrix} -3 & 2 & 3 & 0 \\ -2 & 5 & 3 & 0 \\ 0 & 2 & -2 & -2 \end{pmatrix}$$

For the matrix $Q_0$, there is no flipping time and by taking $t \rightarrow \infty$, we find that, for all $i \in X$,

$$\lim_{t \rightarrow \infty} E(X_t = 1 | X_0 = i) = 0.6937540788.$$