FACULTY OF ENGINEERING
AND ARCHITECTURE

Upper Expectations for Discrete-Time Imprecise Stochastic Processes: In Practice, They Are All the Same!

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Doctoral dissertation submitted to obtain the academic degree of
Doctor of Mathematical Engineering

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June 2022

GHENT
UNIVERSITY

ISBN 978-94-6355-607-1
NUR 919, 921
Wettelijk depot: D/2022/10.500/48

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Preface

This is the end. The last words I put down, a final effort.
The journey is not finished though. It has changed me,
and therefore it is unimaginable to continue without it.

It takes me to places and allows me to see.
It is where I can calmly celebrate.

An immense no-man's-land close to my doorstep.
I will still come and go many times.

The four and a half years that lie behind me have been interesting, to say the least. I will not attempt to express the feelings that I have felt, or the lessons that I have learned. At this point, I mainly feel privileged for having been allowed to spend four years of my life on studying 'nature'. I had the time and resources to engross myself in a field, to a level that I was able to push the boundaries of what is known. I could share my knowledge and ideas, perhaps—and hopefully—inspired some people, and even earned respect for this. It is a job that has enriched me in so many aspects, and I largely owe it to a good dose of luck: I was born in the right country, raised in a stable environment, enjoyed proper education, suffered no 'major' medical issues and was surrounded by lovely people. Therefore, I feel privileged.

Sure, not all went well. Some who read this might know this better than I do. I have walked some rough roads and took some wrong turns. But in hindsight it is always easy to say such things. On top, it doesn't make me, nor you, any happier, so I'll shut up about it. :)

Although there was a good dose of luck involved, we should of course not discredit the generous efforts of all those lovely people that have sur-

rounded me throughout the years. This entire expedition would have failed dramatically without them. Life would fail dramatically without them. As I am writing this, the words of a great saxophone player echo in my head and reflect how I feel: *'Thank you for living, it is fucking tricky!'*

Let me start off by thanking the two persons that were largely responsible for guiding and aiding me on a professional level during the past four years: my two supervisors, Jasper De Bock and Gert de Cooman.

Jasper for the countless evenings that you've passed, not with your family, but with checking my sloppy proofs and my poorly structured texts—I believe I also owe Annelien an apology for this. For all those midnight calls, and the elaborate discussions we had at the blackboard and in the pub. I have learned a *lot* from our interactions. Many of the ideas found in this book, and also in the rest of my work, owe their existence to you. Perhaps your approach was a bit harsh from time to time, but then again, I always got the feeling that you strongly believed in me. Maybe this was your way of pushing me to my limits, and I can ensure you that it was effective. Thank you for pushing me, for letting me learn and for showing me how to believe in myself.

Gert for allowing me to put things into perspective, both research-wise and on a personal level. For planting a seed in my head four or five years ago, when the idea of doing an applied engineering job seemed rather saddening to me; I like the tree that came out of it, by the way! I have not only become a better thinker due to you, but also, and especially, a better speaker. If there is one trait I would describe you with, it is humour. I'll always remember the contagious smile peeping into my office, saying 'bye bye' in a way that is slightly suspicious and made me wonder 'is he just being kind or did I miss something here?' Thanks for all the awkward funny moments.

Apart from a set of great supervisors, I also had a bunch of great colleagues. The atmosphere created by them is unlike what I have ever experienced before—which is perhaps not that surprising given that this job was my first job ever. It is both enjoyable and stressful, heart-warming and competitive, dead serious and extremely childish. The persons responsible for this are/were, in alphabetical order, Alexander, Arne, Arthur, Floris, Keano, Michiel, Simon, Stavros and Thomas. In particular, I want to thank Alexander, whom I have considered to be a sort of professional older brother. How many stupid mistakes I was able to avoid because of your precious advice, I couldn't count. Due to you, I am now considerably less terrible at working with LaTeX, and I was able to save hours or even days on the design of this thesis layout—thank you for allowing me to borrow your template! Sorry for all the times I walked into your office interrupting your flow of concentration, and sorry for having to hear all my frustrations about silly little things. Thank you for the Hierba's, the tasty ice creams and the evenings at Den

Draak, tjapper; hopefully we can have lots more of them in the future! Another colleagues that deserves some additional praise, is Floris. I've always felt that a good friend brings out both the best and the worst in someone, and that's exactly what you do to me. May we never grow up, darling.

On a more professional note, I would like to express my gratitude towards Vladimir Vovk. Even when strict COVID-precautions were being imposed all over Europe, you made it possible for me to come over and enjoy an interesting six weeks at the Royal Holloway University of London. Thank you for having me; it is a shame that I didn't have more time in between my writing to follow up on your thought-provoking questions.

I would also like to thank all the members of the examination committee for having read and reviewed this entire dissertation. I hope that you have enjoyed the book a bit more than I have enjoyed writing it. Reading your nice words, detailed comments and unexpected questions, did make me feel enthusiastic about the whole thing again, though. I appreciate it.

Next, it is time to thank all those people that have been there for me and that didn't understand anything of what I was doing the last four years: my family and (non-colleague) friends. Let me start with my lovely parents: Steve and Marina. Dad, though your contribution to my mathematical education was rather modest, you have been an endless source of inspiration in so many other aspects of life. You've taught me to work hard, contaminated me with a cycling obsession, and allowed me to listen to some amazing music. I admire your eagerness to learn and your ability to fight for an idea, especially now, given that you are already in your 60s. I'll be proud if I end up the same way as you. Mom, thank you for the unconditional love and care you have given me—it must have been hard, somewhat more than a decade ago, when I was probably the most abominable obnoxious teenager ever. Due to your straightforward way of communicating I am now way to honest when people ask for my opinion, and thanks to your splendid cooking skills I have become way to critical when a dish is served to me—not always the best combination. Hopefully you are proud now that I will finally start a 'real' job.

Also a big thanks to my brother, Elias, for designing the wonderful cover of this book, and, of course, for being a such great influence both musically and artistically.

To all my friends: I apologize for letting this dissertation take up monstrous proportions of time, and for all those moments that I lied to you, claiming that it was going to be finished in just one or two months. Undoubtedly, it would have been better to have spent a bit more of that time with you. I thank the boys from Aalst—Laurian, Nick, Peter, Stef and Viktor—for, among all other things, providing non-stop hilarious nonsense for more than ten years; thank you Lieke for the long walks and intense talks; and thank

you Simon for your words of wisdom and destroying my knee. Of course there are many others that have a special place in my heart, yet, since this preface is already becoming longer than I intended—I guess I didn't learn anything—, I will keep it to a general 'THANK YOU!'

Finally, it remains to express my gratitude to one last person: my love, Sara. There is no one who I can be more myself with than you. You know me as I am, without inhibitions, without distortion, in my purest form. I am well aware of the fact that this is *not* always a good thing: I was stressed out when writing my dissertation, badly tempered all the times we didn't have food on time, depressed when my knee was hurting, selfish when making travel plans, and lazy when lying in the couch and in need of a foot massage. Yet, though I am not always proud of it, I strongly needed—and still need—to be this unfiltered version that you allow me to be. To share my emotions, and to digest thoughts instead of letting them swirl out of control. To not feel alienated in a world that is never to be understood. You are my friend, my companion, my fellow traveller, my warmth in cold days. For all that you have given me, love, I bow to you.

Summary

This dissertation is concerned with the study of discrete-time stochastic processes, which are dynamical systems that change over time in an uncertain way, and for which these changes only occur at discrete time instances. We will specifically focus on those cases where the state of the process at any single time instant can only take a finite number of possible values; the stochastic process is then called a finite-state discrete-time stochastic process. To describe and draw conclusions about the behaviour of such a stochastic process, we use imprecise probability models, and in particular conditional upper expectations. These so-called global upper expectations exist in many different forms and shapes, and it is our aim in this dissertation to study the theoretical aspects of these models, reveal the relations between them, and suggest new suitable global upper expectations of our own invention.

Our narrative starts with the basic setting of a single (unknown) variable taking values in a finite set, and presents three possible mathematical models for quantifying the uncertainty with respect to such a variable. These three models are all called imprecise probability models, because they generalise the standard probability model—a probability measure, charge or mass function—to robustly deal with those situations where it is infeasible or not justified to specify such a 'precise' probability model. The first type of imprecise probability model that we consider is a set of probability mass functions (or credal set), which gathers all probability mass functions that are deemed possible. The second type of imprecise probability model is a set of acceptable gambles, and captures a subject's beliefs by expressing her attitude towards gambling on the value of the uncertain variable considered. Finally, we also look at coherent upper and lower expectations; these can be interpreted behaviourally, as expressing a subject's infimum selling and supremum buying prices, or more traditionally, as representing upper and lower tight bounds on a collection of plausible (linear) expectations. We discuss the well-known relations between these models and present some basic extension procedures.

We then move on to consider the specific setting of discrete-time stochastic processes. The act of modelling such a stochastic process starts at a local

level, where we make assessments about how the process is going to change from one time instant to the next. These assessments can typically be acquired from data or subjective expert opinions, and are then mathematically expressed in terms of one of the three types of imprecise uncertainty models mentioned earlier. Though the local models form what is typically directly available from observation, in the end, we are interested in more global features of the process. These global features involve, for instance, upper and lower (bounds on) expected hitting times, expected time average behaviour, or hitting probabilities. In order to make inferences of such kind, we thus want to extend and combine the local models into a single joint 'global' model, which in our case will always be a global upper expectation. Mathematically speaking, these global models are extended real-valued operators whose first argument is an extended real-valued function or variable on the space of all possible infinite state sequences, and whose second argument is a specific type of conditioning event.

Global upper expectations can be obtained in various different frameworks, using various different techniques. We first consider three so-called finitary global upper expectations. These global upper expectations are characterised by the common property that they extend the local models without the use of any continuity assumptions, which is why we call them finitary. Their definitions are relatively simple and rely on concepts that are well-known in the theory of imprecise probabilities. One is deduced in the behavioural framework of sets of acceptable gambles, one is obtained as the upper envelope of the expectations corresponding to a set of global—finitely additive—probabilities, and one is defined axiomatically, as the natural extension under conditional coherence. We study the mathematical properties of these global operators, present alternative characterisations for them, and show that, if the local models from which these different global upper expectations are derived are chosen in accordance with each other, then all these finitary global upper expectations coincide. It will turn out, however, that these finitary upper expectations only behave well on the domain of bounded finitary variables, which are bounded real-valued functions that depend on the states of the process only at a finite number of time instances. This domain is not large enough for many practical purposes, which is why we are inclined to look at other more involved types of global upper expectations.

One first such type are the game-theoretic upper expectations introduced and, for a large part, studied by Shafer and Vovk [85, 86]. These types of global upper expectations start from sets of acceptable gambles, or sometimes upper expectations, on a local level, and then use allowable betting strategies—supermartingales—to turn these local assessments into global assessments. Concretely, the game-theoretic upper expectation represents a subject's infimum starting capital such that, by using an allowable betting strategy, he can surely hedge the uncertain pay-off corresponding to the considered variable. Multiple different versions of game-theoretic upper expectations have been used in the literature, and we first argue why one of them is to be preferred over the others. We then go on to prove a broad range of properties for this operator, with a strong emphasis on its continuity properties. We show in particular that it satisfies continuity with respect to bounded below increasing sequences, continuity with respect to decreasing sequences of bounded above finitary variables, and continuity with respect to decreasing lower cuts. These properties are considerably stronger than those of the finitary global upper expectations, and they make the game-theoretic upper expectation more suitable for use on general domains of variables.

Next, we introduce and study global upper expectations that are centrally based on the notion of a (countably additive) probability measure. More precisely, we start from local sets of probability mass functions, combine and extend these to form a set of plausible global probability measures, and the associated global upper expectation is then the upper envelope of the expectations corresponding to this set of global probability measures. The domain of this global upper expectation is furthermore extended to also include general, not necessarily measurable variables by relying on upper (Lebesgue) integrals. We again study the properties of this operator, and in particular show that its continuity properties are comparable to those of the game-theoretic upper expectation. These properties then allow us to establish that these two types of global upper expectations are equal on a fairly large domain of variables; large enough to cover most practically relevant inferences.

The final type of global upper expectation that we consider is an axiomatic one, similar to the finitary axiomatic global upper expectation, but where a continuity property is added as one of the defining axioms. Two slightly different versions of this continuity axiom—and thus also of the resulting axiomatic global upper expectation—are considered; one is weaker than the other, but since they both solely apply to sequences of bounded finitary variables they are actually both fairly weak. We show that the axiomatic upper expectation based on the stronger of the two axioms coincides with the game-theoretic upper expectation, and therefore also for a large part with the measure-theoretic upper expectation. The axiomatic upper expectation based on the weaker axiom, on the other hand, can be seen as an 'imprecise-probabilistic' generalisation of the Daniell integral [19]. We show that, though this weaker type of axiomatic upper expectation is sometimes too conservative, it still is equal to its stronger counterpart, and therefore possesses desirable properties, in many practical situations.

Samenvatting

Dit proefschrift richt zich op de studie van stochastische processen in discrete tijd; dynamische systemen die op een onzekere manier veranderen doorheen de tijd, en waarbij deze veranderingen zich enkel voordoen op discrete tijdstippen. We zijn in het bijzonder geïnteresseerd in die gevallen waar de toestand van het proces op elk tijdstip slechts een eindig aantal waarden kan aannemen. Het proces heeft dan een zogenoemde eindige toestandsruimte. Om het gedrag van een dergelijk proces wiskundig te beschrijven, alsook om erover te redeneren, gebruiken we imprecieze-waarschijnlijkheidsmodellen, en meer bepaald, conditionele bovenverwachtingswaardeoperatoren. Er bestaan veel verschillende soorten zulke zogenoemde 'globale' bovenverwachtingswaardeoperatoren, en het is ons doel om de theoretische eigenschappen van deze operatoren te bestuderen, hun onderlinge relaties te onthullen, en nieuwe gepaste conditionele bovenverwachtingswaardeoperatoren in te voeren.

Allereerst beschouwen we het eenvoudige geval van een enkele onzekere veranderlijke die een eindig aantal waarden kan aannemen, en behandelen we drie soorten modellen die ons in staat stellen om de onzekerheid over een dergelijke veranderlijke wiskundig te beschrijven. Deze drie modellen worden alle imprecieze-waarschijnlijkheidsmodellen genoemd, omdat ze het klassieke waarschijnlijkheidsmodel – een waarschijnlijkheidsmaat, -lading, of -massa – veralgemenen om robuust te kunnen handelen in die situaties waar het niet mogelijk is, of niet gerechtvaardigd is, om zo'n klassiek 'precies' waarschijnlijkheidsmodel te specificeren. Het eerste type imprecieze-waarschijnlijkheidsmodel is een verzameling van massafuncties, of ook credale verzameling genoemd. Zo'n verzameling bevat alle massafuncties die we mogelijk achten. Het tweede type imprecieze-waarschijnlijkheidsmodel is een verzameling van aanvaardbare gokken, en tracht iemands overtuigingen voor te stellen door uit te drukken welke gokken over de waarde van een onzekere veranderlijke hij of zij bereid is aan te gaan. Tot slot bekijken we ook coherente boven- en onderverwachtingswaardeoperatoren; zij kunnen gedragsmatig geïnterpreteerd worden, als iemands minimale verkoopsprijzen en maximale aankoopprijzen, of op een meer traditionele manier, als nauwe boven- en ondergrenzen op een verzameling van mogelijke (lineaire) verwachtingswaardeoperatoren. We bespreken de gekende relaties tussen deze drie types modellen en behandelen enkele eenvoudige methoden die ons in staat stellen om deze modellen uit te breiden.

Vervolgens kijken we naar de specifieke context van stochastische processen in discrete tijd met eindige toestandsruimte. Het modelleren van zo'n proces begint typisch op een lokaal niveau, waar we uitspraken doen over hoe (wij geloven dat) het proces zal veranderen van het ene tijdstip naar het volgende. Hiervoor kan men zich vaak baseren op beschikbare data of op de mening van een ervaringsdeskundige. Deze lokale uitspraken worden dan wiskundig voorgesteld door een van de drie imprecieze-waarschijnlijkheidsmodellen die we zojuist hebben beschreven. Hoewel men vaak wel een idee heeft over het lokale gedrag, zijn we uiteindelijk voornamelijk geïnteresseerd in de meer globale kenmerken of eigenschappen van een stochastisch proces. Zulke kenmerken zijn bijvoorbeeld boven- en ondergrenzen op de verwachte tijd tot bereik, en boven- en ondergrenzen op verwacht tijdsgemiddeld gedrag. Om conclusies te kunnen trekken over zulke globale aspecten willen we de lokale modellen uitbreiden en combineren tot een enkel globaal model, dat in ons geval de vorm zal aannemen van een globale bovenverwachtingswaardeoperator. Zo'n globale bovenverwachtingswaardeoperator is wiskundig gezien een uitgebreid-reëlwaardige functionaal wiens eerste argument een uitgebreid-reëlwaardige functie of veranderlijke is op de ruimte van alle mogelijke oneindige toestandsrijen, en wiens tweede argument een specifiek soort conditionerende gebeurtenis is.

Globale bovenverwachtingswaardeoperatoren kunnen verkregen worden op veel verschillende manieren. We behandelen eerst drie soorten zogenoemde finitaire bovenverwachtingswaardeoperatoren. Deze bovenverwachtingswaardeoperatoren worden gekarakteriseerd door de gemeenschappelijke eigenschap dat ze de lokale modellen uitbreiden zonder gebruik te maken van enige continuïteitsaannames – daarom noemen we ze dus finitair. De definities van deze modellen zijn relatief eenvoudig en steunen op concepten die welgekend zijn in de imprecieze waarschijnlijkheidsleer. Er is er een die afgeleid is uit het concept van een verzameling van aanvaardbare gokken, een die de vorm aanneemt van bovengrenzen op de verwachtingswaarden die overeenkomen met een verzameling van globale – eindige additieve – waarschijnlijkheden, en een die axiomatisch gedefinieerd wordt, als de natuurlijke extensie onder conditionele coherentie. We behandelen de wiskundige eigenschappen van deze globale operatoren, presenteren alternatieve karakteriseringen, en tonen aan dat, als de lokale modellen waarvan deze globale modellen zijn afgeleid in overeenstemming zijn met elkaar, alle finitaire bovenverwachtingswaardeoperatoren samenvallen.

Anderzijds zullen we ook zien dat deze finitaire bovenverwachtingswaardeoperatoren enkel geschikt zijn voor gebruik op het domein van begrensde finitaire veranderlijken; dit zijn begrensde reëlwaardige functies die afhangen van de toestanden van het proces op slechts een eindig aantal tijdstippen. Dit domein is niet groot genoeg voor de meeste praktische doeleinden en daarom zijn we ertoe genoopt om andere, meer complexe soorten globale bovenverwachtingswaardeoperatoren te onderzoeken.

Een eerste dergelijke soort globale bovenverwachtingswaardeoperator die we bekijken is de speltheoretische bovenverwachtingswaardeoperator, ingevoerd en bestudeerd door Shafer en Vovk [85, 86]. Zulke globale operatoren starten, op een lokaal niveau, van verzamelingen van aanvaardbare gokken, of soms van lokale bovenverwachtingswaardeoperatoren, en gebruiken dan toelaatbare gokstrategieën – supermartingalen – om deze lokale informatie om te zetten naar globale bovenverwachtingswaarden. Meer concreet geeft de speltheoretische bovenverwachtingswaardeoperator het minimale startkapitaal aan waarmee iemand, door een toelaatbare gokstrategie te kiezen en aan te houden, met zekerheid uiteindelijk meer geld zal hebben dan de onzekere prijs die verbonden is met de beschouwde veranderlijke. Verschillende versies van de speltheoretische bovenverwachtingswaardeoperator zijn in het verleden gebruikt en we tonen eerst aan waarom een specifieke versie te verkiezen is boven alle anderen. Vervolgens bewijzen we een hele reeks eigenschappen voor deze operator, met een bijzondere klemtoon op zijn continuïteitseigenschappen. We tonen onder andere aan dat hij voldoet aan continuïteit ten opzichte van naar onder begrensde stijgende rijen, continuïteit ten opzichte van dalende rijen van naar boven begrensde finitaire veranderlijken, en continuïteit ten opzichte van dalende onder-sneden. Deze eigenschappen zijn beduidend sterker dan die van de finitaire bovenverwachtingswaardeoperatoren, waardoor de speltheoretische bovenverwachtingswaardeoperator geschikter is voor het gebruik op een algemeen domein.

Daarnaast introduceren en behandelen we ook een globale bovenverwachtingswaardeoperator die afgeleid is uit het concept van een (aftelbaar additieve) waarschijnlijkheidsmaat. Meer bepaald starten we van lokale verzamelingen van massafuncties, vervolgens combineren we ze en breiden we ze uit tot een verzameling van globale waarschijnlijkheidsmaten, en tot slot definiëren we de geassocieerde globale bovenverwachtingswaardeoperator als de kleinste bovengrens van de verwachtingswaarden afgeleid uit deze verzameling van globale waarschijnlijkheidsmaten. Door gebruik te maken van (Lebesgue-)boven-integralen, wordt het domein van deze operator bovendien uitgebreid zodat het ook niet noodzakelijk meetbare veranderlijken bevat. We onderzoeken de eigenschappen van deze maattheoretische operator, en tonen in het bijzonder aan dat zijn continuïteitseigenschappen gelijkaardig zijn aan die van de speltheoretische bovenverwachtingswaardeoperator. Deze eigenschappen stellen ons vervolgens in staat om te bewijzen dat deze twee soorten operatoren samenvallen op een betrekkelijk groot domein van veranderlijken; een domein dat de meeste praktisch relevante veranderlijken bevat.

Een laatste soort globale bovenverwachtingswaardeoperator die we bekijken is axiomatisch gedefinieerd, gelijkaardig aan de finitaire axiomatische globale bovenverwachtingswaardeoperator, met dit belangrijk verschil dat er nu een continuïteitsaxioma wordt toegevoegd als een van de karakteriserende axioma's. Twee enigszins verschillende versies van dit continuïteitsaxioma – en dus ook van de resulterende axiomatische globale bovenverwachtingswaardeoperator – worden behandeld; het ene is zwakker dan het andere, maar aangezien ze allebei enkel betrekking hebben op rijen van begrensde finitaire veranderlijken zijn ze eigenlijk beiden relatief zwak. We tonen aan dat de axiomatische globale bovenverwachtingswaardeoperator gebaseerd op het sterkere axioma samenvalt met de speltheoretische bovenverwachtingswaardeoperator, en daardoor ook op een groot gebied samenvalt met de maattheoretische bovenverwachtingswaardeoperator. Anderzijds kan de axiomatische globale bovenverwachtingswaardeoperator gebaseerd op het zwakkere axioma kan gezien worden als een 'imprecies-probabilistische' veralgemening van de Daniell-integraal [19]. We tonen aan dat, hoewel deze globale bovenverwachtingswaardeoperator in sommige gevallen te conservatief is, hij desondanks op een groot gebied samenvalt met zijn axiomatisch sterkere tegenhanger, en dus veel van zijn gewenste eigenschappen overneemt.

— Chapter I —

INTRODUCTION

Probability measures are without doubt amongst the most common and celebrated mathematical tools to quantify beliefs and draw inferences about the uncertain evolution of a discrete-time stochastic process [33, 52, 54, 90]. Recent decades however have seen the rise of an entire family of alternative and more general uncertainty models, which we commonly refer to as imprecise probability models [3, 83, 106, 110]. These models are characterised by the common property that they allow reasoning to be performed in an informative and conservative way, even in those situations where it is infeasible or inappropriate to specify a single probability measure. Such situations may for instance arise when data about the considered stochastic process is scarce, or when expert judgements are conflicting.

However, with the exception of game-theoretic upper and lower expectations [85, 86], most imprecise probability models were not specifically designed with the stochastic processes setting in mind. This setting is somewhat unique, and it is often not clear how imprecise probability models can—and should—be adapted and applied to it. Some of the approaches that we will discuss here are not entirely new, and can be seen as modified versions of already existing approaches. Some others, then, are suggestions of our own invention. Yet, whatever their origin, it is the possible properties of such adapted imprecise probability models that will turn out to be decisive for reaching acceptance amongst a broad audience. Our aim in this respect is to clarify, to shed light on the characteristic properties of these models, and to bring forth the mathematical relations that either tie them together or set them apart.

1.1 Context and motivation

A stochastic process can be roughly described as any system or phenomenon that changes—typically over time—in an uncertain way. Such processes are omnipresent and we have to deal with them in everyday life. Think for instance of the COVID pandemic and the related number of infected people. Or, not unrelated to this, the fluctuations of the stock market. A good understanding of such processes has never been more vital.

This dissertation is not a work on medicine or virology though, and neither is it concerned with finance. We study stochastic processes from a purely mathematical perspective, where they are regarded as collections of uncertain variables or uncertain states indexed by time. We focus on discrete-time stochastic processes [35, 45, 54, 86, 90], which are processes whose time index takes values in the set of natural numbers—this in contrast with continuous-time stochastic processes [33, 52] where time takes values in the set of positive real numbers. Such processes are typically accompanied by a number of parameters that describe—in a non-deterministic manner—the (uncertain) values of the individual uncertain states. From these parameters one aims to draw global inferences such as, for instance, average/ergodic behaviour or expected hitting times. The mathematical research on discrete-time stochastic processes, and in particular on discrete-time Markov chains [45, 48, 54], has been going on for more than a century, and our acquired insights about them are being applied extensively in a wide variety of scientific fields, including mathematical finance [71, 79], queueing theory [2, 71], biology [36, 41] and many more. Moreover, it is worth noting that the mathematical treatment of discrete-time stochastic processes goes back to the pioneering work of Huygens, de Fermat and Pascal [51, 84]—often associated with the very dawn of probability theory—where they first appeared as chance games between two or more players.

The act of modelling a discrete-time stochastic process almost always starts off on a local level, where we quantify beliefs about how the process is likely to change from one time instant to the next. That is to say, for any time instant $k \geq 0$ and for every possible evolution of the process up to time k, we make non-deterministic (or probabilistic) statements about the state X_{k+1} at the next time instant. For instance, in the case of the COVID pandemic, we typically specify the expected number of people that will be infected by tomorrow or next week (being X_{k+1}), based on a growth ratio and the number of infected people during the past week or month. If such local assessments are expressed in terms of probability mass functions or probability distributions on the possible values of the next state X_{k+1}, and if these probabilities only depend on the current state of the process and not on any past states, then we say that the process is a finite-state Markov chain [45, 48, 54]—for instance, for the COVID pandemic, if our probability distribution for tomorrow's number of infected people only depends on today's number of infected people and today's growth ratio, and not on any past values of these variables. Markov chains are only special cases though, and in general, assessments about the incremental change of the process's state may depend on the entire history of the process, and may also be mathematically expressed in terms of various uncertainty models different from probability mass functions or probability distributions. One restricting assumption that we always make, though, is that the process's state X_k at any given time instant k can only take a *finite number* of different possible values.

One starts from local assessments because they are typically what is readily available from data or expert knowledge, yet our eventual interest typically lies in more global features of the stochastic process. This could involve, for instance, the time average of a real-valued function on the states of the process, or the time until the process state will attain a certain value. In the case of the COVID pandemic, a typical question of interest would be: 'what is the expected time until all available intensive care beds in the country are occupied?' The local uncertainty models do not tell us anything about such features—at least not directly. As such, we are confronted with a challenge central in the study of all discrete-time stochastic processes: *'How do we combine and extend local uncertainty models such that we can make informative judgements about global properties of the process?'*

As it often goes with questions of considerable importance, this one too does not have a single all-encompassing answer. There are many possible routes one can follow, each with their own strengths and flaws. Perhaps the most famous—or infamous depending on one's perspective—one is the measure-theoretic approach [5, 32, 81, 90, 112]. In this approach, the local probabilities are combined and extended to a single (countably additive) global probability measure on a sufficiently large domain of global events—subsets of the space of all possible trajectories of the process. These global probability measures then lead, by means of integration, to expectations, which can on their turn be used to draw inferences.

Yet, in spite of its popularity, the measure-theoretic approach has some drawbacks. One, for instance, is the multitude of abstract mathematical concepts on which the theory is founded, which may hinder users to come to grips with the practical meaning of the treated objects. The most important however, we feel, is the fact that the theory assumes that beliefs about the local dynamics can be modelled, for every possible history up until some time instant, by a single probability mass function. When information or data about the process is scarce, for instance due to time or budget restrictions, or when it is inconsistent, for instance due to conflicting expert statements, it is often unwarranted to specify such a single mass function [110]. During the start of the COVID pandemic, for example, there was little epidemiological data on which we could rely and, on top of that, expert opinions were seriously divided. In those situations, we are typically—or should be—inclined to act conservatively and only make partial judgements about probability mass functions, leading us to consider an entire set of 'plausible' probability mass functions. In fact, in some cases, even sets of probability mass functions do not suffice or are not the appropriate tool to express a subject's beliefs.

To model the local dynamics in a more general and robust manner, we instead use three types of so-called *imprecise probability models* [3, 83, 106, 110]. One type of model are the sets of probabilities—also known as credal sets[1]—mentioned above. Another model are sets of acceptable gambles, where a subject expresses its beliefs about an uncertain phenomenon by simply specifying which gambles—uncertain payoffs depending on the outcome(s) of the phenomenon—she is willing to accept (or reject). Lastly, we also consider upper expectations; these are generalisations of the traditional (linear) expectations and can be interpreted either behaviourally, as a subject's infimum acceptable selling prices for gambles, or probabilistically, as upper bounds on the expectations corresponding to a set of probabilities.

Though locally we will consider all three of the above imprecise probability models, in the end, on a global level, we will only be interested in the resulting upper expectations or, better, the *global upper expectations*. Mathematically speaking, such a global upper expectation is an operator that associates with each (possibly extended) real-valued function/variable f on the sample space—the set of all possible infinite paths that the process can follow—and any conditioning event[2] A, an extended real number, which we simply call the upper expectation of f conditional on A. One reason for our focus on global upper expectations is that these operators arise naturally in both a behavioural framework with sets of acceptable gambles and a probabilistic framework with sets of probability charges or measures. More important is that, regardless of the framework one works in, (global) upper expectations are typically of central interest when one aims to draw inferences about a stochastic process. In that respect, they fulfil the same role as traditional linear expectations, with the caveat that they only provide us with upper and lower bounds[3] rather than precise numerical values for inferences such as expected time-averages or expected hitting times. Yet, this should not alarm us, nor surprise us; these partial judgements are only a result of the fact that the local models were designed—in contrast to traditional precise models—with the purpose to correctly distinguish between

[1]The term 'credal set' is used more specifically to refer to sets of finitely additive probability charges that are closed and convex.

[2]Strictly speaking, the conditioning events will not be general subsets of the sample space, but will always assumed to correspond to a (single) possible history of the process up until some finite time instant.

[3]Lower bounds are provided by (global) lower expectations rather than (global) upper expectations. Nonetheless, lower expectations are mathematically speaking equivalent to upper expectations because they can be put in a one-to-one relation—this relation is often called the *conjugacy relation*—hence, why we focus on upper expectations only.

what *is* known and what is *not* known. Hence, the upper and lower bounds provided by global upper expectations allow us, in the end, to reason in a way that is more robust and conservative with respect to our own ignorance.[4]

The field of imprecise probabilities is, compared to traditional probability theory, still in its infancy. For the purpose sketched above, where we want to extend local models to a single global upper expectation, one could employ numerous possible methods. Yet some of them have never been applied in this specific context before. Others are simply badly documented. A first aim of this dissertation is to clarify on this account, by providing an overview of the possible approaches and presenting suitable definitions for each of them—sometimes developed here for the first time. We distinguish between six types of global upper expectations, depending on the type of local model they start from and the extension arguments they rely on.

A first class are the finitary global upper expectations. There are three of them; one based on sets of finitely additive probabilities, one based on sets of acceptable gambles, and one axiomatic type that is based on the notion of conditional coherence for upper expectations [106, 110, 113]. We call them finitary because they extend local models solely employing finitary arguments—no continuity arguments are involved. Their definitions are simple and intuitive, and follow from applying concepts well-known within imprecise probability theory. Their critical shortcoming, however, is that they only work well for functions on the sample space that are bounded and finitary—the latter meaning that the function only depends on the states of the process up to some finite time instant. This domain is insufficient for many practical purposes; the hitting time of a certain state value, for instance, is a function on the sample space that depends on the entire infinite path taken by the process and that may sometimes take the value $+\infty$.

A second and—we think—more interesting class of global upper expectations are those whose definition involves one or more continuity assumptions. Such continuity assumptions allow us to broaden the domain of variables that can be mathematically reasoned with in a meaningful way. More specifically, these continuity-based operators allow us to reason with extended real-valued (not necessarily finitary) functions on the sample space, including hitting times/probabilities or limiting time averages. The catch, though, is that their analysis is considerably more challenging.

A first such type of global upper expectation can be seen as a generalisation of the measure-theoretic (linear) expectations that we spoke about earlier. The starting point are sets of local probability mass functions—one

[4]Apart from Walley's seminal work [110], we also recommend [57, Chapter 1] for a short but excellent read on the motivation for upper expectations in stochastic processes.

set for each possible finite history of the process. We pick a single probability mass function from each of these sets and then construct a corresponding global probability measure by using techniques familiar from classical measure theory.[5] By repeating this for every possible selection of local probability mass functions, we obtain an entire set of compatible global probability measures. The resulting *measure-theoretic upper expectation* is then obtained by taking an upper envelope over the expectations associated with this set of global probability measures.

A second type of continuity-based global upper expectation that we study are the *game-theoretic upper expectations* introduced and advocated by Shafer and Vovk [85, 86]. As the name itself suggests, these operators are derived from a game-theoretic type of reasoning, where the specifications of the local models, in the form of sets of acceptable gambles or upper expectations, characterise the moves of a first player—'Forecaster'—and where a second player—'Skeptic'—aims to become rich by betting against Forecaster's moves. The possible evolutions of Skeptic's capital form (super)martingales, and the corresponding global upper expectation of a function f is the smallest possible value for which there is a (super)martingale that starts in this value and eventually hedges—exceeds the value of—f.[6] Game-theoretic upper expectations are attractive because they combine a high level of generality with an easy-to-use constructive flavour, while still satisfying many powerful limit laws and continuity properties. These advantages have already led to game-theoretic upper expectations being applied in a multitude of occasions [8, 26, 58, 60, 88].

Last in our list of continuity-based upper expectations is a suggestion of our own. It is an axiomatic model that aims to combine the intuitive and universal elements of the finitary global upper expectations, and the powerful mathematical properties of the continuity-based global upper expectations. The axioms on which its construction is based are clear and simple; it is a combination of the well-known coherence properties and a single, rather weak continuity property. A conservativity argument is furthermore used to determine, amongst all the global upper expectations that satisfy these properties, our unique desired axiomatic model.

Apart from presenting, motivating and developing possible approaches to arrive at a global upper expectation, a second aim of this dissertation—which of course partially influences the first—is to study the properties of these global upper expectations. In particular, a considerable part of our work is devoted to proving—and disproving—continuity properties. Such

[5]The continuity assumption in this case comes disguised under the form of countable additivity, which is by definition satisfied for a probability measure.

[6]The continuity assumption here comes disguised under the fact that we allow supermartingales to hedge the considered variable at 'infinity'.

	local model	global upper expectation	
		finitary	continuity-based
behavioural	sets of acceptable gambles	upper expectations from global sets of acceptable gambles	game-theoretic upper expectations from supermartingales
axiomatic	upper expectations	most conservative extension under coherence	most conservative extension under coherence + continuity
probabilistic	sets of probability mass functions	upper expectations from sets of finitely additive probabilities	measure-theoretic upper expectations from sets of probability measures

Figure 1.1 Overview of the approaches treated in this dissertation.

continuity properties are important from both a theoretical and a practical point of view, but their relevance is perhaps best illustrated by simply recalling measure-theoretic expectations and how two of their most celebrated properties—the dominated convergence theorem and the monotone convergence theorem—have contributed to the success of measure theory in modern probability theory.

A final aim of the dissertation is to establish relations between the different types of global upper expectations. We will show that all finitary global upper expectations coincide, at least if their respective local models are chosen in accordance with each other. More importantly, we will show that the same is, to a large extent, true for the continuity-based global upper expectations. The merit of such connections is obvious as they allow us to take results, properties and algorithms developed for only one type of global upper expectation and apply them to all other equivalent global upper expectations. Even more important is that, due to such results, we may arrive at a consensus about the proper choice of an imprecise global model; any of the three is suitable because, in the end, it does not matter which is chosen.

1.2 Related work

As already mentioned, the study of discrete-time stochastic processes goes back to the work of Christiaan Huygens [51] in 1657, who himself was inspired by the conversations between Blaise Pascal and Pierre de Fermat [20, 84]. The early 20th century saw a burst of developments in the field of measure theory, which were synthesized and used by Kolmogorov [56] to form the mathematical foundations for his axiomatic approach to probability theory. Since then, probability theory—and the study of discrete-time stochastic processes in specific—has been largely based on measure-theoretic principles.

In spite of this popular status, it seems that relatively little attention was devoted to rigorously developing a measure-theoretic approach in a stochastic processes setting where initial local models come in the form of sets of probabilities. Some considerable effort has already been put into generalising probability-based precise models for specific types of processes [10, 45, 92], but this research typically only involves global (upper and lower) expectations obtained from finitely additive probabilities (rather than countably additive probabilities) and on the domain of finitary variables—in that sense, they can hardly be called 'measure-theoretic'. On the other hand, the study of Miranda & Zaffalon [66] is in line with what we will do, in the sense that it examines the continuity properties of upper and lower envelopes over sets of countably additive probabilities (or previsions). Unfortunately though, this study is not adapted to the stochastic processes setting. Lopatatzidis [62] proposes a measure-theoretic model that is very similar to our measure-theoretic global upper expectation, yet his results focus mainly on the domain of finitary bounded variables. A recently discovered contribution is that of Cohen et al. [7]; their extended sub-linear expectations seem closely related to our global measure-theoretic upper expectations, yet the work in [7] seems to be mainly concerned with integrability conditions and martingale properties, rather than properties of the global sub-linear expectation operator—a more thorough examination is required before we can do precise statements though. Finally, there is also the well-established theory of capacities and Choquet integration as introduced by Choquet [6] and further developed by Dellacherie [28], Denneberg [31], and Greco [42, 43]. This theory generalises the classical measure-theoretic picture to deal with imprecision by using a capacity instead of a single probability measure. The corresponding extension procedures are not specifically designed for the setting of stochastic processes, yet the adaptation to this setting is rather immediate and the resulting method is then close in spirit to what we will do here. Nonetheless, the notion of a capacity—a specific type of non-additive measure—is less general than the sets of probability charges/measures that we will consider [14, 31, 106].

The relative lack of interest in measure-theoretic models within the field of imprecise stochastic processes—and imprecise probabilities in general—is most likely due to the fact that imprecise probabilities has its roots in the work of de Finetti [27], P. M. Williams [113] and Walley [110], who all took betting behaviour rather than probability measures as a primitive notion. The notion of coherence—central in the betting based approach of [113] and [110]—has not been applied very often within the context of discrete-time stochastic processes, except for those instances where the process involves finitely many time steps, or where the variables of interest are of the finitary type [9, 11, 25]. A more popular tool seems to be the game-theoretic upper expectations developed by Shafer and Vovk [85, 86, 109]. Shafer and Vovk themselves drew inspiration from the work of Ville [107], whose ideas did not receive immediate recognition and were unjustly overlooked by many. Since the release of Shafer and Vovk's first book [86] however, game-theoretic probabilities and functionals have become significantly more popular, leading to a multitude of advances [8, 9, 60, 62, 85, 88, 101]. Our contributions to the field of game-theoretic upper expectations can be found in, among others, [95, 97, 98].

Finally, a related line of research that came under our attention only recently, is that on the non-linear expectations introduced by Peng [73]. Especially the contributions of Denk et al. [30], Nendel [68] and, as mentioned earlier, Cohen et al. [7] bear a close connection with our work and deserve to be further investigated.

1.3 Overview of the chapters

We start our narrative in Chapter 2_{17} with the introduction of three imprecise probability models: sets of probabilities, sets of acceptable gambles and coherent upper (and lower) expectations. We consider a single uncertain variable taking values in a finite possibility space, and show that a subject's beliefs about such a variable can be suitably expressed in either of these three models. We employ well-known results by P. M. Williams [113] and Walley [110] to establish close connections between the three models, and also briefly discuss some extension methods.

The possibility space in Chapter 2_{17} is assumed finite, because the models introduced there are used in Chapter 3_{45} to define the local models of our discrete-time stochastic processes, the state space of which we assume to be finite. After we have done so, Chapter 3_{45} splits into three major sections; each of them is devoted to a single type of local model, and shows how these local models can be extended to a global upper expectation. These extensions will largely rely on rather well-established—finitary—notions such as conditional coherence and conditional probability charges [18, 34, 106, 110, 113]. Due to the specific context of stochastic processes, elegant alternative characterisations can be given for these three finitary global upper expectations. One is that the acceptability-based global upper expectation can be seen to coincide with a modified type of game-theoretic upper expectation where supermartingales are required to hedge at a finite time horizon. On the other hand, we also establish a convenient axiomatisation for the notion of conditional coherence (for global upper expectations) and that of a conditional probability charge. Finally, we show that all the different types of finitary global upper expectations coincide—if their respective local models are chosen in accordance with each other—and prove or disprove some important properties. Crucially, we show that these finitary global upper expectations lack basic continuity properties and are therefore unsuitable to be applied on a general domain of variables.

The structure of Chapter 3_{45} is then more or less repeated for the continuity-based global upper expectations, but on a larger scale; Chapter 4_{129} is devoted to game-theoretic upper expectations, Chapter 5_{217} is devoted to measure-theoretic upper expectations, and Chapter 6_{283} studies axiomatic continuity-based upper expectations. These three chapters form the core of this dissertation, as most of our novel ideas and results are presented therein.

Chapter 4_{129} starts with a discussion of the different possible definitions for a global game-theoretic upper expectation. We reason in a stepwise manner, always enlarging the domain of variables, and making modifications to the original—most basic—definition in order to fit our needs. This part also involves procedures for extending the local models—sets of acceptable gambles, but also upper expectations—to deal with extended real-valued local variables. Eventually, we end up with a version of the global game-theoretic upper expectation that is equivalent to Shafer and Vovk's latest version in [85, Part II], but where our local models need not be expressed in terms of upper expectations, but can also be expressed in terms of sets of acceptable gambles. We then continue to present a series of fundamental results for this global upper expectation; some of them have already been stated elsewhere, and then we simply adapt their proofs to our setting; some of them are entirely of our own invention. Among many other results, we prove a law of iterated upper expectations, continuity from below, continuity from above with respect to finitary gambles and Fatou's lemma. At the end of the chapter, we come back to the different possible definitions of the game-theoretic upper expectation, and show that, in retrospect, the version adopted by us in the preceding part—and thus also the one adopted by Shafer and Vovk—could have been replaced by an equivalent but more intuitive and direct version.

Chapter 5_{217} starts off as one would expect, by introducing some standard measure-theoretic notions and terminology. We define countable additivity for the global (conditional) probability charges introduced in Chapter 3_{45}, and use the acquired so-called global probability measures to define conditional linear expectations on measurable extended real-valued variables. A variant of the upper Lebesgue integral will then provide us with a suitable—non-linear—extension to the domain of all extended real-valued variables. In this precise setting, where we consider only a single local probability mass function for every possible history of the process, we show that this measure-theoretic approach is entirely equivalent to the game-theoretic approach. Afterwards, we consider the more general imprecise setting where local models are given by sets of probability mass functions, and define the corresponding global measure-theoretic upper expectation as an upper envelope over the compatible 'precise' measure-theoretic upper expectations. We then show that this imprecise measure-theoretic upper expectation satisfies several types of continuity, which on its turn allows us to infer that measure-theoretic upper expectations and game-theoretic upper expectations coincide on a fairly large domain; it includes all bounded measurable variables and, for closed local models, all monotone limits of finitary gambles.

In Chapter 6_{283}, we propose to modify the finitary coherence-based approach from Chapter 3_{45} by simply adding an extra continuity axiom (and a straightforward monotonicity axiom). We discuss several possibilities, and come up with a specific continuity axiom that suffices for obtaining a global upper expectation that is equally powerful as game-theoretic and measure-theoretic upper expectations. In fact, we will show that this axiomatic global upper expectation is always equal to the game-theoretic upper expectation, and thus to a large extent also equal to the measure-theoretic upper expectation. We moreover prove a series of alternative characterisations for this axiomatic model, and show that it bears a close relationship with Daniell's notion of an upper integral [19].

The dissertation is concluded in Chapter 7_{323}, where we look at the larger picture and explain what role our work might play in further research on stochastic processes.

1.4 Publications

This manuscript contains much of what I—with the help of many fellow researchers—have developed during the past four years as a PhD student. Many of the presented results can already be found elsewhere, yet this book aims to synthesize them into a single all-encompassing picture. Specifically, this dissertation gathers results from the following publications.

(i) Natan T'Joens, Gert de Cooman & Jasper De Bock. Continuity of the Shafer-Vovk-Ville operator. In: **Proceedings of the 9th International**

Conference on Soft Methods in Probability and Statistics. Vol. 832. 2018, pp. 200–207

(ii) Natan T'Joens, Jasper De Bock & Gert de Cooman. In search of a global belief model for discrete-time uncertain processes. In: **Proceedings of the 11th International Symposium on Imprecise Probabilities: Theories and Applications**. Vol. 103. 2019, pp. 377–385

(iii) Natan T'Joens, Jasper De Bock & Gert de Cooman. Game-theoretic upper expectations for discrete-time finite-state uncertain processes. In: **Journal of Mathematical Analysis and Applications 504**.2 (2021)

(iv) Natan T'Joens, Jasper De Bock & Gert de Cooman. A particular upper expectation as global belief model for discrete-time finite-state uncertain processes. In: **International Journal of Approximate Reasoning 131** (2021), pp. 30–55

(v) Natan T'Joens & Jasper De Bock. Global upper expectations for discrete-time stochastic processes: in practice, they are all the same! In: **Proceedings of the 12th International Symposium on Imprecise Probabilities: Theories and Applications**. Vol. 147. 2021, pp. 310–319

The present work also includes various new ideas and results that have never been published before. The most significant among these, we believe, are:

- The connections between the different possible axiomatic (continuity-based) approaches, and how these on their turn relate to an imprecise Daniell-like approach; see Section 6.3_{294} and Section 6.4_{302}.

More modest, but still noteworthy unpublished contributions are:

- Many of the definitions and results presented in Chapter 3_{45}, including the equivalence between acceptability-based and (finitary) game-theoretic global upper expectations [Section $3.2.3_{61}$], the axiomatisation of conditional coherence [Section $3.4.1_{81}$].
- The connections between local sets of acceptable extended real-valued gambles and extended(-real valued) local upper expectations in Section 4.3_{152}.
- The results in Section 5.4_{240} and Section 5.5_{249}, which generalise many of those in Publication (v) above to deal with local sets of probability mass functions that are not necessarily closed and convex.

During my time as a PhD student, I have also engrossed myself in the theory of imprecise discrete-time Markov chains. Imprecise Markov chains are generalisations of classical 'precise' Markov chains, where local transition probabilities are replaced by sets of local probabilities and where the Markov assumption applies to these sets as a whole—so these sets are assumed to only depend on the current state of the process and not on any past states.[7] The work presented here in this dissertation is intended for a setting with general, not necessarily memoryless, local models—whether that be sets of probabilities or other types of local models—and so it in particular applies to the setting of imprecise Markov chains.

My work on imprecise Markov chains has focused on the long-term time average behaviour—or ergodic behaviour—of these stochastic processes, and on practical algorithms for computing certain types of inferences. I have chosen not to include this work in this dissertation though, because I feel that a more coherent and convincing story can be told by restricting myself solely to the study of global upper expectations.

My work on imprecise Markov chains can be found in the following articles.

(vi) Natan T'Joens, Thomas Krak, Jasper De Bock & Gert de Cooman. A recursive algorithm for computing inferences in imprecise Markov chains. In: **Proceedings of the 15th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty**. Vol. 11726. 2019, pp. 455–465

(vii) Thomas Krak, Natan T'Joens & Jasper De Bock. Hitting times and probabilities for imprecise Markov chains. In: **Proceedings of the 11th International Symposium on Imprecise Probabilities: Theories and Applications**. Vol. 103. 2019, pp. 265–275

(viii) Natan T'Joens & Jasper De Bock. Limit behaviour of upper and lower expected time averages in discrete-time imprecise Markov chains. In: **Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU2020, Proceedings**. Vol. 1237. 2020, pp. 224–238

(ix) Natan T'Joens & Jasper De Bock. Average behaviour in discrete-time imprecise Markov chains: a study of weak ergodicity. In: **International Journal of Approximate Reasoning 132** (2021), pp. 181–205

(x) Jasper De Bock & Natan T'Joens. Average behaviour of imprecise Markov chains: a single pointwise ergodic theorem for six different models. In: **Proceedings of the 12th International Symposium on Imprecise Probabilities: Theories and Applications**. Vol. 147. 2021, pp. 90–99

[7] Strictly speaking, these are called imprecise Markov chains *under epistemic irrelevance* [8, 46].

1.5 Navigating this dissertation

This work is divided into seven chapters—the current introductory chapter included—and is provided with a list of symbols and a bibliography near the end. Chapters are divided into sections, which are themselves sometimes further divided into subsections or even subsubsections—the latter will not be numbered. Chapters are often accompanied by one or more appendix sections, where we then gather results and proofs that we believe would otherwise, due to their technical nature or considerable size, obscure some of our arguments in the main text.

External references are denoted by a number between square brackets; the corresponding number in the bibliography at the end of this manuscript then gives the full reference to the appropriate piece of literature. So, for instance, [33] is a book on stochastic processes written by Doob and published in 1953. This dissertation also includes a multitude of internal references to theorems, lemmas, propositions, equations,... To enhance readability, we accompany these internal references with a subscript number that indicates the page on which the referred content can be found; so Theorem $4.4.4_{166}$ can be found on page 166. If the content to which we refer is on the previous or subsequent page, then the subscript number is replaced by the symbols $\curvearrowleft$ and $\curvearrowright$ respectively; if the content is on the same double-page spread, then the subscript is omitted. For instance, the next section is Section 1.6.

1.6 Some mathematical notations

We finish this initial chapter with the introduction of some basic notions that will be used throughout the entire manuscript.

Number sets

We use $\mathbb{N}$ to denote the set of all natural numbers (without zero), and we let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. $\mathbb{R}$ is the set of all real numbers, and $\mathbb{R}_\geq$, $\mathbb{R}_>$ and $\mathbb{R}_<$ are the subsets of, respectively, all non-negative, positive and negative ones. We let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$, $\overline{\mathbb{R}}_> := \mathbb{R}_> \cup \{+\infty\}$ and $\overline{\mathbb{R}}_\geq := \mathbb{R}_\geq \cup \{+\infty\}$, and we extend the strict total order relation $<$ on $\mathbb{R}$ to $\overline{\mathbb{R}}$ by positing that $-\infty < c < +\infty$ for all $c \in \mathbb{R}$. We furthermore endow $\overline{\mathbb{R}}$ with the usual topology corresponding to the two-point compactification [40, Example 1 in Appendix C]. The open sets in $\overline{\mathbb{R}}$ are then the open sets in $\mathbb{R}$, the sets $\{x \in \overline{\mathbb{R}} : x > c\}$ and $\{x \in \overline{\mathbb{R}} : x < c\}$ for all $c \in \mathbb{R}$, and any union of these sets.

Most of the arithmetical operations in $\mathbb{R}$ are extended to $\overline{\mathbb{R}}$ in a trivial way; we let $c + \infty = +\infty$ and $c - \infty = -\infty$ for all real c, $+\infty + \infty = +\infty$ and $-\infty - \infty = -\infty$, $\lambda\,(+\infty) = (-\lambda)\,(-\infty) = +\infty$ and $(-\lambda)\,(+\infty) = \lambda\,(-\infty) = -\infty$ for all $\lambda \in \mathbb{R}_{>}$. Two important and perhaps less obvious conventions are that $0\,(+\infty) = 0\,(-\infty) = 0$ and that $+\infty - \infty = -\infty + \infty = +\infty$. The latter is a typical convenient choice when working with upper expectations; see [85, 'Terminology and Notation'], where the same convention is adopted, and [8], where the dual convention ($+\infty - \infty = -\infty + \infty = -\infty$) is adopted because it considers lower expectations as the primary objects. So with our conventions, for example, $a \geq b$ implies that $a - b \geq 0$, but not necessarily $0 \geq b - a$ for any two a and b in $\overline{\mathbb{R}}$.

We say that a sequence $\{c_n\}_{n\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ is **increasing** if $c_n \leq c_{n+1}$ for all $n \in \mathbb{N}$, and **decreasing** if $c_n \geq c_{n+1}$ for all $n \in \mathbb{N}$—so a sequence that remains constant is both increasing and decreasing. We say that $\{c_n\}_{n\in\mathbb{N}}$ is strictly increasing or strictly decreasing if similar but strict versions of the respective inequalities hold.

The infimum and supremum of the empty set $\emptyset$ are assumed to be equal to $+\infty$ and $-\infty$, respectively.

Extended real-valued functions

For any two sets $\mathscr{X}$ and $\mathscr{Y}$, we use $\mathscr{X}^{\mathscr{Y}}$ to denote the set of all functions $f\colon \mathscr{Y} \to \mathscr{X}$. We let $\overline{\mathscr{L}}(\mathscr{Y}) := \overline{\mathbb{R}}^{\mathscr{Y}}$ denote the set of all extended real-valued functions on $\mathscr{Y}$,[8] and let $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ be the subset of all the bounded below ones; that is, $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ is the set of all functions $f \in \overline{\mathscr{L}}(\mathscr{Y})$ for which there is a $c \in \mathbb{R}$ such that $f(y) \geq c$ for all $y \in \mathscr{Y}$.

For any $f \in \overline{\mathscr{L}}(\mathscr{Y})$, we let $\inf f := \inf_{y\in\mathscr{Y}} f(y)$ and $\sup f := \sup_{y\in\mathscr{Y}} f(y)$. The binary relations $=$, $\leq$, $\geq$, $>$ and $<$ on the set $\overline{\mathscr{L}}(\mathscr{Y})$ of gambles are always intended to be taken point-wise, unless mentioned otherwise. So, for any two $f, g \in \overline{\mathscr{L}}(\mathscr{Y})$, we write $f \leq g$ if $f(y) \leq g(y)$ for all $y \in \mathscr{Y}$. Limits of extended real-valued functions are also intended to be taken pointwise, unless mentioned otherwise; so, for any f and $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathscr{L}}(\mathscr{Y})$, we write that $\lim_{n\to+\infty} f_n = f$ if $\lim_{n\to+\infty} f_n(y) = f(y)$ for all $y \in \mathscr{Y}$. Similar conventions are adopted for the limit inferior and limit superior of a sequence of extended real-valued functions.

Furthermore, a sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathscr{L}}(\mathscr{Y})$ is called **increasing** if it is pointwise increasing and **decreasing** if it is pointwise decreasing, and similarly for the strict versions of these notions. Equivalently, we can say that

[8] We will often refer to extended real-valued functions as extended real-valued 'variables'; see Chapter 3_{45}. However, this choice of terminology is somewhat tricky, as it may get confused with the notion of an 'uncertain variable' introduced in Chapter 2_{17}, and so we therefore prefer not to use it already here.

$(f_n)_{n\in\mathbb{N}}$ is increasing if $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$, and decreasing if $f_n \geq f_{n+1}$ for all $n \in \mathbb{N}$.

Gambles

Given any non-empty set $\mathscr{Y}$, a **gamble** f on $\mathscr{Y}$ [75, 106, 110] is a(n) (extended) real-valued function on $\mathscr{Y}$ that is **bounded**, meaning that there is a real number $c \geq 0$ such that $-c \leq f(y) \leq c$ for all $y \in \mathscr{Y}$. The set of all gambles on $\mathscr{Y}$ is denoted by $\mathscr{L}(\mathscr{Y})$. A specific type of gamble that will often be encountered is the indicator $\mathbb{I}_A$ of a set $A \subseteq \mathscr{Y}$; it takes the value 1 for all $y \in A$ and 0 otherwise.

The conventions introduced above for extended real-valued functions apply in particular to gambles. In this respect, it behoves us to mention that it is somewhat unconventional to let $>$ and $<$ be point-wise operators between gambles; typically, the expression $f > g$ for any $f, g \in \mathscr{L}(\mathscr{Y})$ is taken to mean that $f \geq g$ and $f \neq g$, and similarly for the relation $<$. This alternative relation between two gambles will never be used in this dissertation though—the only exception is in the notations that will be introduced next.

The set $\mathscr{L}_{\geq}(\mathscr{Y})$ denotes the set of all gambles $f \in \mathscr{L}(\mathscr{Y})$ such that $f \geq 0$, and similarly for $\mathscr{L}_{\leq}(\mathscr{Y})$, $\mathscr{L}_{>}(\mathscr{Y})$ and $\mathscr{L}_{<}(\mathscr{Y})$. Moreover, we let $\mathscr{L}_{\lneqq}(\mathscr{Y}) \coloneqq \mathscr{L}_{\leq}(\mathscr{Y}) \setminus \{0\}$ and $\mathscr{L}_{\gneqq}(\mathscr{Y}) \coloneqq \mathscr{L}_{\geq}(\mathscr{Y}) \setminus \{0\}$. In summary, we thus have that

$$\mathscr{L}_{>}(\mathscr{Y}) \subseteq \mathscr{L}_{\gneqq}(\mathscr{Y}) \subseteq \mathscr{L}_{\geq}(\mathscr{Y}) \text{ and } \mathscr{L}_{<}(\mathscr{Y}) \subseteq \mathscr{L}_{\lneqq}(\mathscr{Y}) \subseteq \mathscr{L}_{\leq}(\mathscr{Y}).$$

The uniform closure $\operatorname{cl}(\mathscr{A})$ of a set of gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{Y})$ is equal to the set of all uniform limits of sequences in $\mathscr{A}$ [111, Theorem 11.7], [106, Section 1.6]:

$$\operatorname{cl}(\mathscr{A}) \coloneqq \left\{ f \in \mathscr{L}(\mathscr{Y}) \colon \lim_{n\to+\infty} \sup|f - f_n| = 0 \text{ for a sequence } (f_n)_{n\in\mathbb{N}} \text{ in } \mathscr{A} \right\}.$$

Furthermore, the positive linear span of a set of gambles $\mathscr{A}$ is denoted by $\operatorname{posi}(\mathscr{A})$;

$$\operatorname{posi}(\mathscr{A}) \coloneqq \{\textstyle\sum_{i=1}^{n} \lambda_i f_i \colon n \in \mathbb{N}, \lambda_i \in \mathbb{R}_{>}, f_i \in \mathscr{A}\}.$$

— Chapter II —

Modelling uncertainty for finite possibility spaces

Consider a subject who is uncertain about the value that a variable Y takes in some set $\mathcal{Y}$, referred to as the possibility space of Y. The subject here can be anyone/anything showing some form of intelligent or rational behaviour; be it You, Your Computer, or even Dear Mr. President. The variable Y might, for instance, be the state of the weather tomorrow—in which case $\{\text{Sunny}, \text{Cloudy}, \text{Rainy}\}$ could be an appropriate choice for $\mathcal{Y}$—or the stock price of Apple Inc. for a given time instant in the future—in this case, $\mathcal{Y} = \mathbb{R}_{\geq}$, where any real number $y \in \mathcal{Y}$ is a price expressed in some currency. For obvious reasons, Y is then referred to as an 'uncertain' or 'random' variable.

Given this general setting where we have a subject that is uncertain but still has some beliefs about the value of an uncertain variable Y, how do we quantify these beliefs? This is far from a trivial task and, essentially, this simple question lies at the hart of every mathematical theory of uncertainty. In general, one may choose from a broad spectrum of formalisms and mathematical languages to do so. Most commonly, probabilities are used for this purpose, but as we will argue in this chapter, and as we have already briefly mentioned in Chapter 1_1, a single probability distribution or mass function is in many cases too restrictive to correctly assess a subject's beliefs. We will therefore instead use three more general types of models; sets of probabilities, (coherent) sets of acceptable gambles and coherent upper (and lower) expectations. They are part of a larger family of so-called **imprecise probability models**, which aim to describe uncertainty in a robust and informative manner in situations where, loosely speaking, it is unwarranted to specify a single probability distribution. Within this family, the three previously mentioned types of models are among the most popular and general ones; see, for instance, [12, 15, 16, 17, 22, 46, 75, 91, 106, 110].

In the current chapter, we introduce these three types of imprecise probability models in a gentle manner and only consider the simple case where the uncertain variable Y takes values in a **finite** non-empty possibility space $\mathscr{Y}$. Our focus is on finite possibility spaces because the—unconditional—uncertainty models treated here will be used in the following chapters as local models in stochastic processes with a finite state space; see Section $3.1.2_{48}$. Most of what we will present is directly borrowed from the work of P. M. Williams [113] and Walley [110]. We introduce coherence for unconditional upper expectations, discuss its connections with coherent sets of acceptable gambles and sets of probabilities, and present the notion of natural extension under coherence. Since much more is known about these concepts than what we will present here, the current chapter may perhaps appear rather dull or unimportant, especially to the better informed reader. Yet, unimportant as it may seem, the simple set-up in this chapter forms a perfect basis for some of our more involved arguments later on. Coherence, for instance, will attain a more complex form in Chapter 3_{45} when we apply it to global upper expectations in a stochastic process. Most importantly, this chapter already beautifully illustrates the special role that will be reserved for (coherent) upper expectations in our entire story. They arise as objects of interest in both behavioural frameworks such as that of sets of acceptable gambles or game-theoretic probability, and probabilistic frameworks such as that of finitely additive probabilities or measure-theoretic probability, therefore forming the intersection between two complementary schools of thought.

2.1 Modelling uncertainty with probabilities

One of the most common ways to model the beliefs of a subject is by means of probabilities. A (finitely additive) **probability** on a finite possibility space $\mathscr{Y}$ is a function P that associates with each subset $A \subseteq \mathscr{Y}$ a value $\mathrm{P}(A)$ in the interval $[0,1]$. This value $\mathrm{P}(A)$ expresses the degree to which our subject believes that Y will take a value in A; the closer this value is to 1, the more likely our subject deems it that $Y \in A$. The occurrence that Y takes a value in A is called an **event**; in fact, we will henceforth leave the interpretation implicit, and simply call any subset $B \subseteq \mathscr{Y}$ an event. The value $\mathrm{P}(A)$ for any event A is called **the probability of** A.

Probability charges and probability mass functions

Apart from the fact that a finitely additive probability P—also called a **probability charge**—should take values in $[0,1]$, it is also required to be normalised and (finitely) additive. Normalisation says that the trivial event

$\mathcal{Y}$—that is, Y taking a value in $\mathcal{Y}$—happens with probability 1. Additivity says that, for any two disjoint events A and B, the probability of the event $A \cup B$—that is, Y taking a value in either A or B—is the sum of the probabilities $\mathrm{P}(A)$ and $\mathrm{P}(B)$. The following definition can be found in [5, 77, 89, 106], and uses the notation $\wp(\mathcal{Y}) := 2^{\mathcal{Y}}$ to denote the **powerset** of $\mathcal{Y}$; the set of all subsets of $\mathcal{Y}$.

Definition 2.1 (Probabilities/probability charges)**.** For any finite non-empty set $\mathcal{Y}$, we call $\mathrm{P}\colon \wp(\mathcal{Y}) \to \mathbb{R}$ a (finitely additive) probability or a probability charge on $\mathcal{Y}$ if, for all $A, B \in \wp(\mathcal{Y})$,

P1. $0 \leq \mathrm{P}(A)$ [lower bounds];

P2. $\mathrm{P}(\mathcal{Y}) = 1$ [normalisation];

P3. $A \cap B = \emptyset \Rightarrow \mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ [finite additivity]. ⊚

Since $\mathcal{Y}$ is assumed finite, it can easily be seen that probability charges on $\mathcal{Y}$ are one-to-one—there is a bijective relation—with **probability mass functions** on $\mathcal{Y}$. The latter can simply be seen as probability charges restricted to the singletons in $\mathcal{Y}$.

Definition 2.2 (Probability mass functions)**.** For any finite non-empty set $\mathcal{Y}$, we call $p\colon \mathcal{Y} \to \mathbb{R}$ a probability mass function on $\mathcal{Y}$ if it takes values in $[0, 1]$ and is such that $\sum_{y \in \mathcal{Y}} p(y) = 1$. ⊚

We use $\mathbb{P}(\mathcal{Y})$ to denote the set of all probability mass functions on $\mathcal{Y}$. We leave it to the reader to check that any probability mass function $p \in \mathbb{P}(\mathcal{Y})$ defines a probability charge $\mathrm{P}\colon \wp(\mathcal{Y}) \to \mathbb{R}$ by the relation $\mathrm{P}(A) := \sum_{y \in A} p(y)$ for all $A \in \wp(\mathcal{Y})$, and that any probability charge P has a unique probability mass function p to which it is related in this way. Since, probability mass functions are equally as general as probability charges on $\mathcal{Y}$, we prefer to use probability mass functions because of their simplicity.

Example 2.1.1. Consider a possibility space $\mathcal{Y} = \{a, b, c\}$ consisting of three elements—these may for instance be three possible answers to a question, and Y is then the (unknown) correct answer. Let $p\colon \mathcal{Y} \to \mathbb{R}$ be such that $p(a) = p(b) = p(c) = \frac{1}{3}$. Then p is a probability mass function on $\mathcal{Y}$; more specifically, it models the situation where our subject deems it equally likely that the correct answer Y is a, b or c. Then p is also sometimes called the **uniform distribution**. Note that, for the probability charge P corresponding to p, we have that $\mathrm{P}(\{a\}) = \mathrm{P}(\{b\}) = \mathrm{P}(\{c\}) = \frac{1}{3}$, that $\mathrm{P}(\{a, b\}) = \mathrm{P}(\{b, c\}) = \mathrm{P}(\{a, c\}) = \frac{2}{3}$, that $\mathrm{P}(\{a, b, c\}) = 1$ and that $\mathrm{P}(\emptyset) = 0$. ◇

A note on the interpretation of probabilities

We always interpret probabilities in a **subjective** way [27, 39, 110]; they represent the beliefs of a person or a machine that is uncertain with respect to some entity or phenomenon. These beliefs may be based on collected data or experience, or just mere facts; e.g. COVID-vaccines make people less susceptible to become infected by COVID. Yet, any two subjects need not to assess the same probabilities to the different possible outcomes; for their beliefs may be based on classified data or personal experience, or sometimes an incomprehensible reasoning. The subjective interpretation of a probability is also called the **epistemic** interpretation, and this interpretation can further be divided into different sub-interpretations [110]. One of them is the **behavioural** interpretation that was advocated by de Finetti [27]; probabilities then represent betting behaviour or preferences between a number of possible actions. This behavioural interpretation is similar to the philosophy underlying the sets of acceptable gambles framework, which we will introduce shortly in Section 2.3_{26}.

In contrast with the subjective approach above, we could also interpret probabilities in a **frequentist** or **aleatoric** way [39, 69]. Probabilities are then considered to be physical properties of the system, not depending on an observer. More precisely, the frequentist probability of an event is the relative frequency of the times that this event occurs in a long series of observations; e.g. the probability of landing Heads when tossing a fair coin is assumed to be $\frac{1}{2}$, because we land Heads half of the time when performing this experiment over and over again—for a sufficiently long time. This frequentist interpretation of probability has a few drawbacks though; it is highly unpractical since probabilities are to be derived from empirical study only—for instance, we cannot rely on the (subjective) judgements of an expert—and, perhaps more alarming, the interpretation only holds under the assumption that experiments can be repeated—e.g. 'what is the 50th decimal digit of π?'. On the other hand, note that the subjective interpretation always allows a subject himself/itself to employ a frequentist interpretation. Because of this higher level of flexibility, we choose to adopt the subjective interpretation. Nonetheless, none of our mathematical results actually hinge on this interpretation.

Gambles and expectations

A real-valued function f on $\mathscr{Y}$ is called a **gamble** on $\mathscr{Y}$; note that, since we are considering finite $\mathscr{Y}$, any gamble on $\mathscr{Y}$ is always **bounded**, and so our choice of terminology here is in accordance with the traditional terminology; see Section 1.6_{16}. A gamble f may represent an actual gamble; in that case the uncertain—possibly negative—reward is equal to $f(y)$ if the value of Y is $y \in \mathcal{Y}$, with $f(y)$ being a price expressed in some given currency. In general, however, we do not restrict ourselves to this interpretation and interpret a gamble f as an abstract uncertain quantity that depends on the value of Y. The set of all gambles on $\mathcal{Y}$ is denoted by $\mathcal{L}(\mathcal{Y})$.

Given a probability mass function p on $\mathcal{Y}$, we use the corresponding **expectation** E_p to make statements about gambles in $\mathcal{L}(\mathcal{Y})$; it is defined by

$$\mathrm{E}_p(f) \coloneqq \sum_{y \in \mathcal{Y}} p(y)f(y) \text{ for all } f \in \mathcal{L}(\mathcal{Y}). \tag{2.1}$$

The value $\mathrm{E}_p(f)$ for any $f \in \mathcal{L}$ is then called the **expected value** of, or simply the **upper expectation** of f. Using a frequentist interpretation, such an expected value $\mathrm{E}_p(f)$ then represents the average value of f taken over a large number of observations of Y. Our subjective interpretation, however, leaves room for whatever interpretation one prefers; for instance, an expected value $\mathrm{E}_p(f)$ may represent a subject's fair price for the gamble f, where f is then interpreted as an actual uncertain reward depending on the value of Y. We will come back to the interpretation as fair prices in Section 2.3.229.

Expectations are often an object of interest in probability theory because they allow us to draw general inferences about a system, which can on its turn lead to making decisions.

Example 2.1.2. Taking into account the mental state of your girlfriend, wife or husband—Happy (H), Tired (T) or Emotionally unstable (E)—, should you continue discussing a delicate topic, or stop and go read a book? We represent the former action by the gamble $f_1 = (f_1(H), f_1(T), f_1(E)) = (1, 0, -5)$ that turns out well ($f_1(H) = 1$) if Y is equal to H, but turns out very bad ($f_1(E) = -5$) if $Y = E$. The latter action is represented by the gamble $f_2 = (f_2(H), f_2(T), f_2(E)) = (0, 1, -1)$ that turns out well ($f_2(T) = 1$) if Y is equal to T, but also turns out somewhat bad ($f_2(E) = -1$) if $Y = E$. If we assess—purely hypothetical—the probabilities of H, T and E as respectively $\frac{3}{6}$, $\frac{2}{6}$ and $\frac{1}{6}$, then we obtain that $\mathrm{E}(f_1) = -\frac{2}{6}$ and $\mathrm{E}(f_2) = \frac{1}{6}$. Hence, the expectation of f_2 is higher than that of f_1, and you should therefore definitely continue reading this book. ◇

The following straightforward proposition says that expectations derived from probability mass functions are always linear operators, which is why we often call them **linear expectations**.

Proposition 2.1.3. *For any probability mass function p on $\mathcal{Y}$, any $f, g \in \mathcal{L}(\mathcal{Y})$ and any $\lambda \in \mathbb{R}$,*

(i) $\inf f \leq \mathrm{E}_p(f) \leq \sup f$[1] *[bounds];*

(ii) $\mathrm{E}_p(f+g) = \mathrm{E}_p(f) + \mathrm{E}_p(g)$ *[additivity];*

(iii) $\mathrm{E}_p(\lambda f) = \lambda \mathrm{E}_p(f)$ *[homogeneity].*

Proof. These properties can be straightforwardly derived from the definitions of p [Definition 2.2$_{19}$] and E_p [Eq. (2.1)$_{\curvearrowleft}$]. □

Conversely, given an expectation E satisfying the properties (i)–(iii) above, the restriction of this expectation operator to the space of all indicators forms a probability charge. Indeed, it can easily be checked that the set function $\mathrm{P}\colon \wp(\mathcal{Y}) \to \mathbb{R}$ defined by $\mathrm{P}(A) := \mathrm{E}(\mathbb{I}_A)$ for all $A \in \wp(\mathcal{Y})$ then satisfies P1$_{19}$–P3$_{19}$, and is therefore a probability charge on $\mathcal{Y}$—recall Section 1.6$_{16}$ for the definition of an indicator $\mathbb{I}_A$.

2.2 Modelling uncertainty with sets of probabilities

In the previous section, we have modelled a subject's beliefs with a single probability mass function on $\mathcal{Y}$ or, equivalently, a single probability charge on $\mathcal{Y}$. However, it may well be that a subject is unable or not willing to specify such a single probability mass function. This especially occurs when there is a lack of data or information about the system at hand, or when our subject bases himself on conflicting expert advise (or other information). Forcing our subject to choose a single probability mass function may then lead to unwarranted decisions, as is illustrated by the next example.

Example 2.2.1. Consider a container with (possibly) three types of coloured balls contained in it; red ones, green ones and blue ones. There are 6 balls in total, and exactly 2 of them are red. There is no other information given—the remaining 4 balls may be all green, all blue, or any combination of these colours. If Y is the (uncertain) colour of a ball drawn from this container, what is the probability of drawing Red, Blue or Green?

We have that $\mathcal{Y} = \{R, B, G\}$ and we can assume the probability $p(R)$ to be equal to $\frac{2}{6}$. Yet, we have absolutely no idea about the values of the probabilities $p(G)$ and $p(B)$ apart from the fact that they lie between 0 and $\frac{4}{6}$. In standard 'precise' probability theory this lack of knowledge or beliefs is typically modelled by assuming the probabilities $p(G)$ and $p(B)$ to be equal. The expectation of the gamble $f = (f(R), f(G), f(B)) = (-1, -1, 3)$ would then become

$$\mathrm{E}(f) = f(R)p_u(R) + f(G)p_u(G) + f(B)p_u(B) = -1\tfrac{2}{6} - 1\tfrac{2}{6} + 3\tfrac{2}{6} = \tfrac{2}{6} > 0$$

[1] Since the possibility space $\mathcal{Y}$ is finite here, the infimum and supremum can be replaced by a minimum and a maximum.

Hence, according to the reasoning above, f is a gamble with positive expected pay-off—if for a minute we interpret its values as actual pay-offs—and we are therefore inclined to accept the gamble f. However, this conclusion is clearly inconsistent with reality; only few people would actually be willing to accept f because there might be no blue balls in the container, and then there is no chance of gaining any money at all. ◇

2.2.1 Sets of probability mass functions

In order to model the beliefs of a subject in a more flexible and robust way, we can use **sets of probability mass functions** $\mathscr{P}$ instead of a single probability mass function; such a set can then loosely be interpreted as the set of all probability mass functions our subject deems 'possible'. One of the most common and simple ways of obtaining such a set $\mathscr{P}$ is by specifying upper and lower bounds on individual probabilities; e.g. our beliefs in Example 2.2.1$_{\leftarrow}$ are correctly represented by the set

$$\mathscr{P} = \{p \in \mathbb{P}(\mathscr{Y}) \colon p(R) = \tfrac{2}{6}\} = \{p \in \mathbb{P}(\mathscr{Y}) \colon \tfrac{2}{6} \leq p(R) \leq \tfrac{2}{6}\}.$$

In general, however, such upper and lower bounds on individual probabilities do not suffice to characterise a set $\mathscr{P} \subseteq \mathbb{P}(\mathscr{Y})$.

Example 2.2.2. Consider again the container from Example 2.2.1$_{\leftarrow}$ but where it is now given that there are an equal amount of green and blue balls, and where nothing is said (or known) about how many red balls there are in the container. Our beliefs are in that case correctly represented by the set

$$\mathscr{P} = \{p \in \mathbb{P}(\mathscr{Y}) \colon p(G) = p(B)\}.$$

Such a set $\mathscr{P}$ is called a comparative probability model [65]. It can be checked that this set can never be characterised by only using upper and lower bounds on the individual probabilities. ◇

We will often call any set $\mathscr{P} \subseteq \mathbb{P}(\mathscr{Y})$ an **imprecise** probability model, or simply an imprecise model, because it is a generalisation of the traditional probability mass function—a 'precise' model—where the individual parameters/probabilities are only partially specified. Concerning the interpretation of our sets $\mathscr{P} \subseteq \mathbb{P}(\mathscr{Y})$, let us mention that we do not regard the set $\mathscr{P}$ to be an **exhaustive** or complete representation of our subject's beliefs; that is, the set $\mathscr{P}$ may include more than what our subject actually deems possible. This seems sensible because in most realistic situations we can only gather or represent part of a subject's beliefs, simply because of constraints on time, money, and so forth. We refer to Walley [110, Chapter 2] for more details on such interpretational aspects, and for a more elaborate motivation for imprecise probability models.

2.2.2 Upper and lower expectations from sets of probability mass functions

Of course, since we are now considering sets of probability mass functions $\mathscr{P}$, we cannot specify a single expected value for each gamble any more. Instead, we will have an entire set of such expected values. The upper and lower bounds of these sets are what we call the **upper** and **lower expectations** corresponding to $\mathscr{P}$.

Definition 2.3. For any non-empty set of probability mass functions $\mathscr{P}$, the corresponding upper and lower expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\underline{\mathrm{E}}_{\mathscr{P}}$ are defined, for all $f \in \mathscr{L}(\mathscr{Y})$, by

$$\overline{\mathrm{E}}_{\mathscr{P}}(f) := \sup_{p\in\mathscr{P}} \mathrm{E}_p(f) = \sup_{p\in\mathscr{P}} \sum_{y\in\mathscr{Y}} p(y)f(y) \text{ and}$$
$$\underline{\mathrm{E}}_{\mathscr{P}}(f) := \inf_{p\in\mathscr{P}} \mathrm{E}_p(f) = \inf_{p\in\mathscr{P}} \sum_{y\in\mathscr{Y}} p(y)f(y). \qquad ⊚$$

Upper and lower expectations are of major importance because, just as their precise counterparts, they allow us to draw various non-trivial conclusions, which on their turn may lead to decisions. Decision making becomes somewhat more delicate than in the precise case though, since we can now come up with several different methods that each have their own advantages and disadvantages; see e.g. [110, Section 3.9] and [50].

Example 2.2.3. Reconsider the situation from Example 2.2.1$_{22}$, where $\mathscr{P} = \{p \in \mathbb{P}(\mathscr{Y})\colon p(R) = \frac{2}{6}\}$ is the set that represents our beliefs about the colour Y of the ball that is drawn next. Then the corresponding upper and lower expectation $\overline{\mathrm{E}}_{\mathscr{P}}(f)$ and $\underline{\mathrm{E}}_{\mathscr{P}}(f)$ of the gamble $f = (-1, -1, 3)$ are respectively $\frac{10}{6}$ and -1. If we wish to remain conservative, then we should base ourselves on the worst-case scenario, which is represented by the lower expectation $\underline{\mathrm{E}}_{\mathscr{P}}(f) = -1 < 0$. Hence, in that case, we should not accept the gamble f—indeed, this is similar to the conclusion that we made at the end of Example 2.2.1$_{22}$. This corresponds to the Γ-maximin approach from [50]. On the other hand, the best-case scenario is represented by the upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}(f) = \frac{5}{3} > 0$. Hence, if potential negative pay-offs are little of an issue, then based on this information we might want to accept the gamble f. This would then corresponds to the Γ-maximax approach from [50]. ◇

Upper and lower expectations also satisfy several basic properties that will turn out convenient later on. We list the most important ones.

Proposition 2.2.4. *Consider any non-empty set $\mathscr{P} \subseteq \mathbb{P}(\mathscr{Y})$, and let $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\underline{\mathrm{E}}_{\mathscr{P}}$ be the upper and lower expectations corresponding to $\mathscr{P}$ according to Definition 2.3. Then, for any $f, g \in \mathscr{L}(\mathscr{Y})$ and $\lambda \in \mathbb{R}_{\geq}$,* (i). $\overline{\mathrm{E}}_{\mathscr{P}}(-f) = -\underline{\mathrm{E}}_{\mathscr{P}}(f)$ *[conjugacy];*

(ii). $\inf f \leq \underline{\mathrm{E}}_{\mathscr{P}}(f) \leq \overline{\mathrm{E}}_{\mathscr{P}}(f) \leq \sup f$ *[bounds];*

(iii). $\overline{\mathrm{E}}_{\mathscr{P}}(f+g) \leq \overline{\mathrm{E}}_{\mathscr{P}}(f) + \overline{\mathrm{E}}_{\mathscr{P}}(g)$,
$\underline{\mathrm{E}}_{\mathscr{P}}(f+g) \geq \underline{\mathrm{E}}_{\mathscr{P}}(f) + \underline{\mathrm{E}}_{\mathscr{P}}(g)$ *[sub-/super-additivity];*

(iv). $\overline{\mathrm{E}}_{\mathscr{P}}(\lambda f) = \lambda \overline{\mathrm{E}}_{\mathscr{P}}(f)$; $\underline{\mathrm{E}}_{\mathscr{P}}(\lambda f) = \lambda \underline{\mathrm{E}}_{\mathscr{P}}(f)$ *[non-negative homogeneity].*

Proof. All these properties can be easily deduced from Definition 2.3$_{\leftarrow}$. □

Note in particular that, due to (i) above, upper and lower expectations are related by **conjugacy**, and so it actually suffices to only study either upper expectations or lower expectations; we will focus on upper expectations.

A special case of a set of probability mass functions is when $\mathscr{P}$ consists of only a single probability mass function p. The upper and lower expectations are then equal, and their common values are then given by the linear expectation E_p. The corresponding upper and lower expectations are then called **self-conjugate**.

In contrast to the above, if $\mathscr{P}$ consists of all the probability mass functions in $\mathbb{P}(\mathscr{Y})$, then we call $\mathscr{P}$ the **vacuous** model; it represents a complete lack of knowledge or beliefs about the value of Y. The corresponding upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}(f)$ for any $f \in \mathscr{L}(\mathscr{Y})$ is equal to $\sup f$, and the corresponding lower expectation $\underline{\mathrm{E}}_{\mathscr{P}}(f)$ is equal to $\inf f$.

2.2.3 Upper and lower probabilities from sets of probability mass functions

By restricting upper and lower expectations to the domain of all indicators we obtain so-called **upper** and **lower probabilities** $\overline{\mathrm{P}}_{\mathscr{P}}$ and $\underline{\mathrm{P}}_{\mathscr{P}}$, respectively; so, for any set $\mathscr{P} \subseteq \mathbb{P}(\mathscr{Y})$, they are defined by $\overline{\mathrm{P}}_{\mathscr{P}}(A) := \overline{\mathrm{E}}_{\mathscr{P}}(\mathbb{I}_A)$ and $\underline{\mathrm{P}}_{\mathscr{P}}(A) := \underline{\mathrm{E}}_{\mathscr{P}}(\mathbb{I}_A)$ for all $A \in \wp(\mathscr{Y})$. Alternatively, it follows from Definition 2.3$_{\leftarrow}$ that, for all $A \in \wp(\mathscr{Y})$,

$$\overline{\mathrm{P}}_{\mathscr{P}}(A) := \sup_{p \in \mathscr{P}} \mathrm{P}_p(A) = \sup_{p \in \mathscr{P}} \sum_{y \in A} p(y);$$
$$\underline{\mathrm{P}}_{\mathscr{P}}(A) := \inf_{p \in \mathscr{P}} \mathrm{P}_p(A) = \inf_{p \in \mathscr{P}} \sum_{y \in A} p(y).$$

So upper and lower probabilities give tight upper and lower bounds on the individual probabilities associated with a set $\mathscr{P}$—as was to be expected from their name—and are therefore an object of interest in many applications. As already mentioned, such upper and lower bounds on individual probabilities can be used to derive a compatible set $\mathscr{P}$ of probability mass functions from, but not all sets $\mathscr{P}$ can be obtained in this way; see also [110, Section 2.7.3].[2] We therefore typically regard upper and lower probabilities as secondary objects derived from sets of probabilities/probability mass functions.

2.3 Modelling uncertainty with sets of acceptable gambles

It was briefly mentioned in Section 2.1_{18} that probabilities can be given a behavioural interpretation, yet there actually exists a full-fledged theory that is entirely build on the idea that a subject's betting behaviour ought to be regarded as the fundamental primary object, rather than probabilities; the theory of sets of acceptable gambles (or sets of desirable gambles) [75, 76, 106, 110, 113]. This theory has grown largely from the ideas presented by P. M. Williams [113] and Walley [110]. We next outline some basic but important concepts in this field, and show how they naturally lead us to define corresponding upper and lower expectations.

2.3.1 Sets of acceptable gambles

In the current framework, any gamble $f \in \mathscr{L}(\mathscr{Y})$ is interpreted as an uncertain—possibly negative—reward $f(Y)$ that depends on the value of the variable Y. A central assumption here is that the (real-valued) rewards or pay-offs $f(y)$ associated with such a gamble f represent **linear utilities** for a subject, in the sense that, for any $c \in \mathbb{R}_{\geq}$, the price $cf(y)$ is worth c times as much as $f(y)$. If a subject specifies that she finds a gamble $f \in \mathscr{L}(\mathscr{Y})$ **acceptable**, then we simply take this to mean that she is willing to accept the uncertain reward associated with the gamble f [76, 106, 113]. Accepting a gamble is weaker than finding it desirable [110] or preferring it above the status quo; our subject accepting a gamble f may also entail that she is indifferent with respect to exchanging f for the zero gamble 0—which represents the status quo—and vice versa; see also Axiom $D1_{\rightarrow}$ in the definition of coherence.

The beliefs that a subject has about Y will lead her to make statements about which gambles $f \in \mathscr{L}(\mathscr{Y})$ she finds acceptable. For instance, if she is completely certain that Y takes—or, will take—the value $y \in \mathscr{Y}$, then she will typically accept any gamble $f \in \mathscr{L}(\mathscr{Y})$ that gives her non-negative pay-off $f(y) \geq 0$ when $Y = y$. Or, conversely, in case she is completely uncertain about the value of Y, she might only accept those gambles that are sure—whatever the value of Y—not to give her a negative pay-off. In this way, one

[2]For closed and convex sets of mass functions, a sufficient condition for being fully characterised by the corresponding upper and lower probabilities is that the associated upper (or lower) expectation is 2-monotone; see [106, Theorem 6.22].

can thus see that a subject's beliefs can be modelled by considering her **set of acceptable gambles** $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$.

Of course, our notion of acceptability has little meaning on its own; if our subject chooses to be completely irrational in her specification of $\mathcal{D}$—for instance, she includes gambles $f \in \mathcal{D}$ that give her a (strictly) negative reward irrespectively of what happens—it should not be expected that, using whatever system of logical reasoning, this information will allow us to come up with any sensible conclusions. Hence, in order to be practically meaningful, we shall want to impose some minimal properties on $\mathcal{D}$; properties that translate our idea of rational behaviour.[3] These properties are what we call **coherence**.

Definition 2.4 (Coherent sets of acceptable gambles)**.** We say that a set of acceptable gambles $\mathcal{D} \subseteq \mathcal{L}(\mathcal{Y})$ is coherent if, for any two $f, g \in \mathcal{L}(\mathcal{Y})$ and any $\lambda \in \mathbb{R}_{>}$,

D1. $\mathcal{L}_{\geq}(\mathcal{Y}) \subseteq \mathcal{D}$ [accepting non-negative rewards];

D2. $\mathcal{L}_{\lneqq}(\mathcal{Y}) \cap \mathcal{D} = \emptyset$ [avoiding partial loss];

D3. $f, g \in \mathcal{D} \Rightarrow f + g \in \mathcal{D}$ [combination];

D4. $f \in \mathcal{D} \Rightarrow \lambda f \in \mathcal{D}$ [scale invariance]. ◎

The definition above is entirely the same as that of [106, Definition 3.2], apart from the fact that the scaling axiom [BA3] in [106, Definition 3.2] is with non-negative λ, whereas we only allow λ to be positive [D4]; our choice is in that sense more in line with that of P. M. Williams [113, Footnote 1] and Quaeghebeur et al. [76, Sections 2.10.7 and 2.4]. Nevertheless, this clearly makes no difference, mathematically speaking, because the zero gamble is always included in a set of acceptable gambles that is coherent according to Definition 2.4 due to D1—which is in fact the reason why we prefer to keep D4 as weak as possible.

Let us briefly clarify the meaning of the coherence axioms above. Property D1 requires that a coherent set of acceptable gambles $\mathcal{D}$ should include all the $f \in \mathcal{L}_{\geq}(\mathcal{Y})$ that surely give a non-negative reward; in other words, our subject should always accept the status quo, and any gamble that never makes her lose money, and that in some cases makes her receive money. Property D2 on the other hand requires that a coherent set of acceptable gambles $\mathcal{D}$ should never include any gambles $f \in \mathcal{L}_{\lneqq}(\mathcal{Y})$ from which our subject can never gain anything, and that in some cases make her lose money—this requirement is called avoiding partial loss. The motivation for

[3] One could question, however, whether it is reasonable to assume that all subjects, if, representing actual persons, indeed act rationally.

$\mathrm{D1}_{\curvearrowleft}$ and $\mathrm{D2}_{\curvearrowleft}$ is self-evident.[4]

Axioms $\mathrm{D3}_{\curvearrowleft}$ and $\mathrm{D4}_{\curvearrowleft}$, on the other hand, are concerned with how acceptability of certain gambles lead us to conclusions about the acceptability of other gambles. Axiom $\mathrm{D3}_{\curvearrowleft}$ says that, if two gambles $f, g \in \mathscr{L}(\mathscr{Y})$ are acceptable, then their sum $f+g$ should also be acceptable. Axiom $\mathrm{D4}_{\curvearrowleft}$ says that, if a gamble $f \in \mathscr{L}(\mathscr{Y})$ is acceptable, then any scaled version λf with $\lambda > 0$ should also be acceptable. The motivation for $\mathrm{D3}_{\curvearrowleft}$ and $\mathrm{D4}_{\curvearrowleft}$ relies on the fact that pay-offs corresponding to gambles are assumed to be linear utilities for our subject; we refer to [110, Section 2.2.4] for more details. Note that $\mathrm{D3}_{\curvearrowleft}$ and $\mathrm{D4}_{\curvearrowleft}$ together imply that $\mathscr{D}$ should form a convex cone; that is, $\operatorname{posi}(\mathscr{D}) = \mathscr{D}$.

It can moreover easily be deduced from $\mathrm{D1}_{\curvearrowleft}$ and $\mathrm{D3}_{\curvearrowleft}$ that coherent sets of acceptable gambles satisfy the following monotonicity property.

Corollary 2.3.1. *For any set of acceptable gambles $\mathscr{D} \subseteq \mathscr{L}(\mathscr{Y})$ that is coherent, and any two $f, g \in \mathscr{L}(\mathscr{Y})$,*

D5. *if $f \in \mathscr{D}$ and $f \leq g$, then $g \in \mathscr{D}$.*

Furthermore, for the same reasons as for sets of probability mass functions $\mathscr{P}$ in Section 2.2$_{22}$, we again do not assume a coherent set of acceptable gambles $\mathscr{D}$ to be an **exhaustive** representation of our subject's beliefs. Mathematically speaking, this means that $\mathscr{D}$ may actually include less gambles than the set of all gambles deemed acceptable by our subject.

Example 2.3.2. A special set of acceptable gambles $\mathscr{D}$ is the first orthant $\mathscr{L}_{\geq}(\mathscr{Y})$. It can be checked that $\mathscr{D} = \mathscr{L}_{\geq}(\mathscr{Y})$ is coherent, and more specifically that it is the smallest possible set of acceptable gambles that is coherent. Since no gambles are included in $\mathscr{D}$ apart from the trivially acceptable ones, this set $\mathscr{D}$ can be seen to model the case where our subject is not willing to make any non-trivial commitments with regard to the uncertain value of Y. It is the acceptability-counterpart of the sets $\mathscr{P}$ of probability mass functions consisting of all possible probability mass functions (recall the end of Section 2.2.2$_{24}$), and is also referred to as the **vacuous** model. ◇

Example 2.3.3. In utter contrast to the vacuous model described above, we can also consider **maximal** coherent sets of acceptable gambles [12]; coherent sets of acceptable gambles for which there exists no coherent set

[4] Axioms $\mathrm{D1}_{\curvearrowleft}$ and $\mathrm{D2}_{\curvearrowleft}$ can sometimes be found in a slightly different form; some authors prefer to exclude 0 from the set of trivially acceptable gambles, or sometimes only include the strictly positive gambles in the set of trivially acceptable gambles; similar observations can be made for the trivially non-acceptable gambles. Such weaker versions of Axioms $\mathrm{D1}_{\curvearrowleft}$ and $\mathrm{D2}_{\curvearrowleft}$ are typically used when interpreting acceptability in a stronger way—in that case, it is often called desirability or strict desirability. We refer to [76] for an overview on this matter.

of acceptable gambles that is strictly larger. For instance, if $\mathscr{Y} = \{R, G, B\}$ are the three possible colours of a ball drawn from a container—Red, Green and Blue, respectively—then the sets $\{f \in \mathscr{L}(\mathscr{Y})\colon f(R) + f(G) + f(B) \geq 0\}$ and $\{f \in \mathscr{L}(\mathscr{Y})\colon f(R) > 0\} \cup \{f \in \mathscr{L}(\mathscr{Y})\colon f(R) = 0, f(G) + f(B) \geq 0\}$ are maximal coherent sets of acceptable gambles. The former corresponds to the case where our subject deems the colours Red, Green and Blue all equally likely to come up; the latter corresponds to case where our subject is (almost) certain that the ball will be Red, and that, if it is known that the ball is not Red or if he is not allowed to put stakes on the colour Red, he deems it equally likely that either Green or Blue will come up. ◇

2.3.2 Upper and lower expectations from sets of acceptable gambles

Given a coherent set of acceptable gambles $\mathscr{D}$ that models the beliefs of our subject, we typically want to make judgements about the 'value' or 'price' of certain gambles of interest. To do so, we associate **upper** and **lower expectations** with $\mathscr{D}$.

Definition 2.5. For any coherent set of acceptable gambles $\mathscr{D}$, the corresponding upper and lower expectations $\overline{\mathrm{E}}_{\mathscr{D}}$ and $\underline{\mathrm{E}}_{\mathscr{D}}$ are real-valued operators on $\mathscr{L}(\mathscr{Y})$ defined, for all $f \in \mathscr{L}(\mathscr{Y})$, by

$$\overline{\mathrm{E}}_{\mathscr{D}}(f) := \inf\{\alpha \in \mathbb{R}\colon \alpha - f \in \mathscr{D}\} \text{ and } \underline{\mathrm{E}}_{\mathscr{D}}(f) := \sup\{\alpha \in \mathbb{R}\colon f - \alpha \in \mathscr{D}\}. \circledcirc$$

So, for any $f \in \mathscr{L}(\mathscr{Y})$, the value $\overline{\mathrm{E}}_{\mathscr{D}}(f)$ represents the infimum acceptable selling price corresponding to $\mathscr{D}$; indeed, for any $\alpha \in \mathbb{R}$, accepting $\alpha - f$ is the same as accepting the transaction of selling the uncertain reward f for the fixed price α. Conversely, $\underline{\mathrm{E}}_{\mathscr{D}}(f)$ represents the supremum buying price corresponding to $\mathscr{D}$. Observe that $\overline{\mathrm{E}}_{\mathscr{D}}(f)$ and $\underline{\mathrm{E}}_{\mathscr{D}}(f)$ always lie between $\inf f$ and $\sup f$, and therefore are indeed both real, due to Axioms D1$_{27}$ and D2$_{27}$.

Example 2.3.4. Reconsider the situation from Example 2.3.3$_{\leftarrow}$, and let $\mathscr{D}_1 := \mathscr{L}_{\geq}(\mathscr{Y})$ and $\mathscr{D}_2 := \{f \in \mathscr{L}(\mathscr{Y})\colon f(R) + f(G) + f(B) \geq 0\}$. Then it can be inferred that the vacuous model $\mathscr{D}_1$ gives as upper and lower expectations $\overline{\mathrm{E}}_{\mathscr{D}_1}(f) = \sup f$ and $\overline{\mathrm{E}}_{\mathscr{D}_2}(f) = \inf f$ for all $f \in \mathscr{L}(\mathscr{Y})$. On the other hand, for the model $\mathscr{D}_2$, which considers Red, Green and Blue equally likely to come up, the upper and lower expectations coincide and are equal to $\overline{\mathrm{E}}_{\mathscr{D}_2}(f) = \underline{\mathrm{E}}_{\mathscr{D}_2}(f) = (f(R) + f(G) + f(B))/3$ for all $f \in \mathscr{L}(\mathscr{Y})$. ◇

It can be shown that upper and lower expectations corresponding to coherent sets of acceptable gambles satisfy the same convenient properties as upper and lower expectations corresponding sets of probabilities.

Proposition 2.3.5. *Consider any coherent set of acceptable gambles $\mathscr{D}$, and let $\overline{\mathrm{E}}_{\mathscr{D}}$ and $\underline{\mathrm{E}}_{\mathscr{D}}$ be the upper and lower expectations corresponding to $\mathscr{D}$ according to Definition 2.5$_{\frown}$. Then, for any $f, g \in \mathscr{L}(\mathscr{Y})$ and $\lambda \in \mathbb{R}_{\geq}$,*

(i) $\overline{\mathrm{E}}_{\mathscr{D}}(-f) = -\underline{\mathrm{E}}_{\mathscr{D}}(f)$ *[conjugacy];*

(ii) $\inf f \leq \underline{\mathrm{E}}_{\mathscr{D}}(f) \leq \overline{\mathrm{E}}_{\mathscr{D}}(f) \leq \sup f$ *[bounds];*

(iii) $\overline{\mathrm{E}}_{\mathscr{D}}(f+g) \leq \overline{\mathrm{E}}_{\mathscr{D}}(f) + \overline{\mathrm{E}}_{\mathscr{D}}(g)$*;*
$\underline{\mathrm{E}}_{\mathscr{D}}(f+g) \geq \underline{\mathrm{E}}_{\mathscr{D}}(f) + \underline{\mathrm{E}}_{\mathscr{D}}(g)$ *[sub-/super-additivity];*

(iv) $\overline{\mathrm{E}}_{\mathscr{D}}(\lambda f) = \lambda \overline{\mathrm{E}}_{\mathscr{D}}(f)$*;* $\underline{\mathrm{E}}_{\mathscr{D}}(\lambda f) = \lambda \underline{\mathrm{E}}_{\mathscr{D}}(f)$ *[non-negative homogeneity].*

Proof. Property (i) follows straightforwardly from Definition 2.5$_{\frown}$. Indeed, for any $f \in \mathscr{L}(\mathscr{Y})$, we have that

$$\begin{aligned}\overline{\mathrm{E}}_{\mathscr{D}}(-f) = \inf\{\alpha \in \mathbb{R} : \alpha + f \in \mathscr{D}\} &= \inf\{-\alpha \in \mathbb{R} : f - \alpha \in \mathscr{D}\} \\ &= -\sup\{\alpha \in \mathbb{R} : f - \alpha \in \mathscr{D}\} = -\underline{\mathrm{E}}_{\mathscr{D}}(f).\end{aligned}$$

Properties (ii)–(iv) can be easily deduced from Definition 2.5$_{\frown}$ and the coherence of $\mathscr{D}$. Alternatively, they also follow from the fact that $\overline{\mathrm{E}}_{\mathscr{D}}$ and $\underline{\mathrm{E}}_{\mathscr{D}}$ are 'coherent' in the sense of [106, Definition 4.10], and therefore that they satisfy the properties in [106, Theorem 4.13]. □

Note again that, due to the conjugacy property (i) above, we can limit ourselves to only working with upper expectations. In some instances, as was the case for the upper and lower expectations $\overline{\mathrm{E}}_{\mathscr{D}_2}$ and $\underline{\mathrm{E}}_{\mathscr{D}_2}$ in the example above, it can happen that the upper and lower expectation coincide—it can be observed that this will always be the case for upper and lower expectations deduced from maximal coherent sets of acceptable gambles. The upper and lower expectations are then again called **self-conjugate**. They represent a subject's **fair prices** [27]; indeed, for any $f \in \mathscr{L}(\mathscr{Y})$, if $\mathrm{E}_{\mathscr{D}}(f)$ denotes the common value of $\overline{\mathrm{E}}_{\mathscr{D}}(f)$ and $\underline{\mathrm{E}}_{\mathscr{D}}(f)$, then our subject is willing to sell f for any price α higher than $\mathrm{E}_{\mathscr{D}}(f)$, and willing to buy f for any price α lower than $\mathrm{E}_{\mathscr{D}}(f)$.

We do want to stress, however, that we consider the existence of fair prices to be only a special case and that, in general, we allow infimum (acceptable) selling prices to be higher than supremum (acceptable) buying prices; see (ii) above. By doing this, we allow for **indeterminacy** in a subject's gambling behaviour, in the sense that our subject may choose, for any price $\alpha \in \mathbb{R}$ and any gamble $f \in \mathscr{L}(\mathscr{Y})$, to neither sell f for α, nor buy f for α. If we were to restrict ourselves to only working with self-conjugate (linear) expectations or maximal sets of acceptable gambles, then we would always force our subject to either sell f for α, or buy f for α, which seems like a strong and unnatural limitation.[5] The case of maximal indeterminacy is in Example 2.3.4$_{29}$ represented by the vacuous model $\overline{\mathrm{E}}_{\mathcal{D}_1}$, because there $\overline{\mathrm{E}}_{\mathcal{D}_1}(f) = \sup f$ and $\underline{\mathrm{E}}_{\mathcal{D}_1}(f) = \inf f$ for all $f \in \mathcal{L}(\mathcal{Y})$, and so our subject is then only willing to sell or buy gambles in such a way that she can never lose from her transactions. We again refer to Walley [110] for more details.

2.3.3 Upper and lower probabilities from sets of acceptable gambles

Just as we did in Section 2.2.3$_{25}$ for upper and lower expectations associated with sets of probability mass functions, we can associate an **upper** and **lower probability** $\overline{\mathrm{P}}_{\mathcal{D}}$ and $\underline{\mathrm{P}}_{\mathcal{D}}$ with a coherent set of acceptable gambles $\mathcal{D}$ by restricting the upper and lower expectations $\overline{\mathrm{E}}_{\mathcal{D}}$ and $\underline{\mathrm{E}}_{\mathcal{D}}$ to the domain of indicators; so, for any coherent set of acceptable gambles $\mathcal{D}$, we let $\overline{\mathrm{P}}_{\mathcal{D}}(A) := \overline{\mathrm{E}}_{\mathcal{D}}(\mathbb{I}_A)$ and $\underline{\mathrm{P}}_{\mathcal{D}}(A) := \underline{\mathrm{E}}_{\mathcal{D}}(\mathbb{I}_A)$ for all $A \in \wp(\mathcal{Y})$. The upper and lower probability $\overline{\mathrm{P}}_{\mathcal{D}}$ and $\underline{\mathrm{P}}_{\mathcal{D}}$ corresponding to a coherent set $\mathcal{D}$ do not represent upper and lower bounds on possible individual probabilities as in Section 2.2.3$_{25}$, but rather represent the infimum and supremum stakes at which our subject is willing to bet on the occurence of an event. Indeed, it follows from the definition of $\overline{\mathrm{E}}_{\mathcal{D}}$ [Definition 2.5$_{29}$] that $\overline{\mathrm{P}}_{\mathcal{D}}(A)$ for any $A \in \wp(\mathcal{Y})$ is the infimum price $\alpha \in \mathbb{R}$ for which our subject is willing to accept the the uncertain reward that is equal to $\alpha - 1$ if A occurs, and that is equal to α otherwise. Conversely, $\underline{\mathrm{P}}_{\mathcal{D}}(A)$ for any $A \in \wp(\mathcal{Y})$ is the supremum price $\alpha \in \mathbb{R}$ for which our subject is willing to accept the the uncertain reward that is equal to $1 - \alpha$ if A occurs, and that is equal to $-\alpha$ otherwise.

2.4 Coherent upper and lower expectations directly

We have seen in the previous sections that both the framework of (sets of) probabilities and the framework of sets of acceptable gambles naturally lead us to define upper (and lower) expectations, which are often more convenient or interesting than the (sets of) probabilities or sets of acceptable gambles they are derived from, especially when aiming to draw inferences about the system at hand. The upper expectations deduced from these frameworks moreover have some characteristic properties, which were listed in Propositions 2.2.4$_{24}$ and 2.3.5$_{\leftarrow}$. In many cases, these characteristic properties are all we need and care about—for instance, when deriving (other) mathematical properties or when performing calculations—-and then the definitions of the upper expectations given above are rather lengthy

[5] The only exception where we would allow a subject to be indeterminate when working with self-conjugate (linear) expectations, is when the proposed selling or buying price α is exactly equal to $\mathrm{E}_{\mathcal{D}}(f)$.

and indirect. Moreover, they also demand a user to base himself on either the framework of probabilities or the framework of acceptable gambles, and therefore to choose between two contrasting interpretations of an upper expectation. In light of obtaining a more direct and universal definition of an upper expectation, a possible and logical strategy would be to simply start from the properties in Propositions $2.2.4_{24}$ and $2.3.5_{30}$, and propose them as defining axioms. This approach was followed by P. M. Williams [113] and Walley [110]. They showed that the following three simple axioms—which are the same as Proposition $2.2.4(\text{ii})_{25}$–$(\text{iv})_{25}$ and Proposition $2.3.5(\text{ii})_{30}$–$(\text{iv})_{30}$—are enough to fully characterise the upper expectations in Definitions 2.3_{24} and 2.5_{29}; they are specified, for any real-valued operator $\overline{\mathrm{E}}\colon \mathscr{L}(\mathscr{Y}) \to \mathbb{R}$, any $f, g \in \mathscr{L}(\mathscr{Y})$ and $\lambda \in \mathbb{R}_{\geq}$, by

C1. $\overline{\mathrm{E}}(f) \leq \sup f$ [upper bound];

C2. $\overline{\mathrm{E}}(f+g) \leq \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(g)$ [sub-additivity];

C3. $\overline{\mathrm{E}}(\lambda f) = \lambda \overline{\mathrm{E}}(f)$ [non-negative homogeneity].

In accordance with Propositions $2.2.4_{24}$ and $2.3.5_{30}$, the lower expectation $\underline{\mathrm{E}}$ corresponding to an upper expectation $\overline{\mathrm{E}}$ is simply defined by the conjugacy relation;

$$\underline{\mathrm{E}}(f) \coloneqq -\overline{\mathrm{E}}(-f) \text{ for all } f \in \mathscr{L}(\mathscr{Y}). \tag{2.2}$$

The following uses C1–C3 to define the notion of a **coherent upper expectation**, and establishes our claim that these axioms are enough to characterise the upper expectations from Definition 2.3_{24} and Definition 2.5_{29}.

Definition 2.6. For any operator $\overline{\mathrm{E}}\colon \mathscr{L}(\mathscr{Y}) \to \mathbb{R}$ the following conditions are equivalent. If any—and hence all—are satisfied then we call $\overline{\mathrm{E}}$ a coherent upper expectation, and the corresponding conjugate operator $\underline{\mathrm{E}}$ defined by Eq. (2.2) a coherent lower expectation.

(i) $\overline{\mathrm{E}}$ satisfies C1–C3;

(ii) $\overline{\mathrm{E}}$ is the upper envelope $\overline{\mathrm{E}}_{\mathscr{P}}$ corresponding to some non-empty set $\mathscr{P}$ of probability mass functions;

(iii) $\overline{\mathrm{E}}$ is equal to the infimum selling prices $\overline{\mathrm{E}}_{\mathscr{D}}$ corresponding to some coherent set of acceptable gambles $\mathscr{D}$. ⊚

Proof. That (ii) implies (i) follows from Proposition $2.2.4_{24}$. That (iii) implies (i) follows from Proposition $2.3.5_{30}$. That (i) implies (ii) follows from the lower envelope theorem [110, Theorem 3.3.3 (b)] and conjugacy.[6] That (i) implies (iii) follows from [106, Theorem 4.2] and conjugacy. □

[6] A similar result was actually first stated by Huber [49, Section 10.2], and later used by Artzner et al. [1, Proposition 4.1] to characterise coherent risk measures.

Note that no independent explicit definition of a—not necessarily coherent—upper expectation has been given so far. Upper expectations will appear in several different contexts and take multiple different forms throughout this dissertation. In this chapter, they will always assume them to be real-valued operators on $\mathscr{L}(\mathscr{Y})$, yet, in later chapters, they will be modified to also take values in the extended real numbers $\overline{\mathbb{R}}$, or to even take two arguments instead of only one. In general, no additional implicit assumptions are made with respect to the form or properties of upper expectations; they will typically take a more specific form, but this will then be mentioned explicitly at the point of relevance.

Apart from $\mathrm{C1}_{\leftarrow}$–$\mathrm{C3}_{\leftarrow}$, coherent upper and lower expectations additionally satisfy some basic but convenient properties. The following result can be easily deduced from [110, Section 2.6.1.] and the conjugacy relation [Eq. $(2.2)_{\leftarrow}$].

Proposition 2.4.1. *Consider any coherent upper expectation $\overline{\mathrm{E}}$ on $\mathscr{L}(\mathscr{Y})$, let $\underline{\mathrm{E}}$ be defined by conjugacy, and fix any $f,g \in \mathscr{L}(\mathscr{Y})$ and $\mu \in \mathbb{R}$. Then $\overline{\mathrm{E}}$ and $\underline{\mathrm{E}}$ satisfy the following properties:*

C4. $f \leq g \Rightarrow \overline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(g)$ *[monotonicity];*

C5. $\inf f \leq \underline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(f) \leq \sup f$ *[bounds];*

C6. $\overline{\mathrm{E}}(f+\mu) = \overline{\mathrm{E}}(f) + \mu$ *[constant additivity];*

C7. $\underline{\mathrm{E}}(f+g) \leq \overline{\mathrm{E}}(f) + \underline{\mathrm{E}}(g) \leq \overline{\mathrm{E}}(f+g)$ *[mixed super-/sub-additivity];*

C8. *for any sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\mathscr{L}(\mathscr{Y})$:* *[uniform convergence]*

$$\lim_{n\to+\infty} \sup |f - f_n| = 0 \Rightarrow \lim_{n\to+\infty} \overline{\mathrm{E}}(f_n) = \overline{\mathrm{E}}(f).$$

2.5 Upper expectations are slightly less expressive

We have put forward coherent upper expectations as central objects of interest because they can be given a universal meaning, as is established by Definition $2.6_{\leftarrow}$ above, and because they often allow us to conveniently draw inferences about the systems at hand; e.g. recall Examples $2.1.2_{21}$ and $2.2.3_{24}$. One could therefore be tempted to immediately express everything in terms of coherent upper expectations, yet care should be taken here, since upper expectations are actually less expressive than sets of probabilities or coherent sets of acceptable gambles. In other words, for any coherent upper expectation $\overline{\mathrm{E}}$, the set $\mathscr{P}$ in Definition $2.6_{\leftarrow}$ need not to be unique; there may be multiple sets of probabilities leading to the same coherent upper expectation. Similar considerations hold for upper expectations deduced from coherent sets of acceptable gambles. Let us clarify this.

If we want to associate a set of probability mass functions to a coherent upper expectation $\overline{\mathrm{E}}$, one possibility is to consider the set of all probability mass functions for which the associated expectation is smaller than (or equal to) $\overline{\mathrm{E}}$:

$$\mathscr{P}(\overline{\mathrm{E}}) := \left\{p \in \mathbb{P}(\mathscr{Y})\colon (\forall f \in \mathscr{L}(\mathscr{Y}))\ \textstyle\sum_{y\in\mathscr{Y}} p(y)f(y) \leq \overline{\mathrm{E}}(f)\right\}. \tag{2.3}$$

It follows from [110, Section 3.3.3] and conjugacy that the upper envelope—according to Definition 2.3_{24}—over this set $\mathscr{P}(\overline{\mathrm{E}})$ indeed coincides with $\overline{\mathrm{E}}$. Moreover, it also clear from Definition 2.3_{24} that $\mathscr{P}(\overline{\mathrm{E}})$ must always be the largest such set. But $\mathscr{P}(\overline{\mathrm{E}})$ should not necessarily be the only such set as there can also be smaller ones. For instance, removing the points that are not extreme[7] from the convex set $\mathscr{P}(\overline{\mathrm{E}})$—and therefore making it non-convex—does not alter the values of the upper expectation that results from it [110, Theorem 3.6.2]. On the other hand, since we are considering suprema of expectations, the boundary structure of a set of probability mass functions is often irrelevant as well, in the sense that any two sets $\mathscr{P}_1$ and $\mathscr{P}_2$ that have the same closure[8] will have the same resulting upper expectation. This is also the reason why the one-to-one correspondence in [110, Theorem 3.6.1] only involves sets of probabilities that are closed (or compact).

Example 2.5.1. Suppose that the possibility space $\mathscr{Y} = \{a, b\}$ consists of two elements a and b. Let $p_1, p_2 \in \mathbb{P}(\mathscr{Y})$ be two probability mass functions on $\mathscr{Y}$ such that $0 \leq p_1(a) < p_2(a) \leq 1$. Let $\mathscr{P}_1 := \{p_1, p_2\}$,

$$\mathscr{P}_2 := \{p \in \mathbb{P}(\mathscr{Y})\colon p_1(a) \leq p(a) \leq p_2(a)\} \text{ and}$$
$$\mathscr{P}_3 := \{p \in \mathbb{P}(\mathscr{Y})\colon p_1(a) < p(a) < p_2(a)\}.$$

The sets $\mathscr{P}_1$ and $\mathscr{P}_2$ are different but have the same extreme points; namely p_1 and p_2. On the other hand, it can be checked that $\mathscr{P}_3$ is not closed, but that its closure is equal to $\mathscr{P}_2$ (which is trivially closed).

As far as the corresponding upper expectations are concerned, we have that, for any $f \in \mathscr{L}(\mathscr{Y})$,

$$\overline{\mathrm{E}}_{\mathscr{P}_1}(f) = \sup_{p\in\{p_1,p_2\}} p(a)f(a) + [1-p(a)]f(b)$$
$$= \sup_{p\in\{p_1,p_2\}} p(a)[f(a)-f(b)] + f(b)$$

[7] An extreme point of a convex set B is an element in B that cannot be written as a convex combination of other elements in B.

[8] We here mean the closure under uniform convergence or the closure under pointwise convergence; both are equivalent because $\mathscr{Y}$ is finite.

So if $f(a) \geq f(b)$, then by the fact that $p_2(a) > p_1(a)$ we know that $\overline{\mathrm{E}}_{\mathscr{P}_1}(f) = p_2(a)[f(a)-f(b)]+f(b)$. If on the other hand $f(a) < f(b)$, then by $p_2(a) > p_1(a)$ we have that $\overline{\mathrm{E}}_{\mathscr{P}_1}(f) = p_1(a)[f(a)-f(b)]+f(b)$. It can be inferred in analogous way, and by using the definitions of $\mathscr{P}_2$ and $\mathscr{P}_3$, that the same expressions hold for $\overline{\mathrm{E}}_{\mathscr{P}_2}$ and $\overline{\mathrm{E}}_{\mathscr{P}_3}$; that is,

$$\overline{\mathrm{E}}_{\mathscr{P}_2}(f) = \overline{\mathrm{E}}_{\mathscr{P}_3}(f) = p_2(a)[f(a)-f(b)]+f(b) \text{ if } f(a) \geq f(b) \text{ and}$$
$$\overline{\mathrm{E}}_{\mathscr{P}_2}(f) = \overline{\mathrm{E}}_{\mathscr{P}_3}(f) = p_1(a)[f(a)-f(b)]+f(b) \text{ if } f(a) < f(b).$$

◇

In a similar way, we have that the boundary structure of a coherent set of acceptable gambles $\mathscr{D}$ has no impact on the associated upper expectation $\overline{\mathrm{E}}_{\mathscr{D}}$. For instance, it can easily be proved using Definition 2.5₂₉ that two coherent sets of acceptable gambles with the same (uniform) closure will give the same upper expectation. We do not prove this explicitly, but prefer to illustrate this with an example. Furthermore, note that convexity is always satisfied for a coherent set of acceptable gambles due to D3[27] and D4[27], and therefore that there is no extra degree of freedom in this respect.

Example 2.5.2. Reconsider the situation from Example 2.3.4[29]. Let $\mathscr{D}_2 := \{f \in \mathscr{L}(\mathscr{Y})\colon f(R)+f(G)+f(B) \geq 0\}$ be defined as previously, and let $\mathscr{D}_3 := \{f \in \mathscr{L}(\mathscr{Y})\colon f(R)+f(G)+f(B) > 0\} \cup \{0\}$. Then it can easily be checked that $\mathscr{D}_2$ and $\mathscr{D}_3$ are both indeed coherent sets of acceptable gambles. Moreover, by Definition 2.5[29], we also have that, for all $f \in \mathscr{L}(\mathscr{Y})$,

$$\overline{\mathrm{E}}_{\mathscr{D}_2}(f) = \underline{\mathrm{E}}_{\mathscr{D}_2}(f) = (f(R)+f(G)+f(B))/3 = \overline{\mathrm{E}}_{\mathscr{D}_3}(f) = \underline{\mathrm{E}}_{\mathscr{D}_3}(f).$$

Yet, it is clear that $\mathscr{D}_2 \neq \mathscr{D}_3$ as for instance $g = (g(R),g(G),g(B)) := (-1,\frac{1}{2},\frac{1}{2})$ is an element of $\mathscr{D}_3$ but not of $\mathscr{D}_2$. ◇

Given a coherent upper expectation $\overline{\mathrm{E}}$, one possible coherent set of acceptable gambles that can be associated with $\overline{\mathrm{E}}$ is the set $\mathscr{D}(\overline{\mathrm{E}})$ that results from interpreting $\overline{\mathrm{E}}$ as infimum selling prices:

$$\begin{aligned}\mathscr{D}(\overline{\mathrm{E}}) &:= \left\{\alpha - f\colon f \in \mathscr{L}(\mathscr{Y}) \text{ and } \alpha > \overline{\mathrm{E}}(f)\right\} \cup \mathscr{L}_{\geq}(\mathscr{Y}) \\ &= \left\{f \in \mathscr{L}(\mathscr{Y})\colon 0 < \underline{\mathrm{E}}(f)\right\} \cup \mathscr{L}_{\geq}(\mathscr{Y}), \qquad (2.4)\end{aligned}$$

where we immediately used the coherence of $\overline{\mathrm{E}}$ and conjugacy for the second equality. As is pointed out in [114, Section 3.3] this set $\mathscr{D}(\overline{\mathrm{E}})$ is the smallest coherent set of acceptable gambles for which the associated upper expectation—according to Definition 2.5[29]—is equal to $\overline{\mathrm{E}}$.

The increased expressiveness of both coherent sets of acceptable gambles and sets of probabilities compared to coherent upper expectations may in some cases turn out to be practically relevant; we will henceforth devote little attention to it though, and simply refer to [110, Sections 3.7] and [82] for a more elaborate discussion on this matter. However, one of the major reasons for bringing this topic to the fore is that it urges caution when modelling stochastic processes, as we will do in the following chapters. For in such a stochastic processes setting [Section $3.1.2_{48}$], we usually start off from (multiple) local models, given as either sets of probability mass functions, sets of acceptable gambles or upper expectations, and the central aim is then to combine and extend these local models to obtain more global information about the stochastic process at hand. A naive approach would be to immediately, from the start, express all local beliefs in terms of upper expectations and then simply extend from here on—as, in the end, we will be interested in global upper expectations anyway. Yet, it is a priori not given whether this is equivalent to first performing an extension in one of the more general frameworks (sets of probabilities or sets of acceptable gambles), and then afterwards transitioning to (the less expressive) global upper expectations—which is the preferred route if one wishes to preserve the initial given information as much as possible. We will therefore—amongst other reasons—study three separate approaches of constructing a global upper expectation; one for each of the three different types of local models.

2.6 Extension of an uncertainty model

So far, we have established that a subject's beliefs about an uncertain variable Y can be modelled in three different ways; by means of a set of probabilities, a coherent set of acceptable gambles, or a coherent upper expectation. Nonetheless, in practical situations, when eliciting beliefs from a real-life subject, it should not be expected that such a subject will specify an entire set of probabilities that she deems possible, or specify an entire cone of gambles that she deems acceptable. Or, in the framework of upper expectations, it seems unrealistic to ask our subject to immediately specify her infimum selling prices, or her upper bounds on linear expectations, for **all** gambles in $\mathscr{L}(\mathscr{Y})$. Even if her beliefs itself can be expressed by a full-fledged coherent upper expectation, in reality, we often do not have the time, money or tools to gather all the necessary information needed to characterise this upper expectation. Hence, whatever the framework we are considering, the initial assessments of a subject typically do not match the structural conditions of the models discussed before. As a result, in order to draw inferences, we are confronted with the question of how to extend initial partial assessments to fully developed sets of probabilities, coherent sets of acceptable gambles or coherent upper expectations.

2.6.1 Extensions for sets of probabilities

Performing an extension within the framework of sets of probabilities or sets of acceptable gambles is rather straightforward. In fact, for sets of probabilities, there does not really exist a concrete extension mechanism; as briefly mentioned in Section 2.2_{22}, we usually start off with some given bounds on certain probabilities, or more generally, some restrictions on the form of the possible probabilities. Our next step is then simply to consider the **largest** set of probability mass functions that is compatible with these restrictions; so if $\mathscr{R}$ denotes a (not necessarily finite) collection of restrictions, and compatibility of a probability mass function p with the restrictions $\mathscr{R}$ is denoted by $p \sim \mathscr{R}$, then $\{p \in \mathbb{P}(\mathscr{Y})\colon p \sim \mathscr{R}\}$ is the desired set of probability mass functions. For instance, reconsidering Example $2.2.3_{24}$, the collection of restrictions $\mathscr{R}$ consisted only out of the restriction that $p(R) = \frac{2}{6}$. The reason why we consider the largest set among all possible ones is due to conservativity considerations; taking smaller sets essentially means adding more restrictions, and thus more information on top of what is given by our subject. It could also be that the set $\{p \in \mathbb{P}(\mathscr{Y})\colon p \sim \mathscr{R}\}$ is empty and thus that there are no probability mass functions compatible with the restrictions $\mathscr{R}$; in that case, we call $\mathscr{R}$ inconsistent.

2.6.2 Extensions for sets of acceptable gambles

Consider a (not necessarily coherent) set of acceptable gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{Y})$ that represents the initial assessments of our subject. The set $\mathscr{A}$ may take any form, and is most likely to include only a finite number of gambles—and therefore not to be coherent. Adopting Axioms D1_{27}–D4_{27} however allows us to say something about the acceptability of gambles that are not included in $\mathscr{A}$. Concretely, Axioms D1_{27}, D3_{27} and D4_{27} tell us that our subject should accept, apart from the gambles in $\mathscr{A}$ itself, the gambles in the cone

$$\mathscr{E}(\mathscr{A}) \coloneqq \operatorname{posi}(\mathscr{A} \cup \mathscr{L}_{\geq}(\mathscr{Y})) = \left\{ \sum_{i=1}^{n} \lambda_i f_i \colon n \in \mathbb{N}, f_i \in \mathscr{A} \cup \mathscr{L}_{\geq}(\mathscr{Y}), \lambda_i \in \mathbb{R}_{>} \right\},$$

which is obtained by including $\mathscr{L}_{\geq}(\mathscr{Y})$ and taking all positive linear combinations. Note that adding more gambles to the set $\mathscr{E}(\mathscr{A})$ would mean adding more information apart from what $\mathscr{A}$ and Axioms D1_{27}, D3_{27} and D4_{27} tell us. If $\mathscr{E}(\mathscr{A})$ does not include any gambles from $\mathscr{L}_{\lneq}(\mathscr{Y})$—and therefore does not violate D2_{27}—then $\mathscr{E}(\mathscr{A})$ is coherent and, since nothing more can be deduced from Axioms D1_{27}–D4_{27}, it is the smallest coherent set of acceptable gambles extending the assessments $\mathscr{A}$. So, since smaller coherent sets of acceptable gambles are obviously more conservative, $\mathscr{E}(\mathscr{A})$ is then the most conservative coherent set of acceptable gambles that extends $\mathscr{A}$, and is therefore our desired coherent extension of $\mathscr{A}$.

However, it could of course also be that $\mathscr{E}(\mathscr{A}) \cap \mathscr{L}_{\lneq}(\mathscr{Y}) \neq \emptyset$; in that case, we infer from $\mathscr{A}$ that our subject is willing to accept a gamble in $\mathscr{L}_{\lneq}(\mathscr{Y})$. By adopting $D2_{27}$, we have agreed upon the fact that this is irrational, and this prevents us from extending her assessments $\mathscr{A}$ to a coherent set of acceptable gambles. We then call $\mathscr{A}$ inconsistent. These considerations are gathered in the following definition [106, 110].

Definition 2.7. We say that a set $\mathscr{A} \subseteq \mathscr{L}(\mathscr{Y})$ of acceptable gambles is **consistent** if $\mathscr{E}(\mathscr{A}) \cap \mathscr{L}_{\lneq}(\mathscr{Y}) = \emptyset$. If this is the case, then $\mathscr{E}(\mathscr{A})$ is the smallest coherent set of acceptable gambles including $\mathscr{A}$, and it is called the **natural extension** of $\mathscr{A}$. ◎

Proof. It is clear from the definition of $\mathscr{E}(\mathscr{A})$ that this set $\mathscr{E}(\mathscr{A})$ satisfies $D1_{27}$, $D3_{27}$ and $D4_{27}$. If $\mathscr{A}$ is consistent, then $\mathscr{E}(\mathscr{A})$ additionally satisfies $D2_{27}$, and thus $\mathscr{E}(\mathscr{A})$ is then coherent. It is moreover clear from the definition of $\mathscr{E}(\mathscr{A})$ that any other coherent set of acceptable gambles including $\mathscr{A}$ must always include $\mathscr{E}(\mathscr{A})$ too, so $\mathscr{E}(\mathscr{A})$ is indeed the smallest coherent set of acceptable gambles including $\mathscr{A}$. □

2.6.3 Direct extensions of upper expectations

Suppose now that our subject's initial assessments are represented by an upper expectation $\overline{\mathrm{E}} \colon \mathscr{K} \to \mathbb{R}$ on some arbitrary domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$—these can represent infimum selling prices, upper bounds on linear expectations/probabilities or both. Again, it is most likely the subject only specifies a finite number of values, and therefore that $\mathscr{K}$ is finite (but we do not necessarily require $\mathscr{K}$ to be finite). Our aim is to extend $\overline{\mathrm{E}}$ from $\mathscr{K}$ to the entire space $\mathscr{L}(\mathscr{Y})$ such that the resulting extended operator is coherent in the sense of Definition 2.6_{32}. As before, two crucial questions then come to mind; 'Under what conditions can we perform such an extension?' and 'If there are multiple extensions possible, which one do we take?'.

General coherence

The following result gives an answer to our first question. It is stated as a definition though, more specifically as a renewed definition of **coherence**. Coherence was indeed already introduced earlier on, with Definition 2.6_{32}, but the concept is well-known to generalise to upper expectations on general domains $\mathscr{K}$. This generalised notion immediately turns out to be sufficient (and necessary) in order for a coherent extension—in the sense of Definition 2.6_{32}—to exist. The definition below is due to P. M. Williams [113], but we use [106] in our proof because P. M. Williams [113] immediately gives a version of coherence for conditional upper expectations.

Definition 2.8. Consider any upper expectation $\overline{\mathrm{E}}\colon \mathscr{K} \to \mathbb{R}$ on an arbitrary domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$. Then the following conditions are equivalent. If any—and hence all—of them hold, we call $\overline{\mathrm{E}}$ coherent.

(i) $\overline{\mathrm{E}}$ is the restriction of a coherent upper expectation $\overline{\mathrm{E}}'\colon \mathscr{L}(\mathscr{Y}) \to \mathbb{R}$ according to Definition 2.6$_{32}$;

(ii) for all $n \in \mathbb{N}$, $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq}$ and $f_0, f_1, \dots, f_n \in \mathscr{K}$,

$$\sup\left(\lambda_0(f_0 - \overline{\mathrm{E}}(f_0)) - \sum_{i=1}^{n} \lambda_i(f_i - \overline{\mathrm{E}}(f_i))\right) \geq 0.$$

(iii) there is a non-empty set $\mathscr{P}$ of probability mass functions on $\mathscr{Y}$ such that $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}_{\mathscr{P}}$ on $\mathscr{K}$;

(iv) there is a coherent set $\mathscr{D}$ of acceptable gambles such that $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}_{\mathscr{D}}$ on $\mathscr{K}$. ◎

Proof. It is clear that due to Definition 2.6$_{32}$, conditions (i), (iii) and (iv) are equivalent. The fact that (ii) is equivalent to (i) follows rather straightforwardly from [106, Definition 4.10 (B) and (E)] and conjugacy. □

What's perhaps somewhat unfortunate about this general notion of coherence, or at least if we compare it to the simpler version in Definition 2.6$_{32}$, is that it cannot be characterised in terms of the three simple axioms C1$_{32}$–C3$_{32}$ any more; the requirement (i) above only uses C1$_{32}$–C3$_{32}$ in an indirect manner; in general, it does not suffice for an upper expectation $\overline{\mathrm{E}}$ on a general domain $\mathscr{K}$ to satisfy (the restricted versions of) C1$_{32}$–C3$_{32}$ to be coherent. The requirement (ii) above is direct, but the involved expression is, compared to C1$_{32}$–C3$_{32}$, more difficult to grasp and makes mathematical analysis less straightforward. This requirement can nevertheless be intuitively motivated on behavioural grounds; a topic for which we like to refer to [110, Section 2.5]—note that a characterisation similar to (ii) can be found in [110, Section 2.5.4].

The natural extension of an upper expectation

By condition (i) in the definition above it is clear that coherence is necessary and sufficient in order for a coherent extension to the entire space $\mathscr{L}(\mathscr{Y})$ to exist. Yet, even if the initial upper expectation $\overline{\mathrm{E}}\colon \mathscr{K} \to \mathbb{R}$ is coherent, this extension might still not be unique and so the question remains which extension to pick. Once more, we will choose for the most conservative extension; in this case that translates to choosing the (pointwise) **largest** upper expectation. For indeed, that larger upper expectations correspond to more conservative—less committal or less informative—judgements can be argued on the basis of the dual meaning of a coherent upper expectation. Using an interpretation in terms of upper bounds on possible probabilities or expectations, larger (or higher) coherent upper expectations mean higher upper bounds, which are more conservative—less informative. On the other hand, using an interpretation in terms of infimum selling prices, larger coherent upper expectations represent higher infimum selling prices, which are again more conservative.

The most conservative—or thus the pointwise largest—coherent extension among all possible coherent extensions is called the **natural extension under coherence**, or simply the **natural extension** [110, 113].

Definition 2.9. Consider a coherent upper expectation $\overline{\mathrm{E}}\colon \mathscr{K} \to \mathbb{R}$ on an arbitrary domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$. Then there is a—trivially unique—pointwise largest coherent extension $\overline{\mathrm{E}}'$ of $\overline{\mathrm{E}}$ to $\mathscr{L}(\mathscr{Y})$. This extended upper expectation $\overline{\mathrm{E}}'$ is called the natural extension of $\overline{\mathrm{E}}$. ◎

Proof. The existence of $\overline{\mathrm{E}}'$ follows from [106, Theorem 4.26(ii)] and conjugacy. □

Two explicit expressions for the natural extension

Definition 2.9 guarantees that the natural extension of a coherent upper expectation exists, yet the form and properties of this natural extension are still to be derived implicitly from the fact that it satisfies C1$_{32}$–C8$_{33}$ and the fact that it is the largest among all coherent extensions. More explicit and tangible characterisations of this natural extension can nonetheless be obtained by relying on the close connections with the frameworks of sets of probabilities and sets of acceptable gambles.

Suppose once more that we are given a coherent upper expectation $\overline{\mathrm{E}}$ on some domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$. If we regard $\overline{\mathrm{E}}$ as being upper bounds on possible expectations corresponding to probabilities, then the set

$$\mathscr{P}(\overline{\mathrm{E}}) := \left\{p \in \mathbb{P}(\mathscr{Y})\colon (\forall f \in \mathscr{K}) \textstyle\sum_{y\in\mathscr{Y}} p(y)f(y) \leq \overline{\mathrm{E}}(f)\right\}, \tag{2.5}$$

is the largest possible set of probability mass functions that is compatible with the upper bounds represented by $\overline{\mathrm{E}}$. Note moreover that the expression above is simply a generalisation of the expression given for $\mathscr{P}(\overline{\mathrm{E}})$ in Section 2.5$_{33}$, which is why we are allowed to use the same notation. It can be shown that, since $\overline{\mathrm{E}}$ was assumed coherent, the set $\mathscr{P}(\overline{\mathrm{E}})$ is non-empty and has the natural extension $\overline{\mathrm{E}}'$ as its upper envelope. This result was first stated by Walley [110, Section 3.3.3].

Theorem 2.6.1. *For any coherent upper expectation* $\overline{\mathrm{E}}$ *on* $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$*, the set* $\mathscr{P}(\overline{\mathrm{E}})$ *is non-empty. Moreover, the natural extension* $\overline{\mathrm{E}}'$ *of* $\overline{\mathrm{E}}$ *to* $\mathscr{L}(\mathscr{Y})$ *is equal to the upper expectation* $\overline{\mathrm{E}}_{\mathscr{P}(\overline{\mathrm{E}})}$ *deduced from* $\mathscr{P}(\overline{\mathrm{E}})$ *according to Definition 2.3*$_{24}$.

Proof. This follows from [110, Section 3.3.3] or [106, Theorem 4.38] and by applying conjugacy. □

A similar thing can be done for sets of acceptable gambles. Consider a coherent upper expectation $\overline{\mathrm{E}}$ on some domain $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$, and let $\mathscr{A}(\overline{\mathrm{E}})$ be the set of all gambles that are deemed acceptable if we interpret $\overline{\mathrm{E}}$ as infimum selling prices; so we let

$$\mathscr{A}(\overline{\mathrm{E}}) := \left\{\alpha - f : f \in \mathscr{K} \text{ and } \alpha > \overline{\mathrm{E}}(f)\right\}. \tag{2.6}$$

Note that, strictly speaking, $\mathscr{A}(\overline{\mathrm{E}})$ is not the smallest—most conservative—set of acceptable gambles that corresponds to the interpretation of $\overline{\mathrm{E}}$ as representing a subject's infimum selling prices; indeed, this interpretation only implies that, for any $f \in \mathscr{K}$, there are $\alpha > \overline{\mathrm{E}}(f)$ arbitrarily close to $\overline{\mathrm{E}}(f)$ for which $\alpha - f$ is acceptable, and not that this is the case for **all** $\alpha > \overline{\mathrm{E}}(f)$. By starting from $\mathscr{A}(\overline{\mathrm{E}})$ as the initial set of acceptable gambles that corresponds to $\overline{\mathrm{E}}$, we already implicitly adopt the property of coherence that, if $\alpha - f$ is acceptable, then $\beta - f$ must also be acceptable for all $\beta > \alpha$ [D1$_{27}$ and D3$_{27}$]. The expression for $\mathscr{A}(\overline{\mathrm{E}})$ above is also similar to the one for $\mathscr{D}(\overline{\mathrm{E}})$ given in Section 2.5$_{33}$, but $\mathscr{A}(\overline{\mathrm{E}})$ here is not necessarily coherent whereas $\mathscr{D}(\overline{\mathrm{E}})$ always is.

As was first pointed out by P. M. Williams [113, Theorem 1]—though less explicitly and in a somewhat different setting—the set $\mathscr{A}(\overline{\mathrm{E}})$ is consistent if $\overline{\mathrm{E}}$ is coherent, and the upper expectation $\overline{\mathrm{E}}_{\mathscr{E}(\mathscr{A}(\overline{\mathrm{E}}))}$ is then the natural extension of $\overline{\mathrm{E}}$ to $\mathscr{L}(\mathscr{Y})$.

Theorem 2.6.2. *For any coherent upper expectation* $\overline{\mathrm{E}}$ *on* $\mathscr{K} \subseteq \mathscr{L}(\mathscr{Y})$*, the set of acceptable gambles* $\mathscr{A}(\overline{\mathrm{E}})$ *is consistent. Moreover, the natural extension* $\overline{\mathrm{E}}'$ *of* $\overline{\mathrm{E}}$ *to* $\mathscr{L}(\mathscr{Y})$ *is equal to the upper expectation* $\overline{\mathrm{E}}_{\mathscr{E}(\mathscr{A}(\overline{\mathrm{E}}))}$ *according to Definition 2.5*$_{29}$*. It is given, for all* $f \in \mathscr{L}(\mathscr{Y})$*, by*

$$\overline{\mathrm{E}}'(f) = \inf\left\{\alpha \in \mathbb{R} : \alpha - f \geq \sum_{i=1}^{n} \lambda_i \left(\overline{\mathrm{E}}(f_i) - f_i\right), n \in \mathbb{N}, f_i \in \mathscr{K}, \lambda_i \in \mathbb{R}_{>}\right\}.$$

Proof. Suppose that $\overline{\mathrm{E}}$ is coherent, and let $\underline{\mathrm{E}}$ defined through conjugacy; so, for all f such that $-f \in \mathscr{K}$, we let $\underline{\mathrm{E}}(f) := -\overline{\mathrm{E}}(-f)$. Then it can be inferred from Definition 2.8$_{39}$ (ii)$_{39}$, that $\underline{\mathrm{E}}$ is coherent in the sense of [106, Definition 4.10 (E)]. Hence, by [106, Definition 4.10(C)] and [106, Definition 4.6(A)], the set $\mathscr{A}(\overline{\mathrm{E}})$ is consistent in the sense of [106, Definition 3.4(B)], which in turn implies that it is consistent according to our Definition 2.7$_{38}$. That $\overline{\mathrm{E}}_{\mathcal{E}(\mathcal{A}(\overline{\mathrm{E}}))}$ is equal to the natural extension $\overline{\mathrm{E}}'$ then follows from [106, Theorem 4.26(ii)], conjugacy, and the fact that our definition of $\mathcal{E}(\cdot)$ is equivalent to the one of the natural extension in [106, Theorem 3.7]. Finally, the expression for $\overline{\mathrm{E}}'$ in the statement above then follows from [106, Definition 4.8] and conjugacy (and the fact that coherence implies 'avoiding sure loss' [106, Definition 4.10]). □

Extending non-coherent upper expectations

If we interpret upper expectations to be non-exhaustive representations of a subject's beliefs—as we usually do—then the requirement of **extending** an initial upper expectation is in fact unnecessarily stringent. As in such a case, there is nothing that prevents us from updating or sharpening the already given upper expected values. If unnecessary, we prefer not to do this due to conservativity considerations, but in some cases this additional freedom of correcting already specified assessments provides the opportunity to still come up with a coherent 'sharpened' model in cases where a (coherent) extension would simply not exist. In particular, these cases occur if our subject, in the act of specifying her initial upper expectation, does not fully take into account the consequences of her own statements.

So, in such a case, we want our 'extended' upper expectation $\overline{\mathrm{E}}'$ on $\mathcal{L}(\mathcal{Y})$ to dominate or sharpen the initial upper expectation $\overline{\mathrm{E}}\colon \mathcal{K} \to \mathbb{R}$; that is, $\overline{\mathrm{E}}'$ should be equal to **or smaller** than $\overline{\mathrm{E}}$ on $\mathcal{K}$. If there exists at least one such smaller—dominating—coherent upper expectation, then $\overline{\mathrm{E}}$ is called consistent, or is said to avoid sure loss [106, Definition 4.6]. In this case, there must moreover always exist a pointwise largest—most conservative—upper expectation that is coherent and dominates $\overline{\mathrm{E}}$, and this upper expectation is then typically also called the natural extension of $\overline{\mathrm{E}}$; see [110, Section 3.1.2] or [106, Theorem 4.26(i)].[9] As established by [106, Definition 4.10(C)] and the example below, consistency is strictly weaker than coherence. Furthermore, if an upper expectation $\overline{\mathrm{E}}$ is coherent—and therefore also consistent—then the generalised type of natural extension above is the same as the natural extension from Definition 2.9$_{40}$; see [106, Theorem 4.26]. So in that case it is again an actual extension, and thus the freedom to sharpen a subject's assessments only really becomes relevant if we are dealing with non-coherent upper expectations.

Example 2.6.3. Consider any non-empty set $\mathcal{Y}$ with $|\mathcal{Y}| \geq 2$, any $f_1 \in \mathcal{L}(\mathcal{Y})$, let $y \in \mathcal{Y}$ and let $\overline{\mathrm{E}}$ be defined by $\overline{\mathrm{E}}(f_1) := f_1(y)$ and $\overline{\mathrm{E}}(f) := \sup f$ for all $f \in \mathcal{L}(\mathcal{Y}) \setminus \{f_1\}$. Then note that, if $f_1(y) < f_1(z)$ for some $z \in \mathcal{Y}$ such that $z \neq y$, the upper expectation $\overline{\mathrm{E}}$ is not coherent. Indeed, then $f_1 \neq 0$

[9] Though, confusingly enough, it need not necessarily be an actual extension.

and thus $2f_1 \neq f_1$, which implies that

$$\overline{\mathrm{E}}(2f_1) = \sup(2f_1) \geq 2f_1(z) > 2f_1(y) = 2\overline{\mathrm{E}}(f_1),$$

which violates non-negative homogeneity [C3$_{32}$] and sub-additivity [C2$_{32}$]. Yet, it is not difficult to find a coherent upper expectation on $\mathscr{L}(\mathscr{Y})$ that dominates $\overline{\mathrm{E}}$; e.g. consider the (linear) upper expectation $\overline{\mathrm{E}}'$ defined by $\overline{\mathrm{E}}'(f) = f(y)$ for all $f \in \mathscr{L}(\mathscr{Y})$. So $\overline{\mathrm{E}}$ is consistent and therefore, according to [110, Section 3.1.2] or [106, Theorem 4.26(i)], its 'natural extension' exists. ◇

We will never make use of the notions of consistency or sharpening for upper expectations, though. The coming chapters deal with discrete-time stochastic processes and will solely discuss extension procedures that allow us to go from local upper expectations to global upper expectations. These local upper expectations will always be assumed coherent (and defined on the entire domain of all local gambles) from the start anyway, and so, similar as explained here for the present simple finitary context, the extensions that we will be interested in will never sharpen or dominate the initial local upper expectations. Of course, in order to adhere to a more realistic scenario, the coherent local upper expectations can nevertheless be regarded as if they are derived from some initial consistent—but not necessarily coherent—(local) upper expectations as in the way set out above. We refer the interested reader to [106, 110] for more details on the topic of consistency and natural extension for non-coherent upper expectations.

In fact, the extension procedures described in the following chapters will also never explicitly use the methods introduced here in Section 2.6$_{36}$ for finite possibility spaces. This is because the discrete-time stochastic processes setting requires us to deal with (uncountably) infinite possibility spaces and to perform specific types of extensions, for which the methods just presented are inadequate. Our reason for nevertheless devoting an entire section to them is because many of the key ideas underlying these methods are similar but clearer than those that underlie the the more involved procedures discussed in future chapters.

— Chapter III —

Finitary upper expectations in discrete-time stochastic processes

Consider a subject whose beliefs about a—finite or infinite—series X_1, X_2, $X_3, \cdots, X_k, \cdots$ of uncertain variables we want to model. Each of these uncertain variables X_k take values in the same finite non-empty set $\mathscr{X}$, and the index $k \in \mathbb{N}$ can be interpreted as a discrete time indication. For example, $k \in \mathbb{N}$ may denote the k-th day of the year, and the variable X_k may be the state of the weather in Ghent on that k-th day. The state could then involve detailed information about the weather—the average temperature, humidity, air pressure, ...—but it can also be a rough simplification, with the set of possible values $\mathscr{X}$ for instance being $\{\text{Sunny}, \text{Cloudy}, \text{Rainy}\}$. Any such sequence $(X_k)_{k\in\mathbb{N}}$ of uncertain variables, indexed by a discrete time variable $k \in \mathbb{N}$, is what we call a **discrete-time stochastic process**.

In the previous chapter, we have seen how sets of probability mass functions, sets of acceptable gambles, and coherent upper expectations can each provide an appropriate way to model (unconditional) beliefs about a single uncertain variable Y taking values in a finite set. The context of stochastic processes requires a somewhat more complicated set-up though, since we have to deal not with a single, but with a possibly infinite sequence of uncertain variables—the corresponding total possibility space will therefore not necessarily be finite. Moreover, since it is a process—a physical system changing through time—that we are modelling, we need a model that can incorporate new information as it becomes available. If the process advances one time step from k to $k + 1$, and the new value of the state X_{k+1} is presented, this should be taken into account by our uncertainty model. In other words, we want to be able to condition on such newly arrived information, and we therefore need a conditional global uncertainty model.

The current chapter aims to show how the notions of (sets of) probability mass functions, sets of acceptable gambles, and coherent upper and lower expectations as introduced previously for finite possibility spaces, can be suitably adapted and generalised in order to develop joint global models for stochastic processes. We start in Section 3.1→ from local assessments on the variables X_1, X_2, X_3, ... individually. These assessments are expressed in the form of one of the three types of (unconditional) uncertainty models introduced in the previous chapter—this is possible because the local state space $\mathscr{X}$ is always assumed to be finite. We then combine and extend these local models in order to obtain a single global model, which will always take the form of a conditional upper expectation. We will do this in different ways, depending on the type of local model that is started from.

In Section 3.256, we start from local sets of acceptable gambles and propose an extension that is entirely based on the behavioural framework of sets of acceptable gambles [106, 113]. The resulting global upper expectations will then represent infimum (acceptable) selling prices. Furthermore, it will also be shown that this upper expectation can be given an alternative characterisation in terms of super- and submartingales; game-theoretic concepts that represent allowable betting strategies. The game-theoretic upper expectations considered in this chapter, however, are finitary in nature and do not use the concept of superhedging at infinity; this is in contrast with the Shafer and Vovk type of game-theoretic upper expectation [85, 86] treated in Chapter 4129.

Section 3.369 then, considers global upper expectations deduced from local probability mass functions. Our central notion to extend from a local to a global level will be that of a conditional probability charge [18, 34]. From conditional probability charges we will derive global upper expectations by first using (Lebesgue-wise) integration and then taking upper envelopes. Mark however that we do **not** necessarily consider countably additive probability charges in this chapter; global upper expectations based on countably additive probability charges—probability measures—will be discussed in Chapter 5217. Not doing so not only makes our treatment more general, but it also allows us to introduce probability-based global upper expectations in a more tangible way, without having to introduce technical machinery such as σ-algebras or the notion of measurability corresponding to such σ-algebras.

A last type of global upper expectation is discussed in Section 3.480, and is based on the framework of coherent upper and lower expectations. We will first propose four simple axioms, common to both behavioural and probability-based global upper expectations. This axiomatisation will moreover be shown to be equivalent to the usual requirement of conditional coherence [106, 113]—a result that, to our knowledge, has not been stated before. The global upper expectation will then subsequently be defined as the most conservative extension (of the local upper expectations) under these axioms—or, equivalently, the natural extension under conditional coherence. We conclude Section 3.480 by proving two alternative characterisations for the axiomatic/coherence-based global upper expectation; the first is a direct and explicit formula which can be used in practice to do inference; the second is a full axiomatisation—without conservatism—that will turn out to be highly convenient in our further mathematical analysis.

In our penultimate section, Section 3.5_{90}, we study the relation between the global models introduced and discussed in Section 3.2_{56}–3.4_{80}. It will turn out that, if local models are chosen in a logical way to agree with one another, then the corresponding global upper expectations will all be equal. This is a fundamental—and new—result, that we argue to have merit in a number of different ways.

Finally, in Section 3.6_{98}, we expose the Achilles heel of this common global model; it lacks some minimal—weak—continuity properties, therefore sometimes leading to extremely conservative, or even meaningless values for non-finitary variables—variables that depend on an infinite number of process states. This ought not to surprise us entirely since, after all, none of the global upper expectations treated here relies, in their definition, on any continuity assumption. All extension mechanisms are finitary in nature. Our reasons for nevertheless devoting an entire chapter to these 'finitary' global upper expectations is threefold; first and foremost, these finitary global upper expectations are actually perfectly suited if one is solely interested in the domain of finitary bounded variables; secondly, our study of them, and especially our observation that they lack minimal continuity properties, urges the necessity of using the more involved continuity-based global upper expectations in Chapter 4_{129}–6_{283} when dealing with more general domains; and finally, our choice is also due to didactic considerations, since the finitary global upper expectations introduced here allow us to already set many fundamental ideas and concepts in place, before introducing their more complicated continuity-based variants in Chapter 4_{129}–6_{283}.

3.1 Stochastic processes

Before we construct a global model for discrete-time stochastic processes, we first elaborate on the underlying mathematical structure of such a process. We introduce notions that are essential for a suitable treatment of stochastic processes, and clarify what the difference is between local and global uncertainty models.

3.1.1 About event trees, situations and paths

As mentioned earlier, any sequence $(X_k)_{k\in\mathbb{N}}$ of uncertain states taking values in a state space $\mathscr{X}$ is called a discrete-time stochastic process. We will always assume that $\mathscr{X}$ is **finite**, and in this case the stochastic process can be visualised in a so-called event tree; see Figure 3.1$_\rightarrow$. It presents each possible partial (and possibly empty) realisation $X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k$ of the process as a finite sequence of state values $x_{1:k} := x_1x_2 \cdots x_k \in \mathscr{X}^k$. These finite sequences of state values are called **situations**, and we gather all of them in $\mathscr{X}^* := \cup_{k\in\mathbb{N}_0}\mathscr{X}^k$. The empty sequence $\square := x_{1:0} = \emptyset$ is called the **initial situation**, and corresponds to the case where there are no observations about the values of the states $(X_k)_{k\in\mathbb{N}}$. The **length** of a situation $s \in \mathscr{X}^*$ is denoted by $|s|$.

On the other hand, an infinite sequence $\omega = x_1x_2x_3 \cdots$ of state values is called a **path**; such a path ω represents so to speak an entire (idealised) trajectory or realisation of the process, where the k-th state value of ω, denoted by ω_k, represents the value of the state X_k at time k. We write $\Omega := \mathscr{X}^{\mathbb{N}}$ to denote the set of all paths. For any path $\omega \in \Omega$ and any $k, \ell \in \mathbb{N}_0$, we let $\omega_{k:k+\ell} := \omega_k \cdots \omega_{k+\ell}$ be the situation that consists of ω's k-th to $k+\ell$-th state values. We also let $\omega^k := \omega_{1:k}$ for all $\omega \in \Omega$ and all $k \in \mathbb{N}_0$. Furthermore, an important type of event—a subset of Ω—is the cylinder event $\Gamma(s)$ of a situation $s \in \mathscr{X}^*$; it is the set $\{\omega \in \Omega \colon \omega^{|s|} = s\}$ of all paths that go through the situation s.

For any two situations s and t, we write $st \in \mathscr{X}^*$ to denote the concatenation of s and t. The same can be done for the concatenation of a situation s and a path ω. The path $s\omega$ will then clearly go through s and, in fact, we can write the cylinder event $\Gamma(s)$ as $\{s\omega \colon \omega \in \Omega\}$. For any two situations $s, t \in \mathscr{X}^*$, we write that $s \sqsubseteq t$ or $t \sqsupseteq s$, and say that s precedes t or that t follows s, if there is some $u \in \mathscr{X}^*$ such that $su = t$. We then also say that t starts with s. Furthermore, we write that $s \sqsubset t$ or $t \sqsupset s$, if $s \sqsubseteq t$ and $s \neq t$. For any two $s, t \in \mathscr{X}^*$, if $s \sqsubseteq t$ then $\Gamma(s) \supseteq \Gamma(t)$, and if $s \sqsubset t$ then $\Gamma(s) \supset \Gamma(t)$. If neither $s \not\sqsubseteq t$, nor $t \not\sqsubseteq s$, then we say that s and t are incomparable and write that $s \parallel t$. In that case, we have that $\Gamma(s) \cap \Gamma(t) = \emptyset$.

3.1.2 The local description of a stochastic process

Irrespectively of how we construct a conditional (global) uncertainty model for a stochastic process $(X_k)_{k\in\mathbb{N}}$, the starting point is almost always a description of the local behaviour of the process. That is, how the process state X_k is likely to change or develop from one time instant k to the next $k+1$. In most applications, this is what is directly available from a subject, being it either a real person, a bunch of data and/or knowledge about the system. For example, given the stock X_k of face masks in some inventory at the k-th day of the year, we usually have an idea about how many face masks are likely to be produced and consumed—and, perhaps, thrown away—that day, and hence, an idea about how many face masks X_{k+1} will be left by the

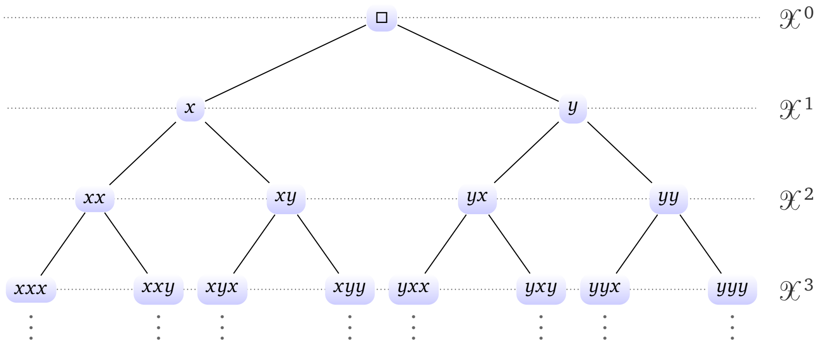

Figure 3.1 The event tree of a discrete-time stochastic process with state space $\mathscr{X} = \{x, y\}$.

next day. Yet, it is less obvious to assess the expected time until the inventory is out of stock, or, more importantly, what the probability is that such a terrible event will ever happen! Such inferences involve the process behaviour on a more global level and, in order to make statements about them, we will need to combine and extend the given local assessments. However, it is far from trivial how we should do so, especially in an imprecise probabilities context and when dealing with an infinite time horizon. In fact, one could argue that this is the prime reason why there exist multiple different types of global uncertainty models for stochastic processes; each of these global models relies on a different set of principles and assumptions to combine and extend the local assessments.

But let us first return to the original problem of describing a stochastic process locally. In a traditional precise context, this is commonly done by specifying a **(precise) probability tree** p; such a tree maps each situation $x_{1:k} \in \mathscr{X}^*$ to a probability mass function $p(\cdot|x_{1:k})$ on the finite state space $\mathscr{X}$. Each so-called local mass function $p(\cdot|x_{1:k})$ then represents beliefs about the value of the next state X_{k+1}, given that the partial realisation $X_{1:k} := X_1 \cdots X_k = x_{1:k}$ of the process was observed. The use of a single probability charge $p(\cdot|s)$ for each $s \in \mathscr{X}^*$ would however considerably restrict the generality of our approach—for recall our considerations from Chapter 2$_{17}$. We therefore instead use either of the following more general 'imprecise' approaches to model the local dynamics of a stochastic process.

First, we can simply consider a generalisation of the concept of a (precise) probability tree; an **imprecise probability tree** $\mathscr{P}_\bullet \colon s \in \mathscr{X}^* \mapsto \mathscr{P}_s$ maps each situation s to a non-empty set of probability mass functions $\mathscr{P}_s$ on $\mathscr{X}$. Each of these sets of probability mass functions $\mathscr{P}_s$ is called a local set of probability mass functions, and should be considered as containing all probability mass functions $p(\cdot|s)$ that a subject deems 'possible', where $p(x|s)$ for each $x \in \mathscr{X}$ represents the probability that $X_{|s|+1}$ takes the value x given that $X_{1:|s|} = s$ was observed. If an imprecise probability tree $\mathscr{P}_\bullet$ consists, for each situation $s \in \mathscr{X}^*$, of only a single probability mass function $p(\cdot|s)$, then we regard the imprecise tree $\mathscr{P}_\bullet$ and the precise tree $p\colon s \in \mathscr{X}^* \mapsto p(\cdot|s)$ as one and the same object, and also simply call $\mathscr{P}_\bullet$ a **precise probability tree**.

A second possible approach to model the local dynamics of a stochastic process is by means of an **acceptable gambles tree**; a map $\mathscr{A}_\bullet\colon s \in \mathscr{X}^* \mapsto \mathscr{A}_s$ that maps each situation s to a **coherent** set of acceptable gambles $\mathscr{A}_s$ on $\mathscr{X}$. Such a set of acceptable gambles $\mathscr{A}_s$ is called a local set of acceptable gambles and, as was the case for a local set of probability mass functions, expresses beliefs about the value of the next state $X_{|s|+1}$. More precisely, it contains the variables $f \in \mathscr{L}(\mathscr{X})$ for which a subject is willing to accept the gamble $f(X_{|s|+1})$ whose uncertain pay-off depends on $X_{|s|+1}$ given that she observed $X_{1:|s|} = s$.

Finally, we can also describe the local dynamics in terms of an **upper expectations tree** $\overline{\mathrm{Q}}_\bullet\colon s \in \mathscr{X}^* \mapsto \overline{\mathrm{Q}}_s$; each situation $s \in \mathscr{X}^*$ is then mapped to a **coherent unconditional** upper expectation $\overline{\mathrm{Q}}_s\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$. Such a coherent upper expectation $\overline{\mathrm{Q}}_s$ is called a local upper expectation and—though $\overline{\mathrm{Q}}_s$ is an unconditional upper expectation in the technical sense—it is interpreted as expressing a subject's beliefs about the value of the next state $X_{|s|+1}$ given that $X_{1:|s|} = s$ was observed. Specifically, in accordance with our considerations from Chapter 217, the value $\overline{\mathrm{Q}}_s(f)$ for any $s \in \mathscr{X}^*$ and any $f \in \mathscr{L}(\mathscr{X})$ can either be regarded as a subject's infimum selling price for the gamble $f(X_{|s|+1})$, or as an upper envelope over a set of linear expectations that a subject deems possible. Moreover, it may also be that our subject did actually not specify an entire coherent upper expectation for each $s \in \mathscr{X}^*$ in the first place, but that $\overline{\mathrm{Q}}_s$ is the natural extension of some general local assessments made by her—which are then assumed to be consistent.

Using the methods from Chapter 217, we can relate acceptable gambles trees, imprecise probability trees and upper expectations trees as follows. For any acceptable gambles tree $\mathscr{A}_\bullet$, the corresponding upper expectations tree $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ is defined, for all $f \in \mathscr{L}(\mathscr{X})$ and $s \in \mathscr{X}^*$, by Definition 2.529:

$$\overline{\mathrm{Q}}_{s,\mathscr{A}}(f) := \inf\left\{\alpha \in \mathbb{R} : \alpha - f \in \mathscr{A}_s\right\}. \tag{3.1}$$

It follows from Definition 2.632 that each $\overline{\mathrm{Q}}_{s,\mathscr{A}}$ is coherent, and therefore that $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ is indeed an upper expectations tree. We say that any two trees $\mathscr{A}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ **agree** if they are related by this Eq. (3.1) or, equivalently, if $\overline{\mathrm{Q}}_\bullet$ is equal to $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$. Recall from Section 2.533 that there are typically multiple—different—coherent sets of acceptable gambles leading to the same coherent upper expectation, and hence, there are also typically multiple different

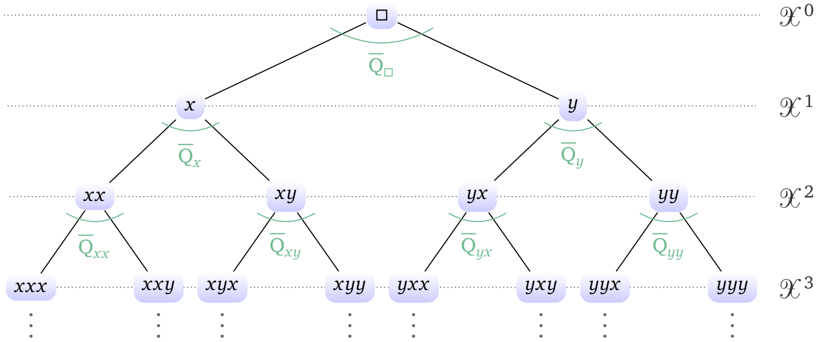

Figure 3.2 The upper expectations tree $\overline{Q}_\bullet$ of a discrete-time stochastic process with state space $\mathscr{X} = \{x, y\}$.

acceptable gambles trees that agree with the same upper expectations tree. Consequently, acceptable gambles trees are somewhat more expressive than upper expectations trees.

Given an upper expectations tree $\overline{Q}_\bullet$, one possible agreeing acceptable gambles tree that can always be chosen is the tree $\mathscr{A}_{\bullet,\overline{Q}}$ for which the set $\mathscr{A}_{s,\overline{Q}}$ for each $s \in \mathscr{X}^*$ is derived from the coherent upper expectation $\overline{Q}_s$ according to Eq. (2.4)$_{35}$:

$$\mathscr{A}_{s,\overline{Q}} := \{f \in \mathscr{L}(\mathscr{X}) \colon 0 < \underline{Q}_s(f)\} \cup \mathscr{L}_{\geq}(\mathscr{X}), \tag{3.2}$$

where $\underline{Q}_s$ for any $s \in \mathscr{X}^*$ is the conjugate lower expectation corresponding to $\overline{Q}_s$. Moreover, for any $s \in \mathscr{X}^*$, as was mentioned in Section 2.5$_{33}$, the coherent set of acceptable gambles $\mathscr{A}_{s,\overline{Q}}$ is the smallest—the most conservative—one for which the associated upper expectation is equal to $\overline{Q}_s$. For any other acceptable gambles tree $\mathscr{A}_\bullet$ that agrees with $\overline{Q}_\bullet$, we thus have that $\mathscr{A}_{s,\overline{Q}} \subseteq \mathscr{A}_s$ for all $s \in \mathscr{X}^*$.

The relations between imprecise probability trees and upper expectations trees can be deduced in a similar way. That is, for any imprecise probability tree $\mathscr{P}_\bullet$, the corresponding upper expectations tree $\overline{Q}_{\bullet,\mathscr{P}}$ is defined, for all $f \in \mathscr{L}(\mathscr{X})$ and $s \in \mathscr{X}^*$, by Definition 2.3$_{24}$:

$$\overline{Q}_{s,\mathscr{P}}(f) := \sup\left\{\sum_{x \in \mathscr{X}} f(x)p(x|s) \colon p(\cdot|s) \in \mathscr{P}_s\right\}. \tag{3.3}$$

Once more, it follows from Definition 2.6$_{32}$ that each $\overline{Q}_{s,\mathscr{P}}$ is coherent, and therefore that $\overline{Q}_{\bullet,\mathscr{P}}$ is indeed an upper expectations tree. A special case is when $\mathscr{P}_\bullet$ is a precise probability tree p; in that case the corresponding upper expectations tree $\overline{Q}_{\bullet,\mathscr{P}}$ is equal to the upper expectations tree $Q_{\bullet,p}$ defined by

$$\mathrm{Q}_{s,p}(f) := \sum_{x \in \mathscr{X}} f(x)p(x|s) \text{ for all } f \in \mathscr{L}(\mathscr{X}) \text{ and all } s \in \mathscr{X}^*. \tag{3.4}$$

Recall from Eq. $(2.1)_{21}$ and Proposition $2.1.3_{21}$ that each $\mathrm{Q}_{s,p}$ is then actually a linear expectation, which is why we will often call $\mathrm{Q}_{\bullet,p}$ a linear expectations tree or simply an expectations tree instead of an upper expectations tree.

We say that an imprecise probability tree $\mathscr{P}_\bullet$ and an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ **agree** if they are related by Eq. $(3.3)_\curvearrowleft$ or, equivalently, if $\overline{\mathrm{Q}}_\bullet$ is equal to $\overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$. It follows once more from the discussion in Section 2.5_{33} that there are (typically) multiple different imprecise probability trees agreeing with a single upper expectations tree, and thus that imprecise probability trees can be regarded as to be somewhat more expressive than upper expectations trees. Given an upper expectations tree $\overline{\mathrm{Q}}_\bullet$, one specific imprecise probability tree that agrees with it is the tree $\mathscr{P}_{\bullet,\overline{\mathrm{Q}}}$ for which the set $\mathscr{P}_{s,\overline{\mathrm{Q}}}$ for each $s \in \mathscr{X}^*$ is derived from $\overline{\mathrm{Q}}_s$ according to Eq. $(2.3)_{34}$:

$$\mathscr{P}_{s,\overline{\mathrm{Q}}} := \Big\{p(\cdot|s) \in \mathbb{P}(\mathscr{X}) \colon (\forall f \in \mathscr{L}(\mathscr{X})) \sum_{x \in \mathscr{X}} f(x)p(x|s) \leq \overline{\mathrm{Q}}_s(f)\Big\}. \tag{3.5}$$

Then for any other imprecise probability tree $\mathscr{P}_\bullet$ that agrees with $\overline{\mathrm{Q}}_\bullet$, it again follows from the discussion in Section 2.5_{33} that $\mathscr{P}_s \subseteq \mathscr{P}_{s,\overline{\mathrm{Q}}}$ for all $s \in \mathscr{X}^*$.

What's important to note at this point is that, by specifying either an acceptable gambles tree, an imprecise probability tree, or an upper expectations tree, we parametrize the dynamics of a stochastic process. In other words, as far as our theoretical study is concerned, we do not distinguish between any two stochastic processes with the same state space $\mathscr{X}$ and the same acceptable gambles tree/imprecise probability tree/upper expectations tree, even though they may describe two completely different physical systems in reality.

3.1.3 Global variables and global upper expectations

Acceptable gambles trees, imprecise probability trees and upper expectations trees all parametrize a stochastic process by telling us how the process is likely to change—or how we believe it to change—from one time instant to the next, yet they do not give information, at least not directly, about many practically relevant inferences such as, for example, the time τ_x it takes until the process is in a given state $x \in \mathscr{X}$—that is, the hitting time of x—or the k-step time average $\frac{1}{k}\sum_{\ell=1}^{k} h(X_\ell)$ of a real-valued function $h(X_\ell)$ that depends on the state X_ℓ at a single time instant ℓ. Such inferences can depend, in the most general case, on the entire infinite trajectory or path ω taken by the process. More specifically, these inferences can always be written as functions from the space of all paths $\Omega = \mathscr{X}^{\mathbb{N}}$ to the extended reals $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$. For instance, the hitting time τ_x is given by $\tau_x(\omega) := \inf\{k \in \mathbb{N} \colon \omega_k = x\}$ for all $\omega \in \Omega$ and the k-step time average of $h(X_k)$ takes the value $\frac{1}{k}\sum_{\ell=1}^{k} h(\omega_\ell)$ for all $\omega \in \Omega$.

For a general possibility space $\mathscr{Y}$, we henceforth call the extended real-valued functions in $\overline{\mathscr{L}}(\mathscr{Y}) = \overline{\mathbb{R}}^{\mathscr{Y}}$ **extended real-valued variables** on $\mathscr{Y}$, and more specifically, we call the functions in $\overline{\mathbb{V}} := \overline{\mathscr{L}}(\Omega)$ **global variables**, and the functions in $\overline{\mathscr{L}}(\mathscr{X})$ **local variables**.[1] A similar distinction is made between global gambles and local gambles, and we gather all of them in the sets $\mathbb{V} := \mathscr{L}(\Omega)$ and $\mathscr{L}(\mathscr{X})$ respectively. We moreover let $\mathbb{V}_{\rhd} := \mathscr{L}_{\rhd}(\Omega)$ and $\mathbb{V}_{\lhd} := \mathscr{L}_{\lhd}(\Omega)$ where $\rhd$ takes the form of any of the relations $>$, $\gneq$ or $\geq$, and where $\lhd$ takes the form of any of the relations $<$, $\lneq$ or $\leq$.

We often regard, for any $k \in \mathbb{N}_0$, the sequence $X_{1:k}$ as the map on Ω that returns the first k state values $\omega_{1:k}$ of a path $\omega \in \Omega$. For any $g \in \overline{\mathscr{L}}(\mathscr{X}^k)$, we then write $g(X_{1:k})$ to denote the global variable formed by the concatenation $g \circ X_{1:k}$. Global variables of this type will play an important role further on, and are called **finitary variables**; so a variable $f \in \overline{\mathbb{V}}$ is called finitary if we can write that $f = g(X_{1:k})$ for some $k \in \mathbb{N}$ and some $g \in \overline{\mathscr{L}}(\mathscr{X}^k)$. Finitary variables thus depend on the process state only at a finite number of time instances. If we want to make clear that a finitary variable $f = g(X_{1:k})$ depends specifically on the first k state values, the variable f will be called ***k*-measurable**. We will then often allow ourselves a slight abuse of notation, and write $f(x_{1:k})$ for any $x_{1:k} \in \mathscr{X}^k$ to denote the constant value $f(\omega) = g(x_{1:k})$ of f for all $\omega \in \Gamma(x_{1:k})$. Note that if a variable f is k-measurable for some $k \in \mathbb{N}_0$, then it is trivially also ℓ-measurable for all $\ell \geq k$; we will be using this property a lot when working with finitary variables. We denote the set of all finitary variables by $\overline{\mathbb{F}}$, and the set of all finitary gambles—bounded finitary variables—by $\mathbb{F}$. An example of a finitary gamble that we will often encounter is the indicator $\mathbb{1}_{x_{1:k}} := \mathbb{I}_{\Gamma(x_{1:k})}$ of the cylinder event $\Gamma(x_{1:k}) = \{\omega \in \Omega \colon \omega^k = x_{1:k}\}$ corresponding to a situation $x_{1:k} \in \mathscr{X}^*$.

In order to express beliefs about global variables, we aim to construct a binary function $\overline{\mathrm{E}} \colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ that maps each global variable $f \in \overline{\mathbb{V}}$ and each situation $s \in \mathscr{X}^*$ to an extended real number $\overline{\mathrm{E}}(f|s)$. Such an operator, and more generally, any function $\overline{\mathrm{E}} \colon \mathscr{K} \to \overline{\mathbb{R}}$ on a subset $\mathscr{K}$ of $\overline{\mathbb{V}} \times \mathscr{X}^*$, will

[1] We referred to Y as an uncertain or random variable in Chapter 2_{17}, yet the term 'variable' is henceforth used in a more formal fashion, to refer to any (not necessarily extended real-valued) map f on the corresponding possibility space $\mathscr{Y}$—which may for example be $\mathscr{X}$ or Ω in the context of stochastic processes. Often, a variable f will have the interpretation of representing the uncertain quantity $f(Y)$ that depends on the value of Y in $\mathscr{Y}$, but we typically leave this interpretation implicit. The uncertain variable Y itself is a variable in this more formal sense by letting it correspond with the identity map on $\mathscr{Y}$.

be called a **global upper expectation**.

Many different sorts of global upper expectations will be deduced throughout the course of this manuscript, and there is no single interpretation that covers them all. We can however categorise them in three different clusters, depending on their starting point and the corresponding interpretational principles from which they result: behavioural global upper expectations deduced from acceptable gambles trees, probability-based global upper expectations deduced from imprecise probability trees, and direct axiomatic global upper expectations deduced from upper expectations trees. This distinction is somewhat similar to the distinction made in Chapter 2_{17} for coherent (unconditional) upper expectations, and once more, for each of these different approaches, global upper expectations are eventually the main objects of interest when drawing inferences. They express an upper 'value' or an infimum selling 'price'—behaviourally—or a tight upper bound on (linear) expectations—probabilistically—for each global variable, which can then in turn be used to make decisions with; see [110, Section 3.9] and [50] or Example $2.2.3_{24}$. It is exactly this role of a global upper expectation as being a universal object of interest that constitutes our main reason for focussing our study on them—instead of global sets of acceptable gambles or global sets of probability charges. Nevertheless, there are some specific problems where the additional expressive power of sets of probabilities and/or sets of acceptable gambles with respect to upper expectations may actually play an important role, and where we are thus somewhat more restricted in our approach; we refer to [82] or [110, Section 3.7] for a discussion of such cases.

Instead of a global upper expectation, one could just as well work with a **global lower expectation** $\underline{\mathrm{E}}\colon \mathscr{K} \to \overline{\mathbb{R}}$ on a subset $\mathscr{K}$ of $\overline{\mathbb{V}} \times \mathscr{X}^*$. Such a global lower expectation often has an interpretation or definition that is complementary to that of a global upper expectation; behavioural global lower expectations represents **lower** 'values' or **supremum buying** 'prices' for global variables, whereas probabilistic global lower expectations represent tight **lower** bounds on possible (linear) expectations of global variables. We say 'just as well', because (as we will see) a global upper expectation $\overline{\mathrm{E}}$ and a global lower expectation $\underline{\mathrm{E}}$—of the same type and deduced from the same local models—will always be related to each other by the conjugacy relation:

$$\underline{\mathrm{E}}(f|s) = -\overline{\mathrm{E}}(-f|s) \text{ for all } (f,s) \in \mathscr{K} \text{ such that } (-f,s) \in \mathscr{K}.$$

Hence, there is no loss in generality by solely focussing on global upper expectations; properties and results for global lower expectations can then simply deduced from the relation above.

	local model	**global upper expectation**	
		finitary	continuity-based
behavioural	$\mathscr{A}_\bullet$ sets of acceptable gambles	$\overline{\mathrm{E}}_{\mathscr{A}}, \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ from sets of acceptable gambles or martingales	Chapter 4
axiomatic	$\overline{\mathrm{Q}}_\bullet$ coherent upper expectations	$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extension under coherence	Chapter 6
probabilistic	$\mathscr{P}_\bullet$ sets of probability mass functions	$\overline{\mathrm{E}}_{\mathscr{P}}$ from finitely additive probabilities	Chapter 5

Figure 3.3 Overview of the global upper expectations in this chapter.

Furthermore, for any global upper expectation $\overline{\mathrm{E}}$ and any global lower expectation $\underline{\mathrm{E}}$ on a domain $\mathscr{K} \subseteq \overline{\mathbb{V}} \times \mathscr{X}^*$, the corresponding **global upper probability** $\overline{\mathrm{P}}$ and **global lower probability** $\underline{\mathrm{P}}$ are defined by restricting $\overline{\mathrm{E}}$ and $\underline{\mathrm{E}}$ to the indicators (and all situations); so, for any $(A, s) \in \wp(\Omega) \times \mathscr{X}^*$ such that $(\mathbb{I}_A, s) \in \mathscr{K}$, we let

$$\overline{\mathrm{P}}(A|s) := \overline{\mathrm{E}}(\mathbb{I}_A|s) \quad \text{and} \quad \underline{\mathrm{P}}(A|s) := \underline{\mathrm{E}}(\mathbb{I}_A|s).$$

In the present chapter, we will limit ourselves to building global upper expectations only on global gambles $\mathbb{V}$ (and situations $\mathscr{X}^*$). Moreover, for each of the three clustered types of global upper expectations described above, we will only discuss the ones that are based on finitary arguments; there will be no limit or continuity arguments involved in the extension from the local uncertainty models to the corresponding global upper expectation. This in contrast with Chapters 4_{129}–6_{283}, where we will modify the finitary global upper expectations to incorporate some type of continuity argument; see Fig. 3.3 for a schematic overview.

Furthermore, care to note that, though local upper expectations are by definition assumed to be coherent, a similar (conditional version of the coherence) condition is not necessarily assumed for global upper expectations, even if they are restricted to the domain $\mathbb{V} \times \mathscr{X}^*$. Furthermore, though the second argument of a global upper expectation is mathematically speaking always a situation $s \in \mathscr{X}^*$, this is actually no more than a shorthand notation and should be understood as taking the (global) upper expectation conditional on the cylinder event $\Gamma(s) \subseteq \Omega$. For any global upper expectation $\overline{\mathrm{E}}$ and any $f \in \overline{\mathbb{V}}$, we will let $\overline{\mathrm{E}}(f) := \overline{\mathrm{E}}(f|\Box)$ and $\underline{\mathrm{E}}(f) := \underline{\mathrm{E}}(f|\Box)$; these should then be thought of as the unconditional global upper and lower expectation of f. The slight misuse of notation where a situation $s \in \mathscr{X}^*$ is used instead of its corresponding cylinder event $\Gamma(s)$, will sometimes also be adopted in other instances when it is clear from the context what we mean; e.g. $\sup(f|s)$ denotes the supremum $\sup(f|\Gamma(s))$. An (in)equality is also sometimes subindexed by a situation, which then means that the (in)equality only holds on the corresponding cylinder event; e.g. $f \leq_s g$ means that $f(\omega) \leq g(\omega)$ for all $\omega \in \Gamma(s)$.

3.2 Global upper expectations from acceptable gambles trees

The purpose of a global upper expectation is always to extend the local uncertainty models, which may come in the form of an acceptable gambles tree, an imprecise probability tree or an upper expectations tree. In the current section, we consider the case where the local uncertainty models are given in the form of an acceptable gambles tree. Moreover, our construction of the corresponding global upper expectation will be entirely based on the framework of sets of acceptable gambles as it was introduced in Chapter 2_{17}.

3.2.1 Extending local assessments to global assessments

Suppose that we are given an acceptable gambles tree $\mathscr{A}_\bullet$. Recall that $\mathscr{A}_{x_{1:k}}$ for any $x_{1:k} \in \mathscr{X}^*$ is then interpreted as the set of all gambles $f \in \mathscr{L}(\mathscr{X})$ for which our subject finds the uncertain reward $f(X_{k+1})$ acceptable, given that she observed $X_{1:k} = x_{1:k}$. This can be translated as saying that our subject finds the **global** gamble $f(X_{k+1})\mathbb{1}_{x_{1:k}}$ acceptable; indeed, this gamble is equal to the uncertain reward $f(X_{k+1})$ unless the path taken by the process does not go through $x_{1:k}$, in which case the gamble is called off—meaning that no money (or units of utility) is (are) exchanged. The set that gathers all these acceptable global gambles is denoted by $\mathscr{D}_{\mathscr{A}}$:

$$\mathscr{D}_{\mathscr{A}} := \{f(X_{k+1})\mathbb{1}_{x_{1:k}} : x_{1:k} \in \mathscr{X}^* \text{ and } f \in \mathscr{A}_{x_{1:k}}\}. \tag{3.6}$$

Note that the set of acceptable gambles $\mathscr{D}_{\mathscr{A}}$ is simply a translation of the local assessments $\mathscr{A}_\bullet$ to a global level without any additional information included.

Next, we can extend $\mathscr{D}_{\mathscr{A}}$ in the usual way, using the coherence axioms

D1$_{27}$, D3$_{27}$ and D4$_{27}$, with Ω taking the role of $\mathscr{Y}$:

$$\mathscr{E}(\mathscr{D}_{\mathscr{A}}) = \operatorname{posi}\left(\mathscr{D}_{\mathscr{A}} \cup \mathbb{V}_{\geq}\right) = \left\{\sum_{i=1}^{n} \lambda_i f_i \colon n \in \mathbb{N}, f_i \in \mathscr{D}_{\mathscr{A}} \cup \mathbb{V}_{\geq}, \lambda_i \in \mathbb{R}_{>}\right\}. \tag{3.7}$$

If $\mathscr{D}_{\mathscr{A}}$ is consistent—that is, if $\mathscr{E}(\mathscr{D}_{\mathscr{A}}) \cap \mathbb{V}_{\lneq} = \emptyset$—then $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ is the smallest coherent set of acceptable global gambles that includes $\mathscr{D}_{\mathscr{A}}$ and is then called the natural extension of $\mathscr{D}_{\mathscr{A}}$ [Definition 2.7$_{38}$].

The following two lemmas reveal the remarkably simple nature of the sets $\operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right)$ and $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$, and will be used shortly to prove that $\mathscr{D}_{\mathscr{A}}$ is consistent.

Lemma 3.2.1. *For any acceptable gambles tree $\mathscr{A}_\bullet$,*

$$\operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right) = \left\{\sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \textit{ finite set } S \subset \mathscr{X}^* \textit{ and } f_s \in \mathscr{A}_s\right\}.$$

Proof. It is clear by the definition of the positive span $\operatorname{posi}(\cdot)$ and the set $\mathscr{D}_{\mathscr{A}}$ [Eq. (3.6)$_\leftarrow$], that

$$\operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right) \supseteq \left\{\sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \text{ non-empty finite set } S \subset \mathscr{X}^* \text{ and } f_s \in \mathscr{A}_s\right\}.$$

Since $0 \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$ by D1$_{27}$, it is also clear that $0 \in \operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right)$, and therefore that

$$\operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right) \supseteq \left\{\sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \text{ finite set } S \subset \mathscr{X}^* \text{ and } f_s \in \mathscr{A}_s\right\}.$$

To prove the converse inclusion, fix any $f \in \operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right)$. Then there is an $n \in \mathbb{N}$, an $h_i \in \mathscr{D}_{\mathscr{A}}$ for all $i \in \{1,\dots,n\}$, and a $\lambda_i \in \mathbb{R}_{>}$ for all $i \in \{1,\dots,n\}$, such that $f = \sum_{i=1}^{n} \lambda_i h_i$. Then, for any $i \in \{1,\dots,n\}$, we have by Eq. (3.6)$_\leftarrow$ that there is an $s_i \in \mathscr{X}^*$ and a $g_i \in \mathscr{A}_{s_i}$ such that $h_i = g_i(X_{|s_i|+1})\mathbb{1}_{s_i}$. Let S be the set of all such situations s_i, and let

$$f_s \coloneqq \sum_{\substack{i\in\{1,\dots,n\}\\ s_i=s}} \lambda_i g_i \text{ for all } s \in S. \tag{3.8}$$

Then we have that

$$\sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s = \sum_{s\in S}\sum_{\substack{i\in\{1,\dots,n\}\\ s_i=s}} \lambda_i g_i(X_{|s|+1})\mathbb{1}_s = \sum_{s\in S}\sum_{\substack{i\in\{1,\dots,n\}\\ s_i=s}} \lambda_i g_i(X_{|s_i|+1})\mathbb{1}_{s_i} = \sum_{s\in S}\sum_{\substack{i\in\{1,\dots,n\}\\ s_i=s}} \lambda_i h_i$$
$$= \sum_{i\in\{1,\dots,n\}} \lambda_i h_i = f$$

Moreover observe that $S \subset \mathscr{X}^*$ is a finite set because the situations s_i are enumerated over the finite index set $\{1,\dots,n\}$. To see that $f_s \in \mathscr{A}_s$ for any $s \in S$, it suffices to observe by Eq. (3.8) that f_s is a positive linear combination of gambles $g_i \in \mathscr{A}_s$ and to take into account that $\mathscr{A}_s$ is a convex cone [because it is coherent]. □

Lemma 3.2.2. *For any acceptable gambles tree $\mathscr{A}_\bullet$,*

$$\mathscr{E}(\mathscr{D}_{\mathscr{A}}) = \left\{g + \sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \textit{ finite set } S \subset \mathscr{X}^*,\ g \in \mathbb{V}_{\geq} \textit{ and } f_s \in \mathscr{A}_s\right\}.$$

Proof. It is clear by the definition of the positive span posi$(\cdot)$ and the set $\mathscr{D}_{\mathscr{A}}$ [Eq. (3.6)$_{56}$], that

$$\begin{aligned}\mathscr{E}(\mathscr{D}_{\mathscr{A}}) &= \operatorname{posi}\left(\mathscr{D}_{\mathscr{A}} \cup \mathbb{V}_{\geq}\right)\\ &\supseteq \left\{g + \sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \text{ finite set } S \subset \mathscr{X}^*,\ g \in \mathbb{V}_{\geq} \text{ and } f_s \in \mathscr{A}_s\right\}.\end{aligned}$$

To prove the converse inclusion, fix any $f \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})$. Then by Eq. (3.7)$_{\frown}$ there is an $n \in \mathbb{N}$, an $f_i \in \mathscr{D}_{\mathscr{A}} \cup \mathbb{V}_{\geq}$ for all $i \in \{1, \dots, n\}$ and a $\lambda_i \in \mathbb{R}_{>}$ for all $i \in \{1, \dots, n\}$, such that $f = \sum_{i=1}^{n} \lambda_i f_i$. Since $\mathbb{V}_{\geq}$ is a convex cone that includes the zero gamble, there is some $g \in \mathbb{V}_{\geq}$ such that

$$f = \sum_{i \in \{1,\dots,n\}} \lambda_i f_i = \sum_{\substack{i \in \{1,\dots,n\}\\ f_i \in \mathbb{V}_{\geq}}} \lambda_i f_i + \sum_{\substack{i \in \{1,\dots,n\}\\ f_i \notin \mathbb{V}_{\geq}}} \lambda_i f_i = g + \sum_{\substack{i \in \{1,\dots,n\}\\ f_i \notin \mathbb{V}_{\geq}}} \lambda_i f_i = g + \sum_{\substack{i \in \{1,\dots,n\}\\ f_i \in \mathscr{D}_{\mathscr{A}} \setminus \mathbb{V}_{\geq}}} \lambda_i f_i, \quad (3.9)$$

where the last step follows from the fact that $f_i \in \mathscr{D}_{\mathscr{A}} \cup \mathbb{V}_{\geq}$ for all $i \in \{1, \dots, n\}$. The latter sum $\sum_{i \in \{1,\dots,n\}, f_i \in \mathscr{D}_{\mathscr{A}} \setminus \mathbb{V}_{\geq}} \lambda_i f_i$ is clearly an element of posi$\left(\mathscr{D}_{\mathscr{A}}\right)$ if the sum is taken over a non-empty set. If the sum is taken over an empty set, then it is equal to 0 and therefore, since $0 \in \operatorname{posi}\left(\mathscr{D}_{\mathscr{A}}\right)$ by Lemma 3.2.1$_{\frown}$, also an element of posi$\left(\mathscr{D}_{\mathscr{A}}\right)$. So, in either case, the sum is an element of posi$\left(\mathscr{D}_{\mathscr{A}}\right)$ and can therefore, by Lemma 3.2.1$_{\frown}$, be written as a sum $\sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s$ for some finite set $S \subset \mathscr{X}^*$ and gambles $f_s \in \mathscr{A}_s$ for all $s \in S$. Hence, we have that

$$f \in \left\{g + \sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s \colon \text{ finite set } S \subset \mathscr{X}^*,\ g \in \mathbb{V}_{\geq} \text{ and } f_s \in \mathscr{A}_s\right\},$$

which implies the desired inclusion. □

The following proposition establishes that, for any acceptable gambles tree $\mathscr{A}_{\bullet}$, the set $\mathscr{D}_{\mathscr{A}}$ of acceptable gambles is consistent and thus that it can be extended to a coherent set of acceptable gambles.

Proposition 3.2.3. *For any acceptable gambles tree $\mathscr{A}_{\bullet}$, the set $\mathscr{D}_{\mathscr{A}}$ defined by Eq.* (3.6)$_{56}$ *is consistent, and so $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ is the natural extension of $\mathscr{D}_{\mathscr{A}}$.*

Proof. According to Definition 2.7$_{38}$, we need to show that $\mathscr{E}(\mathscr{D}_{\mathscr{A}}) \cap \mathbb{V}_{\lneq} = \emptyset$, or equivalently that, for any $f \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})$, we either have that $f = 0$ or that $f(\omega) > 0$ for some $\omega \in \Omega$. From Lemma 3.2.2$_{\frown}$, we know that $f = g + \sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s$ for some $g \in \mathbb{V}_{\geq}$, some finite set $S \subset \mathscr{X}^*$ of situations and local gambles $f_s \in \mathscr{A}_s$ for all $s \in S$. If $\sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s = 0$, then we have that $f \geq 0$ and thus clearly that the desired condition is satisfied. Otherwise, if $\sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s \neq 0$, let $S' \subseteq S$ be the set of situations s in S such that $f_s \neq 0$ [note that S' must then be non-empty]. Fix any $x_{1:k} \in S'$ such that there is no $t \in S'$ such that $t \sqsubset x_{1:k}$ [the existence of such a situation $x_{1:k}$ is guaranteed by the fact that any given situation is clearly preceded by only finitely many other situations]. Then since $f_{x_{1:k}} \in \mathscr{A}_{x_{1:k}}$ and $f_{x_{1:k}} \neq 0$, the coherence [D2$_{27}$] of $\mathscr{A}_{x_{1:k}}$ implies that there must be some $x_{k+1} \in \mathscr{X}$ such that $f_{x_{1:k}}(x_{k+1}) > 0$.

Then, for any path $\omega \in \Gamma(x_{1:k+1})$, we have that

$$\begin{aligned}
f(\omega) &= g(\omega) + \sum_{s \in S} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) = g(\omega) + \sum_{s \in S'} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&\geq \sum_{s \in S'} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) = \sum_{\substack{s \in S' \\ s \sqsubset x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&= f_{x_{1:k}}(x_{k+1}) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) > \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega),
\end{aligned}$$

where the second step follows trivially from the definition of S', the third from the fact that $g \in \mathbb{V}_{\geq}$, the fourth from the fact that $\mathbb{1}_s(\omega) = 0$ for any $s \parallel x_{1:k+1}$ [because then $\Gamma(s) \cap \Gamma(x_{1:k+1}) = \emptyset$], the fifth from the fact that, due to our choice of $x_{1:k}$, there are no situations $t \in S'$ such that $t \sqsubset x_{1:k}$ [and from the fact that $\omega^k = x_{1:k}$], and the last from the fact that $f_{x_{1:k}}(x_{k+1}) > 0$.

Next, let x_{k+2} be any state in $\mathscr{X}$ if $x_{1:k+1} \notin S'$, or otherwise, if $x_{1:k+1} \in S'$, let x_{k+2} be such that $f_{x_{1:k+1}}(x_{k+2}) > 0$; the latter is possible because of the same reasons as before. Then for any $\omega \in \Gamma(x_{1:k+2}) \subset \Gamma(x_{1:k+1})$, we have by the inequality above that

$$\begin{aligned}
f(\omega) > \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) &= \sum_{\substack{s \in S' \\ s = x_{1:k+1}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+2}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&= \sum_{\substack{s \in S' \\ s = x_{1:k+1}}} f_s(x_{k+2}) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+2}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&\geq \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+2}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega),
\end{aligned}$$

where the second equality follows from the fact that $\omega_{|x_{1:k+1}|+1} = \omega_{k+2} = x_{k+2}$ because $\omega \in \Gamma(x_{1:k+2})$, and where the last inequality follows from how we chose x_{k+2}. We can now simply repeat the same reasoning; let x_{k+3} be any state in $\mathscr{X}$ if $x_{1:k+2} \notin S'$, or otherwise, if $x_{1:k+2} \in S'$, let x_{k+3} be such that $f_{x_{1:k+2}}(x_{k+3}) > 0$ [which is again possible because of the same reasons as before]. Then for any $\omega \in \Gamma(x_{1:k+3}) \subset \Gamma(x_{1:k+2})$, we have by the previous inequality that

$$\begin{aligned}
f(\omega) > \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+2}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) &= \sum_{\substack{s \in S' \\ s = x_{1:k+2}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+3}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&= \sum_{\substack{s \in S' \\ s = x_{1:k+2}}} f_s(x_{k+3}) + \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+3}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) \\
&\geq \sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+3}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega).
\end{aligned}$$

We can continue to do this, choosing new consecutive states $x_{k+4}, x_{k+5}, \ldots$ in the same way, until at some point we encounter an x_{k+n} for which there are no situations in $s \in S'$ anymore such that $s \sqsupseteq x_{1:k+n}$, which is guaranteed to happen because S and S' are finite. Then we have that

$$\sum_{\substack{s \in S' \\ s \sqsupseteq x_{1:k+n}}} f_s(\omega_{|s|+1}) \mathbb{1}_s(\omega) = 0,$$

and hence, by the argument above, that $f(\omega) > 0$. So we conclude that $\mathscr{E}(\mathscr{D}_{\mathscr{A}}) \cap \mathbb{V}_{\lneq} = \emptyset$ and therefore that $\mathscr{D}_{\mathscr{A}}$ is consistent. □

3.2.2 The global upper and lower expectation associated with a set of acceptable global gambles

Given an acceptable gambles tree $\mathscr{A}_\bullet$, we have associated with it a global set of acceptable gambles $\mathscr{D}_{\mathscr{A}}$ and shown that the natural extension of this global set $\mathscr{D}_{\mathscr{A}}$ exists—it is equal to $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$. The global upper expectation $\overline{\mathrm{E}}_{\mathscr{A}} : \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ and global lower expectation $\underline{\mathrm{E}}_{\mathscr{A}} : \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ corresponding to $\mathscr{A}_\bullet$ are now defined as follows; for all $(f, s) \in \mathbb{V} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}_{\mathscr{A}}(f|s) := \inf\{\alpha \in \mathbb{R} \colon (\alpha - f)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})\}; \tag{3.10}$$
$$\underline{\mathrm{E}}_{\mathscr{A}}(f|s) := \sup\{\alpha \in \mathbb{R} \colon (f - \alpha)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})\}.$$

The upper and lower expectation $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\underline{\mathrm{E}}_{\mathscr{A}}$ are related by conjugacy, and so it suffices to focus on the upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$.

Corollary 3.2.4 (Conjugacy). *For any acceptable gambles tree $\mathscr{A}_\bullet$ and $(f, s) \in \mathbb{V} \times \mathscr{X}^*$, we have that $\underline{\mathrm{E}}_{\mathscr{A}}(f|s) = -\overline{\mathrm{E}}_{\mathscr{A}}(-f|s)$.*

Proof. Observe that

$$\begin{aligned}-\overline{\mathrm{E}}_{\mathscr{A}}(-f|s) &= -\inf\{\alpha \in \mathbb{R} \colon (\alpha + f)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})\} \\ &= \sup\{-\alpha \in \mathbb{R} \colon (\alpha + f)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})\} \\ &= \sup\{\alpha \in \mathbb{R} \colon (-\alpha + f)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_{\mathscr{A}})\} = \underline{\mathrm{E}}_{\mathscr{A}}(f|s).\end{aligned}$$ □

The definition of $\overline{\mathrm{E}}_{\mathscr{A}}$ is, in spirit, similar to the definition of an unconditional upper expectation that corresponds to a coherent set of acceptable gambles [see Definition 2.5$_{29}$], in the sense that $\overline{\mathrm{E}}_{\mathscr{A}}$ again represents infimum acceptable selling prices corresponding to the set $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$. However, the formula is now somewhat altered to suitably deal with conditioning on situations. More concretely, it is based on the interpretation that if, conditional on any situation $s \in \mathscr{X}^*$, a subject finds it acceptable to sell a gamble $f \in \mathbb{V}$ for the price $\alpha \in \mathbb{R}$, this is taken to mean that she finds the gamble $(\alpha - f)\mathbb{1}_s$ acceptable, because this gamble is equal to $\alpha - f$ if the process goes through s, and is called off otherwise (in which case no money—or units of utility—is exchanged). Note that this interpretation is in line with Walley's contingent interpretation of conditional upper expectations (or previsions) [110, Section 6.1.1]. The definition above moreover agrees with traditional definitions, which are usually stated for general conditioning events instead of only situations; see for instance [114, Eq. (2)].

3.2.3 The global upper expectation corresponding to an acceptable gambles tree: It's only a game!

The expression for the upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$ given in Eq. (3.10)$_{\leftarrow}$ is not very elegant, and quite impractical if one desires to compute its values; it indirectly characterises $\overline{\mathrm{E}}_{\mathscr{A}}$ based on the set $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$, which itself ought to be deduced from Eq. (3.7)$_{57}$ or Lemma 3.2.2$_{57}$. A more direct characterisation can however be given if we make use of the notions of super- and submartingales. As we will see, this alternative characterisation arises naturally from game-theoretic principles, and will therefore allow us to interpret $\overline{\mathrm{E}}_{\mathscr{A}}$ in a game-theoretic way. We start by introducing the concepts of a real process, a betting process and a sub-/supermartingale.

Real processes, betting processes, sub- and supermartingales

A **real process** $\mathscr{C}$ is simply a real-valued map on the situations $\mathscr{X}^*$. We will often use it to describe the evolution of a subject's capital as he gambles on the subsequent values of the process state X_k; in that case, we will sometimes call it a capital process. A **betting process** $\mathscr{G}$ is a map that associates with each situation $s \in \mathscr{X}^*$ a local gamble $\mathscr{G}(s) \in \mathscr{L}(\mathscr{X})$. The value of the local gamble $\mathscr{G}(s)$ in $x \in \mathscr{X}$ is then denoted by $\mathscr{G}(s)(x)$. With any betting process $\mathscr{G}$, we associate a real process $\mathscr{C}^{\mathscr{G}}$ defined by

$$\mathscr{C}^{\mathscr{G}}(x_{1:k}) := \sum_{\ell=0}^{k-1} \mathscr{G}(x_{1:\ell})(x_{\ell+1}) \text{ for all } x_{1:k} \in \mathscr{X}^*.$$

So $\mathscr{C}^{\mathscr{G}}$ denotes the evolution of a subject's capital if he starts with zero capital and gambles according to $\mathscr{G}$ on the subsequent state values. Conversely, with any real process $\mathscr{C}$, we can also associate a betting process $\Delta\mathscr{C}$ given by

$$\Delta\mathscr{C}(x_{1:k}) := \mathscr{C}(x_{1:k}\cdot) - \mathscr{C}(x_{1:k}) \text{ for all } x_{1:k} \in \mathscr{X}^*,$$

where $\mathscr{C}(x_{1:k}\cdot)$ is the local gamble that assumes the value $\mathscr{C}(x_{1:k}y)$ in $y \in \mathscr{X}$. We call $\Delta\mathscr{C}$ the **process difference** of $\mathscr{C}$. Note that, for any real process $\mathscr{C}$, the process difference $\Delta\mathscr{C}$ is the unique betting process $\mathscr{G}$ such that $\mathscr{C} = \mathscr{C}(\square) + \mathscr{C}^{\mathscr{G}}$.

Now, given an acceptable gambles tree $\mathscr{A}_{\bullet}$, we say that a betting process $\mathscr{G}$ is **acceptable** if $\mathscr{G}(s) \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$. A real process $\mathscr{M}$ is then called a **submartingale** according to $\mathscr{A}_{\bullet}$, if the corresponding process difference $\Delta\mathscr{M}$ is acceptable—or, equivalently, if there is an acceptable betting process $\mathscr{G}$ such that $\mathscr{M} = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}$. A real process $\mathscr{M}$ is called a **supermartingale** according to $\mathscr{A}_{\bullet}$ if $-\Delta\mathscr{M}(s) \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$ (or, equivalently, if there is an betting process $\mathscr{G}$ such that $-\mathscr{G}$ is acceptable and $\mathscr{M} = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}})$.[2] Hence, $\mathscr{M}$ is a supermartingale if and only if $-\mathscr{M}$ is a submartingale.

A submartingale $\mathscr{M}$ thus describes the possible evolutions of a subject's capital if he adopts a strategy that, for each possible situation $s \in \mathscr{X}^*$, picks a local gamble $\Delta\mathscr{M}(s) \in \mathscr{A}_s$ among the ones that are regarded acceptable by the local model $\mathscr{A}_s$. Or, if the sets $\mathscr{A}_\bullet$ are specified by the subject himself, a submartingale describes the possible evolutions of his capital if he chooses to gamble simply according to his own beliefs. The notion of a supermartingale, which will be more central in our treatment than that of a submartingale, can be interpreted in an equally intuitive way yet requires us to consider a second subject that gambles against the beliefs $\mathscr{A}_s$ of our first subject. To make this more concrete, we imagine the following game between two subjects—or, better, two players—in which we use terminology that is similar to that of Shafer and Vovk [85].

The first player specifies, for each $s \in \mathscr{X}^*$, the set of gambles $\mathscr{A}_s$ that he finds acceptable (which should be coherent); this player is called **Forecaster**. The second player, called **Skeptic**, will take Forecaster up on his commitments. More concretely, Skeptic plays according to the following protocol, where $\mathscr{M}(\square) \in \mathbb{R}$ denotes his (arbitrary) starting capital, where $k \in \mathbb{N}_0$ is any point in time and $X_{1:k} = x_{1:k}$ is any possible history up until time k:

i. Skeptic chooses a local gamble $g \in -\mathscr{A}_{x_{1:k}} := \{-f : f \in \mathscr{A}_{x_{1:k}}\}$.
ii. The value x_{k+1} of the next state X_{k+1} is revealed.
iii. Skeptic's capital becomes $\mathscr{M}(x_{1:k+1}) = \mathscr{M}(x_{1:k}) + g(x_{k+1})$.

Observe that, at any given situation $x_{1:k} \in \mathscr{X}^*$, Skeptic is required to choose from the gambles in $-\mathscr{A}_{x_{1:k}}$ and not the gambles in $\mathscr{A}_{x_{1:k}}$. The reason is that Forecaster accepts any gamble $f(X_{k+1})$ for which $f \in \mathscr{A}_{x_{1:k}}$, or equivalently, is willing to offer the gamble $-f(X_{k+1})$ to someone else. So, since Skeptic plays against Forecaster, he can only choose gambles $g(X_{k+1}) = -f(X_{k+1})$ for which $g \in -\mathscr{A}_{x_{1:k}}$.

Now note that supermartingales can be interpreted as the possible evolutions of Skeptic's capital if he uses a strategy to play against Forecaster in the protocol above. Indeed, any map $\Delta\mathscr{M} : s \in \mathscr{X}^* \mapsto \Delta\mathscr{M}(s) \in -\mathscr{A}_s$ is a possible gambling strategy for Skeptic; by the definition of a supermartingale this implies that the real process $\mathscr{M}$ given by $\mathscr{M}(\square) + \sum_{\ell=0}^{k-1} \Delta\mathscr{M}(x_{1:\ell})(x_{\ell+1})$ for all $x_{1:k} \in \mathscr{X}^*$, which describes the corresponding evolution of Skeptic's capital if he starts with an initial capital of $\mathscr{M}(\square)$, is a supermartingale.

[2] This alternative characterisation in terms of acceptable betting processes will be used in Section 4.2.3$_{145}$.

The set of all supermartingales for a given acceptable gambles tree $\mathscr{A}_\bullet$ will be denoted by $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$; the corresponding set of submartingales by $\underline{\mathbb{M}}(\mathscr{A}_\bullet)$. Note that the initial value $\mathscr{C}(\square)$ of a process $\mathscr{C}$—which can be seen as our subject's initial capital—is irrelevant for whether $\mathscr{C}$ is a supermartingale or not; if $\mathscr{C} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ then also $\alpha + \mathscr{C} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ for all $\alpha \in \mathbb{R}$.

The following lemma describes the monotone behaviour of a sub- or supermartingale along a path; it will be used later on in various proofs.

Lemma 3.2.5. *Consider any acceptable gambles tree $\mathscr{A}_\bullet$, any $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, $x_{1:k} \in \mathscr{X}^*$ and $\ell \geq k$. Then we have that*

$$\mathscr{M}(x_{1:k}) \leq \mathscr{M}(x_{1:k+1}) \leq \cdots \leq \mathscr{M}(x_{1:\ell}) \text{ for some } x_{k+1:\ell} \in \mathscr{X}^{\ell-k}.$$

Furthermore, the converse inequalities hold for any supermartingale $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$.

Proof. To prove the statement for a submartingale $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, first observe that, for any $s \in \mathscr{X}^*$, there is an $x \in \mathscr{X}$ such that $\mathscr{M}(s) \leq \mathscr{M}(sx)$. Indeed, assume **ex absurdo** that this is not the case, and so $\mathscr{M}(s) > \mathscr{M}(sx)$ for all $x \in \mathscr{X}$. This implies that $\Delta\mathscr{M}(s) < 0$. But this is in contradiction with $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, because the latter implies that $\Delta\mathscr{M}(s) \in \mathscr{A}_s$ and thus, by the coherence [D2$_{27}$] of $\mathscr{A}_s$, that $\Delta\mathscr{M}(s) \notin \mathscr{L}_<(\mathscr{X})$. As a result, we must have that $\mathscr{M}(s) \leq \mathscr{M}(sx)$ for some $x \in \mathscr{X}$. Since this holds for any general $s \in \mathscr{X}^*$, we can iteratively apply this implication starting from $x_{1:k}$ to infer that there is indeed some $x_{k+1:\ell} \in \mathscr{X}^{\ell-k}$ such that $\mathscr{M}(x_{1:k}) \leq \mathscr{M}(x_{1:k+1}) \leq \cdots \leq \mathscr{M}(x_{1:\ell})$. The second statement about supermartingales in $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$ then follows trivially from the statement about submartingales and the definition of a supermartingale. □

Global game-theoretic upper expectations

For any real process $\mathscr{C}$ and any $k \in \mathbb{N}_0$, let $\mathscr{C}(X_{1:k})$ denote the k-measurable gamble that assumes the value $\mathscr{C}(\omega^k)$ in $\omega \in \Omega$; so $\mathscr{C}(X_{1:k})$ is the gamble obtained by stopping $\mathscr{C}$ at time k. Then, given a set $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$ of supermartingales corresponding to an acceptable gambles tree $\mathscr{A}_\bullet$, we define the corresponding **finitary game-theoretic upper expectation** $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}} \colon \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ as follows:[3] for any $f \in \mathbb{V}$ and $s \in \mathscr{X}^*$,

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) := \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, \mathscr{M}(X_{1:k}) \geq_s f\right\}. \tag{3.11}$$

Taking into account the interpretation of a supermartingale as described above, the upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ can be given a straightforward meaning: for any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, $\overline{\mathrm{E}}_{\mathscr{A}}(f|s)$ is the infimum starting capital for Skeptic

[3]The superscript 'f' in $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ is intended to refer to 'finitary', whereas the subscript 'V' is intended to refer to Jean Ville [107] as pointed out in Section 4.1$_{131}$.

in the situation s such that, by playing against Forecaster according to the protocol above, he can declare a finite point in time $k \geq |s|$ where he will **surely** have more money than the reward associated with the global gamble f. Remark that the 'surely' here refers to the fact that this is the case **irrespectively of the path** ω **taken through the situation** s. In other words, any starting capital in s larger than $\overline{\mathrm{E}}_{\mathscr{A}}(f|s)$ allows Skeptic to (super-)hedge the global gamble f at a finite time point in the future.

The **finitary game-theoretic lower expectation** $\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ can be defined in a complementary way as $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ using submartingales instead of supermartingales; for any $f \in \mathbb{V}$ and $s \in \mathscr{X}^*$,

$$\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) := \sup\left\{\mathscr{M}(s) \colon \mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, \mathscr{M}(X_{1:k}) \leq_s f\right\}.$$

We do not go into full detail on how the definition above can be interpreted in a game-theoretic setting but, roughly speaking, submartingales can be seen as the possible evolutions of Forecaster's capital when Skeptic is betting against him, and the lower expectation $\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$ is then Forecaster's supremum starting capital in s such that it is still possible for Skeptic to—surely—prevent Forecaster from ending up with more money than the payoff corresponding to the gamble f. The lower expectation $\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ is once more related to the upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ by conjugacy, which is why we will henceforth focus on $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$.

Corollary 3.2.6 (Conjugacy). *For any acceptable gambles tree $\mathscr{A}_\bullet$ and $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, we have that $\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = -\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(-f|s)$.*

Proof. Since, for any real process $\mathscr{M}$, $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A})$ if and only if $-\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A})$, we have that

$$\begin{aligned}
&-\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(-f|s)\\
&\quad= -\inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, \mathscr{M}(X_{1:k}) \geq_s -f\right\}\\
&\quad= \sup\left\{-\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, \mathscr{M}(X_{1:k}) \geq_s -f\right\}\\
&\quad= \sup\left\{\mathscr{M}(s) \colon (-\mathscr{M}) \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, (-\mathscr{M})(X_{1:k}) \geq_s -f\right\}\\
&\quad= \sup\left\{\mathscr{M}(s) \colon \mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\, \mathscr{M}(X_{1:k}) \leq_s f\right\} = \underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s). \qquad \square
\end{aligned}$$

At this point, readers that are familiar with the global game-theoretic upper expectations suggested by Glenn Shafer and Vladimir Vovk [85, 86, 109] will surely have noticed the close relation with our expression for $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ in Eq. (3.11)↶. In fact, the conceptual ideas that we have presented above are entirely the same and largely due to them.[4] There are various important technical differences, though, which we will currently not discuss in

[4] The close resemblance with Shafer and Vovk's game-theoretic framework in [86] clarifies the title of this section.

detail. The most important difference, at this point, is the fact that Shafer and Vovk's game-theoretic upper expectations involve supermartingales that (super-)hedge the considered gamble only at an infinite time horizon and not necessarily at a finite time horizon as is the case for $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$. These types of game-theoretic upper expectations, where (super-)hedging happens at an infinite time horizon, will be the subject of Chapter 4_{129}. We will there also discuss in great detail what the correspondences and differences are with Shafer and Vovk's framework.

An equivalence between $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$

The following theorem establishes that, for any acceptable gambles tree $\mathscr{A}_\bullet$, the upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$ obtained from $\mathscr{A}_\bullet$ using the standard coherence arguments is entirely the same as the finitary game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$; see also Fig. $3.4_{\curvearrowright}$. Due to its considerable length, we relegate the proof to Appendix $3.\mathrm{A}_{102}$.

Theorem 3.2.7. *For any acceptable gambles tree* $\mathscr{A}_\bullet$*,*

$$\overline{\mathrm{E}}_{\mathscr{A}}(f|s) = \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

The game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ can thus be seen as an alternative characterisation for $\overline{\mathrm{E}}_{\mathscr{A}}$, yet more intuitive and with a more constructive flavour. Furthermore, it is exactly this game-theoretic upper expectation that will be later on, in Chapter 4_{129}, suitably adapted to involve a continuity argument. This will then give us a definition of the game-theoretic upper expectation that is in line with Shafer and Vovk their game-theoretic upper expectations. Moreover, note that the equality above also holds for the lower expectations $\underline{\mathrm{E}}_{\mathscr{A}}$ and $\underline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ because they are both related to their respective upper expectations by conjugacy [see Corollary $3.2.4_{60}$ and Corollary $3.2.6_{\leftarrow}$].

3.2.4 The law of iterated upper expectations

We conclude this section on behavioural upper expectations with proving a **law of iterated upper expectations** [8, 62, 86]; a generalised version of the law of iterated expectations or law of total probability in measure-theoretic probability [89, Section 1.3.2]. The law will be crucial later on, in Section 3.5_{90}, when we relate $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ to other types of global upper expectations.

We start with two lemmas, of which the first establishes a slightly modified expression for $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and the second implies that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ is real-valued on $\mathbb{V} \times \mathscr{X}^*$.

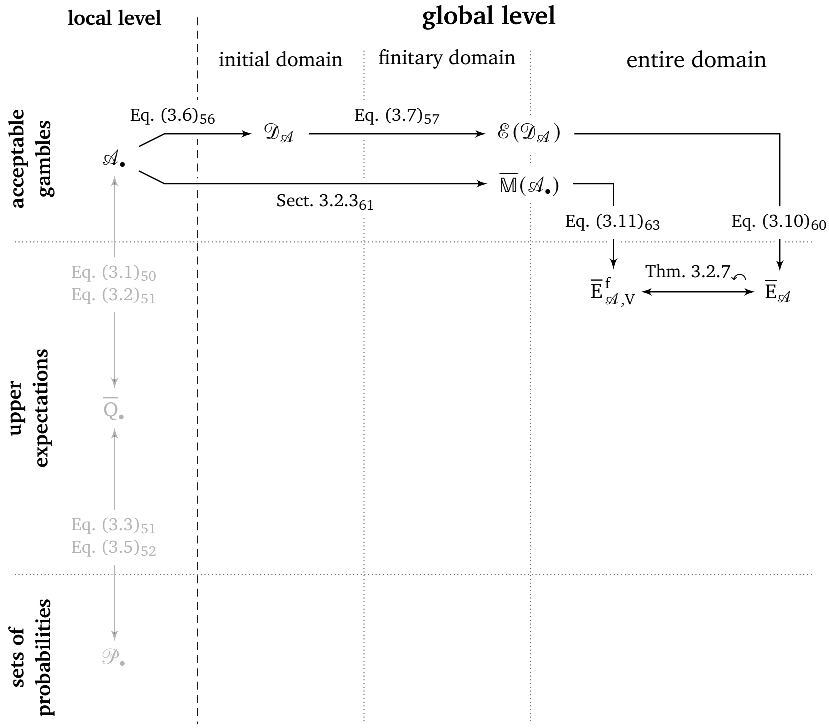

Figure 3.4 Schematic overview of the finitary behavioural approaches.

Lemma 3.2.8. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that, for all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$,*

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = \inf\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)(\forall \ell \geq k)\mathscr{M}(X_{1:\ell}) \geq_s f\}.$$

Proof. Fix any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$. That

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \inf\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)(\forall \ell \geq k)\mathscr{M}(X_{1:\ell}) \geq_s f\}$$

is clear from the definition of $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ [Eq. (3.11)$_{63}$]. To prove the converse inequality, fix any $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:k}) \geq_s f$ for some $k \geq |s|$. Let $\mathscr{M}'$ be the real process defined, for all $x_{1:\ell} \in \mathscr{X}^*$, by $\mathscr{M}'(x_{1:\ell}) := \mathscr{M}(x_{1:\ell})$ if $\ell \leq k$, and by $\mathscr{M}'(x_{1:\ell}) := \mathscr{M}(x_{1:k})$ if $\ell > k$. Then, for any $x_{1:\ell} \in \mathscr{X}^*$, $\Delta\mathscr{M}'(x_{1:\ell})$ is either equal to $\Delta\mathscr{M}(x_{1:\ell})$ or equal to 0. Since $-\Delta\mathscr{M}(x_{1:\ell}) \in \mathscr{A}_{x_{1:\ell}}$ because $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$, and since $0 \in \mathscr{A}_{x_{1:\ell}}$ due to D1$_{27}$, we have that $-\Delta\mathscr{M}'(x_{1:\ell}) \in \mathscr{A}_{x_{1:\ell}}$. Since this holds for any $x_{1:\ell} \in \mathscr{X}^*$, we have that $\mathscr{M}' \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$. Moreover, it is clear from the definition of $\mathscr{M}'$ that $\mathscr{M}'(X_{1:\ell}) = \mathscr{M}(X_{1:k})$ for all $\ell \geq k$, which by the fact that $\mathscr{M}(X_{1:k}) \geq_s f$ implies that $\mathscr{M}'(X_{1:\ell}) \geq_s f$ for all $\ell \geq k$. Hence, we have that

$$\inf\{\tilde{\mathscr{M}}(s) \colon \tilde{\mathscr{M}} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists \tilde{k} \geq |s|)(\forall \tilde{\ell} \geq k)\tilde{\mathscr{M}}(X_{1:\tilde{\ell}}) \geq_s f\} \leq \mathscr{M}'(s) = \mathscr{M}(s),$$

where the last equality follows from the definition of $\mathscr{M}'$ and the fact that $k \geq |s|$. Since the inequality above holds for any $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:k}) \geq_s f$ for some $k \geq |s|$, we have by Eq. (3.11)$_{63}$ that

$$\inf\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)(\forall \ell \geq k)\mathscr{M}(X_{1:\ell}) \geq_s f\} \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s),$$

as desired. □

Lemma 3.2.9. *For any acceptable gambles tree $\mathscr{A}_\bullet$ and any $(f, s) \in \mathbb{V} \times \mathscr{X}^*$, we have that*

$$\inf(f|s) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \sup(f|s).$$

In particular, we have that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \in \mathbb{R}$.

Proof. To see that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \sup(f|s)$, simply observe that the real process $\mathscr{M}$ that is equal to the constant $\sup(f|s)$ everywhere is, by D1$_{27}$ of the local sets $\mathscr{A}_s$, a supermartingale in $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$, and is clearly such that $\mathscr{M}(X_{1:k}) \geq_s f$ for some $k \geq |s|$ [in fact it holds for all $k \in \mathbb{N}_0$]. On the other hand, assume **ex absurdo** that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) < \inf(f|s)$. Then according to Eq. (3.11)$_{63}$ there is a $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(s) < \inf(f|s)$ and $\mathscr{M}(X_{1:k}) \geq_s f$ for some $k \geq |s|$. But Lemma 3.2.5$_{63}$ says that there is a $x_{|s|+1:k} \in \mathscr{X}^{k-|s|}$ such that

$$\mathscr{M}(sx_{|s|+1:k}) \leq \mathscr{M}(s) < \inf(f|s).$$

This is in contradiction with the fact that $\mathscr{M}(X_{1:k}) \geq_s f \geq_s \inf(f|s)$. As a result, we must have that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \geq \inf(f|s)$. □

For any $f \in \mathbb{V}$, any $x_{1:k} \in \mathscr{X}^*$ and any $\ell \in \mathbb{N}$, we use the notation $\overline{\mathrm{E}}_{\mathscr{A}}(f|x_{1:k}X_{k+1:k+\ell})$ to denote the $k+\ell$-measurable variable taking the value $\overline{\mathrm{E}}_{\mathscr{A}}(f|x_{1:k}x'_{k+1:k+\ell})$ for all $x'_{k+1:k+\ell} \in \mathscr{X}^\ell$, and the notation $\overline{\mathrm{E}}_{\mathscr{A}}(\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k+1})|X_{1:k})$ to denote the k-measurable variable taking the value $\overline{\mathrm{E}}_{\mathscr{A}}(\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k+1})|x_{1:k})$ for all $x_{1:k} \in \mathscr{X}^k$. Note that $\overline{\mathrm{E}}_{\mathscr{A}}(\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k+1})|x_{1:k})$ is well-defined, because $\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k+1})$ is a gamble due to Lemma 3.2.9 and Theorem 3.2.7$_{65}$. Analogous notations will be used for the global upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, and for all other global upper expectations in this dissertation.

Proposition 3.2.10 (Law of iterated upper expectations)**.** *For any acceptable gambles tree $\mathscr{A}_\bullet$, any $f \in \mathbb{V}$ and any $k \in \mathbb{N}_0$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k}) = \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|X_{1:k}),$$

and

$$\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k}) = \overline{\mathrm{E}}_{\mathscr{A}}(\overline{\mathrm{E}}_{\mathscr{A}}(f|X_{1:k+1})|X_{1:k}).$$

Proof. Fix any $f \in \mathbb{V}$ and any $x_{1:k} \in \mathscr{X}^*$. We first establish the statement for the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, and to that end, we begin by proving the inequality $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k})$. If $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k}) = +\infty$, then this inequality is trivially satisfied. If $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k}) < +\infty$ then, for any real number $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k})$, there is according to Lemma 3.2.8$_{66}$ a supermartingale $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(x_{1:k}) \leq \alpha$ and $\mathscr{M}(X_{1:\ell}) \geq_{x_{1:k}} f$ for all ℓ larger or equal than some $k' \geq k$. Then it is clear that, for any $x_{k+1} \in \mathscr{X}$, $\mathscr{M}(X_{1:\ell}) \geq_{x_{1:k+1}} f$ for all ℓ larger or equal than $k'+1 \geq k+1$, and hence, again by Lemma 3.2.8$_{66}$, we have that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1})$. Since this holds for any $x_{k+1} \in \mathscr{X}$, we obtain that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1}) \leq_{x_{1:k}} \mathscr{M}(X_{1:k+1})$. Let $\mathscr{M}'$ be the real process that is equal to $\mathscr{M}$ for all situations that precede $x_{1:k}$ or are incomparable with $x_{1:k}$, and that is equal to the constant $\mathscr{M}(x_{1:k+1})$ for all situations that follow $x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Then, because $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1}) \leq_{x_{1:k}} \mathscr{M}(X_{1:k+1})$, we also have that

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1}) \leq_{x_{1:k}} \mathscr{M}'(X_{1:k+1}) =_{x_{1:k}} \mathscr{M}'(X_{1:k+\ell}) \text{ for all } \ell \geq 1.$$

Moreover, observe that $\mathscr{M}' \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$. Indeed, for any $t \in \mathscr{X}^*$, $\Delta\mathscr{M}'(t)$ is either equal to $\Delta\mathscr{M}(t)$ or equal to zero. If it is equal to $\Delta\mathscr{M}(t)$, then since $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ we have that $-\Delta\mathscr{M}'(t) \in \mathscr{A}_t$. If it is equal to zero, then by the coherence [D1$_{27}$] of $\mathscr{A}_t$ we also have that $-\Delta\mathscr{M}'(t) \in \mathscr{A}_t$. Hence, in both cases, $-\Delta\mathscr{M}'(t) \in \mathscr{A}_t$, and since this holds for any $t \in \mathscr{X}^*$ we indeed have that $\mathscr{M}' \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$. Recalling that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1}) \leq_{x_{1:k}} \mathscr{M}'(X_{1:k+\ell})$ for all $\ell \geq 1$, this implies by Lemma 3.2.8$_{66}$ and the fact that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})$ is a (bounded) gamble by Lemma 3.2.9$_\frown$ that

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k}) \leq \mathscr{M}'(x_{1:k}) = \mathscr{M}(x_{1:k}) \leq \alpha.$$

Since this holds for any real $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k})$, we indeed have that

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k}).$$

To prove the converse inequality, start by recalling that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})$ is bounded [due to Lemma 3.2.9$_\frown$]. Then Lemma 3.2.9$_\frown$ says that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k})$ is real. Fix any real $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k})$ and any $\epsilon \in \mathbb{R}_>$. Then by Lemma 3.2.8$_{66}$ there must be a real process $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(x_{1:k}) \leq \alpha$ and $\mathscr{M}(X_{1:\ell}) \geq_{x_{1:k}} \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})$ for all ℓ larger or equal than some $k' \geq k$. Fix any $x_{k+1} \in \mathscr{X}$. Then it follows that $\mathscr{M}(X_{1:\ell}) \geq_{x_{1:k+1}} \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k+1})$ for all ℓ larger or equal than $k'+1 \geq k+1$, which by Lemma 3.2.5$_{63}$ implies that $\mathscr{M}(x_{1:k+1}) \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k+1})$. Since $\mathscr{M}(x_{1:k+1})$ is real—because it is a real process—and since $\mathscr{M}(x_{1:k+1}) \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k+1})$, it follows from Lemma 3.2.8$_{66}$ that there is a real process $\mathscr{M}_{x_{1:k+1}} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}_{x_{1:k+1}}(x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1}) + \epsilon$ and $\mathscr{M}_{x_{1:k+1}}(X_{1:\ell}) \geq_{x_{1:k+1}} f$ for all ℓ larger or equal than some $k_{x_{k+1}} \geq k+1$. This holds for any $x_{k+1} \in \mathscr{X}$, so since $\mathscr{X}$ is finite, there is moreover a finite $k'' \coloneqq \max_{x_{k+1} \in \mathscr{X}} k_{x_{k+1}} \geq k+1$ such that $\mathscr{M}_{x_{1:k+1}}(X_{1:\ell}) \geq_{x_{1:k+1}} f$ for all $\ell \geq k''$ and $x_{k+1} \in \mathscr{X}$.

Let $\mathscr{M}^*$ be the process that is equal to $\mathscr{M} + \epsilon$ for all situations that precede or are incomparable with $x_{1:k}$, and that is equal to $\mathscr{M}_{x_{1:k+1}}$ for all situations that follow $x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Since, as we have just shown previously, $\mathscr{M}_{x_{1:k+1}}(X_{1:\ell}) \geq_{x_{1:k+1}} f$ for all $x_{k+1} \in \mathscr{X}$ and all $\ell \geq k'' \geq k+1 \geq k$, and since $\mathscr{M}^*(X_{1:\ell}) =_{x_{1:k+1}} \mathscr{M}_{x_{1:k+1}}(X_{1:\ell})$ for all $x_{k+1} \in \mathscr{X}$ and all $\ell \geq k+1 \geq k$, we have that $\mathscr{M}^*(X_{1:\ell}) \geq_{x_{1:k}} f$ for all $\ell \geq k'' \geq k+1 \geq k$. We furthermore show that $\mathscr{M}^* \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$.

For any $x_{k+1} \in \mathscr{X}$, we have that $\mathscr{M}^*(x_{1:k+1}) = \mathscr{M}_{x_{1:k+1}}(x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1}) + \epsilon$, implying that $\mathscr{M}^*(x_{1:k}\cdot) \leq \mathscr{M}(x_{1:k}\cdot) + \epsilon$ and therefore that

$$\begin{aligned} -\Delta\mathscr{M}^*(x_{1:k}) = -\mathscr{M}^*(x_{1:k}\cdot) + \mathscr{M}^*(x_{1:k}) &\geq -\mathscr{M}(x_{1:k}\cdot) + \mathscr{M}^*(x_{1:k}) - \epsilon \\ &= -\mathscr{M}(x_{1:k}\cdot) + \mathscr{M}(x_{1:k}) = -\Delta\mathscr{M}(x_{1:k}). \end{aligned}$$

Since $-\Delta\mathscr{M}(x_{1:k}) \in \mathscr{A}_{x_{1:k}}$ by the fact that $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$, we have by D1$_{27}$ and D3$_{27}$ that $-\Delta\mathscr{M}^*(x_{1:k}) \in \mathscr{A}_{x_{1:k}}$. Moreover, for all situations $t \not\sqsupseteq x_{1:k}$, we have that $\Delta\mathscr{M}^*(t) = \Delta\mathscr{M}(t)$ and therefore, since $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$, that $-\Delta\mathscr{M}^*(t) \in \mathscr{A}_t$. For all situations $t \in \mathscr{X}^*$ such that $t \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$, we have that $\Delta\mathscr{M}^*(t) = \Delta\mathscr{M}_{x_{1:k+1}}(t)$ and therefore—again since $\mathscr{M}_{x_{1:k+1}} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$—also that $-\Delta\mathscr{M}^*(t) \in \mathscr{A}_t$. All together, we have that $-\Delta\mathscr{M}^*(t) \in \mathscr{A}_t$ for all $t \in \mathscr{X}^*$, and therefore that $\mathscr{M}^* \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$.

Recalling that moreover $\mathscr{M}^*(X_{1:\ell}) \geq_{x_{1:k}} f$ for all $\ell \geq k'' \geq k$ and $\mathscr{M}^*(x_{1:k}) = \mathscr{M}(x_{1:k}) + \epsilon \leq \alpha + \epsilon$, Lemma 3.2.8$_{66}$ implies that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k}) \leq \mathscr{M}^*(x_{1:k}) \leq \alpha + \epsilon$. This holds for any $\epsilon \in \mathbb{R}_>$ and any real $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k})$, so we conclude that indeed $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|x_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|X_{1:k+1})|x_{1:k})$.

This proves the desired statement for the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$. The desired statement for the upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$ then follows from Theorem 3.2.7$_{65}$. □

3.3 Global upper expectations from imprecise probability trees

In this section, we take imprecise probability trees as the primary object that parametrizes our stochastic process. Starting from these trees, we use concepts such as global probability charges and global linear expectations to build a corresponding global upper expectation. Later on, in Section 3.5$_{90}$, we will then moreover see that these probability-based global upper expectations are equivalent to the previously discussed behavioural global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$.

3.3.1 Global probability charges

Our construction of the probability-based global upper expectation will fundamentally rely on the notion of a conditional probability charge. To introduce such conditional probability charges, we require the notion of an algebra. An **algebra** or **field** $\mathcal{A}$ on a non-empty—general—set $\mathscr{Y}$ is a collection of subsets of $\mathscr{Y}$ that contains the empty subset $\emptyset \subset \mathscr{Y}$ and that is closed under finite unions and complementation; that is, for any $A, B \subseteq \mathcal{A}$, we have that $A \cup B \in \mathcal{A}$ and that $A^c := \mathscr{Y} \setminus A \in \mathcal{A}$. It then follows that $\mathcal{A}$ also contains the set $\mathscr{Y}$ and that it is closed under finite intersections. The largest algebra on $\mathscr{Y}$ is the powerset $\wp(\mathscr{Y}) := \{A \subseteq \mathscr{Y}\}$ and the smallest algebra is $\{\mathscr{Y}, \emptyset\}$. For any algebra $\mathcal{A} \subseteq \wp(\mathscr{Y})$, we use the notation $\mathcal{A}^\circ$ to denote $\mathcal{A} \setminus \{\emptyset\}$.

The following definition of a **conditional probability charge** is due to Dubins [34, Section 3] and Regazzini [78, Definition 2], who simply call this object a 'conditional probability'.

Definition 3.1 (Conditional probability charges)**.** Let $\mathcal{A}, \mathcal{B}$ be any two algebras (or fields) on $\mathcal{Y}$ such that $\mathcal{B} \subseteq \mathcal{A}$. Then we call $\mathrm{P}\colon \mathcal{A} \times \mathcal{B}^\circ \to \mathbb{R}$ a conditional probability charge if, for all $A, C \in \mathcal{A}$ and $B, D \in \mathcal{B}^\circ$,

CP1. $0 \leq \mathrm{P}(A|B)$ [lower bounds];

CP2. $B \subseteq A \Rightarrow \mathrm{P}(A|B) = 1$ [normalisation];

CP3. $A \cap C = \emptyset \Rightarrow \mathrm{P}(A \cup C|B) = \mathrm{P}(A|B) + \mathrm{P}(C|B)$ [finite additivity];

CP4. $D \cap B \neq \emptyset \Rightarrow \mathrm{P}(A \cap D|B) = \mathrm{P}(A|D \cap B)\mathrm{P}(D|B)$ [Bayes' rule]. ◎

The definition above will in our context only be applied to the case where the possibility space $\mathcal{Y}$ takes the form of the sample space Ω. Moreover, we are also not interested in probabilities conditional on general events in $\wp(\Omega)$, but rather only in probabilities conditional on situations $s \in \mathcal{X}^*$—or better, on their corresponding cylinder events $\Gamma(s)$. Therefore, we propose the following definition of a **global probability charge**, where we use $\Gamma(\mathcal{X}^*) := \{\Gamma(s)\colon s \in \mathcal{X}^*\}$ to denote the set of all cylinder events.

Definition 3.2 (Global probability charges)**.** For any algebra $\mathcal{A}$ on Ω such that $\Gamma(\mathcal{X}^*) \subseteq \mathcal{A}$, we say that $\mathrm{P}\colon \mathcal{A} \times \mathcal{X}^* \to \mathbb{R}$ is a global probability charge if, for all $A, C \in \mathcal{A}$ and all $s, t \in \mathcal{X}^*$ such that $s \sqsubseteq t$,

GP1. $0 \leq \mathrm{P}(A|s)$ [lower bounds];

GP2. $\Gamma(s) \subseteq A \Rightarrow \mathrm{P}(A|s) = 1$ [normalisation];

GP3. $A \cap C = \emptyset \Rightarrow \mathrm{P}(A \cup C|s) = \mathrm{P}(A|s) + \mathrm{P}(C|s)$ [finite additivity];

GP4. $\mathrm{P}(A \cap \Gamma(t)|s) = \mathrm{P}(A|t)\mathrm{P}(t|s)$ [Bayes' rule]. ◎

Axiom GP2 indicates that we take conditioning on a situation $s \in \mathcal{X}^*$ to mean the same as conditioning on its cylinder event $\Gamma(s)$. We will sometimes also let a situation be the first argument of a global probability charge; it then simply refers to its corresponding cylinder event—see e.g. Eq. $(3.12)_{72}$.

It can easily be seen that, for any two algebras $\mathcal{A}, \mathcal{B}$ on $\mathcal{Y}$ with $\Gamma(\mathcal{X}^*) \subseteq \mathcal{B} \subseteq \mathcal{A}$, the restriction of a conditional probability charge on $\mathcal{A} \times \mathcal{B}^\circ$ to $\mathcal{A} \times \Gamma(\mathcal{X}^*)$—or $\mathcal{A} \times \mathcal{X}^*$—satisfies GP1–GP4, and therefore that it is a global probability charge. The converse implication is less trivial however; the following result establishes that it holds nonetheless. Moreover, it also shows—see point $(\text{iii})_{\rightarrow}$ below—that any global probability charge according to Definition 3.2 is a 'conditional probability' in the sense of [4] or a 'coherent conditional probability' in the sense of [62]. Especially the version in [62, Definition 5] is important to us because we will often use results from [62, Section 3.4] in future derivations.

Proposition 3.3.1. *For any algebra* $\mathcal{A}$ *on* Ω *such that* $\Gamma(\mathcal{X}^*) \subseteq \mathcal{A}$*, and any map* $\mathrm{P}\colon \mathcal{A} \times \mathcal{X}^* \to \mathbb{R}$*, the following statements are equivalent:*

(i) P *is a global probability charge;*

(ii) P *is the restriction of a conditional probability charge* P' *defined on* $\wp(\Omega) \times \wp(\Omega)^\circ$*;*

(iii) *for all* $n \in \mathbb{N}$*, all* $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ *and all* $(A_1, s_1), \dots, (A_n, s_n) \in \mathcal{A} \times \mathcal{X}^*$.

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{P}(A_i|s_i)\right)\Big|\cup_{i=1}^{n}\Gamma(s_i)\Big) \geq 0.$$ [5]

As far as the proof of the result above is concerned; we are convinced that it follows from the statements in [4, p. 73], since a global probability charge according to Definition $3.2_{\leftarrow}$ seems to always satisfy conditions (a), (b_1) and (b_2) in [4, p. 73]. We nonetheless choose to give an independent and self-contained proof in Appendix $3.\mathrm{B}_{105}$, since we can deduce it fairly easily from our axiomatisation of coherence for global upper expectations further on [Theorem $3.4.3_{84}$].

As a consequence of the result above, global probability charges inherit the same basic properties satisfied by conditional probability charges—the proof of the following result is left as an exercise for the reader.

Proposition 3.3.2. *For any global probability charge* P *on* $\mathcal{A}\times\mathcal{X}^*$*, any* $A \in \mathcal{A}$ *and* $s \in \mathcal{X}^*$*, we have that*

GP5. $\mathrm{P}(\Omega|s) = 1$;

GP6. $\mathrm{P}(\emptyset|s) = 0$;

GP7. $0 \leq \mathrm{P}(A|s) \leq 1$;

GP8. $\mathrm{P}(A|s) = \mathrm{P}(A \cap \Gamma(s)|s)$.

It readily follows from $\mathrm{GP1}_{\leftarrow}$, $\mathrm{GP3}_{\leftarrow}$ and GP5 that, for any global probability charge P on $\mathcal{A} \times \mathcal{X}^*$ and any fixed situation $s \in \mathcal{X}^*$, the set function $\mathrm{P}(\cdot|s)\colon \mathcal{A} \to \mathbb{R}$ is a(n) (unconditional) probability charge on $\mathcal{A}$ in the sense of [77, 106], or a finitely additive probability measure in the sense of [5]. In

[5] The definition in [4] additionally requires the infimum of $\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{P}(A_i|s_i)\right)$ over $\cup_{i=1}^{n}\Gamma(s_i)$ to be smaller than or equal to 0, but it can be observed that this is implied by (iii); it suffices to switch the signs of the λ_i's. On the other hand, [62, Definition 5] uses a maximum instead of a supremum, and in our case too the supremum could actually be replaced by a maximum because it is taken over a finite number of (finite) values.

particular, if we speak about **the** unconditional (global) probability charge corresponding to P, we always mean $\mathrm{P}(\cdot) := \mathrm{P}(\cdot|\square)$.

3.3.2 Global probability charges from precise probability trees

Before building a global upper expectation from a general imprecise probability tree, we first restrict ourselves to the special case that our stochastic process is described by a **precise** probability tree $p\colon s \in \mathscr{X}^* \mapsto p(\cdot|s)$. We intend to use the local probability mass functions $p(\cdot|s)$ for all $s \in \mathscr{X}^*$ to make assertions about global probability charges. These assertions can straightforwardly be deduced from the interpretation of the mass functions $p(\cdot|s)$: we will want to impose, for any global probability charge P corresponding to p, the condition that

$$\mathrm{P}(x_{1:k+1}|x_{1:k}) = p(x_{k+1}|x_{1:k}) \text{ for all } k \in \mathbb{N}_0 \text{ and all } x_{1:k+1} \in \mathscr{X}^{k+1}. \quad (3.12)$$

Note that Eq. (3.12) is the same as the condition that is being imposed in [62, Eq. (3.3)]. In fact, in general, much of what we will do in the current section can be seen to be similar to—and inspired by—the work in [62, Section 3.4].

We start by establishing the existence of a global probability charge that respects Eq. (3.12) and that is defined on the smallest possible domain: the set $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$, with $\langle\mathscr{X}^*\rangle$ the (smallest) algebra generated by the cylinder events $\Gamma(\mathscr{X}^*)$. That the algebra $\langle\mathscr{X}^*\rangle$ exists follows from the discussion in [5, p. 27–28].

Lemma 3.3.3. *The algebra $\langle\mathscr{X}^*\rangle$ exists and, for any $A \subseteq \Omega$, we have that $A \in \langle\mathscr{X}^*\rangle$ if and only if it is a finite union of (disjoint) cylinder events. If this is the case, then we moreover have that $A = \cup_{z_{1:\ell}\in C}\Gamma(z_{1:\ell})$ for some $\ell \in \mathbb{N}_0$ and $C \subseteq \mathscr{X}^\ell$.*

Proof. To establish the existence, first note that a 'cylinder event of rank n' according to [5] is in our language a finite union $\cup_{s_i}\Gamma(s_i)$ of cylinder events of situations s_i of length n. Any such 'cylinder event of rank n' must clearly be an element of any algebra that includes $\Gamma(\mathscr{X}^*)$, and so since [5, p. 27–28] says that the set of 'cylinder events of all ranks' form an algebra, this algebra must be the smallest algebra including $\Gamma(\mathscr{X}^*)$ and thus be equal to $\langle\mathscr{X}^*\rangle$. This establishes existence, and also immediately that any $A \in \langle\mathscr{X}^*\rangle$ is the finite union $\cup_{z_{1:\ell}\in C}\Gamma(z_{1:\ell})$ for some $\ell \in \mathbb{N}_0$ and $C \subseteq \mathscr{X}^\ell$, and thus that A is the finite union of (disjoint) cylinder events. It then remains to show the converse; that any finite union of (disjoint) cylinder events is an element of $\langle\mathscr{X}^*\rangle$. This follows straightforwardly from the definition of the algebra $\langle\mathscr{X}^*\rangle$. ☐

The next proposition not only confirms the existence of a global probability charge on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ satisfying Eq. (3.12), it also says that this probability charge is unique and that its values can be computed in an intuitive way. The result is entirely the same as [62, Lemma 14], apart from the fact that [62, Lemma 14] uses conditional probability measures—see [62, Definition 6 and below]—instead of global probability charges. The familiarized reader, however, may notice that on the current domain of interest $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ there is no difference between the two notions, therefore allowing us to nevertheless use [62, Lemma 14]. A formal proof of this result can be found in Appendix $3.\mathrm{B}_{105}$.

Proposition 3.3.4. *For any precise probability tree p, there is a unique global probability charge* P_p *on* $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ *that respects Eq.* $(3.12)_{\leftarrow}$. *Specifically, for any* $x_{1:k} \in \mathscr{X}^*$, *any* $\ell \in \mathbb{N}_0$ *and any* $C \subseteq \mathscr{X}^\ell$,[6]

$$\mathrm{P}_p(\cup_{z_{1:\ell} \in C} \Gamma(z_{1:\ell}) | x_{1:k}) = \sum_{z_{1:\ell} \in C} \mathrm{P}_p(z_{1:\ell} | x_{1:k}),$$

$$\textit{with} \;\; \mathrm{P}_p(z_{1:\ell}|x_{1:k}) = \begin{cases} \prod_{i=k}^{\ell-1} p(z_{i+1}|z_{1:i}) & \textit{if } k < \ell \textit{ and } z_{1:k} = x_{1:k} \\ 1 & \textit{if } k \geq \ell \textit{ and } z_{1:\ell} = x_{1:\ell} \\ 0 & \textit{otherwise.} \end{cases}$$

The domain $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ is also immediately the largest domain that we will consider for a global probability charge satisfying Eq. $(3.12)_{\leftarrow}$—at least, in this chapter. The reason is that, on a domain that is larger than $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$—e.g. $\wp(\Omega) \times \mathscr{X}^*$ or $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ with $\sigma(\mathscr{X}^*)$ the σ-algebra generated by $\langle \mathscr{X}^* \rangle$ (see Chapter 5_{217})—there is not necessarily one unique global probability charge satisfying Eq. $(3.12)_{\leftarrow}$. Uniqueness can however be preserved on the larger domain $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$, if we would additionally impose countable additivity or σ-additivity on a global probability charge; see Chapter 5_{217}. As mentioned in the introduction of this chapter, we choose not to do so yet and first study an approach that is solely based on finitely additive probability charges. Finitely additive probabilities were advocated by [27, 35, 77], and are more popular than their countably additive variants in the field of imprecise probabilities, due to their generality and their strong relationship with coherent upper expectations [105, 106, 110, 113]. Nonetheless, we will see in Section 3.6_{98} that the resulting finitary global upper expectation is not satisfactory for a general domain $\mathbb{V} \times \mathscr{X}^*$, and so we will continue in Chapter 5_{217} to study a more involved approach that is based on countably additive global probability charges.

Furthermore, observe that there is another important aspect in which our approach here differs from more traditional measure-theoretic approaches [5, 89]: we consider conditional (global) probabilities to be

[6] By Lemma $3.3.3_{\leftarrow}$, any $A \in \langle \mathscr{X}^* \rangle$ can be written as $\cup_{z_{1:\ell} \in C} \Gamma(z_{1:\ell})$ for some $\ell \in \mathbb{N}_0$ and $C \subseteq \mathscr{X}^\ell$.

equally as fundamental as unconditional (global) probabilities, whereas the typical measure-theoretic practices consider unconditional probabilities to be primary objects and then deduce conditional probabilities from unconditional probabilities by means of Bayes' rule. That is, for an unconditional probability charge P on an algebra $\mathcal{A}$, the corresponding probability $\mathrm{P}(A|B)$ conditional on any $B \in \mathcal{A}^{\circ}$ is in that case defined by $\mathrm{P}(A|B) := \mathrm{P}(A \cap B)/\mathrm{P}(B)$. Of course, the latter is ill-defined if $\mathrm{P}(B) \neq 0$, a problem that is then usually 'solved' by allowing $\mathrm{P}(A|B)$ to take an arbitrary value. Such an approach is inadequate when one wishes to retain information about given conditional probabilities—the local ones in our case. Indeed, the unconditional probability charge $\mathrm{P}(\cdot) = \mathrm{P}(\cdot|\mathscr{Y})$ corresponding to a given conditional probability charge $\mathrm{P}(\cdot|\cdot)$ does in general not allow us to recover $\mathrm{P}(\cdot|\cdot)$. Hence the reason why we prefer to use conditional probability charges instead.

Lastly, though we choose not to extend a global probability charge beyond the domain $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$, it is nevertheless possible to do so—this can be seen to follow from [78, Theorem 4] and the fact that, as pointed out by Proposition 3.3.1$_{71}$, global probability charges are equivalent to 'coherent conditional probabilities' restricted to a particular domain. Since in general this means giving up unicity, we would have to work with a (non-empty) set of global probability charges instead of a single one. By defining a (linear) expectation for each global probability charge in this set—using the integration techniques described below—and then subsequently taking an upper envelope over this set, we could come up with an alternative version of the global upper expectation $\overline{\mathrm{E}}_p$ defined here, in Definition 3.5$_{78}$. Such an approach, however, is conceptually rather different from what is done in classical (measure-theoretic) probability theory—and also from what we will do in Chapter 5$_{217}$—where a single probability charge or measure always forms the central starting point.[7]

3.3.3 Global linear/upper expectations from precise probability trees

To obtain global linear expectations and upper expectations from global probability charges, we will use the S-integral [47, 77, 105, 106]. We choose to use this integral because (i) it is a conceptually easy type of integral, popular among those who study general (finitely additive) probability charges, and (ii) it is equivalent to the well-known Lebesgue integral that we

[7]A reader that is familiar with Williams's notion of conditional coherence [72, 114] may nevertheless infer from the 'envelope theorem' [72, Section 3.1] and the fact that our probability-based global upper expectation $\overline{\mathrm{E}}_p$ is coherent (see Corollary 3.5.6$_{94}$ and Theorem 3.4.3$_{84}$), that this alterative approach leads to a global upper expectation that is equivalent to $\overline{\mathrm{E}}_p$.

will use later on—see Definition 5.3228—but adapted to deal with general algebras and probability charges.

The measure-theoretic concepts that will be introduced in the following sections are taken largely from [106, Section 1.8 and Chapter 8]. These concepts deviate somewhat from the standard definitions, because they are adapted to deal with general algebras and general (finitely additive) probability charges on algebras. Measurability, for instance, is in the standard case typically only associated with σ-algebras, not with general algebras. One may easily observe from [106, Section 1.8 and Chapter 8] that the adapted notions that we use here are simple generalisations of the standard measure-theoretic concepts.

Simple global gambles and measurable global gambles

For any algebra (or field) $\mathcal{A}$ on Ω, we say that a gamble $f \in \mathbb{V}$ is **$\mathcal{A}$-simple** [106, Definition 1.16] if $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ for some $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathcal{A}$, and $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ is then called a **representation** of f. So the linear span

$$\operatorname{span}(\mathcal{A}) \coloneqq \left\{ \sum_{i=1}^{n} a_i \mathbb{I}_{A_i} : n \in \mathbb{N}, a_i \in \mathbb{R}, A_i \in \mathcal{A} \right\}$$

is the set of all $\mathcal{A}$-simple gambles. A gamble $f \in \mathbb{V}$ is then called **$\mathcal{A}$-measurable** [106, Definition 1.17 (B)] if it is in the uniform closure of $\operatorname{span}(\mathcal{A})$, meaning that there is a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{A}$-simple gambles such that

$$\lim_{n\to+\infty} \sup |f - f_n| = 0.$$

If it is clear from the context which algebra $\mathcal{A}$ we are considering, we will simply call gambles simple or measurable, instead of respectively $\mathcal{A}$-simple and $\mathcal{A}$-measurable.

In this chapter, we will mainly be concerned with $\langle \mathscr{X}^* \rangle$-simple gambles, which can easily be seen to be equal to finitary gambles.

Lemma 3.3.5. *We have that* $\mathbb{F} = \operatorname{span}(\langle \mathscr{X}^* \rangle)$.

Proof. Since for any finitary gamble $f \in \mathbb{F}$ there is a $k \in \mathbb{N}_0$ such that we can write $f = \sum_{x_{1:k} \in \mathscr{X}^k} f(x_{1:k}) \mathbb{1}_{x_{1:k}}$, and since all cylinder events are by definition included in $\langle \mathscr{X}^* \rangle$, it follows that $\mathbb{F} \subseteq \operatorname{span}(\langle \mathscr{X}^* \rangle)$. Conversely, consider any $f \in \operatorname{span}(\langle \mathscr{X}^* \rangle)$. Let $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ be any representation of f. For any $i \in \{1, \dots, n\}$, we have by Lemma 3.3.372 that A_i is a finite union $\cup_{j=1}^{m_i} \Gamma(s_{i,j})$ of (disjoint) cylinder events $\Gamma(s_{i,j})$. So then we have that $\mathbb{I}_{A_i} = \sum_{j=1}^{m_i} \mathbb{1}_{s_{i,j}}$ for all $i \in \{1, \dots, n\}$, and therefore that

$$f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i} = \sum_{i=1}^{n} \sum_{j=1}^{m_i} a_i \mathbb{1}_{s_{i,j}}.$$

Hence, it is clear that f can thus be written as $\sum_{k=1}^{\ell} b_k \mathbb{1}_{t_k}$ for some $\ell \in \mathbb{N}_0, b_1, \dots, b_\ell \in \mathbb{R}$ and $t_1, \dots, t_\ell \in \mathscr{X}^*$. So if p is the maximum of the lengths of the situations $t_1, \dots, t_\ell$, it is clear that $f = \sum_{k=1}^{\ell} b_k \mathbb{1}_{t_k}$ depends only on the states $X_{1:p}$, and thus that f is finitary. □

The S-integral or the Lebesgue integral

For the definition of the S-integral, we follow [106, Definition 8.24] but immediately state a version adapted to the context of stochastic processes.

Definition 3.3. Consider any probability charge P on the algebra $\langle \mathscr{X}^* \rangle$, and any $f \in \mathbb{V}$. Then the S-integral of f exists if the **upper** and **lower S-integral** of f, respectively given by

$$\overline{\int} f \mathrm{dP} := \inf \left\{ \sum_{i=1}^{n} \sup(f|A_i)\mathrm{P}(A_i) \colon A_i \in \langle \mathscr{X}^* \rangle \text{ and } (A_i)_{i=1}^{n} \text{ partitions } \Omega \right\};$$

$$\underline{\int} f \mathrm{dP} := \sup \left\{ \sum_{i=1}^{n} \inf(f|A_i)\mathrm{P}(A_i) \colon A_i \in \langle \mathscr{X}^* \rangle \text{ and } (A_i)_{i=1}^{n} \text{ partitions } \Omega \right\},$$

coincide. In that case, the **S-integral** of f is given by the common value $\int f \mathrm{dP} := \overline{\int} f \mathrm{dP} = \underline{\int} f \mathrm{dP}$. ◎

The following result gives an alternative characterisation for the S-integral in terms of $\langle \mathscr{X}^* \rangle$-simple functions, and allows us to easily relate the integral with other—perhaps more familiar—types of integrals; e.g. the Lebesgue integral.

Proposition 3.3.6. *For any probability charge* P *on the algebra* $\langle \mathscr{X}^* \rangle$*, the following statements hold.*

(i) *For any* $\langle \mathscr{X}^* \rangle$*-simple* $f \in \mathbb{V}$ *and any representation* $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ *of* f*,*

$$\int f \mathrm{dP} = \sum_{i=1}^{n} a_i \mathrm{P}(A_i).$$

(ii) *For any general* $f \in \mathbb{V}$*, we have that*

$$\overline{\int} f \mathrm{dP} = \inf \left\{ \int g \mathrm{dP} \colon g \in \operatorname{span}(\langle \mathscr{X}^* \rangle) \text{ and } g \geq f \right\} \text{ and}$$

$$\underline{\int} f \mathrm{dP} = \sup \left\{ \int g \mathrm{dP} \colon g \in \operatorname{span}(\langle \mathscr{X}^* \rangle) \text{ and } g \leq f \right\}.$$

Proof. Property (i) follows from the definition of the D-integral on all $\langle \mathscr{X}^* \rangle$-simple gambles [106, Definition 8.13] and its equivalence with the S-integral [106, Theorem 8.32]. Property (ii) follows from the equivalence between the S-integral and the Lebesgue integral [106, Theorem 8.28], and the fact that, as we have just stated, the D-integral inside the definition of the Lebesgue integral [106, Definition 8.27] is equal to the S-integral on the domain of $\langle\mathscr{X}^*\rangle$-simple gambles. □

This result shows that indeed, as claimed above, the (upper/lower) S-integral is completely equivalent to the (upper/lower) Lebesgue integral given in [106, Definition 8.27]; see also [106, Theorem 8.28]. So why do we not use the (upper/lower) Lebesgue integral as our main integral here? One reason is that we find the definition of the S-integral more direct and elegant. Another is that we already want to adhere to the choice of integral made in Chapter 5_{217} when we will deal with (σ-additive) probability spaces. We will there use Billingsley's [5] version of the Lebesgue integral which, amusingly enough, is much more similar to Definition $3.3_{\leftarrow}$ than the definition of the Lebesgue integral given in [106, Definition 8.27]. Because of the mathematical equivalence, and because we do not want our treatment to be obscured by semantic delicacies, we henceforth simply refer to the (upper/lower) S-integral Definition $3.3_{\leftarrow}$—or thus Lebesgue (upper/lower) integral—as **the (upper/lower) integral**.

We next use this integral to define the global expectation corresponding to a precise probability tree.

Definition 3.4. Consider any precise probability tree p, let P_p be the unique global probability charge from Proposition $3.3.4_{73}$, and let $\mathrm{P}_p^{|s} := \mathrm{P}_p(\cdot|s)$ for any $s \in \mathscr{X}^*$. Then the global expectation E_p is defined by $\mathrm{E}_p(f|s) := \int f\mathrm{d}\mathrm{P}_p^{|s}$ for all $(f, s) \in \mathbb{V} \times \mathscr{X}^*$ such that $\int f\mathrm{d}\mathrm{P}_p^{|s}$ exists. ◎

It can be inferred from [106, Proposition 8.17] and [106, Theorem 8.32] that, for all $\langle\mathscr{X}^*\rangle$-measurable gambles, the integral exists and thus also the expectation E_p. Moreover, by [106, Theorem 8.32] and [77, Theorem 4.4.13 (ii)], for any $s \in \mathscr{X}^*$, we have that the integral $\int \cdot\, \mathrm{d}\mathrm{P}_p^{|s}$, and thus the expectation $\mathrm{E}_p(\cdot|s)$, is a linear operator on the domain where it exists, which is why we sometimes call E_p a global linear expectation.[8]

For gambles that are not $\langle\mathscr{X}^*\rangle$-measurable, upper and lower integrals may not coincide anymore. A possible alternative would then be to work with either the upper or the lower integral itself—for these are defined on all gambles. The reason why this is usually not done in standard 'precise' probability theory, is that this cannot be done without giving up linearity of the resulting global operator. We do not consider this to be a problem though; we will in general consider imprecise local models instead of the

[8]In the field of coherent upper and lower expectations, the term '(conditional) linear expectation' or '(conditional) linear prevision' is typically used in a more general sense to refer to a self-conjugate coherent (conditional) upper expectation; see e.g. [106, Definition 13.28].

mass functions $p(\cdot|s)$ that we are now considering, so it should not be expected that linearity of the resulting global (upper) expectation can be preserved anyway.

Definition 3.5. Consider any precise probability tree p, let P_p be the unique global probability charge from Proposition 3.3.473, and let $\mathrm{P}_p^{|s} := \mathrm{P}_p(\cdot|s)$ for any $s \in \mathscr{X}^*$. Then the global upper expectation $\overline{\mathrm{E}}_p$ and global lower expectation $\underline{\mathrm{E}}_p$ are defined, for all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, by

$$\begin{aligned}\overline{\mathrm{E}}_p(f|s) &:= \overline{\int} f \mathrm{d}\mathrm{P}_p^{|s}\\ &= \inf\left\{\sum_{i=1}^{n} \sup(f|A_i)\mathrm{P}_p^{|s}(A_i) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}.\\ \underline{\mathrm{E}}_p(f|s) &:= \underline{\int} f \mathrm{d}\mathrm{P}_p^{|s}\\ &= \sup\left\{\sum_{i=1}^{n} \inf(f|A_i)\mathrm{P}_p^{|s}(A_i) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}. \quad \circledcirc\end{aligned}$$

Obviously, we define our global upper expectation $\overline{\mathrm{E}}_p$ as an upper integral and not as a lower integral, because the former is always larger or equal than the latter, and conversely for the global lower expectation $\underline{\mathrm{E}}_p$. The lower expectation $\underline{\mathrm{E}}_p$ is once more related to the upper expectation $\overline{\mathrm{E}}_p$ by conjugacy.

Corollary 3.3.7 (Conjugacy)**.** *For any precise probability tree p, we have that* $\underline{\mathrm{E}}_p(f|s) = -\overline{\mathrm{E}}_p(-f|s)$ *for all* $(f,s) \in \mathbb{V} \times \mathscr{X}^*$.

Proof. For any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, we have that

$$\begin{aligned}-\overline{\mathrm{E}}_p(-f|s) &= -\inf\left\{\sum_{i=1}^{n} \sup(-f|A_i)\mathrm{P}_p^{|s}(A_i) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}\\ &= \sup\left\{\sum_{i=1}^{n} -\sup(-f|A_i)\mathrm{P}_p^{|s}(A_i) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}\\ &= \sup\left\{\sum_{i=1}^{n} \inf(f|A_i)\mathrm{P}_p^{|s}(A_i) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}\\ &= \underline{\mathrm{E}}_p(f|s).\end{aligned}$$ □

This allows us to henceforth again focus mainly on upper expectations $\overline{\mathrm{E}}_p$.

Observe by Definition 3.4↶ and Definition 3.376 that $\overline{\mathrm{E}}_p$ is an extension of E_p. In particular, $\overline{\mathrm{E}}_p$ retains the basic form that the integral assumes on $\langle\mathscr{X}^*\rangle$-simple gambles—or, by Lemma 3.3.575, on finitary gambles. Combined with Proposition 3.3.473, this leads to the following result.

Proposition 3.3.8. *Consider any precise probability tree p. Then, for all* $(f, x_{1:k}) \in \mathbb{F} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}_p(f|x_{1:k}) = \mathrm{E}_p(f|x_{1:k}) = \sum_{z_{1:\ell} \in \mathscr{X}^\ell} f(z_{1:\ell})\mathrm{P}_p(z_{1:\ell}|x_{1:k}) \\ = \sum_{x_{k+1:\ell} \in \mathscr{X}^{\ell-k}} f(x_{1:\ell}) \prod_{i=k}^{\ell-1} p(x_{i+1}|x_{1:i}),$$

where P_p *on* $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ *is related to p according to Proposition 3.3.4*$_{73}$*, and where* $\ell > k$ *is any natural number such that f is* ℓ*-measurable.*

Proof. Fix any $f \in \mathbb{F}$, any $x_{1:k} \in \mathscr{X}^*$ and any $\ell > k$ such that f is ℓ-measurable—this is always possible because f is finitary. Then we have that $f = \sum_{z_{1:\ell} \in \mathscr{X}^\ell} f(z_{1:\ell}) \mathbb{1}_{z_{1:\ell}}$, and so that f is $\langle \mathscr{X}^* \rangle$-simple and that $\sum_{z_{1:\ell} \in \mathscr{X}^\ell} f(z_{1:\ell}) \mathbb{1}_{z_{1:\ell}}$ is a representation of f. Hence, due to Proposition 3.3.6(i)$_{76}$ and Definition 3.4$_{77}$, we have that $\mathrm{E}_p(f|x_{1:k}) = \sum_{z_{1:\ell} \in \mathscr{X}^\ell} f(z_{1:\ell})\mathrm{P}_p(z_{1:\ell}|x_{1:k})$. Since $\overline{\mathrm{E}}_p$ is an extension of E_p, we also have that

$$\overline{\mathrm{E}}_p(f|x_{1:k}) = \mathrm{E}_p(f|x_{1:k}) = \sum_{z_{1:\ell} \in \mathscr{X}^\ell} f(z_{1:\ell})\mathrm{P}_p(z_{1:\ell}|x_{1:k}).$$

The last equality in statement above then follows immediately from Proposition 3.3.4$_{73}$. □

3.3.4 Global upper and lower expectations from imprecise probability trees

Suppose that we are now given a general imprecise probability tree $\mathscr{P}_\bullet$ that associates with each situation s a set of probability mass functions $\mathscr{P}_s$. Then we can apply the extension procedure from the previous section to every precise probability tree p that is constructed by selecting, in each situation s, a probability mass function $p(\cdot|s)$ from the set $\mathscr{P}_s$. Any such precise probability tree p is called **compatible** with $\mathscr{P}_\bullet$ and we make this clear by writing $p \sim \mathscr{P}_\bullet$. The upper (resp. lower) envelope over the global upper (lower) expectations $\overline{\mathrm{E}}_p$ ($\underline{\mathrm{E}}_p$) corresponding to the compatible precise trees $p \sim \mathscr{P}_\bullet$ is then what we refer to as the global upper (lower) expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ ($\underline{\mathrm{E}}_{\mathscr{P}}$) corresponding to $\mathscr{P}_\bullet$.

Definition 3.6. Consider any imprecise probability tree $\mathscr{P}_\bullet$. The global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ and global lower expectation $\underline{\mathrm{E}}_{\mathscr{P}}$ are defined, for all $(f, s) \in \mathbb{V} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}_{\mathscr{P}}(f|s) := \sup\{\overline{\mathrm{E}}_p(f|s) : p \sim \mathscr{P}_\bullet\} \text{ and } \underline{\mathrm{E}}_{\mathscr{P}}(f|s) := \inf\{\underline{\mathrm{E}}_p(f|s) : p \sim \mathscr{P}_\bullet\},$$

with $\overline{\mathrm{E}}_p$ and $\underline{\mathrm{E}}_p$ for any $p \sim \mathscr{P}_\bullet$ given by Definition 3.5$_\leftarrow$. ⊚

The probability-based upper and lower expectations $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\underline{\mathrm{E}}_{\mathscr{P}}$ are once more related by conjugacy, therefore allowing us to focus mainly on upper expectations in the sequel.

Corollary 3.3.9 (Conjugacy). *For any imprecise probability tree $\mathscr{P}$, we have that $\underline{\mathrm{E}}_{\mathscr{P}}(f|s) = -\overline{\mathrm{E}}_{\mathscr{P}}(-f|s)$ for all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$.*

Proof. For any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, we have that

$$-\overline{\mathrm{E}}_{\mathscr{P}}(-f|s) = -\sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_p(-f|s) = \inf_{p\sim\mathscr{P}_\bullet} -\overline{\mathrm{E}}_p(-f|s) = \inf_{p\sim\mathscr{P}_\bullet} \underline{\mathrm{E}}_p(f|s) = \underline{\mathrm{E}}_{\mathscr{P}}(f|s),$$

where the penultimate step follows from the conjugacy between $\overline{\mathrm{E}}_p$ and $\underline{\mathrm{E}}_p$ for any precise tree p; recall Corollary 3.3.7$_{78}$. □

If we restrict ourselves to the finitary domain $\mathbb{F} \times \mathscr{X}^*$, the values of the global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ can be obtained from the following simple expression.

Corollary 3.3.10. *Consider any imprecise probability tree $\mathscr{P}_\bullet$. Then, for all $(f, x_{1:k}) \in \mathbb{F} \times \mathscr{X}^*$,*

$$\begin{aligned}\overline{\mathrm{E}}_{\mathscr{P}}(f|x_{1:k}) &= \sup\left\{\mathrm{E}_p(f|x_{1:k}) \colon p \sim \mathscr{P}_\bullet\right\} \\ &= \sup\left\{ \textstyle\sum_{x_{k+1:\ell}\in\mathscr{X}^{\ell-k}} f(x_{1:\ell}) \prod_{i=k}^{\ell-1} p(x_{i+1}|x_{1:i}) \colon p \sim \mathscr{P}_\bullet\right\},\end{aligned}$$

where $\ell > k$ is any natural number such that f is ℓ-measurable.

Proof. This follows immediately from Proposition 3.3.8$_\frown$ and Definition 3.6$_\frown$. □

3.4 Global upper expectations from upper expectations trees

Lastly, we consider the case where local dynamics are described by an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and ask ourselves the question how this tree can be extended to a global upper expectation. Of course, we could associate with $\overline{\mathrm{Q}}_\bullet$ an agreeing acceptable gambles tree $\mathscr{A}_\bullet$ or an agreeing imprecise probability tree $\mathscr{P}_\bullet$, and then subsequently use the global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$ or $\overline{\mathrm{E}}_{\mathscr{P}}$ from the respective frameworks. Yet this would raise the question of which framework to pick and, if we would have chosen one, which agreeing tree to choose; for recall that Eq. (3.2)$_{51}$ and Eq. (3.5)$_{52}$ respectively provide expressions for an agreeing acceptable gambles tree and an agreeing imprecise probability tree, but that these are not necessarily the only agreeing acceptable gambles tree or agreeing imprecise probability tree. Moreover, any of these approaches is rather indirect, both philosophically and mathematically speaking, because we would start from the framework of upper expectations, we would then switch to either the framework of acceptable gambles or the framework of probability charges, only to switch back in the end to the framework of upper expectations.

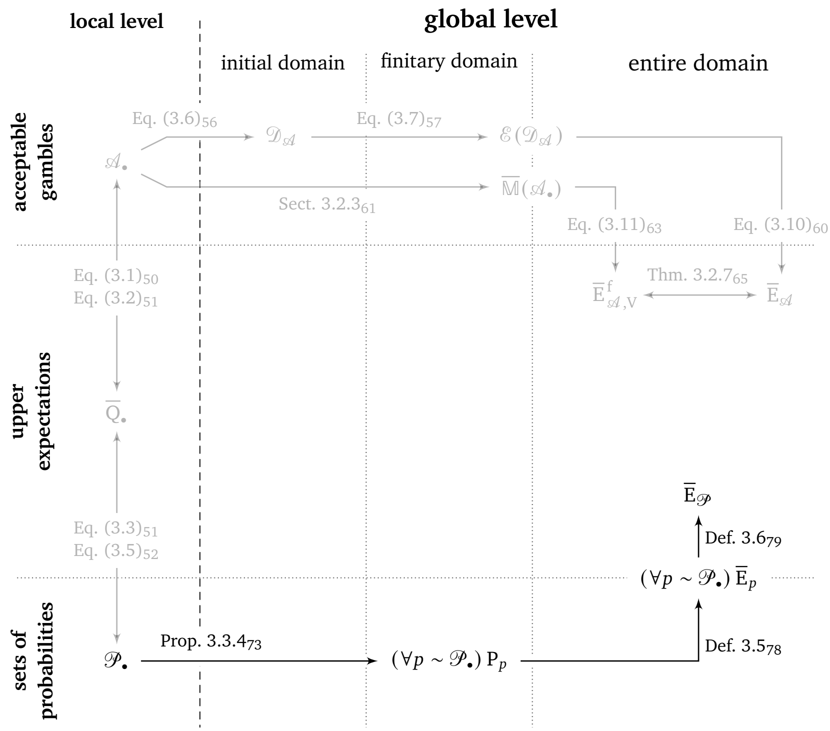

Figure 3.5 Schematic overview of the finitary probabilistic approach.

Instead, to propose a more direct and interpretationally neutral global upper expectation, we will put forward some basic axioms common to both behavioural and probability-based global upper expectations. These axioms, together, will turn out to be equivalent to the well-known requirement of (conditional) coherence [106, 113]. We will use them as a tool to extend local upper expectations to a global upper expectation, and the resulting operator will be equivalent to the natural extension under coherence.[9]

3.4.1 An axiomatisation of coherence for global upper expectations

Consider any global upper expectation $\overline{\mathrm{E}}$ on a domain $\mathscr{K} = \mathscr{I} \times \mathscr{X}^* \subseteq \mathbb{V} \times \mathscr{X}^*$ where $\mathscr{I}$ is a linear space of global gambles containing all constants and is such that $f\mathbb{1}_s \in \mathscr{I}$ for all $f \in \mathscr{I}$ and $s \in \mathscr{X}^*$. Consider also the following basic axioms for $\overline{\mathrm{E}}$; for all $f, g \in \mathscr{I}$, all $\lambda \in \mathbb{R}_{\geq}$ and all $s, t \in \mathscr{X}^*$ such that $s \sqsubseteq t$,

[9] More specifically, the natural extension of the upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ defined by Eq. (3.13)85.

WC1. $\overline{\mathrm{E}}(f|s) \leq \sup(f|s)$ [upper bound];

WC2. $\overline{\mathrm{E}}(f+g|s) \leq \overline{\mathrm{E}}(f|s) + \overline{\mathrm{E}}(g|s)$ [sub-additivity];

WC3. $\overline{\mathrm{E}}(\lambda f|s) = \lambda \overline{\mathrm{E}}(f|s)$ [non-negative homogeneity];

WC4. $\overline{\mathrm{E}}\big((f - \overline{\mathrm{E}}(f|t))\mathbb{1}_t\big|s\big) = 0$ [Bayes' rule].

It can then be observed that the behavioural global upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$—and thus also $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—and the probability-based global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ both satisfy the axioms above. We will not explicitly prove this result here, though, because it will later on simply follow from a more powerful result; see Corollary 3.5.6$_{94}$.

Proposition 3.4.1. *For any acceptable gambles tree $\mathscr{A}_\bullet$ and any imprecise probability tree $\mathscr{P}_\bullet$, the global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfy* WC1–WC4.

Note that Proposition 3.4.1 on itself is already a motivation for adopting WC1–WC4 as axioms to impose on a global upper expectation $\overline{\mathrm{E}}$; on the one hand, the axioms follow from a behavioural interpretation of $\overline{\mathrm{E}}$ as being the upper expectation $\overline{\mathrm{E}}_{\mathscr{A}}$ or $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ (or their restriction to $\mathscr{I} \times \mathscr{X}^*$) corresponding to some acceptable gambles tree $\mathscr{A}_\bullet$, and on the other hand, the axioms also follow from a probability-based interpretation of $\overline{\mathrm{E}}$ as being the upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ (or its restriction) corresponding to some imprecise probability tree $\mathscr{P}_\bullet$.

But there is more to WC1–WC4 than meets the eye; they are actually equivalent to the requirement of conditional coherence for global upper expectations. This requirement is similar to, but more general than, the notion of coherence that was introduced in Section 2.4$_{31}$ and Section 2.6.3$_{38}$ for unconditional upper expectations on a finite possibility space—recall for instance that the local upper expectations $\overline{\mathrm{Q}}_s$ are assumed to be coherent. Compare for instance the expression below to that in Definition 2.8(ii)$_{39}$. Conditional coherence was first formally introduced by P. M. Williams [113], yet his original definition required a particular structure on the domain of an upper expectation, and so we immediately adopt the generalised version that was established later on [72, 104, 106]. We also immediately apply this definition to our context where we consider global upper expectations instead of general conditional upper expectations.

Definition 3.7 (coherence for global upper expectations)**.** A global upper expectation $\overline{\mathrm{E}}$ on $\mathscr{K} \subseteq \mathbb{V} \times \mathscr{X}^*$ is coherent if, for all $n \in \mathbb{N}_0$, all $\lambda_0, \lambda_1, \ldots, \lambda_n \in$

$\mathbb{R}_{\geq}$ and all $(f_0, s_0), (f_1, s_1), \ldots, (f_n, s_n) \in \mathscr{K}$,

$$\sup \left(\lambda_0 \mathbb{1}_{s_0} \left(f_0 - \overline{\mathrm{E}}(f_0|s_0) \right) - \sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(f_i - \overline{\mathrm{E}}(f_i|s_i) \right) \middle| \cup_{i=0}^{n} \Gamma(s_i) \right) \geq 0. \quad ◎$$

The following corollary establishes that coherent global upper expectations are real-valued, which makes our definition above agree with traditional definitions of coherence, which immediately apply to real-valued conditional upper expectations [72, 113].

Corollary 3.4.2. *Any coherent global upper expectation $\overline{\mathrm{E}}$ on a domain $\mathscr{K} \subseteq \mathbb{V} \times \mathscr{X}^*$ is real-valued.*

Proof. Assume **ex absurdo** that $\overline{\mathrm{E}}(f|s) = +\infty$ for some $(f, s) \in \mathscr{K}$. Then we have that

$$\sup \left(\mathbb{1}_s (f - \overline{\mathrm{E}}(f|s)) | s \right) = \sup \left(f - \overline{\mathrm{E}}(f|s) | s \right) = \sup (f - \infty | s) = \sup (-\infty | s) = -\infty.$$

This is clearly in contradiction with the definition of coherence. In an analogous way, we can check that $\overline{\mathrm{E}}(f|s) \neq -\infty$ for all $(f, s) \in \mathscr{K}$, because then $\sup \left(-\mathbb{1}_s (f - \overline{\mathrm{E}}(f|s)) | s \right) = -\infty$. As a result, $\overline{\mathrm{E}}$ must be real-valued. □

Though the definition of coherence may look abstract, it can actually be argued for on strong grounds and in a similar way as its unconditional counterpart; it is once more based on the dual interpretation that $\overline{\mathrm{E}}$, on the one hand, can represent the infimum selling prices corresponding to a set of acceptable gambles, and on the other hand, that it can represent the upper envelope of the (conditional) expectations corresponding to a set of possible (conditional) probability charges. These connections are made mathematically firm by well-known results such as [113, Prop. 2, Theorems 1 and 2]—note the analogy of these results with the unconditional variants in Section 2.431 and Section 2.636. In view of these results, it ought not to surprise us that $\overline{\mathrm{E}}_{\mathscr{A}}$—and thus $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—for any $\mathscr{A}_\bullet$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ for any $\mathscr{P}_\bullet$ are both coherent; the former is obtained as the infimum selling prices corresponding to the coherent set $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ of acceptable gambles; the latter is obtained as the upper envelope of the global expectations E_p—if, for the sake of conceptual ease, we restrict ourselves to a domain where the expectations E_p all exist—corresponding to the probability charges P_p with $p \sim \mathscr{P}_\bullet$. We do not go into further detail concerning the motivation and interpretation of conditional coherence, and simply leave it to the current conceptual treatment; for a more elaborate discussion of the topic, we refer to the abundance of literature hereof [72, 106, 110, 113].

The following theorem now says that the basic axioms WC1$_{\leftarrow}$–WC4$_{\leftarrow}$ are indeed equivalent to the requirement of coherence for global upper expectations, at least if the domain of the considered global upper expectations has a structure that is rich enough. Therefore, and because of Proposition 3.4.1$_{82}$, $\overline{\mathrm{E}}_{\mathscr{A}}$—and thus $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—and $\overline{\mathrm{E}}_{\mathscr{P}}$ are indeed coherent for any acceptable gambles tree $\mathscr{A}_\bullet$ and any imprecise probability tree $\mathscr{P}_\bullet$.

Theorem 3.4.3. *Consider any global upper expectation $\overline{\mathrm{E}}$ on a domain $\mathscr{K} = \mathscr{I} \times \mathscr{X}^* \subseteq \mathbb{V} \times \mathscr{X}^*$ where $\mathscr{I}$ is a linear space of global gambles containing all constants and is such that $f\mathbb{1}_s \in \mathscr{I}$ for all $f \in \mathscr{I}$ and $s \in \mathscr{X}^*$. Then $\overline{\mathrm{E}}$ is coherent if and only if it satisfies* WC1$_{82}$–WC4$_{82}$.

It is rather well-known that (conditional) coherence for general conditional upper or lower expectations can be axiomatised in a similar way as what is done here; see e.g. [106, Theorem 13.33]. The main difference, however, is that such results always require the domain of a conditional upper expectation to have a particular structure; a structure that is not satisfied by the domain $\mathscr{I} \times \mathscr{X}^*$—or $\mathscr{I} \times \Gamma(\mathscr{X}^*)$—of the global upper expectations in the theorem above. Indeed, in [106, Theorem 13.33] for instance, the set of conditioning events is assumed to be closed under finite unions, which is clearly not satisfied by the set of conditioning events $\Gamma(\mathscr{X}^*)$ here. The proof of Theorem 3.4.3 is therefore independent and does not rely on results such as [106, Theorem 13.33]; it can be found in Appendix 3.C$_{107}$.

We next give an extensive list of properties that are satisfied by any global upper expectation that satisfies WC1$_{82}$–WC4$_{82}$, or, equivalently, that is coherent. The proof of this result, too, is relegated to Appendix 3.C$_{107}$.

Proposition 3.4.4. *Consider any global upper expectation $\overline{\mathrm{E}}$ on a domain $\mathscr{I} \times \mathscr{X}^* \subseteq \mathbb{V} \times \mathscr{X}^*$ where $\mathscr{I}$ is a linear space of global gambles containing all the constants, and let $\underline{\mathrm{E}} \colon \mathscr{I} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ be the corresponding conjugate lower expectation. If $\overline{\mathrm{E}}$ satisfies* WC1$_{82}$–WC3$_{82}$*, then for any $f, g \in \mathscr{I}$, $\mu \in \mathbb{R}$, and $s \in \mathscr{X}^*$,*

WC5. $f \leq_s g \Rightarrow \overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(g|s)$ *[monotonicity];*

WC6. $\inf(f|s) \leq \underline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(f|s) \leq \sup(f|s)$ *[bounds];*

WC7. $\overline{\mathrm{E}}(f+\mu|s) = \overline{\mathrm{E}}(f|s) + \mu$ *[constant additivity];*

WC8. $\underline{\mathrm{E}}(f+g|s) \leq \overline{\mathrm{E}}(f|s) + \underline{\mathrm{E}}(g|s) \leq \overline{\mathrm{E}}(f+g|s)$ *[mixed super-/sub-additivity];*

WC9. *for any sequence $\{f_n\}_{n\in\mathbb{N}_0}$ in $\mathscr{I}$:* *[uniform convergence]*

$$\lim_{n\to+\infty} \sup\left(|f-f_n|\,|s\right) = 0 \;\Rightarrow\; \lim_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s).$$

Furthermore, if we assume that $\mathscr{I}$ is moreover such that $f\mathbb{1}_s \in \mathscr{I}$ for any $f \in \mathscr{I}$ and $s \in \mathscr{X}^*$, and that $\overline{\mathrm{E}}$ satisfies WC4$_{82}$*, then $\overline{\mathrm{E}}$ additionally satisfies the following properties; for any $f \in \mathscr{I}$, $s, t \in \mathscr{X}^*$ such that $s \sqsubseteq t$, and $x_{1:k} \in \mathscr{X}^*$,* WC10. $\underline{\mathrm{E}}\left((f-\underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right)=0$;

WC11. $\overline{\mathrm{E}}\,(f|s)=\overline{\mathrm{E}}(f\mathbb{1}_s|s)$ *and* $\underline{\mathrm{E}}\,(f|s)=\underline{\mathrm{E}}(f\mathbb{1}_s|s)$;

WC12. $\overline{\mathrm{E}}\,(f|x_{1:k})\leq\overline{\mathrm{E}}(\overline{\mathrm{E}}\,(f|x_{1:k}X_{k+1})\,|x_{1:k})$ *and*
$\underline{\mathrm{E}}\,(f|x_{1:k})\geq\underline{\mathrm{E}}\left(\underline{\mathrm{E}}\,(f|x_{1:k}X_{k+1})\,|x_{1:k}\right)$;

WC13. $\overline{\mathrm{E}}\,(f|X_{1:k})\leq\overline{\mathrm{E}}(\overline{\mathrm{E}}\,(f|X_{1:k+1})\,|X_{1:k})$ *and*
$\underline{\mathrm{E}}\,(f|X_{1:k})\geq\underline{\mathrm{E}}\left(\underline{\mathrm{E}}\,(f|X_{1:k+1})\,|X_{1:k}\right)$;

WC14. $\underline{\mathrm{E}}(f|t)\geq 0\Rightarrow\underline{\mathrm{E}}(f\mathbb{1}_t|s)\geq 0$;

WC15. $\underline{\mathrm{E}}(f\mathbb{1}_t|s)>0\Rightarrow\underline{\mathrm{E}}(f|t)>0$.

In particular, all the properties above hold for any global upper expectation $\overline{\mathrm{E}}$ *that satisfies* WC1$_{82}$–WC4$_{82}$ *and that is defined on the domain* $\mathbb{V}\times\mathscr{X}^*$ *or* $\mathbb{F}\times\mathscr{X}^*$.

3.4.2 From local to global upper expectations using coherence

Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$. We now want to use Axioms WC1$_{82}$–WC4$_{82}$ (or coherence), to extend the local upper expectations $\overline{\mathrm{Q}}_s$ to a single global upper expectation. However, since these two types of upper expectations involve completely different domains, it is mathematically not entirely clear what it means for a global upper expectation to 'extend' the local upper expectations $\overline{\mathrm{Q}}_s$. To formalise this, we first transform the local assessments $\overline{\mathrm{Q}}_s$ into equivalent assessments about a preliminary global upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ on a subset of $\mathbb{V}\times\mathscr{X}^*$—similar to how we went from an acceptable gambles tree $\mathscr{A}_\bullet$ to the set $\mathscr{D}_{\mathscr{A}}$; see Eq. (3.6)$_{56}$.

We do this in accordance with the interpretation of the local upper expectations $\overline{\mathrm{Q}}_s$; recall from Section 3.1.2$_{48}$ that $\overline{\mathrm{Q}}_s(f)$ for any $s\in\mathscr{X}^*$ and any $f\in\mathscr{L}(\mathscr{X})$ is interpreted as the (coherent) upper expectation—which can itself be interpreted behaviourally or in terms of probabilities—of $f(X_{|s|+1})$ given that the history of the process is $X_{1:|s|}=s$. The translation to a global level is thus straightforward: we define the global upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ by

$$\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}(f(X_{k+1})|x_{1:k}):=\overline{\mathrm{Q}}_{x_{1:k}}(f)\text{ for all }(f(X_{k+1}),x_{1:k})\in\mathscr{K}_{\mathrm{pre}},\qquad(3.13)$$

with $\mathscr{K}_{\mathrm{pre}}:=\{(f(X_{k+1}),x_{1:k})\colon k\in\mathbb{N}_0,x_{1:k}\in\mathscr{X}^k,f\in\mathscr{L}(\mathscr{X})\}$. Note that this transformation from $\overline{\mathrm{Q}}_\bullet$ to $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ does not add, nor remove information; it is merely a different representation. Hence, we will often speak of $\overline{\mathrm{Q}}_\bullet$ and $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ as being one and the same object. For instance, if we say that a global upper expectation $\overline{\mathrm{E}}$ on $\mathscr{K}\subseteq\mathbb{V}\times\mathscr{X}^*$ extends the tree $\overline{\mathrm{Q}}_\bullet$, then, mathematically speaking, we take this to mean that $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$; that is to say, that the domain of $\overline{\mathrm{E}}$ includes $\mathscr{K}_{\mathrm{pre}}$ and that $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ on $\mathscr{K}_{\mathrm{pre}}$.

The natural extension under coherence

The global upper expectation that we are after is thus required to satisfy, on the one hand, Eq. (3.13)↶, and on the other hand, Axioms WC1$_{82}$–WC4$_{82}$ (or coherence). A question that already comes to mind then, is whether such a global upper expectation always exists. It will soon be shown that the answer is positive, however, let us already think ahead and pose ourselves the question: if there are multiple global upper expectations satisfying these conditions, which one do we choose?

It is a question similar to the one raised in Section 2.6$_{36}$, and our answer follows once more from conservativity considerations. To that end, we start from the interpretation that higher/larger upper expectations are more conservative—or less informative—uncertainty models. This can be justified on the basis of all the global upper expectations that we have previously seen—$\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$—but, actually, it also plainly follows from interpreting a global upper expectation $\overline{\mathrm{E}}$ as either representing infimum selling prices for global gambles—not necessarily those resulting from a tree $\mathscr{A}_{\bullet}$—or representing upper bounds on possible global expectations corresponding to possible global probability charges—not necessarily those resulting from a tree $\mathscr{P}_{\bullet}$. Indeed, under the first interpretation, higher upper expectations mean higher selling prices, which is clearly more conservative; under the second interpretation, higher upper expectations correspond to higher upper bounds on the possible global expectations or probability charges, which is again less informative and hence more conservative. Recall moreover that the notion of coherence—and thus also Axioms WC1$_{82}$–WC4$_{82}$—is also motivated by this same dual interpretation, so it indeed makes sense to combine Axioms WC1$_{82}$–WC4$_{82}$ with the interpretation that larger global upper expectations are more conservative.

Now, given an upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, our model of interest will be the most conservative global upper expectation on $\mathbb{V} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfies WC1$_{82}$–WC4$_{82}$ (or equivalently, that is coherent). In the field of imprecise probabilities, and just as we did in Chapter 2$_{17}$, the most conservative extension of an upper expectation (conditional or not) under some set A of properties is, if it exists, typically called **the natural extension** under A [106, 110]. If A is the requirement of coherence, then the 'under A' is often dropped and it is then simply called the natural extension without further ado. In this chapter, we are thus considering the standard natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, yet later on in Chapter 6$_{283}$ we will also consider natural extensions under more involved conditions.

Definition 3.8. For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the global upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is, if it exists, the natural—the pointwise largest—extension of

$\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ to $\mathbb{V} \times \mathscr{X}^*$ under WC1$_{82}$–WC4$_{82}$. Furthermore, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is, if it exists, the natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ to $\mathbb{F}\times\mathscr{X}^*$ under WC1$_{82}$–WC4$_{82}$. The corresponding lower expectations $\underline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{fin}}$ are defined by conjugacy. ◎

Though we have defined the lower expectations $\underline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{fin}}$ using the conjugacy relation, one could also define them as most conservative—now in the sense of being the lowest—global lower expectations that satisfy axioms similar but complementary to WC1$_{82}$–WC4$_{82}$ (see for instance [106, Theorem 13.11]), and that extend local—conjugate—lower expectations. Both approaches are entirely equivalent, but for the sake of brevity and since we are following an axiomatic approach anyway, we have chosen to go with the former.

The following corollary shows that WC1$_{82}$–WC4$_{82}$ in the definition above can be replaced by the notion of (conditional) coherence.

Corollary 3.4.5. *For any upper expectations tree* $\overline{\mathrm{Q}}_{\bullet}$*,* $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ *is (if it exists) the natural extension of* $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ *to* $\mathbb{V} \times \mathscr{X}^*$ *under coherence, and* $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ *is (if it exists) the natural extension of* $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ *to* $\mathbb{F} \times \mathscr{X}^*$ *under coherence.*

Proof. This follows immediately from Definition 3.8$_{\leftarrow}$ and Theorem 3.4.3$_{84}$. □

Before we establish the existence and the—trivial—uniqueness of these global upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, it still behoves us to clarify why we want to choose, among all the global upper expectations extending $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfying WC1$_{82}$–WC4$_{82}$, the most conservative global upper expectation. Our reason is simple; choosing any other—smaller—global upper expectation would mean adding 'information'—or assumptions—not given by $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ nor by WC1$_{82}$–WC4$_{82}$. We are not necessarily arguing that it is undesirable to impose more than WC1$_{82}$–WC4$_{82}$ though—we ourselves will impose an additional property later on in Chapter 6$_{283}$—but since adding assumptions impacts generality in the negative, it seems logical to start with a study of the global upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$) that is solely based on the minimal requirements WC1$_{82}$–WC4$_{82}$ (or coherence) and nothing more. For if $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ then turns out to be a suitable global upper expectation with desirable features, the better. If not, and one desires to impose additional properties, then $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ will still provide a conservative upper bound.

3.4.3 Existence, uniqueness, and an axiomatisation

Unlike the definitions of $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$[10], the definitions of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ are based on a non-constructive argument; they are simply operators

[10] Note that by Proposition 3.3.4$_{73}$ the transition from a precise probability tree p to the corresponding global probability charge P_p , and thus also the rest of the definition of $\overline{\mathrm{E}}_{\mathscr{P}}$, can be regarded as constructive.

satisfying a bunch of properties. Hence, there is no guarantee yet that these operators exist, nor that they are unique—though the latter is trivial. In the current section, we establish this existence and uniqueness.

One possible way to prove the existence and uniqueness, is by showing that $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ itself is coherent; it can then be derived from well-known results such as [106, Definition 13.25] that the natural extensions $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ under coherence both exist (and are trivially unique). The coherence of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, and the existence and form of the natural extensions $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, could perhaps also be derived from the Marginal Extension Theorem in [64, Theorem 2]. The issue however is that, apart from the fact that using this result would first require us to extend the domain of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ in a basic yet specific way, and require us to work with new concepts such as separate coherence and conditioning on partitions, the result is only valid when we would want to extend a family of conditional upper or lower expectations corresponding to a finite series of partitions of the sample space Ω. The partitions would in our case be formed by the sets $\{\Gamma(s) \colon |s| = n\}$ consisting of all cylinder events of a certain length $n \in \mathbb{N}_0$, yet there are infinitely many such partitions and so [64, Theorem 2] cannot be immediately applied here.

Instead of proving the coherence of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ or using [64, Theorem 2], we opt for the following approach; we present a set of axioms—aimed to be as weak as possible—that is sufficient for being equal to natural extensions $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, and show that there always is a global upper expectation satisfying these axioms. This guarantees the existence and trivial uniqueness, and it also immediately provides an axiomatisation—without conservativity arguments—for the upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. We gather these findings in one result, the proof of which can be found in Appendix 3.D$_{114}$.

Theorem 3.4.6. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ exist. Furthermore, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is the unique global upper expectation on $\mathbb{F} \times \mathscr{X}^*$ satisfying the following axioms (stated for a general global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F} \times \mathscr{X}^*$):*

NE1. $\overline{\mathrm{E}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}_{x_{1:k}}(f)$ *for all* $f \in \mathscr{L}(\mathscr{X})$ *and* $x_{1:k} \in \mathscr{X}^*$.

NE2. $\overline{\mathrm{E}}(f|s) = \overline{\mathrm{E}}(f\mathbb{1}_s|s)$ *for all* $f \in \mathbb{F}$ *and* $s \in \mathscr{X}^*$.

NE3. $\overline{\mathrm{E}}(f|X_{1:k}) = \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k})$ *for all* $f \in \mathbb{F}$ *and* $k \in \mathbb{N}_0$ *such that* $\overline{\mathrm{E}}(f|X_{1:k+1})$ *is real-valued.*

Moreover, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is the unique global upper expectation on $\mathbb{V} \times \mathscr{X}^*$ satisfying NE1–NE3 *and the following axiom (with $\overline{\mathrm{E}}$ any global upper expectation on $\mathbb{V} \times \mathscr{X}^*$):*

NE4. $\overline{\mathrm{E}}(f|s) = \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\}$ *for all* $f \in \mathbb{V}$ *and all* $s \in \mathscr{X}^*$.

The axiomatisation above is rather simple; NE1$_\leftarrow$ demands compatibility with the local models, and NE2$_\leftarrow$ guarantees that the second argument of a global upper expectation plays the role of a conditioning event. NE2$_\leftarrow$ is furthermore the same as WC11$_{85}$, so it is satisfied by any coherent global upper expectation. NE3$_\leftarrow$ says that Property WC13$_{85}$—which is satisfied by any coherent global upper expectation—holds with equality. In other words, NE3$_\leftarrow$ says that the law of iterated upper expectations—recall Section 3.2.4$_{65}$—holds on the domain $\mathbb{F}\times\mathscr{X}^*$. Note that the axiom only applies to those instances where $\overline{\mathrm{E}}(f|X_{1:k+1})$ is real-valued, because then $\overline{\mathrm{E}}(f|X_{1:k+1})$ is a (finitary) gamble and it can therefore be considered as an argument for the upper expectation $\overline{\mathrm{E}}(\cdot|X_{1:k})$. If $\overline{\mathrm{E}}(f|X_{1:k+1})$ would not be real-valued—which is possible for a general global upper expectation $\overline{\mathrm{E}}$—then the expression $\overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k})$ would be meaningless. Of course, since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ are always real-valued due to Proposition 3.4.4 [WC6$_{84}$] and their definitions, NE3$_\leftarrow$ holds for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ in all cases. In fact, though NE3$_\leftarrow$ only involves the domain $\mathbb{F}\times\mathscr{X}^*$, it will be shown in Section 3.5.3$_{93}$ that the law of iterated upper expectations holds for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ on the entire domain $\mathbb{V}\times\mathscr{X}^*$.

Together, Axioms NE1$_\leftarrow$–NE3$_\leftarrow$ uniquely determine the values of a global upper expectation on all finitary gambles; more specifically, it straightforwardly leads to the form of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ stated in Proposition 3.5.9$_{96}$—see also Lemma 3.D.5$_{116}$. Axiom NE4$_\leftarrow$, then, imposes that a global upper expectation's value on a general gamble in $\mathbb{V}$ can be approximated arbitrarily closely from above by its values on the finitary gambles in $\mathbb{F}$.

It can be observed rather straightforwardly from Theorem 3.4.6$_\leftarrow$ above that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is an extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. But, in fact, we can even prove more.

Corollary 3.4.7. *For any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ *is the natural—pointwise largest—extension of* $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ *under monotonicity [*WC5$_{84}$*].*

Proof. That $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ follows from Theorem 3.4.6$_\leftarrow$. That $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, and thus also $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, is monotone [WC5$_{84}$] follows from Proposition 3.4.4$_{84}$ and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ satisfies WC1$_{82}$–WC4$_{82}$ by definition. So it remains to show that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) \geq \overline{\mathrm{E}}(f|s)$ for any $(f,s) \in \mathbb{V}\times\mathscr{X}^*$ and any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{V}\times\mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and that is monotone [WC5$_{84}$]. And indeed, since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ satisfies NE4$_\leftarrow$ by Theorem 3.4.6$_\leftarrow$,

$$\begin{aligned}\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) &= \inf\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(g|s) : g\in\mathbb{F} \text{ and } g \geq_s f\right\}\\ &= \inf\left\{\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g|s) : g\in\mathbb{F} \text{ and } g \geq_s f\right\}\\ &= \inf\left\{\overline{\mathrm{E}}(g|s) : g\in\mathbb{F} \text{ and } g \geq_s f\right\} \geq \overline{\mathrm{E}}(f|s),\end{aligned}$$

where the last step follows from WC5$_{84}$. □

It is well-known that the natural extension (under coherence) is transitive [106, Section 13.7.3] and therefore that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is automatically the natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ under coherence. However, what is striking about Corollary 3.4.7$_{\curvearrowleft}$ is that, as far as this extension from $\mathbb{F}\times\mathscr{X}^*$ to $\mathbb{V}\times\mathscr{X}^*$ is concerned, we can replace coherence by the much weaker property of monotonicity. As we will explain in Section 3.6$_{98}$, the fact that monotonicity is the only property relating the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ on $(\mathbb{V}\setminus\mathbb{F})\times\mathscr{X}^*$ to those of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$) on $\mathbb{F}\times\mathscr{X}^*$ is somewhat problematic when interested in general, non-finitary inferences.

3.5 Relation between the three approaches

The current section is concerned with how the axiomatic global upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ relate to, on the one hand, the behavioural global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, and on the other hand, the probability-based global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$. As it will turn out, these global upper expectations are all equal if, respectively, the tree $\mathscr{A}_\bullet$ agrees with $\overline{\mathrm{Q}}_\bullet$, and the tree $\mathscr{P}_\bullet$ agrees with $\overline{\mathrm{Q}}_\bullet$.

3.5.1 Relation between axiomatic and behavioural global upper expectations

We first show that, for any two trees $\mathscr{A}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ that agree according to Eq. (3.1)$_{50}$, the global set of acceptable gambles $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ and the global upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ also 'agree', in the sense that the infimum selling prices $\overline{\mathrm{E}}_{\mathscr{A}}$ deduced from $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ coincide with the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. One may have noticed, however, when reading through Appendix 3.D$_{114}$, that our proof of Theorem 3.4.6$_{88}$ was fundamentally based on the result that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ (or $\overline{\mathrm{E}}_{\mathscr{A}}$) satisfies NE1$_{88}$–NE4$_{88}$, and therefore that the equality with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ was essentially already proved there.

Theorem 3.5.1. *For any acceptable gambles tree $\mathscr{A}_\bullet$ and upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agree according to Eq.* (3.1)$_{50}$*, we have that*

$$\overline{\mathrm{E}}_{\mathscr{A}}(f|s)=\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)=\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s)\text{ for all } f\in\mathbb{V}\text{ and all } s\in\mathscr{X}^*.$$

Proof. Lemma 3.D.3$_{115}$ says that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies NE1$_{88}$–NE4$_{88}$. Hence, by Theorem 3.4.6$_{88}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. That this also holds for $\overline{\mathrm{E}}_{\mathscr{A}}$ then follows from Theorem 3.2.7$_{65}$. □

3.5.2 Relation with probability-based global upper expectations

Next, we show that for any two agreeing trees $\mathscr{P}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$, the global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ [Definition 3.6$_{79}$] obtained by taking an upper envelope over all the global upper expectations $\overline{\mathrm{E}}_p$ corresponding to a compatible

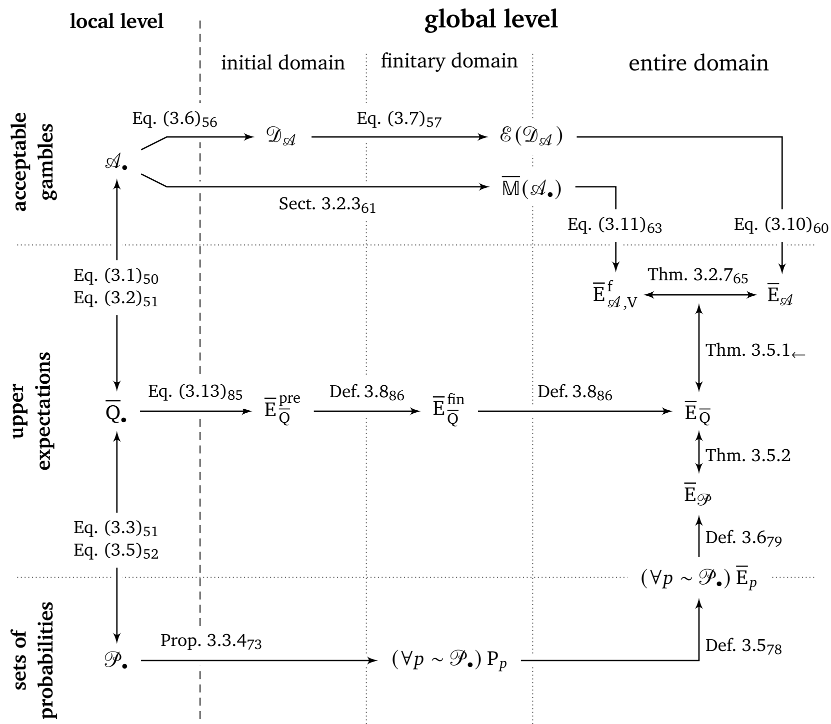

Figure 3.6 Schematic overview of the possible finitary approaches and their connections.

precise probability tree $p \sim \mathscr{P}_\bullet$, is equal to the axiomatic upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. The proof of this result will be given at the end of this Section 3.5.2$_{\leftarrow}$.

Theorem 3.5.2. *Consider any imprecise probability tree* $\mathscr{P}_\bullet$ *and any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$ *that agree according to Eq.* (3.3)$_{51}$. *Then we have that*

$$\overline{\mathrm{E}}_{\mathscr{P}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) \text{ for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

It then also immediately follows from Theorem 3.5.1$_{\leftarrow}$ that $\overline{\mathrm{E}}_{\mathscr{P}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ coincide if $\mathscr{P}_\bullet$ and $\mathscr{A}_\bullet$ agree with a common upper expectations tree $\overline{\mathrm{Q}}_\bullet$ according to Eq. (3.3)$_{51}$ and Eq. (3.1)$_{50}$, respectively. See Fig. 3.6 for a schematic overview of the connections between the global upper expectations $\overline{\mathrm{E}}_{\mathscr{P}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$.

As a special interesting case, Theorem 3.5.2 says that $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ coincide if $\mathscr{P}_\bullet$ consists of only a single precise probability tree p. Note that $\overline{\mathrm{E}}_{\mathscr{P}}$, as defined by Definition 3.6$_{79}$, then simply reduces to $\overline{\mathrm{E}}_p$, given by Definition 3.5$_{78}$. Furthermore, using Proposition 3.3.8$_{79}$ and restricting to the domain to $\mathbb{F} \times \mathscr{X}^*$, it follows that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ can be expressed as a simple finite weighted sum.

Corollary 3.5.3. *Consider any precise probability tree p and let $\mathrm{Q}_\bullet$ be the agreeing expectations tree according to Eq. (3.4)$_{52}$. Then, for any $(f, s) \in \mathbb{F} \times \mathscr{X}^*$, we have that*

$$\overline{\mathrm{E}}_{\mathrm{Q}}(f|s) = \mathrm{E}_p(f|s) = \textstyle\sum_{x_{k+1:\ell} \in \mathscr{X}^{\ell-k}} f(x_{1:\ell}) \prod_{i=k}^{\ell-1} p(x_{i+1}|x_{1:i}),$$

where $\ell > k$ is any natural number such that f is ℓ-measurable.

In order to prove Theorem 3.5.2$_{\curvearrowleft}$, we show that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE1$_{88}$–NE4$_{88}$, and then subsequently use Theorem 3.4.6$_{88}$ to infer that $\overline{\mathrm{E}}_{\mathscr{P}}$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. We start by establishing that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE1$_{88}$–NE3$_{88}$, which can almost immediately be seen to follow from [62, Theorem 21]. In the proof of this result, and also further on in this dissertation, we rely on the following notation (which was also already used in Appendix 3.D$_{114}$). For any situation $x_{1:k} \in \mathscr{X}^*$ and any $(k+1)$-measurable gamble $g(X_{1:k+1})$, let $g(x_{1:k}\cdot)$ be the local gamble on $\mathscr{X}$ that assumes the value $g(x_{1:k+1})$ in $x_{k+1} \in \mathscr{X}$. Then, for any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $k \in \mathbb{N}_0$ and any $(k+1)$-measurable gamble $g(X_{1:k+1})$, we use $\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))$ to denote the k-measurable gamble defined by

$$\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))(x_{1:k}) := \overline{\mathrm{Q}}_{x_{1:k}}(g(x_{1:k}\cdot)) \text{ for all } x_{1:k} \in \mathscr{X}^k.$$

Observe that $\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))$ is indeed bounded and therefore a gamble, because the local upper expectation $\overline{\mathrm{Q}}_{x_{1:k}}(g(x_{1:k}\cdot))$ for all $x_{1:k} \in \mathscr{X}^k$ is real due to coherence [C5$_{33}$].

Proposition 3.5.4. *For any imprecise probability tree $\mathscr{P}_\bullet$ and the agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet$ according to Eq. (3.3)$_{51}$, the global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies* NE1$_{88}$–NE3$_{88}$.

Proof. [62, Theorem 21] says that, for any $k, \ell \in \mathbb{N}_0$ such that $\ell > k$, and any $(\ell+1)$-measurable gamble h,

$$\overline{\mathrm{E}}'_{\mathscr{P}}(h|X_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots\, \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(h)\big)\cdots\big)\big), \tag{3.14}$$

with $\overline{\mathrm{E}}'_{\mathscr{P}}$ defined according to [62, Eq. (3.27)]. The latter is different from our definition of $\overline{\mathrm{E}}_{\mathscr{P}}$ [Definition 3.6$_{79}$] because [62, Eq. (3.27)] involves probability measures—countably additive probability charges—instead of general probability charges. This makes no difference in the expression above though, because the two global upper expectations coincide on finitary gambles; indeed, it can be checked from [62, Eqs. (3.18) and (3.27)] that, for any finitary gamble $f \in \mathbb{F}$ and any $x_{1:k} \in \mathscr{X}^*$,

$$\overline{\mathrm{E}}'_{\mathscr{P}}(f|x_{1:k}) = \sup\big\{\textstyle\sum_{x_{k+1:\ell} \in \mathscr{X}^{\ell-k}} f(x_{1:\ell}) \prod_{i=k}^{\ell-1} p(x_{i+1}|x_{1:i}) \colon p \sim \mathscr{P}_\bullet\big\},$$

where $\ell \geq k$ is any natural number for which f is ℓ-measurable. Hence, according to Corollary 3.3.10$_{80}$, we indeed have that $\overline{\mathrm{E}}'_{\mathscr{P}}(f|x_{1:k}) = \overline{\mathrm{E}}_{\mathscr{P}}(f|x_{1:k})$ for all $(f, x_{1:k}) \in \mathbb{F} \times \mathscr{X}^*$. Then it follows from Eq. (3.14)$_{\leftarrow}$ and Lemma 3.D.5$_{116}$ that $\overline{\mathrm{E}}_{\mathscr{P}}$ coincides [on $\mathbb{F} \times \mathscr{X}^*$] with any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F} \times \mathscr{X}^*$ that satisfies NE1$_{88}$–NE3$_{88}$. Lemma 3.D.3$_{115}$ gaurantees the existence of such a global upper expectation $\overline{\mathrm{E}}$, so we indeed find that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE1$_{88}$–NE3$_{88}$. □

The proof of the fact that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE4$_{88}$ is rather technical and therefore relegated to Appendix 3.E$_{120}$.

Proposition 3.5.5. *For any imprecise probability tree $\mathscr{P}_\bullet$, the global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies* NE4$_{88}$.

Proof of Theorem 3.5.2$_{91}$. Proposition 3.5.4$_{\leftarrow}$ says that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE1$_{88}$–NE3$_{88}$. Proposition 3.5.5 says that $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies NE4$_{88}$. Hence, we infer by Theorem 3.4.6$_{88}$ that $\overline{\mathrm{E}}_{\mathscr{P}}$ is equal to $\overline{\mathrm{E}}_{\mathrm{Q}}$. □

3.5.3 Implications of the equality between the three types of upper expectations

The fact that the three types of global upper expectations—$\overline{\mathrm{E}}_{\mathscr{A}}$ (or $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$), $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\overline{\mathrm{E}}_{\mathrm{Q}}$—are all equal if the respective trees agree has a number of interesting consequences. First of all, it is clear that the three different types of global upper expectations each rely on their own set of methods and ideas to extend local models to global models, and so it is remarkable from a philosophical point of view that, whether one uses gambling, probabilities or axioms as a tool, one always ends up with the same global upper expectation. Moreover, the fact that all these approaches lead to the same global upper expectation significantly broadens the scope of this common upper expectation; for a user may choose whatever framework suits him the best, depending on e.g. his background knowledge or on practical considerations. Finally, there are also a number of important mathematical consequences of which we will now highlight the most important ones.

Coherence properties

Since the different types of global upper expectations are all equal (if the corresponding trees agree), it follows that all these global upper expectations share the same properties. Hence, any of the properties that were previously proved to hold for one type of global upper expectation, can now immediately be established for all other global upper expectations. In particular, since $\overline{\mathrm{E}}_{\mathrm{Q}}$ is coherent and satisfies WC1$_{82}$–WC15$_{85}$, this is also true for $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$. The fact that $\overline{\mathrm{E}}_{\mathscr{A}}$ is coherent (or satisfies WC1$_{82}$–WC4$_{82}$), however, also straightforwardly follows from Lemma 3.C.1$_{109}$ or Lemma 3.D.6$_{117}$

(whose proofs can be seen to follow from standard results such as [113, Proposition 2]). That $\overline{\mathrm{E}}_{\mathscr{P}}$ is coherent could also be shown using a result such as [106, Proposition 13.42], which says that the lower (upper) envelope of a set of coherent conditional lower (upper) expectations is itself a coherent conditional lower (upper) expectation. In order to use this result, however, we would thus first be required to prove that $\overline{\mathrm{E}}_p$ is a coherent global upper expectation for any precise probability tree p.

Corollary 3.5.6. *For any acceptable gambles tree $\mathscr{A}_\bullet$, the global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathbb{V}}$ are coherent and satisfy* WC1$_{82}$–WC15$_{85}$. *The same holds for the upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ corresponding to any imprecise probability tree $\mathscr{P}_\bullet$.*

Proof. The fact that $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathbb{V}}$ satisfy WC1$_{82}$–WC15$_{85}$ follows from Theorem 3.5.1$_{90}$, the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ for any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agrees with $\mathscr{A}_\bullet$ according to Eq. (3.1)$_{50}$ satisfies WC1$_{82}$–WC4$_{82}$ by definition, and Proposition 3.4.4$_{84}$. The coherence of these global upper expectations then moreover follows from Theorem 3.4.3$_{84}$. The statement about $\overline{\mathrm{E}}_{\mathscr{P}}$ follows from Theorem 3.5.2$_{91}$, and again the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ [for any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agrees with $\mathscr{P}_\bullet$ according to Eq. (3.3)$_{51}$], Proposition 3.4.4$_{84}$ and Theorem 3.4.3$_{84}$. □

Note that the corollary above also establishes Proposition 3.4.1$_{82}$ stated earlier on in Section 3.4.1$_{81}$. One can easily check that Proposition 3.4.1$_{82}$ was never used as a tool in any of the proofs so far, and thus that there is no possibility that we have been adopting any type of circular reasoning. The order was chosen in this particular fashion simply because, as mentioned in Section 3.4.1$_{81}$, we regard Proposition 3.4.1$_{82}$ as part of our motivation to impose WC1$_{82}$–WC4$_{82}$ on a global upper expectation.

Law of iterated upper expectations

Another important consequence is that the global upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ are guaranteed to satisfy the law of iterated upper expectations on the entire domain $\mathbb{V} \times \mathscr{X}^*$. It was already shown above that these upper expectations satisfy NE3$_{88}$, and thus (since they are real-valued due to coherence [WC6$_{84}$]) that they satisfy the law of iterated upper expectations on the restricted domain $\mathbb{F} \times \mathscr{X}^*$. The fact that this property can be extended to the entire domain $\mathbb{V} \times \mathscr{X}^*$ follows from Proposition 3.2.10$_{67}$ which says that $\overline{\mathrm{E}}_{\mathscr{A}}$ satisfies the law of iterated upper expectations on $\mathbb{V} \times \mathscr{X}^*$.

Corollary 3.5.7 (Law of iterated upper expectations). *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \mathbb{V}$ and any $k \in \mathbb{N}_0$, we have that*

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|X_{1:k}) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|X_{1:k+1})|X_{1:k}).$$

The same holds for the upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ *corresponding to any imprecise probability tree* $\mathscr{P}_\bullet$*, and for* $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ *if* $f \in \mathbb{F}$.

Proof. The first statement follows from Theorem $3.5.1_{90}$ and Proposition $3.2.10_{67}$. The second statement follows from Theorem $3.5.2_{91}$, Theorem $3.5.1_{90}$ [applied to the upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agrees with $\mathscr{P}_\bullet$ according to Eq. $(3.3)_{51}$] and Proposition $3.2.10_{67}$. The last statement about $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ follows from the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ [Corollary $3.4.7_{89}$]. □

Overdimensioned local models

Perhaps the most remarkable consequence of all is that, even though acceptable gambles trees and imprecise probability trees are both more expressive than upper expectations trees when it comes to parametrising the local dynamics of a stochastic process [Section $3.1.2_{48}$], this additional expressive power vanishes when solely looking at the resulting global upper expectations.

Corollary 3.5.8. *For any two acceptable gambles trees* $\mathscr{A}_\bullet$ *and* $\mathscr{A}'_\bullet$ *with the same agreeing upper expectations tree* $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}} = \overline{\mathrm{Q}}_{\bullet,\mathscr{A}'}$*,*

$$\overline{\mathrm{E}}_{\mathscr{A}}(f|s) = \overline{\mathrm{E}}_{\mathscr{A}'}(f|s) \ \ \textit{and} \ \ \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A}',\mathrm{V}}(f|s) \ \ \textit{for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

Similarly, for any two imprecise probability trees $\mathscr{P}_\bullet$ *and* $\mathscr{P}'_\bullet$ *with the same agreeing upper expectations tree* $\overline{\mathrm{Q}}_{\bullet,\mathscr{P}} = \overline{\mathrm{Q}}_{\bullet,\mathscr{P}'}$*, we have that*

$$\overline{\mathrm{E}}_{\mathscr{P}}(f|s) = \overline{\mathrm{E}}_{\mathscr{P}'}(f|s) \ \ \textit{for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

That acceptable gambles trees and imprecise probability trees are more expressive than upper expectations trees was already discussed in Section $3.1.2_{48}$. More precisely, as mentioned and illustrated in Section 2.5_{33}, any two local sets of acceptable gambles with the same (uniform) closure—so not necessarily with the same border structure—have the same agreeing local upper expectation. Similarly, any two local sets of probability mass functions lead to the same agreeing local upper expectation if their convex closures are equal. According to the corollary above, analogous considerations hold on a global level, when we consider the global upper expectations deduced from either acceptable gambles trees, imprecise probability trees and upper expectations trees. In other words, it does not matter whether we first—already on a local level—transition to the less expressive framework of upper expectations and do all the extensions in this framework, or first remain in one of the more expressive frameworks, do all the extensions here, and only transition in the end to the framework of upper expectations—the resulting global upper expectations will always be the same; see Fig. 3.6_{91}

for a visual representation. This is not trivial because, in general, the border structure of a coherent set of acceptable gambles or a set of probability charges may in fact impact the corresponding **conditional** upper expectations; see e.g. [75, Section 1.6.6]. In fact, later on when we have introduced global upper expectations based on countably additive probability charges and defined on a domain of extended real-valued variables (and situations), we will encounter instances where the boundary structure of the local sets of probabilities $\mathscr{P}_\bullet$ impacts the corresponding global upper expectation greatly; see Section $5.4.2_{245}$.

Direct explicit expressions

The axiomatic characterisations for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ given in Theorem $3.4.6_{88}$ are elegant and universal in nature, but they are on the other hand inconvenient in practice, when we need to compute the actual values of these operators. Moreover, we know by Theorem $3.5.1_{90}$ and Theorem $3.5.2_{91}$ that $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ coincide with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ (and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$) for appropriately chosen local models, and so that the expressions in Eq. $(3.10)_{60}$, Eq. $(3.11)_{63}$ and Definition 3.6_{79} can be used as alternative tools to compute the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ (and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$), yet these expressions are still rather indirect and not the most practical to work with. To address this, we next present explicit expressions for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ that allow us to straightforwardly compute their values, starting from the values of the—initially given—local upper expectations $\overline{\mathrm{Q}}_\bullet$. Of course, by Theorem $3.5.1_{90}$ and Theorem $3.5.2_{91}$, these expressions can also be used to compute the values of $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$.

We start with establishing the expression for the upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on the finitary domain $\mathbb{F} \times \mathscr{X}^*$.

Proposition 3.5.9. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $x_{1:k} \in \mathscr{X}^*$, and any $(\ell+1)$-measurable gamble $f(X_{1:\ell+1}) \in \mathbb{F}$ with $\ell \geq k$,*

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f(X_{1:\ell+1})|x_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f(X_{1:\ell+1}))\big)\cdots\big)\big)(x_{1:k}).$$

Furthermore, the same expression holds for $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$, with $\mathscr{A}_\bullet$ and $\mathscr{P}_\bullet$ any two trees that agree with $\overline{\mathrm{Q}}_\bullet$.

Proof. The expression for $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ follows immediately from Lemma $3.\mathrm{D}.5_{116}$ and Theorem $3.4.6_{88}$. The remaining statement then follows from Theorem $3.5.1_{90}$ and Theorem $3.5.2_{91}$. □

In order to use the expression above to compute the value of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f(X_{1:\ell+1})|x_{1:k})$, it should be read backwards; one should start with the inner term $g_\ell(X_{1:\ell}) := \overline{\mathrm{Q}}_{X_{1:\ell}}(f(X_{1:\ell+1}))$, and compute $g_\ell(x_{1:k}z_{k+1:\ell}) =$

$\overline{\mathrm{Q}}_{x_{1:k}z_{k+1:\ell}}(f(x_{1:k}z_{k+1:\ell}\cdot))$ for all $z_{k+1:\ell} \in \mathscr{X}^{\ell-k}$; subsequently, one should consider $g_{\ell-1}(X_{1:\ell-1}) := \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f(X_{1:\ell+1}))\big)$ and use the previously obtained values of $g_\ell(X_{1:\ell})$ to compute its values in $x_{1:k}z_{k+1:\ell-1}$ for all $z_{k+1:\ell-1} \in \mathscr{X}^{\ell-k-1}$ according to

$$g_{\ell-1}(x_{1:k}z_{k+1:\ell-1}) = \overline{\mathrm{Q}}_{x_{1:k}z_{k+1:\ell-1}}\big(g_\ell(x_{1:k}z_{k+1:\ell-1}\cdot)\big).$$

By repeating this procedure, one eventually arrives at the value of $g_k(x_{1:k}) := \overline{\mathrm{Q}}_{X_{1:k}}\big(\cdots\overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f(X_{1:\ell+1}))\big)\cdots\big)(x_{1:k})$. We refer to [100] for a more detailed explanation of comparable methods for imprecise Markov chains.

Once we have obtained the values of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, those of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ can easily be derived from it by means of the following expression. Note moreover that this expression is similar to the expression of the natural extension under coherence stated in [106, Theorem 13.55].

Proposition 3.5.10. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, and any $(f,s) \in \mathbb{V}\times\mathscr{X}^*$,*

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) = \inf\Big\{\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\Big\}.$$

Furthermore, the equality above remains to hold if we replace $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and/or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ by either $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}_{\mathscr{P}}$, with $\mathscr{A}_\bullet$ and $\mathscr{P}_\bullet$ any two trees that agree with $\overline{\mathrm{Q}}_\bullet$.

Proof. The expression for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ follows immediately from the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ satisfies NE4$_{88}$ by Theorem 3.4.6$_{88}$, and the fact that, also due to Theorem 3.4.6$_{88}$, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. The remaining statement then follows from Theorem 3.5.1$_{90}$ and Theorem 3.5.2$_{91}$, and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F}\times\mathscr{X}^*$ due to Theorem 3.4.6$_{88}$. □

The result above already establishes a fairly direct and simple characterisation for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ as being the infimum value that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ takes on all dominating finitary gambles. However, this way of expressing the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ can be simplified even further: we can restrict ourselves to taking the infimum value of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on a single specific sequence of dominating finitary gambles—instead of on **all** dominating ones. In fact, as is established by the following lemma, it can be seen that this is true for any global upper expectation that is monotone on $\mathbb{F}\times\mathscr{X}^*$ and satisfies NE4$_{88}$.

Lemma 3.5.11. *Consider any global upper expectation $\overline{\mathrm{E}}\colon \mathbb{V}\times\mathscr{X}^* \to \overline{\mathbb{R}}$ that is monotone [*WC5$_{84}$*] on $\mathbb{F}\times\mathscr{X}^*$ and satisfies* NE4$_{88}$*. Then, for any $f \in \mathbb{V}$ and $s \in \mathscr{X}^*$, we have that*

$$\overline{\mathrm{E}}(f|s) = \lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s) = \inf_{n\in\mathbb{N}}\overline{\mathrm{E}}(g_n|s),$$

where $(g_n)_{n\in\mathbb{N}}$ is the decreasing sequence of finitary gambles defined, for all $n \in \mathbb{N}$, by

$$g_n(\omega) = g_n(\omega^n) := \sup_{\tilde{\omega}\in\Gamma(\omega^n)} f(\tilde{\omega}) \text{ for all } \omega \in \Omega.$$

Proof. Note that $\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$ exists and that $\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)=\inf_{n\in\mathbb{N}}\overline{\mathrm{E}}(g_n|s)$, because $(g_n)_{n\in\mathbb{N}}$ is decreasing and $\overline{\mathrm{E}}$ is monotone on $\mathbb{F}\times\mathscr{X}^*$. So it suffices to prove that $\overline{\mathrm{E}}(f|s)=\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$. Due to the definition of $(g_n)_{n\in\mathbb{N}}$, we have that $f\leq g_n$ for all $n\in\mathbb{N}$, and therefore by NE4$_{88}$ that also $\overline{\mathrm{E}}(f|s)\leq\overline{\mathrm{E}}(g_n|s)$ for all $n\in\mathbb{N}$. Hence, we have that $\overline{\mathrm{E}}(f|s)\leq\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$. To see that the converse inequality is also true, observe that, for any real $a>\inf\{\overline{\mathrm{E}}(g|s)\colon g\in\mathbb{F}\text{ and }g\geq_s f\}$, there is a $g'\in\mathbb{F}$ such that $g'\geq_s f$ and $a\geq\overline{\mathrm{E}}(g'|s)$. Since g' is finitary, it is m-measurable for some $m\geq|s|$. Consider any $\omega\in\Gamma(s)$ and note that $\omega^m\sqsupseteq s$ and therefore that $\Gamma(\omega^m)\subseteq\Gamma(s)$. Then, since $g'\geq_s f$, we also have that $g'(\tilde{\omega})\geq f(\tilde{\omega})$ for all $\tilde{\omega}\in\Gamma(\omega^m)\subseteq\Gamma(s)$. But g' is m-measurable, so $g'(\tilde{\omega})=g'(\omega^m)$ is constant for all $\tilde{\omega}\in\Gamma(\omega^m)$. Hence, $g'(\omega^m)\geq f(\tilde{\omega})$ for all $\tilde{\omega}\in\Gamma(\omega^m)$, and therefore

$$g'(\omega^m)\geq\sup_{\tilde{\omega}\in\Gamma(\omega^m)}f(\tilde{\omega})=g_m(\omega^m).$$

This holds for any $\omega\in\Gamma(s)$, so we have that $g'\geq_s g_m$ and therefore, by the monotonicity of $\overline{\mathrm{E}}$ on $\mathbb{F}\times\mathscr{X}^*$, that $\overline{\mathrm{E}}(g'|s)\geq\overline{\mathrm{E}}(g_m|s)$. Since $(g_n)_{n\in\mathbb{N}}$ is decreasing and, again, $\overline{\mathrm{E}}$ is monotone on $\mathbb{F}\times\mathscr{X}^*$, this implies that $\overline{\mathrm{E}}(g'|s)\geq\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$. By the fact that $a\geq\overline{\mathrm{E}}(g'|s)$, we have thus that $a\geq\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$. Since this holds for any real $a>\inf\{\overline{\mathrm{E}}(g|s)\colon g\in\mathbb{F}\text{ and }g\geq_s f\}$, and since $\overline{\mathrm{E}}(f|s)=\inf\{\overline{\mathrm{E}}(g|s)\colon g\in\mathbb{F}\text{ and }g\geq_s f\}$ by NE4$_{88}$, we infer that $\overline{\mathrm{E}}(f|s)\geq\lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s)$. □

Corollary 3.5.12. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f\in\mathbb{V}$ and $s\in\mathscr{X}^*$, we have that*

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s)=\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s)=\inf_{n\in\mathbb{N}}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s),$$

where $(g_n)_{n\in\mathbb{N}}$ is the decreasing sequence of finitary gambles defined, for all $n\in\mathbb{N}$, by

$$g_n(\omega)=g_n(\omega^n):=\sup_{\tilde{\omega}\in\Gamma(\omega^n)}f(\tilde{\omega})\text{ for all }\omega\in\Omega.$$

Furthermore, the statement above remains to hold if we replace $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and/or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ by either $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}_{\mathscr{P}}$, with $\mathscr{A}_\bullet$ and $\mathscr{P}_\bullet$ any two trees that agree with $\overline{\mathrm{Q}}_\bullet$.

Proof. The statement for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ follows immediately from Lemma 3.5.11$_\frown$, the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ satisfies WC5$_{84}$ by Proposition 3.4.4$_{84}$ [and because it satisfies WC1$_{82}$–WC4$_{82}$ by definition], the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ satisfies NE4$_{88}$ by Theorem 3.4.6$_{88}$, and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ by Corollary 3.4.7$_{89}$. The remaining statement then follows from Theorem 3.5.1$_{90}$ and Theorem 3.5.2$_{91}$, and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F}\times\mathscr{X}^*$ due to Theorem 3.4.6$_{88}$. □

3.6 Finitary global upper expectations are not enough

From all we know so far—or rather, all what has been told so far—it seems that any of the global upper expectations discussed above, whether that is $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}_{\mathscr{P}}$ or $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, seems to have all features one could possible desire; due to its equivalence with other models, it can interpreted and motivated in various ways; it has an abundance of convenient mathematical properties, including WC1$_{82}$–WC15$_{85}$ and the law of iterated upper expectations; and its values can in practice be straightforwardly computed using the explicit expressions in Lemma 3.D.5$_{116}$ and Lemma 3.5.11$_{97}$. Unfortunately however, as the title of this section predicts, there is a price to pay—a price that we are not willing to accept. To get a clue of where the deficit lies, let us go back to Corollary 3.4.7$_{89}$ and the discussion below it. There, it was pointed out that the extension of the global upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$—which we can now also equivalently regard as $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}_{\mathscr{P}}$—from the domain $\mathbb{F} \times \mathscr{X}^*$ to $\mathbb{V} \times \mathscr{X}^*$ did not entail much; this extension is solely based on evoking the monotonicity property. It is therefore to be expected that this extension will sometimes lead to rather uninformative conclusions about the global upper expectation of some specific global non-finitary gambles. The following simple example confirms our suspicion.

The example is expressed in terms of a(n) (im)precise probability tree p and the corresponding global upper expectation $\overline{\mathrm{E}}_p$, however, one could just as well repeat the same reasoning with an agreeing upper expectations tree or an agreeing acceptable gambles tree, and their respective global upper expectations. The example will also involve the upper probability $\overline{\mathrm{P}}_p$ corresponding to the upper expectation $\overline{\mathrm{E}}_p$; recall from Section 3.1.3$_{52}$ that $\overline{\mathrm{P}}_p$ is simply obtained from $\overline{\mathrm{E}}_p$ by restricting to the indicators.

Example 3.6.1. Consider a state space $\mathscr{X} := \{a, b\}$ consisting of two elements, and a precise probability tree p consisting of probability mass functions that put all mass on a; so $p(a|s) = 1$ and $p(b|s) = 0$ for all $s \in \mathscr{X}^*$. Now consider the event H_b of ever hitting the state b; so $H_b := \Omega \setminus \{aaa\cdots\}$. Since, at each time instant $k \in \mathbb{N}_0$ and for any possible history $z_{1:k} \in \mathscr{X}^k$, the local probability mass function $p(\cdot|z_{1:k})$ assigns probability 1 to a, we also expect that probability 1 is assigned to the event that the first k states are all equal to a, with $k \in \mathbb{N}_0$ any arbitrary time instant—and this in fact follows from Proposition 3.3.4$_{73}$. By idealisation, we would thus expect that probability 1 is assigned to the path $\omega = aaa\cdots$, and therefore that probability 0 is assigned to H_b. But this is not what happens.

Indeed, recall Lemma 3.5.11$_{97}$—which holds for $\overline{\mathrm{E}}_p$ because of Proposition 3.5.5$_{93}$ and Corollary 3.5.6$_{94}$—which says that, for the sequence $(g_n)_{n\in\mathbb{N}}$ defined by $g_n(\omega) := \sup_{\tilde{\omega}\in\Gamma(\omega^n)} \mathbb{I}_{H_b}(\tilde{\omega})$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$,

$$\overline{\mathrm{P}}_p(H_b) = \overline{\mathrm{E}}_p(\mathbb{I}_{H_b}) = \lim_{n\to+\infty} \overline{\mathrm{E}}_p(g_n).$$

Since every cylinder event $\Gamma(s)$ includes a path for which b appears at least one time, and thus a path that is in H_b, we obtain that $g_n(\omega) = \sup_{\tilde{\omega}\in\Gamma(\omega^n)} \mathbb{I}_{H_b}(\tilde{\omega}) = 1$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}$. So by WC6$_{84}$ [which we can apply due to Corollary 3.5.6₉₄], we have that $\overline{\mathrm{E}}_p(g_n) = 1$ for all $n \in \mathbb{N}$. Hence, by the equality above, we find that $\overline{\mathrm{P}}_p(H_b) = 1$; the upper probability of H_b is 1.

The fact that this outcome $\overline{\mathrm{P}}_p(H_b) = 1$ is not in line with what we would desire, is only strengthened if we additionally take into account the values of the 'stopped' upper hitting probabilities. Indeed, let H_b^k for any $k \in \mathbb{N}_0$ be the event of hitting b before time $k+1$;

$$H_b^k := \Omega \setminus \Gamma(a^k) = \cup_{z_{1:k} \in \mathscr{X}^k \setminus \{a^k\}} \Gamma(z_{1:k}), \tag{3.15}$$

where a^k simply denotes the situation consisting of k times a. One may check that $H_b^k = \cup_{\ell=0}^{k-1} \Gamma(a^\ell b)$ and $H_b = \cup_{\ell \in \mathbb{N}_0} \Gamma(a^\ell b)$, and therefore that $\lim_{k \to +\infty} H_b^k = H_b$. On the other hand, for any $k \in \mathbb{N}_0$, we have by Proposition 3.3.8₇₉ and Eq. (3.15) that

$$\overline{\mathrm{P}}_p(H_b^k) = \overline{\mathrm{E}}_p(\mathbb{I}_{H_b^k}) = \sum_{z_{1:k} \in \mathscr{X}^k \setminus \{a^k\}} \mathrm{P}_p(z_{1:k}) = \sum_{z_{1:k} \in \mathscr{X}^k \setminus \{a^k\}} \prod_{i=0}^{k-1} p(z_{i+1} | z_{1:i}) = 0,$$

where the last equality follows from the fact that, for any $z_{1:k} \in \mathscr{X}^k$ such that $z_{i+1} = b$ for some $i \in \{0, \ldots, k-1\}$, we have that $p(z_{i+1}|z_{1:i}) = p(b|z_{1:i}) = 0$ by assumption. As a result, we obtain that

$$\lim_{k \to +\infty} \overline{\mathrm{P}}_p(H_b^k) = 0 \neq 1 = \overline{\mathrm{P}}_p(H_b).$$

In summary, we thus have that the upper probability $\overline{\mathrm{P}}_p(H_b^k)$ of hitting b **before** time $k+1$ is equal to zero **for all** $k \in \mathbb{N}_0$, but that the upper probability $\overline{\mathrm{P}}_p(H_b)$ of hitting b over an infinite time interval is one. ◇

In the example above, we see that the global upper expectation $\overline{\mathrm{E}}_p$ takes values on the finitary gambles $\mathbb{I}_{H_b^k}$ that we would expect. For the non-finitary gamble $\mathbb{I}_{H_b}$, however, this is not the case; its resulting value is extremely conservative—even vacuous—and seems to disregard any information given by the local models. Unfortunately, this phenomenon is not unique to the specific example above, nor to the probability-based global upper expectation $\overline{\mathrm{E}}_p$; as already mentioned under Corollary 3.4.7₈₉, and as can also be seen from Corollary 3.5.12₉₈, the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$—and thus by Theorem 3.5.2₉₁ and Theorem 3.5.1₉₀ the values of $\overline{\mathrm{E}}_{\mathscr{P}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—on non-finitary gambles are derived from the values of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, $\overline{\mathrm{E}}_{\mathscr{P}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$) on $\mathbb{F} \times \mathscr{X}^*$ by only relying on monotonicity; a basic property that is often not powerful enough to make informative statements about non-finitary gambles. This is our first concern with the use of $\overline{\mathrm{E}}_{\mathscr{P}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ as a global model.

Secondly, and equally as important: the domain of these global upper expectations. They are only defined for (global) gambles, so real-valued functions that are bounded. However, many of the inferences encountered in practice require the computation of (upper and lower) expectations of functions that are not bounded, and often not even real-valued. For one, the hitting time τ_a of a state $a \in \mathscr{X} = \{a, b\}$ [see Section 3.1.3$_{52}$] takes the value $+\infty$ in the path $\omega = bbb\cdots$.

The reason that we have chosen, for now, to only consider global gambles (and situations) as a domain for a global upper expectation, is largely because the frameworks of sets of acceptable gambles and coherent upper/lower expectations as initially developed by P. M. Williams [113] and Walley [110] did not involve extended real-valued functions. For they were built on the idea that gambles—bounded real-valued functions—can be interpreted as uncertain pay-offs, and that upper expectations can be interpreted as infimum selling prices for such gambles. Such an interpretation of course becomes somewhat less obvious if uncertain pay-offs can be infinite in value. The problem of extending this theory beyond the domain of bounded real-valued functions was already addressed by Troffaes & De Cooman [106, Part Two], yet still only to deal with unbounded real-valued functions and not extended real-valued functions. On the other hand, most references [77, 106] on finitely additive probability charges that we are aware of, also only involve integration over real-valued functions. All together, it thus seemed as a natural choice to first consider and study the domain of gambles and situations.

In the coming chapters, it is our aim to put forward other global models that deal with these two issues each in their own distinct way. Once more, we shall consider three different types: a (non-finitary) game-theoretic model, a measure-theoretic model, and an axiomatic model. Philosophically speaking, they can be seen as continuations of, respectively, the finitary game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, the probability-based upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$, and the axiomatic upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ presented in the current chapter. The main difference is that the new models will not extend the local models solely using finitary principles, but also using one or more continuity arguments. To clarify this distinction, we will often refer—and already have been referring—to $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}_{\mathscr{P}}$, and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ all together as the **finitary** global upper expectations.

As a final remark before we move on, note that the finitary global upper expectations introduced in this chapter actually behave in a satisfactory way if we are solely interested in the finitary domain $\mathbb{F} \times \mathscr{X}^*$. Indeed, the undesirable behaviour that was illustrated by Example 3.6.1$_{99}$ only occurs if we look at non-finitary gambles. This observation will be used in Chapter 6$_{283}$, where we will argue for the use of an axiomatic continuity-based global upper expectation, and where $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ will serve as our starting point.

— APPENDICES —

3.A Proof of Theorem 3.2.7

The proof of Theorem 3.2.7$_{65}$ relies on the following lemma, which uses the notation $\mathbb{S}_n(\mathscr{A}_\bullet)$ for any $n \in \mathbb{N}_0$ to denote the set of all **sub**martingales stopped at some time k larger than n, and with zero initial value:

$$\mathbb{S}_n(\mathscr{A}_\bullet) := \left\{\mathscr{M}(X_{1:k}) \colon \mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet), \mathscr{M}(\square) = 0 \text{ and } k \geq n\right\}.$$

Lemma 3.A.1. posi $\left(\mathscr{D}_{\mathscr{A}}\right) = \mathbb{S}_n(\mathscr{A}_\bullet)$ *for any acceptable gambles tree* $\mathscr{A}_\bullet$ *and any* $n \in \mathbb{N}_0$.

Proof. To prove that posi $\left(\mathscr{D}_{\mathscr{A}}\right) \subseteq \mathbb{S}_n(\mathscr{A}_\bullet)$, fix any $f \in$ posi $\left(\mathscr{D}_{\mathscr{A}}\right)$. Then, due to Lemma 3.2.1$_{57}$, there is a finite set $S \subset \mathscr{X}^*$ of situations and corresponding gambles $f_s \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$, such that $f = \sum_{s \in S} f_s(X_{|s|+1})\mathbb{1}_s$. Let $\Delta\mathscr{M}$ be the betting process defined, for all $s \in \mathscr{X}^*$, by $\Delta\mathscr{M}(s) = f_s$ if $s \in S$, and $\Delta\mathscr{M}(s) = 0$ otherwise. Then since $f_s \in \mathscr{A}_s$ for all $s \in S$, and $0 \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$ because $\mathscr{A}_s$ satisfies D1$_{27}$, we have that $\Delta\mathscr{M}$ is acceptable and thus that the corresponding real process $\mathscr{M}$ [that starts with initial value $\mathscr{M}(\square) := 0$] is a submartingale in $\underline{\mathbb{M}}(\mathscr{A}_\bullet)$. Let $k \in \mathbb{N}_0$ be such that $k \geq n$ and such that $k-1$ is larger than or equal to the maximum of the lengths of the situations in S [which is a natural number because S is finite]. Then, to show that $\mathscr{M}(X_{1:k}) = f$, consider any $\omega \in \Omega$ and note that

$$\begin{aligned}\mathscr{M}(X_{1:k})(\omega) = \mathscr{M}(\omega_{1:k}) &= \sum_{\ell=0}^{k-1} \Delta\mathscr{M}(\omega_{1:\ell})(\omega_{\ell+1}) \\ &= \sum_{\substack{\ell \in \{0,\ldots,k-1\} \\ \omega_{1:\ell} \in S}} \Delta\mathscr{M}(\omega_{1:\ell})(\omega_{\ell+1}) + \sum_{\substack{\ell \in \{0,\ldots,k-1\} \\ \omega_{1:\ell} \notin S}} \Delta\mathscr{M}(\omega_{1:\ell})(\omega_{\ell+1}) \\ &= \sum_{\substack{\ell \in \{0,\ldots,k-1\} \\ \omega_{1:\ell} \in S}} \Delta\mathscr{M}(\omega_{1:\ell})(\omega_{\ell+1}) = \sum_{\substack{\ell \in \{0,\ldots,k-1\} \\ \omega_{1:\ell} \in S}} f_{\omega_{1:\ell}}(\omega_{\ell+1}) \end{aligned} \tag{3.16}$$

where the penultimate and last step follow from our definition of $\Delta\mathscr{M}$. The last term involves a sum over the situations $\{\omega_{1:\ell} \in S \colon \ell \in \{0,\ldots,k-1\}\}$, but note that, since $k-1$ is larger than or equal to the largest possible length of a situation in S, we could equivalently write it as a sum over

$$\{\omega_{1:\ell} \in S \colon \ell \in \mathbb{N}_0\} = \{s \in S \colon (\exists \ell \in \mathbb{N}_0)\, s = \omega_{1:\ell}\} = \{s \in S \colon \omega \in \Gamma(s)\}.$$

So we get that

$$\sum_{\substack{\ell \in \{0,\ldots,k-1\} \\ \omega_{1:\ell} \in S}} f_{\omega_{1:\ell}}(\omega_{\ell+1}) = \sum_{\substack{s \in S \\ \omega \in \Gamma(s)}} f_s(\omega_{|s|+1}) = \sum_{s \in S} f_s(\omega_{|s|+1})\mathbb{1}_s(\omega) = f(\omega).$$

Combined with Eq. (3.16), this gives us that $\mathscr{M}(X_{1:k})(\omega) = f(\omega)$. Since this holds for any $\omega \in \Omega$, we obtain that $\mathscr{M}(X_{1:k}) = f$ as desired. Together with the fact that

$\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, $\mathscr{M}(\square) = 0$ and $k \geq n$, we find that $f \in \mathbb{S}_n(\mathscr{A}_\bullet)$. Since this is true for any $f \in \text{posi}\left(\mathscr{D}_\mathscr{A}\right)$, it follows that $\text{posi}\left(\mathscr{D}_\mathscr{A}\right) \subseteq \mathbb{S}_n(\mathscr{A}_\bullet)$.

The converse inclusion can be proved in a similar but easier fashion. Fix any $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(\square) = 0$, and any $k \geq n$. Let $S := \{s \in \mathscr{X}^*: |s| \leq k-1\}$ and let $f_s := \Delta\mathscr{M}(s)$ for all $s \in S$. Then S is a finite set of situations because $\mathscr{X}$ is finite. Moreover, $f_s \in \mathscr{A}_s$ for all $s \in S$, because $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$ and therefore $\Delta\mathscr{M}(s) \in \mathscr{A}_s$. So it suffices to prove that $\mathscr{M}(X_{1:k}) = \sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s$, because it then follows from Lemma 3.2.157 that $\mathscr{M}(X_{1:k}) \in \text{posi}\left(\mathscr{D}_\mathscr{A}\right)$ as desired. To that end, observe that

$$\mathscr{M}(x_{1:k}) = \mathscr{M}(\square) + \sum_{\ell=0}^{k-1} \Delta\mathscr{M}(x_{1:\ell})(x_{\ell+1}) = \sum_{\ell=0}^{k-1} \Delta\mathscr{M}(x_{1:\ell})(x_{\ell+1}) \text{ for all } x_{1:k} \in \mathscr{X}^k,$$

because $\mathscr{M}(\square) = 0$ by assumption. Hence, since the above holds for all $x_{1:k} \in \mathscr{X}^k$,

$$\begin{aligned}\mathscr{M}(X_{1:k}) = \sum_{\ell=0}^{k-1} \sum_{x_{1:\ell}\in\mathscr{X}^\ell} \Delta\mathscr{M}(x_{1:\ell})(X_{\ell+1})\mathbb{1}_{x_{1:\ell}}(X_{1:\ell}) &= \sum_{s\in S} \Delta\mathscr{M}(s)(X_{|s|+1})\mathbb{1}_s(X_{1:|s|}) \\ &= \sum_{s\in S} f_s(X_{|s|+1})\mathbb{1}_s,\end{aligned}$$

where the second equality follows from the definition of S, and the last from the definition of the gambles f_s. □

*Proof of Theorem 3.2.7*65. Fix any $f \in \mathbb{V}$ and $s \in \mathscr{X}^*$. First note that, if we combine Lemmas 3.2.157 and 3.2.257, we clearly get that $\mathscr{E}(\mathscr{D}_\mathscr{A}) = \mathbb{V}_\geq + \text{posi}\left(\mathscr{D}_\mathscr{A}\right)$, which on its turn implies by Lemma 3.A.1← that $\mathscr{E}(\mathscr{D}_\mathscr{A}) = \mathbb{V}_\geq + \mathbb{S}_n(\mathscr{A}_\bullet)$ for all $n \in \mathbb{N}_0$.[11] As a result, by Eq. (3.10)60, for all $n \in \mathbb{N}_0$,

$$\overline{\mathrm{E}}_\mathscr{A}(f|s) = \inf\{\alpha \in \mathbb{R}: (\alpha - f)\mathbb{1}_s \in \mathscr{E}(\mathscr{D}_\mathscr{A})\} = \inf\{\alpha \in \mathbb{R}: (\alpha - f)\mathbb{1}_s \in (\mathbb{V}_\geq + \mathbb{S}_n(\mathscr{A}_\bullet))\}. \tag{3.17}$$

Let us now establish that

$$\overline{\mathrm{E}}_\mathscr{A}(f|s) \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s). \tag{3.18}$$

To that end, fix any $n \geq |s|$ and any $\alpha \in \mathbb{R}$ such that $(\alpha - f)\mathbb{1}_s \in (\mathbb{V}_\geq + \mathbb{S}_n(\mathscr{A}_\bullet))$. Then, by the definition of $\mathbb{S}_n(\mathscr{A}_\bullet)$, there is some $g \in \mathbb{V}_\geq$, some $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$ with $\mathscr{M}(\square) = 0$ and some $k \geq n$, such that

$$(\alpha - f)\mathbb{1}_s = g + \mathscr{M}(X_{1:k}). \tag{3.19}$$

Then we certainly have that $\alpha - f =_s g + \mathscr{M}(X_{1:k})$, which by the fact that $g \in \mathbb{V}_\geq$ in turn implies that $\alpha - f \geq_s \mathscr{M}(X_{1:k})$, and therefore that

$$\alpha - \mathscr{M}(X_{1:k}) \geq_s f. \tag{3.20}$$

Next, let $\mathscr{M}'$ be the real process defined by $\mathscr{M}'(s) := \alpha - \mathscr{M}(s)$ for all $s \in \mathscr{X}^*$. Then, for any $s \in \mathscr{X}^*$, we have that $-\Delta\mathscr{M}'(s) = \Delta\mathscr{M}(s)$ and so, since $\mathscr{M} \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, that $-\Delta\mathscr{M}'(s) \in \mathscr{A}_s$. Hence, $\mathscr{M}'$ is a supermartingale in $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$. Moreover, by

[11] The sum $\mathscr{K}_1 + \mathscr{K}_2$ between two sets $\mathscr{K}_1, \mathscr{K}_2 \subseteq \mathbb{V}$ is defined as usually, as the set $\{f + g: f \in \mathscr{K}_1, f \in \mathscr{K}_2\}$.

Eq. (3.20)$_{\frown}$, we have that $\mathscr{M}'(X_{1:k}) \geq_s f$. Let us next show that $\mathscr{M}(\tilde{s}) \geq 0$ for all $\tilde{s} \sqsubseteq s$.

Suppose **ex absurdo** that $\mathscr{M}(\tilde{s}) < 0$ for some $\tilde{s} \sqsubseteq s$. Since we know that $\mathscr{M}(\square) = 0$, we infer that there must be some $t \in \mathscr{X}^*$ and $x \in \mathscr{X}$ such that $tx \sqsubseteq \tilde{s}$ for which $\mathscr{M}(t) \geq 0$ and $\mathscr{M}(tx) < 0$. Then we have that $\Delta\mathscr{M}(t)(x) < 0$. This implies by the coherence [D2$_{27}$] of $\mathscr{A}_t$ that there is some $y \in \mathscr{X} \setminus \{x\}$ such that $\Delta\mathscr{M}(t)(y) > 0$, and so, since $\mathscr{M}(t) \geq 0$, that $\mathscr{M}(ty) > 0$. By Lemma 3.2.5$_{63}$, this implies that, for all $\ell \geq |ty|$, there is some $x_{|ty|+1:\ell} \in \mathscr{X}^{\ell-|ty|}$ such that $\mathscr{M}(tyx_{|ty|+1:\ell}) > 0$. Hence, since $k \geq n \geq |s| \geq |\tilde{s}| \geq |tx| = |ty|$, there is some path $\omega \in \Gamma(ty)$ such that $\mathscr{M}(\omega^k) > 0$. Furthermore, recall that $g \in \mathbb{V}_{\geq}$, so we also have that $\mathscr{M}(\omega^k) + g(\omega) > 0$. Due to Eq. (3.19)$_{\frown}$, this in turn implies that $(\alpha - f(\omega))\mathbb{1}_s(\omega) > 0$. In particular, this implies that $\omega \in \Gamma(s)$. Recalling that $tx \sqsubseteq \tilde{s} \sqsubseteq s$, this would imply that $\omega \in \Gamma(tx)$, yet this is in contradiction with the earlier assumption that $\omega \in \Gamma(ty)$ [and the fact that $y \neq x$]. Hence, we must indeed have that $\mathscr{M}(\tilde{s}) \geq 0$ for all $\tilde{s} \sqsubseteq s$.

The consideration above implies that $\mathscr{M}'(s) = \alpha - \mathscr{M}(s) \leq \alpha$. Combining this with the earlier considerations that $\mathscr{M}' \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$, that $\mathscr{M}'(X_{1:k}) \geq_s f$ and that $k \geq n \geq |s|$, we find by Eq. (3.11)$_{63}$ that

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}'(s) \leq \alpha.$$

Since this holds for all $\alpha \in \mathbb{R}$ such that $(\alpha - f)\mathbb{1}_s \in (\mathbb{V}_{\geq} + \mathbb{S}_n(\mathscr{A}_\bullet))$, we infer by Eq. (3.17)$_{\frown}$ that Eq. (3.18)$_{\frown}$ holds.

To prove the converse inequality—that $\overline{\mathrm{E}}_{\mathscr{A}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$—fix any $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:k}) \geq_s f$ with $k \geq |s|$. Let $\mathscr{M}'$ be the real process defined by

$$\mathscr{M}'(t) := \begin{cases} \mathscr{M}(s) - \mathscr{M}(t) \text{ for all } t \sqsupseteq s \\ \mathscr{M}(s) - \mathscr{M}(s) = 0 \text{ for all } t \not\sqsupseteq s. \end{cases}$$

Then note that $\Delta\mathscr{M}'(t) = -\Delta\mathscr{M}(t)$ for all $t \sqsupseteq s$, and $\Delta\mathscr{M}'(t) = 0$ otherwise. Hence, since $-\Delta\mathscr{M}(t) \in \mathscr{A}_t$ for all $t \in \mathscr{X}^*$, and since $0 \in \mathscr{A}_t$ [because $\mathscr{A}_t$ satisfies D1$_{27}$] for all $t \in \mathscr{X}^*$, we have that $\mathscr{M}' \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$. Moreover, by the definition of $\mathscr{M}'$, and since $k \geq |s|$, we also have that $\mathscr{M}'(X_{1:k}) =_s \mathscr{M}(s) - \mathscr{M}(X_{1:k})$. So, by our assumptions about $\mathscr{M}$, we infer that

$$\mathscr{M}'(X_{1:k}) =_s \mathscr{M}(s) - \mathscr{M}(X_{1:k}) \leq_s \mathscr{M}(s) - f.$$

This means that $\mathscr{M}'(X_{1:k})\mathbb{1}_s \leq (\mathscr{M}(s) - f)\mathbb{1}_s$. Note that, since $k \geq |s|$ and due to the definition of $\mathscr{M}'$, the gamble $\mathscr{M}'(X_{1:k})$ is zero outside $\Gamma(s)$. Hence, the latter inequality can be simplified to $\mathscr{M}'(X_{1:k}) \leq (\mathscr{M}(s) - f)\mathbb{1}_s$. So the variable $g := (\mathscr{M}(s) - f)\mathbb{1}_s - \mathscr{M}'(X_{1:k})$ is non-negative, and also bounded because f is bounded and $\mathscr{M}'(X_{1:k})$ is bounded [because $\mathscr{X}$ is finite]. As a consequence, we have that $g \in \mathbb{V}_{\geq}$ and that

$$g + \mathscr{M}'(X_{1:k}) = (\mathscr{M}(s) - f)\mathbb{1}_s.$$

Since $\mathscr{M}'(\square) = 0$ [by the definition of $\mathscr{M}'$], and since $\mathscr{M}' \in \underline{\mathbb{M}}(\mathscr{A}_\bullet)$, we have that, with $n = k$,

$$(\mathscr{M}(s) - f)\mathbb{1}_s \in (\mathbb{V}_{\geq} + \mathbb{S}_n(\mathscr{A}_\bullet)).$$

This in turn implies that

$$\inf\{\alpha \in \mathbb{R} : (\alpha - f)\mathbb{1}_s \in (\mathbb{V}_{\geq} + \mathbb{S}_n(\mathscr{A}_{\bullet}))\} \leq \mathscr{M}(s). \tag{3.21}$$

Hence, it follows from Eq. $(3.17)_{103}$ that $\overline{\mathrm{E}}_{\mathscr{A}}(f|s) \leq \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_{\bullet})$ such that $\mathscr{M}(X_{1:k}) \geq_s f$ with $k \geq |s|$, we infer from Eq. $(3.11)_{63}$ that $\overline{\mathrm{E}}_{\mathscr{A}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$ as desired. □

3.B Proofs of the results in Section 3.3

Proof of Proposition $3.3.1_{71}$. We show that $(\mathrm{i})_{71} \Rightarrow (\mathrm{iii})_{71} \Rightarrow (\mathrm{ii})_{71} \Rightarrow (\mathrm{i})_{71}$. So let us start by proving that $(\mathrm{iii})_{71}$ holds for any global probability charge P on $\mathcal{A} \times \mathscr{X}^*$. In order to do so we will associate a global (upper) expectation E_{P} with P, and show that this expectation E_{P} satisfies $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$ on its domain. This will then imply by Theorem $3.4.3_{84}$ that E_{P} is coherent according to Definition 3.7_{82}, and $(\mathrm{iii})_{71}$ will then straightforwardly follow by restricting the domain of this expectation E_{P}. Furthermore, we want to point out that the proof of Theorem $3.4.3_{84}$ is independent of the current result [Proposition $3.3.1_{71}$] or any other results in Section 3.3_{69}. This guarantees that there can be no misunderstanding about whether we have adopted any type of circular reasoning.

First note that, for any $s \in \mathscr{X}^*$, since the functional $\mathrm{P}(\cdot|s) : \mathcal{A} \to \mathbb{R}$ satisfies $\mathrm{GP1}_{70}$–$\mathrm{GP3}_{70}$ and, by Proposition $3.3.2_{71}$, $\mathrm{GP5}_{71}$, it is a probability charge according to [106, Definition 1.15]. Hence, according to [106, Definition 8.13], we can define $\mathrm{E}_{\mathrm{P}}(\cdot|s) : \mathrm{span}(\mathcal{A}) \to \mathbb{R}$ by

$$\mathrm{E}_{\mathrm{P}}(f|s) := \sum_{i=1}^{n} a_i \mathrm{P}(A_i|s) \text{ for any } f \in \mathrm{span}(\mathcal{A}),$$

with $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ any representation of f; so $n \in \mathbb{N}$, $a_1, \dots, a_n \in \mathbb{R}$ and $A_1, \dots, A_n \in \mathcal{A}$ are such that $f = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$. Let us check that E_{P} satisfies $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$.

To see that $\mathrm{WC1}_{82}$ holds, start by observing that, since $\mathrm{E}_{\mathrm{P}}(\cdot|s)$ for any $s \in \mathscr{X}^*$ is defined according to [106, Definition 8.13], [106, Theorem 8.15] says that $\mathrm{E}_{\mathrm{P}}(\cdot|s)$ is a 'linear prevision' on $\mathrm{span}(\mathcal{A})$, and thus by [106, Corollary 4.14(i)] that $\mathrm{E}_{\mathrm{P}}(g|s) \leq \sup g$ for all $g \in \mathrm{span}(\mathcal{A})$. Now fix any $f \in \mathrm{span}(\mathcal{A})$ and let $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ be any representation of f. Then we have that

$$\begin{aligned}\mathrm{E}_{\mathrm{P}}(f|s) = \sum_{i=1}^{n} a_i \mathrm{P}(A_i|s) = \sum_{i=1}^{n} a_i \mathrm{P}(A_i \cap \Gamma(s)|s) &= \mathrm{E}_{\mathrm{P}}\Big(\sum_{i=1}^{n} a_i \mathbb{I}_{A_i \cap \Gamma(s)} \Big| s\Big) \\ &= \mathrm{E}_{\mathrm{P}}\Big(\sum_{i=1}^{n} a_i \mathbb{I}_{A_i} \mathbb{1}_s \Big| s\Big) = \mathrm{E}_{\mathrm{P}}(f\mathbb{1}_s|s),\end{aligned}$$

where the second equality follows from Proposition $3.3.2_{71}$[$\mathrm{GP8}_{71}$]. In a similar way, since $g := f\mathbb{1}_s + \sup(f|s)\mathbb{I}_{\Gamma(s)^c}$ is in $\mathrm{span}(\mathcal{A})$ [because $\Gamma(s)$ and $\Gamma(s)^c$ are in $\mathcal{A} \supseteq \langle \mathscr{X}^* \rangle$], we also have that $\mathrm{E}_{\mathrm{P}}(g|s) = \mathrm{E}_{\mathrm{P}}(g\mathbb{1}_s|s) = \mathrm{E}_{\mathrm{P}}(f\mathbb{1}_s|s)$. Now, since $\mathrm{E}_{\mathrm{P}}(g|s) \leq \sup g$ by our considerations above, and since $\sup g = \sup(f|s)$, we indeed find that $\mathrm{E}_{\mathrm{P}}(f|s) = \mathrm{E}_{\mathrm{P}}(f\mathbb{1}_s|s) = \mathrm{E}_{\mathrm{P}}(g|s) \leq \sup(f|s)$. This establishes $\mathrm{WC1}_{82}$.

We leave it for the reader to check that $\mathrm{E_P}$ also satisfies WC2$_{82}$–WC3$_{82}$. To prove WC4$_{82}$, consider any $f \in \operatorname{span}(\mathcal{A})$ and $s, t \in \mathscr{X}^*$ such that $s \sqsubseteq t$. Let $\sum_{i=1}^n a_i \mathbb{I}_{A_i}$ be any representation of f. Then we have that

$$(f - \mathrm{E_P}(f|t))\mathbb{1}_t = (\sum_{i=1}^n a_i \mathbb{I}_{A_i} - \mathrm{E_P}(f|t))\mathbb{1}_t = \sum_{i=1}^n a_i \mathbb{I}_{A_i}\mathbb{1}_t - \mathrm{E_P}(f|t)\mathbb{1}_t$$
$$= \sum_{i=1}^n a_i \mathbb{I}_{A_i \cap \Gamma(t)} - \mathrm{E_P}(f|t)\mathbb{I}_{\Gamma(t)}.$$

Since $\Gamma(t) \in \mathcal{A}$ and $A_i \cap \Gamma(t) \in \mathcal{A}$ for all $i \in \{1, \dots, n\}$ because $\mathcal{A}$ is by assumption an algebra that includes $\langle \mathscr{X}^* \rangle$, and thus all cylinder events $\Gamma(\mathscr{X}^*)$, $\sum_{i=1}^n a_i \mathbb{I}_{A_i \cap \Gamma(t)} - \mathrm{E_P}(f|t)\mathbb{I}_{\Gamma(t)}$ is a representation of the gamble $(f - \mathrm{E_P}(f|t))\mathbb{1}_t$ [and thus also $(f - \mathrm{E_P}(f|t))\mathbb{1}_t \in \operatorname{span}(\mathcal{A})$]. So by the definition of $\mathrm{E_P}$, we have that

$$\begin{aligned}\mathrm{E_P}\big((f - \mathrm{E_P}(f|t))\mathbb{1}_t|s\big) &= \textstyle\sum_{i=1}^n a_i \mathrm{P}(A_i \cap \Gamma(t)|s) - \mathrm{E_P}(f|t)\mathrm{P}(t|s) \\ &= \textstyle\sum_{i=1}^n a_i \mathrm{P}(A_i|t)\mathrm{P}(t|s) - \mathrm{E_P}(f|t)\mathrm{P}(t|s) \\ &= \mathrm{E_P}(f|t)\mathrm{P}(t|s) - \mathrm{E_P}(f|t)\mathrm{P}(t|s) = 0,\end{aligned}$$

where the second equality follows from GP4$_{70}$, and the third follows once more from the definition of $\mathrm{E_P}$ and the fact that $\sum_{i=1}^n a_i \mathbb{I}_{A_i}$ is a representation of f.

So $\mathrm{E_P}$ satisfies WC1$_{82}$–WC4$_{82}$, and is therefore coherent by Theorem 3.4.3$_{84}$. Fix any $n \in \mathbb{N}$, any $m \in \mathbb{N}_0$ such that $m \leq n$, any $\lambda_1, \dots, \lambda_m \in \mathbb{R}_{\geq}$, any $\lambda_{m+1}, \dots, \lambda_n \in \mathbb{R}_{<}$, and any $(A_1, s_1), \dots, (A_n, s_n) \in \mathcal{A} \times \mathscr{X}^*$. Then, for all $\lambda_0 \in \mathbb{R}_{\geq}$ and all $(A_0, s_0) \in \mathcal{A} \times \mathscr{X}^*$, by Definition 3.7$_{82}$ and since $\lambda_1, \dots, \lambda_m, -\lambda_{m+1}, \dots, -\lambda_n$ are all non-negative,

$$\sup\Big(\lambda_0 \mathbb{1}_{s_0}\left(\mathbb{I}_{A_0} - \mathrm{E_P}(\mathbb{I}_{A_0}|s_0)\right) - \sum_{i=1}^m \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right)$$
$$- \sum_{i=m+1}^n (-\lambda_i)\mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) \Big| \cup_{i=0}^n \Gamma(s_i)\Big) \geq 0.$$

In particular, by letting λ_0 be equal to 0 and s_0 be equal to one of the situations $s_1, \cdots, s_n$, we have that

$$\sup\Big(-\sum_{i=1}^m \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) + \sum_{i=m+1}^n \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) \Big| \cup_{i=1}^n \Gamma(s_i)\Big) \geq 0,$$

or equivalently,

$$\begin{aligned}0 &\leq \sup\Big(-\textstyle\sum_{i=1}^m \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) + \sum_{i=m+1}^n \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) \Big| \cup_{i=1}^n \Gamma(s_i)\Big) \\ &= \sup\Big(\textstyle\sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(1 - \mathbb{I}_{A_i} - (1 - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i))\right) \\ &\qquad + \textstyle\sum_{i=m+1}^n \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{E_P}(\mathbb{I}_{A_i}|s_i)\right) \Big| \cup_{i=1}^n \Gamma(s_i)\Big) \\ &= \sup\Big(\textstyle\sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\Big(\mathbb{I}_{A_i^c} - (1 - \mathrm{P}(A_i|s_i))\Big) + \sum_{i=m+1}^n \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{P}(A_i|s_i)\right) \Big| \cup_{i=1}^n \Gamma(s_i)\Big) \\ &= \sup\Big(\textstyle\sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\Big(\mathbb{I}_{A_i^c} - \mathrm{P}(A_i^c|s_i)\Big) + \sum_{i=m+1}^n \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{P}(A_i|s_i)\right) \Big| \cup_{i=1}^n \Gamma(s_i)\Big),\end{aligned}$$

where the second equality follows from the definition of $\mathrm{E_P}$, and the last from GP2$_{70}$ and GP3$_{70}$. Since the above holds for any $n \in \mathbb{N}$, any sequence of reals $\lambda_1, \dots, \lambda_n \in \mathbb{R}$

[we simply distinguished between the non-negative and the negative ones] and all $(A_1, s_1), \dots, (A_n, s_n) \in \mathcal{A} \times \mathscr{X}^*$, and since $\mathcal{A}$ is an algebra and thus closed under complementation, we indeed have that (iii)$_{71}$ in Proposition 3.3.1$_{71}$ holds.

In order to prove that (iii)$_{71} \Rightarrow$ (ii)$_{71}$, suppose that $\mathrm{P}\colon \mathcal{A} \times \mathscr{X}^* \to \mathbb{R}$ satisfies (iii)$_{71}$. Then [62, Theorem 8] guarantees that we can extend P to the domain $\wp(\Omega) \times \wp(\Omega)^\circ$ such that it remains to satisfy (iii)$_{71}$. This extension $\mathrm{P}'\colon \wp(\Omega) \times \wp(\Omega)^\circ \to \mathbb{R}$ satisfies CP1$_{70}$–CP4$_{70}$ according to [62, Theorem 7]. Hence, P is the restriction of a conditional probability charge P′ on $\wp(\Omega) \times \wp(\Omega)^\circ$ according to Definition 3.1$_{70}$.

Finally, the fact that (ii)$_{71} \Rightarrow$ (i)$_{71}$ follows straightforwardly from the definitions of a conditional probability charge [Definition 3.1$_{70}$] and a global probability charge [Definition 3.2$_{70}$]. □

Proof of Proposition 3.3.4$_{73}$. [62, Lemma 14] establishes the existence of a unique 'conditional probability measure' P on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ satisfying Eq. (3.12)$_{72}$ and being of the appropriate form as described in Proposition 3.3.4$_{73}$. According to [62, Definition 6], such a 'conditional probability measure' P is simply a 'coherent conditional probability' [62, Definition 5] for which $\mathrm{P}(\cdot|s)\colon \langle \mathscr{X}^* \rangle \to \mathbb{R}$ for any $s \in \mathscr{X}^*$ is σ-additive; see also Definition 5.1$_{221}$ further below. As established by Proposition 3.3.1$_{71}$, a 'coherent conditional probability' on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ is a global probability charge and vice versa, and so the notion of a 'conditional probability measure' [62] on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ is equivalent to the notion a global probability charge P′ on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ for which $\mathrm{P}'(\cdot|s)\colon \langle \mathscr{X}^* \rangle \to \mathbb{R}$ for any $s \in \mathscr{X}^*$ is σ-additive. Yet, for any global probability charge P′ on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ and any $s \in \mathscr{X}^*$, we have by [5, Theorem 2.3] that the finite additivity of $\mathrm{P}'(\cdot|s)$ on $\langle \mathscr{X}^* \rangle$ automatically implies its σ-additivity, so our notion of a (general) global probability charge on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ is equivalent to the notion of a 'conditional probability measure' [62] on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$. Our proposition thus indeed follows from [62, Lemma 14]. □

3.C Proofs of the results in Section 3.4.1

Proof of Proposition 3.4.4$_{84}$. We first prove WC6$_{84}$. Consider any $s \in \mathscr{X}^*$, and note that $\overline{\mathrm{E}}(0|s) = 0$ because of WC3$_{82}$ and our convention that $0\,(+\infty) = 0\,(-\infty) = 0$. Therefore, for all $f \in \mathscr{I}$, it follows from WC2$_{82}$ that $0 \leq \overline{\mathrm{E}}(f|s) + \overline{\mathrm{E}}(-f|s)$, or equivalently [since $+\infty - \infty = +\infty$], that $-\overline{\mathrm{E}}(-f|s) \leq \overline{\mathrm{E}}(f|s)$. Applying WC1$_{82}$ to both sides, we find that $\inf(f|s) = -\sup(-f|s) \leq -\overline{\mathrm{E}}(-f|s) \leq \overline{\mathrm{E}}(f|s) \leq \sup(f|s)$. Property WC6$_{84}$ now follows readily from the definition of $\underline{\mathrm{E}}$.

To see that the remaining properties hold, note that WC6$_{84}$ implies that the map $\overline{\mathrm{E}}$ (and therefore also $\underline{\mathrm{E}}$) is real-valued on $\mathscr{I} \times \mathscr{X}^*$. Properties WC5$_{84}$ and WC7$_{84}$–WC9$_{84}$ then follow from WC1$_{82}$–WC3$_{82}$ using arguments well-known in the field of coherent upper and lower expectations; see, for instance, the proof of [106, Lemma 13.13], where they use lower expectations instead of upper expectations, and where 'gambles' have a more general meaning because there they can also be unbounded.

Now suppose that $\mathscr{I}$ is, apart from a linear space of global gambles that contains all constants, also such that $f\mathbb{1}_s \in \mathscr{I}$ for all $f \in \mathscr{I}$ and $s \in \mathscr{X}^*$. Moreover suppose that $\overline{\mathrm{E}}$ additionally satisfies WC4$_{82}$. To prove the remaining properties, we will implicitly make use of the fact that, due to WC6$_{84}$, $\overline{\mathrm{E}}$ is real-valued. Consider any $f \in \mathscr{I}$ and any two $s, t \in \mathscr{X}^*$ such that $s \sqsubseteq t$. That WC10$_{85}$ holds, follows immediately from conjugacy; indeed, we have that

$$\underline{\mathrm{E}}\left((f - \underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right) = -\overline{\mathrm{E}}\left((-f + \underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right) = -\overline{\mathrm{E}}\left(\left((-f) - \overline{\mathrm{E}}((-f)|t)\right)\mathbb{1}_t|s\right) = 0,$$

where the last step follows from applying WC4$_{82}$ to the gamble $(-f)$.

To prove WC11$_{85}$, note that, for any $\mu \in \mathbb{R}$, we have that $\overline{\mathrm{E}}(\mu\mathbb{1}_s|s) = \underline{\mathrm{E}}(\mu\mathbb{1}_s|s) = \mu$ due to WC6$_{84}$. Hence, from WC4$_{82}$ we obtain in particular that

$$0 = \overline{\mathrm{E}}\left((f - \overline{\mathrm{E}}(f|s))\mathbb{1}_s|s\right) \overset{\text{WC2}_{82}}{\leq} \overline{\mathrm{E}}(f\mathbb{1}_s|s) + \overline{\mathrm{E}}\left(-\overline{\mathrm{E}}(f|s)\mathbb{1}_s|s\right) = \overline{\mathrm{E}}(f\mathbb{1}_s|s) - \overline{\mathrm{E}}(f|s).$$

But on the other hand, WC4$_{82}$ also implies that

$$0 = \overline{\mathrm{E}}\left((f - \overline{\mathrm{E}}(f|s))\mathbb{1}_s|s\right) \overset{\text{WC8}_{84}}{\geq} \overline{\mathrm{E}}(f\mathbb{1}_s|s) + \underline{\mathrm{E}}\left(-\overline{\mathrm{E}}(f|s)\mathbb{1}_s|s\right) = \overline{\mathrm{E}}(f\mathbb{1}_s|s) - \overline{\mathrm{E}}(f|s).$$

So we find that $\overline{\mathrm{E}}(f|s) = \overline{\mathrm{E}}(f\mathbb{1}_s|s)$ as desired. The expression for the lower expectations then follows from conjugacy.

We continue by proving WC12$_{85}$. Consider any $f \in \mathscr{I}$ and any $x_{1:k} \in \mathscr{X}^*$, and note that WC4$_{82}$ implies that

$$\overline{\mathrm{E}}\left((f - \overline{\mathrm{E}}(f|x_{1:k+1}))\mathbb{1}_{x_{1:k+1}}|x_{1:k}\right) = 0 \text{ for all } x_{k+1} \in \mathscr{X}.$$

As a consequence, we have that

$$\begin{aligned}0 &= \textstyle\sum_{x_{k+1}\in\mathscr{X}} \overline{\mathrm{E}}\left((f - \overline{\mathrm{E}}(f|x_{1:k+1}))\mathbb{1}_{x_{1:k+1}}|x_{1:k}\right)\\ &\geq \overline{\mathrm{E}}\left(\textstyle\sum_{x_{k+1}\in\mathscr{X}}(f - \overline{\mathrm{E}}(f|x_{1:k+1}))\mathbb{1}_{x_{1:k+1}}|x_{1:k}\right)\\ &= \overline{\mathrm{E}}\left(\textstyle\sum_{x_{k+1}\in\mathscr{X}} f\mathbb{1}_{x_{1:k+1}} - \sum_{x_{k+1}\in\mathscr{X}} \overline{\mathrm{E}}(f|x_{1:k+1})\mathbb{1}_{x_{1:k+1}}|x_{1:k}\right),\end{aligned}$$

where the inequality follows from WC2$_{82}$ and the fact that $\mathscr{X}$ is finite. But note that

$$\textstyle\sum_{x_{k+1}\in\mathscr{X}} f\mathbb{1}_{x_{1:k+1}} = f\mathbb{1}_{x_{1:k}} \text{ and } \sum_{x_{k+1}\in\mathscr{X}} \overline{\mathrm{E}}(f|x_{1:k+1})\mathbb{1}_{x_{1:k+1}} = \overline{\mathrm{E}}(f|x_{1:k}X_{k+1})\mathbb{1}_{x_{1:k}},$$

so we get that

$$0 \geq \overline{\mathrm{E}}\left(f\mathbb{1}_{x_{1:k}} - \overline{\mathrm{E}}(f|x_{1:k}X_{k+1})\mathbb{1}_{x_{1:k}}|x_{1:k}\right) \geq \overline{\mathrm{E}}\left(f\mathbb{1}_{x_{1:k}}|x_{1:k}\right) - \overline{\mathrm{E}}\left(\overline{\mathrm{E}}(f|x_{1:k}X_{k+1})\mathbb{1}_{x_{1:k}}|x_{1:k}\right),$$

where the last inequality follows from WC8$_{84}$ and conjugacy. It now only remains to apply WC11$_{85}$ to each of the above terms, to find that indeed

$$\overline{\mathrm{E}}\left(\overline{\mathrm{E}}(f|x_{1:k}X_{k+1})|x_{1:k}\right) \geq \overline{\mathrm{E}}\left(f|x_{1:k}\right).$$

The inequality for the lower expectations then follows readily from conjugacy.

Next, WC13$_{85}$ can be seen as a fairly straightforward consequence of WC12$_{85}$. Indeed, the expression for the upper expectations holds, by our definition of $\overline{\mathrm{E}}\left(f|X_{1:k}\right)$

and $\overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|X_{1:k+1}\right)|X_{1:k}\right)$, if $\overline{\mathrm{E}}\left(f|x_{1:k}\right) \leq \overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|X_{1:k+1}\right)|x_{1:k}\right)$ for all $x_{1:k} \in \mathscr{X}^k$. But this follows from applying WC11$_{85}$ twice, and then using WC12$_{85}$:

$$\begin{aligned}\overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|X_{1:k+1}\right)|x_{1:k}\right) = \overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|X_{1:k+1}\right)\mathbb{1}_{x_{1:k}}|x_{1:k}\right) &= \overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|x_{1:k}X_{k+1}\right)\mathbb{1}_{x_{1:k}}|x_{1:k}\right)\\ &= \overline{\mathrm{E}}\left(\overline{\mathrm{E}}\left(f|x_{1:k}X_{k+1}\right)|x_{1:k}\right)\\ &\geq \overline{\mathrm{E}}\left(f|x_{1:k}\right).\end{aligned}$$

The inequality for the lower expectations then follows once more from conjugacy.

To see that WC14$_{85}$ holds, suppose that $\underline{\mathrm{E}}(f|t) \geq 0$, and observe that

$$\begin{aligned}\underline{\mathrm{E}}\left((f-\underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right) &\overset{\mathrm{WC8_{84}}}{\leq} \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right) + \overline{\mathrm{E}}\left(-\underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right)\\ &= \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right) - \underline{\mathrm{E}}\left(\underline{\mathrm{E}}(f|t)\mathbb{1}_t|s\right)\\ &\overset{\mathrm{WC3_{82}}}{=} \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right) - \underline{\mathrm{E}}(f|t)\underline{\mathrm{E}}(\mathbb{1}_t|s) \leq \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right),\end{aligned}$$

where we are allowed to use WC3$_{82}$ in the third step because $\underline{\mathrm{E}}(f|t) \geq 0$, and where the last step follows from the fact that $\underline{\mathrm{E}}(f|t) \geq 0$ and, because of $\mathbb{1}_t \geq 0$ and WC6$_{84}$, that $\underline{\mathrm{E}}(\mathbb{1}_t|s) \geq 0$. Since the left-hand side of the expression above is equal to 0 due to WC10$_{85}$, we find that $0 \leq \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right)$ as desired.

Finally, in order to prove WC15$_{85}$, assume that $\underline{\mathrm{E}}(f\mathbb{1}_t|s) > 0$ and observe that

$$\begin{aligned}\underline{\mathrm{E}}\left((f-\underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right) \geq \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right) + \underline{\mathrm{E}}\left(-\underline{\mathrm{E}}(f|t))\mathbb{1}_t|s\right) &= \underline{\mathrm{E}}\left(f\mathbb{1}_t|s\right) - \overline{\mathrm{E}}\left(\underline{\mathrm{E}}(f|t)\mathbb{1}_t|s\right)\\ &> -\overline{\mathrm{E}}\left(\underline{\mathrm{E}}(f|t)\mathbb{1}_t|s\right),\end{aligned}$$

where the first step follows from WC2$_{82}$ and conjugacy, and where the last inequality follows from the fact that $\underline{\mathrm{E}}(f\mathbb{1}_t|s) > 0$. The left-hand side of this expression is again equal to 0 [due to WC10$_{85}$], and so we have that

$$0 < \overline{\mathrm{E}}\left(\underline{\mathrm{E}}(f|t)\mathbb{1}_t|s\right) \overset{\mathrm{WC6_{84}}}{\leq} \sup(\underline{\mathrm{E}}(f|t)\mathbb{1}_t|s) \leq \max\{0, \underline{\mathrm{E}}(f|t)\}.$$

As a result, we have that $\underline{\mathrm{E}}(f|t) > 0$.

The last statement, the fact that all the properties WC5$_{84}$–WC15$_{85}$ hold for a global upper expectation $\overline{\mathrm{E}}$ that satisfies WC1$_{82}$–WC4$_{82}$ and that is defined on $\mathbb{V} \times \mathscr{X}^*$ or $\mathbb{F} \times \mathscr{X}^*$, is trivial because it can easily be checked that both $\mathbb{V}$ and $\mathbb{F}$ are linear spaces of global gambles containing all the constants and are invariant under multiplication with indicators of situations. □

Lemma 3.C.1. *Consider any global upper expectation $\overline{\mathrm{E}}$ on a domain $\mathscr{I} \times \mathscr{X}^* \subseteq \mathbb{V} \times \mathscr{X}^*$ such that $\mathscr{I}$ is a linear space of global gambles. Then $\overline{\mathrm{E}}$ is coherent if there is a coherent set of acceptable global gambles $\mathscr{D}$ such that*

$$\overline{\mathrm{E}}(f|s) = \inf\{\alpha \in \mathbb{R} : (\alpha - f)\mathbb{1}_s \in \mathscr{D}\} \text{ for all } (f,s) \in \mathscr{I} \times \mathscr{X}^*.$$

Proof. This result follows from [114, Proposition 1] as a special case. Indeed, suppose that there is a coherent set of acceptable global gambles $\mathscr{D}$ such that

$$\overline{\mathrm{E}}(f|s) = \inf\{\alpha \in \mathbb{R} : (\alpha - f)\mathbb{1}_s \in \mathscr{D}\} \text{ for all } (f,s) \in \mathscr{I} \times \mathscr{X}^*. \tag{3.22}$$

Then since $\mathscr{D}$ satisfies D1$_{27}$ and D2$_{27}$ by definition, it also satisfies properties C1' and C2' in [114, Section 3]; care to note that the conditional events E in [114] are in our case subsets of the sample space Ω, and the 'random quantities' in [114] here take the form of global gambles f on Ω. Then if we let the conditional events E_i in [114, Proposition 1] be of the form $\Gamma(s)$ with $s \in \mathscr{X}^*$, then this result says that, if $\mathscr{I}_s$ is a linear subspace of $\mathbb{V}$ for all $s \in \mathscr{X}^*$, the global upper expectation $\overline{\mathrm{E}}^*$ defined, for all $s \in \mathscr{X}^*$ and all $f \in \mathscr{I}_s$, by

$$\overline{\mathrm{E}}^*(f|s) := \inf\{\alpha \in \mathbb{R} : (\alpha - f)\mathbb{1}_s \in \mathscr{D}\},$$

is coherent [Definition 3.7$_{82}$]. In particular, this is true if we let $\mathscr{I}_s$ for all $s \in \mathscr{X}^*$ be equal to the fixed linear space $\mathscr{I}$, and then $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}$ due to Eq. (3.22)$_{\frown}$. Hence, since $\overline{\mathrm{E}}^*$ is coherent, $\overline{\mathrm{E}}$ is coherent too. □

Proof of Theorem 3.4.3$_{84}$. Necessity of WC1$_{82}$–WC4$_{82}$ can be inferred from [114, Section 3.1]. In particular, note that any coherent global upper expectation $\overline{\mathrm{E}}$ [Definition 3.7$_{82}$] on $\mathscr{I} \times \mathscr{X}^*$, is a (specific) 'upper conditional prevision' according to [114, Definition 1], where the linear spaces $\mathscr{X}_E$ in [114, Definition 1] are here all equal to the fixed linear space $\mathscr{I}$, and where the real-valuedness of $\overline{\mathrm{E}}$ is guaranteed by Corollary 3.4.2$_{83}$.

To prove sufficiency, suppose that $\overline{\mathrm{E}}$ satisfies WC1$_{82}$–WC4$_{82}$ and let

$$\mathscr{D} := \left\{\sum_{i=1}^{n} g_i \mathbb{1}_{s_i} + h : n \in \mathbb{N}_0, g_i \in \mathscr{I}, s_i \in \mathscr{X}^*, \underline{\mathrm{E}}(g_i|s_i) > 0, h \in \mathscr{L}_{\geq}(\Omega)\right\}.$$

We will show that $\mathscr{D}$ is coherent and that its corresponding infimum selling prices operator $\overline{\mathrm{E}}_{\mathscr{D}}$, defined by

$$\overline{\mathrm{E}}_{\mathscr{D}}(f|s) := \inf\{\alpha \in \mathbb{R} : (\alpha - f)\mathbb{1}_s \in \mathscr{D}\} \text{ for all } (f, s) \in \mathbb{V} \times \mathscr{X}^*, \tag{3.23}$$

coincides with $\overline{\mathrm{E}}$ on $\mathscr{I} \times \mathscr{X}^*$. Lemma 3.C.1$_{\frown}$ will then imply that $\overline{\mathrm{E}}$ is coherent.

Let us first check that $\mathscr{D}$ is coherent. It is clear from the definition of $\mathscr{D}$ that D1$_{27}$ and D3$_{27}$ are satisfied. To see that D4$_{27}$ is satisfied, consider any $\lambda > 0$ and any $f \in \mathscr{D}$, and let us check that $\lambda f \in \mathscr{D}$. Since $f \in \mathscr{D}$, we can write that $f = \sum_{i=1}^n g_i \mathbb{1}_{s_i} + h$ for some $n \in \mathbb{N}_0$, $g_1, \dots, g_n \in \mathscr{I}$, $s_1, \dots, s_n \in \mathscr{X}^*$, with $\underline{\mathrm{E}}(g_i|s_i) > 0$ for all $i \in \{1, \dots, n\}$, and some $h \in \mathscr{L}_{\geq}(\Omega)$. Multiplying with $\lambda > 0$ gives us $\lambda f = \sum_{i=1}^n \lambda g_i \mathbb{1}_{s_i} + \lambda h$. For all $i \in \{1, \dots, n\}$, we have that $\underline{\mathrm{E}}(g_i|s_i) > 0$, which by WC3$_{82}$ implies that

$$0 < \lambda \underline{\mathrm{E}}(g_i|s_i) = \underline{\mathrm{E}}(\lambda g_i|s_i),$$

and where $\underline{\mathrm{E}}(\lambda g_i|s_i)$ is well-defined because $g_i \in \mathscr{I}$ and $\mathscr{I}$ is a linear space—and thus $\lambda g_i \in \mathscr{I}$. Since moreover $h \in \mathscr{L}_{\geq}(\Omega)$, and therefore $\lambda h \in \mathscr{L}_{\geq}(\Omega)$, it follows that, indeed, by the definition of $\mathscr{D}$,

$$\lambda f = \sum_{i=1}^{n} \lambda g_i \mathbb{1}_{s_i} + \lambda h \in \mathscr{D}.$$

Finally, to prove that D2$_{27}$ holds, assume **ex absurdo** that there is a gamble $f \in \mathscr{D}$ such that $f \in \mathscr{L}_{\lneq}(\Omega)$. The fact that $f \in \mathscr{D}$ again implies that $f = \sum_{i=1}^n g_i \mathbb{1}_{s_i} + h$ for some $n \in \mathbb{N}_0$, $g_1, \dots, g_n \in \mathcal{I}$, $s_1, \dots, s_n \in \mathscr{X}^*$, with $\underline{\mathrm{E}}(g_i|s_i) > 0$ for all $i \in \{1, \dots, n\}$, and some $h \in \mathscr{L}_{\geq}(\Omega)$. Since $f \in \mathscr{L}_{\lneqq}(\Omega)$ and $h \in \mathscr{L}_{\geq}(\Omega)$, we know that $n \geq 1$. Note that the gamble $\sum_{i=1}^n g_i \mathbb{1}_{s_i} = f - h$ is then an element of $\mathcal{I}$ because of our assumptions about $\mathcal{I}$ [linear space and closed under multiplying with indicators of situations] and because $g_i \in \mathcal{I}$ for all $i \in \{1, \dots, n\}$. Moreover, since $f \in \mathscr{L}_{\lneqq}(\Omega)$ and since $h \geq 0$, we have that $f - h \leq 0$ and $f - h \neq 0$. Then, for all $1 \leq i \leq n$, we have by WC6$_{84}$ that $\underline{\mathrm{E}}(f - h|s_i) \leq 0$. But on the other hand, consider any $i \in \{1, \dots, n\}$ such that s_i has minimal length among the situations $s_1, \dots, s_n$ [it is clear that there is such an i]. Then there are no $j \in \{1, \dots, n\}$ such that $s_j \sqsubset s_i$, and therefore

$$\begin{aligned}
\underline{\mathrm{E}}(f - h|s_i) = \underline{\mathrm{E}}(\textstyle\sum_{j=1}^n g_j \mathbb{1}_{s_j}|s_i) &\overset{\text{WC2}_{82}}{\geq} \sum_{j=1}^n \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) \\
&= \sum_{\substack{1 \leq j \leq n \\ s_j \sqsupseteq s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) + \sum_{\substack{1 \leq j \leq n \\ s_j \sqsubset s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) + \sum_{\substack{1 \leq j \leq n \\ s_j \parallel s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) \\
&= \sum_{\substack{1 \leq j \leq n \\ s_j \sqsupseteq s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) + \sum_{\substack{1 \leq j \leq n \\ s_j \parallel s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i).
\end{aligned}$$

By WC11$_{85}$ and WC6$_{84}$, we have that $\underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) = \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j} \mathbb{1}_{s_i}|s_i) = \underline{\mathrm{E}}(0|s_i) = 0$ for any $s_j \parallel s_i$, and so the above implies that

$$\underline{\mathrm{E}}(f - h|s_i) = \sum_{\substack{1 \leq j \leq n \\ s_j \sqsupseteq s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) = \sum_{\substack{1 \leq j \leq n \\ s_j = s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_j) + \sum_{\substack{1 \leq j \leq n \\ s_j \sqsupset s_i}} \underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i)$$

Note that the first sum in the expression above is strictly larger than zero because, by assumption, we have that $\underline{\mathrm{E}}(g_j|s_j) > 0$ for all $1 \leq j \leq n$, and so by WC11$_{85}$ that $\underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_j) > 0$ for all $1 \leq j \leq n$. The second sum is larger or equal than zero, because the fact that $\underline{\mathrm{E}}(g_j|s_j) > 0$ for all $1 \leq j \leq n$ implies by WC14$_{85}$ that $\underline{\mathrm{E}}(g_j \mathbb{1}_{s_j}|s_i) \geq 0$ for all $1 \leq j \leq n$ such that $s_j \sqsupset s_i$. Hence, both sums taken together, we should have that $\underline{\mathrm{E}}(f - h|s_i) > 0$. Yet, this is in contradiction with our earlier conclusion. As a result, the set of acceptable gambles $\mathscr{D}$ satisfies D2$_{27}$, and together with D1$_{27}$ and D3$_{27}$–D4$_{27}$, we obtain that $\mathscr{D}$ is coherent. It now remains to prove that $\overline{\mathrm{E}}$ coincides with the upper expectation $\overline{\mathrm{E}}_{\mathscr{D}}$ corresponding to $\mathscr{D}$ according to Eq. (3.23)$_{\leftarrow}$.

As a first step, we prove that for any $f \in \mathbb{V}$ and any $t \in \mathscr{X}^*$,

$$f \mathbb{1}_t \in \mathscr{D} \Rightarrow f \mathbb{1}_t \in \mathscr{D}_{|t}, \tag{3.24}$$

with

$$\mathscr{D}_{|t} := \left\{ \sum_{i=1}^n g_i \mathbb{1}_{ts_i} + h \colon n \in \mathbb{N}_0, g_i \in \mathcal{I}, s_i \in \mathscr{X}^*, \underline{\mathrm{E}}(g_i|ts_i) > 0, h \in \mathscr{L}_{\geq}(\Omega) \right\}.$$

So fix any $f \in \mathbb{V}$ and any $t \in \mathscr{X}^*$ such that $f \mathbb{1}_t \in \mathscr{D} = \mathscr{D}_{|\square}$. Let $k \in \mathbb{N}_0$ and $x_{1:k} \in \mathscr{X}^k$ be such that $x_{1:k} = t$. If $k = 0$, then the desired implication is trivially true because in that case $f \mathbb{1}_t \in \mathscr{D}_{|\square} = \mathscr{D}_{|t}$. So consider the case that $k \geq 1$ and let $\ell < k$. We show that

$$f \mathbb{1}_t \in \mathscr{D}_{|x_{1:\ell}} \Rightarrow f \mathbb{1}_t \in \mathscr{D}_{|x_{1:\ell+1}}. \tag{3.25}$$

Suppose that $f\mathbb{1}_t \in \mathcal{D}_{|x_{1:\ell}}$ and therefore that, for some $n \in \mathbb{N}_0$, $g_1, \dots, g_n \in \mathcal{I}$, $s_1, \dots, s_n \in \mathcal{X}^*$, with $\underline{\mathrm{E}}(g_i|x_{1:\ell}s_i) > 0$ for all $i \in \{1, \dots, n\}$, and some $h \in \mathcal{L}_{\geq}(\Omega)$,

$$f\mathbb{1}_t = \sum_{i=1}^{n} g_i \mathbb{1}_{x_{1:\ell}s_i} + h.$$

The situations $s_1, \dots, s_n \in \mathcal{X}^*$ can be categorised in three different groups [where the order of the situations within these individual groups is of no importance]; let $u_1, \dots, u_m \in \mathcal{X}^*$ be those situations in $s_1, \dots, s_n$ that start with $x_{\ell+1}$; let $v_1, \dots, v_p \in \mathcal{X}^*$ be those situations in $s_1, \dots, s_n$ that start with a different state than $x_{\ell+1}$; and let $w_1, \dots, w_{n-m-p} \in \mathcal{X}^*$ all be equal to the empty situation $\square$. Let $\zeta_1, \dots, \zeta_m, \eta_1, \dots, \eta_p$ and $\theta_1, \dots, \theta_{n-m-p}$ be the corresponding categorisation of the gambles $g_1, \dots, g_n$; so we have that $\underline{\mathrm{E}}(\zeta_i|x_{1:\ell}u_i) > 0$, $\underline{\mathrm{E}}(\eta_i|x_{1:\ell}v_i) > 0$ and $\underline{\mathrm{E}}(\theta_i|x_{1:\ell}) = \underline{\mathrm{E}}(\theta_i|x_{1:\ell}w_i) > 0$. Then we get that

$$f\mathbb{1}_t = \sum_{i=1}^{m} \zeta_i \mathbb{1}_{x_{1:\ell+1}u_i'} + \sum_{i=1}^{p} \eta_i \mathbb{1}_{x_{1:\ell}v_i} + \sum_{i=1}^{n-m-p} \theta_i \mathbb{1}_{x_{1:\ell}} + h, \tag{3.26}$$

where, for all $i \in \{1, \dots, m\}$, the situation u_i' is such that $u_i = x_{\ell+1}u_i'$. Note that $x_{1:\ell+1} \parallel x_{1:\ell}v_i$ for any v_i, and therefore that $\mathbb{1}_{x_{1:\ell+1}}\mathbb{1}_{x_{1:\ell}v_i} = 0$. Since moreover $x_{1:\ell+1} \sqsubseteq t$, and therefore that $\mathbb{1}_{x_{1:\ell+1}}\mathbb{1}_t = \mathbb{1}_t$, multiplying Eq. (3.26) with $\mathbb{1}_{x_{1:\ell+1}}$ gives us

$$f\mathbb{1}_t = \sum_{i=1}^{m} \zeta_i \mathbb{1}_{x_{1:\ell+1}u_i'} + \sum_{i=1}^{n-m-p} \theta_i \mathbb{1}_{x_{1:\ell+1}} + h\mathbb{1}_{x_{1:\ell+1}}. \tag{3.27}$$

By assumption, we have that $\underline{\mathrm{E}}(\zeta_i|x_{1:\ell+1}u_i') = \underline{\mathrm{E}}(\zeta_i|x_{1:\ell}u_i) > 0$ for all $i \in \{1, \dots, m\}$. Since also $h \geq 0$, and therefore $h\mathbb{1}_{x_{1:\ell+1}} \geq 0$, it follows from the expression above and the definition of $\mathcal{D}_{|x_{1:\ell+1}}$ that then $f\mathbb{1}_t \in \mathcal{D}_{|x_{1:\ell+1}}$ if $n - m - p = 0$, or, in case that $n - m - p \geq 1$, if

$$\underline{\mathrm{E}}\left(\textstyle\sum_{i=1}^{n-m-p} \theta_i | x_{1:\ell+1}\right) > 0. \tag{3.28}$$

To see that these conditions are met—that is, that $n-m-p \geq 1$ implies Eq. (3.28)—we substract Eq. (3.27) from Eq. (3.26); this gives us

$$0 = \sum_{i=1}^{p} \eta_i \mathbb{1}_{x_{1:\ell}v_i} + \sum_{i=1}^{n-m-p} \theta_i (\mathbb{1}_{x_{1:\ell}} - \mathbb{1}_{x_{1:\ell+1}}) + h(1 - \mathbb{1}_{x_{1:\ell+1}}),$$

or, equivalently, that

$$\sum_{i=1}^{n-m-p} \theta_i \mathbb{1}_{x_{1:\ell+1}} = \sum_{i=1}^{p} \eta_i \mathbb{1}_{x_{1:\ell}v_i} + \sum_{i=1}^{n-m-p} \theta_i \mathbb{1}_{x_{1:\ell}} + h(1 - \mathbb{1}_{x_{1:\ell+1}}).$$

Since each θ_i and each η_i is by assumption an element of $\mathcal{I}$, it follows from the assumptions about $\mathcal{I}$ that also each $\theta_i\mathbb{1}_{x_{1:\ell+1}}$, each $\eta_i\mathbb{1}_{x_{1:\ell}v_i}$ and each $\theta_i\mathbb{1}_{x_{1:\ell}}$ in the expression above is an element of $\mathcal{I}$. Since $\mathcal{I}$ is moreover a linear space, it therefore follows from the expression above that $h(1 - \mathbb{1}_{x_{1:\ell+1}}) \in \mathcal{I}$. Hence, we can take the lower expectation of both sides conditional on $x_{1:\ell}$ and immediately apply WC2$_{82}$, to find that

$$\begin{aligned}
\underline{\mathrm{E}}\left(\textstyle\sum_{i=1}^{n-m-p} \theta_i \mathbb{1}_{x_{1:\ell+1}} | x_{1:\ell}\right) &\geq \sum_{i=1}^{p} \underline{\mathrm{E}}\left(\eta_i \mathbb{1}_{x_{1:\ell}v_i} | x_{1:\ell}\right) + \sum_{i=1}^{n-m-p} \underline{\mathrm{E}}\left(\theta_i \mathbb{1}_{x_{1:\ell}} | x_{1:\ell}\right) \\
&\qquad + \underline{\mathrm{E}}\left(h(1 - \mathbb{1}_{x_{1:\ell+1}}) | x_{1:\ell}\right) \\
&\geq \sum_{i=1}^{p} \underline{\mathrm{E}}\left(\eta_i \mathbb{1}_{x_{1:\ell}v_i} | x_{1:\ell}\right) + \sum_{i=1}^{n-m-p} \underline{\mathrm{E}}\left(\theta_i | x_{1:\ell}\right),
\end{aligned} \tag{3.29}$$

where the second inequality follows from WC11$_{85}$ and the fact that $\underline{\mathrm{E}}\left(h(1-\mathbb{1}_{x_{1:\ell+1}})|x_{1:\ell}\right) \geq 0$, which is itself a result of WC6$_{84}$ and the fact that $h(1-\mathbb{1}_{x_{1:\ell+1}}) \geq 0$. Recall that, for all $i \in \{1,\dots,p\}$, we have that $\underline{\mathrm{E}}\,(\eta_i|x_{1:\ell}v_i) > 0$, and therefore by WC14$_{85}$ that $\underline{\mathrm{E}}\left(\eta_i\mathbb{1}_{x_{1:\ell}v_i}|x_{1:\ell}\right) \geq 0$. Furthermore, we also have that $\underline{\mathrm{E}}\,(\theta_i|x_{1:\ell}) > 0$ for all $i \in \{1,\dots,n-m-p\}$. Hence, if $n-m-p \geq 1$—implying that $\{1,\dots,n-m-p\}$ is non-empty—it follows from Eq. (3.29)$_\leftarrow$ that

$$\underline{\mathrm{E}}\left(\textstyle\sum_{i=1}^{n-m-p}\theta_i\mathbb{1}_{x_{1:\ell+1}}|x_{1:\ell}\right) > 0,$$

and therefore, because of WC15$_{85}$, that Eq. (3.28)$_\leftarrow$ indeed holds. As a result, $f\mathbb{1}_t \in \mathscr{D}_{|x_{1:\ell+1}}$, which in turn establishes Eq. (3.25)$_{111}$. But recall that $\ell < k$ was arbitrary, so we can start from the fact $f\mathbb{1}_t \in \mathscr{D} = \mathscr{D}_{|\square}$ and apply Eq. (3.25)$_{111}$ iteratively to eventually find that $f\mathbb{1}_t \in \mathscr{D}_{|x_{1:k}} = \mathscr{D}_{|t}$. Hence, Eq. (3.24)$_{111}$ holds.

Next, we use Eq. (3.24)$_{111}$ to show that $\overline{\mathrm{E}}_{\mathscr{D}}(f|t) = \overline{\mathrm{E}}(f|t)$ for all $f \in \mathscr{I}$ and all $t \in \mathscr{X}^*$, where $\overline{\mathrm{E}}_{\mathscr{D}}$ is defined by $\mathscr{D}$ according to Eq. (3.23)$_{110}$. That $\overline{\mathrm{E}}_{\mathscr{D}}(f|t) \leq \overline{\mathrm{E}}(f|t)$ for all $f \in \mathscr{I}$ and all $t \in \mathscr{X}^*$, follows from the fact that, for all real $\alpha > \overline{\mathrm{E}}(f|t)$,

$$\underline{\mathrm{E}}(\alpha - f|t) \overset{\text{WC7}_{84}}{=} \alpha + \underline{\mathrm{E}}(-f|t) = \alpha - \overline{\mathrm{E}}(f|t) > 0,$$

where the leftmost term is well-defined because $\mathscr{I}$ is a linear space that includes the constants. Indeed, this implies by the definition of $\mathscr{D}$ that $(\alpha - f)\mathbb{1}_t \in \mathscr{D}$ for all real $\alpha > \overline{\mathrm{E}}(f|t)$, and so by Eq. (3.23)$_{110}$ that

$$\overline{\mathrm{E}}_{\mathscr{D}}(f|t) = \inf\{\alpha \in \mathbb{R}\colon (\alpha - f)\mathbb{1}_t \in \mathscr{D}\} \leq \overline{\mathrm{E}}(f|t).$$

It remains to prove that $\overline{\mathrm{E}}_{\mathscr{D}}(f|t) \geq \overline{\mathrm{E}}(f|t)$.

Fix any $f \in \mathscr{I}$, any $t \in \mathscr{X}^*$, and any $\alpha \in \mathbb{R}$ such that $(\alpha - f)\mathbb{1}_t \in \mathscr{D}$. Then Eq. (3.24)$_{111}$ implies that also $(\alpha - f)\mathbb{1}_t \in \mathscr{D}_{|t}$. By the definition of $\mathscr{D}_{|t}$, we then have that, for some $n \in \mathbb{N}_0$, $g_1,\dots,g_n \in \mathscr{I}$, $s_1,\dots,s_n \in \mathscr{X}^*$, with $\underline{\mathrm{E}}(g_i|ts_i) > 0$ for all $i \in \{1,\dots,n\}$, and some $h \in \mathscr{L}_{\geq}(\Omega)$,

$$(\alpha - f)\mathbb{1}_t = \sum_{i=1}^{n} g_i\mathbb{1}_{ts_i} + h. \tag{3.30}$$

Hence,

$$\underline{\mathrm{E}}\left((\alpha - f)\mathbb{1}_t|t\right) = \underline{\mathrm{E}}\left(\textstyle\sum_{i=1}^{n} g_i\mathbb{1}_{ts_i} + h|t\right) \overset{\text{WC2}_{82}}{\geq} \sum_{i=1}^{n}\underline{\mathrm{E}}\left(g_i\mathbb{1}_{ts_i}|t\right) + \underline{\mathrm{E}}\,(h|t) \geq 0, \tag{3.31}$$

where one may again check that each of the considered lower expectations is well-defined because of Eq. (3.30) and the assumptions about $\mathscr{I}$, and where the last inequality follows, on the one hand, from the fact that, for all $i \in \{1,\dots,n\}$, $\underline{\mathrm{E}}\,(g_i|ts_i) > 0$ and therefore by WC14$_{85}$ that $\underline{\mathrm{E}}\left(g_i\mathbb{1}_{ts_i}|t\right) \geq 0$, and on the other hand, from the fact that $h \geq 0$ and therefore by WC6$_{84}$ that $\underline{\mathrm{E}}\,(h|t) \geq 0$. The left-hand side of Eq. (3.31) can also be seen to be equal to

$$\underline{\mathrm{E}}\left((\alpha - f)\mathbb{1}_t|t\right) \overset{\text{WC11}_{85}}{=} \underline{\mathrm{E}}\,(\alpha - f|t) \overset{\text{WC7}_{84}}{=} \alpha - \overline{\mathrm{E}}\,(f|t)\,.$$

Hence, by Eq. (3.31), we have that $\alpha \geq \overline{\mathrm{E}}(f|t)$. But this holds for any $\alpha \in \mathbb{R}$ such that $(\alpha - f)\mathbb{1}_t \in \mathscr{D}$, so

$$\overline{\mathrm{E}}(f|t) \leq \inf\{\alpha \in \mathbb{R}\colon (\alpha - f)\mathbb{1}_t \in \mathscr{D}\} = \overline{\mathrm{E}}_{\mathscr{D}}(f|t),$$

which is the inequality that we were after. So $\overline{\mathrm{E}}$ coincides with the upper expectation $\overline{\mathrm{E}}_{\mathscr{D}}$ corresponding to the coherent set of acceptable gambles $\mathscr{D}$, which implies by Lemma 3.C.1$_{109}$ that $\overline{\mathrm{E}}$ is coherent. This establishes the sufficiency of WC1$_{82}$–WC4$_{82}$, and therefore concludes the proof. □

3.D Proof of Theorem 3.4.6

The proof of Theorem 3.4.6$_{88}$ is divided into two parts: showing that Axioms NE1$_{88}$–NE4$_{88}$ (or NE1$_{88}$–NE3$_{88}$) are sufficient for a global upper expectation to be the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$), and showing that there always is a global upper expectation satisfying Axioms NE1$_{88}$–NE4$_{88}$. We start with the latter; more specifically, we show that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—or equivalently, $\overline{\mathrm{E}}_{\mathscr{A}}$—satisfies NE1$_{88}$–NE4$_{88}$ for any acceptable gambles tree $\mathscr{A}_\bullet$.

3.D.1 Axioms NE1–NE4 are consistent

The following two results establish respectively NE1$_{88}$ and NE4$_{88}$ for $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$.

Lemma 3.D.1. *For any acceptable gambles tree $\mathscr{A}_\bullet$ and upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agree according to Eq.* (3.1)$_{50}$*, we have that*

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}_{x_{1:k}}(f) \text{ for all } f \in \mathscr{L}(\mathscr{X}) \text{ and all } x_{1:k} \in \mathscr{X}^*.$$

Proof. Since $\mathscr{A}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ agree, we have that $\overline{\mathrm{Q}}_\bullet = \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ and so we will make no particular distinction in notation between $\overline{\mathrm{Q}}_\bullet$ and $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ in the following reasoning.

Consider any $f \in \mathscr{L}(\mathscr{X})$ and any $x_{1:k} \in \mathscr{X}^*$. Fix any $\epsilon > 0$ and observe that by Eq. (3.1)$_{50}$ there is an $\alpha \in \mathbb{R}$ such that $\alpha \leq \overline{\mathrm{Q}}_{x_{1:k}}(f) + \epsilon$ and $\alpha - f \in \mathscr{A}_{x_{1:k}}$. Let $\mathscr{M}$ be the real process that is equal to the constant α for all situations s such that $s \not\sqsupset x_{1:k}$, and that is equal to $f(x_{k+1})$ for all situations s such that $s \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Then for all $s \neq x_{1:k}$, we have that $\Delta\mathscr{M}(s) = 0$ and therefore, since $0 \in \mathscr{A}_s$ due to coherence [D1$_{27}$], that $-\Delta\mathscr{M}(s) \in \mathscr{A}_s$. For the situation $x_{1:k}$ itself, we have that $\Delta\mathscr{M}(x_{1:k}) = f - \alpha$ and thus, because of how we have chosen α, that $-\Delta\mathscr{M}(x_{1:k}) \in \mathscr{A}_{x_{1:k}}$. So we obtain that $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$. Since by the definition of $\mathscr{M}$ we clearly also have that $\mathscr{M}(X_{1:k+1}) \geq_{x_{1:k}} f(X_{k+1})$, the definition of $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ [Eq. (3.11)$_{63}$] implies that

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k}) \leq \mathscr{M}(x_{1:k}) = \alpha \leq \overline{\mathrm{Q}}_{x_{1:k}}(f) + \epsilon.$$

This holds for any $\epsilon > 0$, so we find that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k}) \leq \overline{\mathrm{Q}}_{x_{1:k}}(f)$.

Conversely, consider any $\mathscr{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:\ell}) \geq_{x_{1:k}} f(X_{k+1})$ for some $\ell \geq k$. First consider the case that $\ell = k$. Then we have that $\mathscr{M}(x_{1:k}) \geq_{x_{1:k}} f(X_{k+1})$, and thus that $\mathscr{M}(x_{1:k}) \geq \sup f(X_{k+1}) = \sup f$. By the coherence [C1$_{32}$] of $\overline{\mathrm{Q}}_{x_{1:k}}$, this implies that

$$\mathscr{M}(x_{1:k}) \geq \overline{\mathrm{Q}}_{x_{1:k}}(f).$$

We now proceed to prove the same for the case that $\mathcal{M}(X_{1:\ell}) \geq_{x_{1:k}} f(X_{k+1})$ with $\ell > k$. Then, for any $x_{k+1} \in \mathcal{X}$, we also have that $\mathcal{M}(x_{1:k+1}X_{k+2:\ell}) \geq_{x_{1:k+1}} f(x_{k+1})$. By Lemma 3.2.5$_{63}$, we have that $\mathcal{M}(x_{1:k+1}) \geq \mathcal{M}(x_{1:\ell})$ for some $x_{k+2:\ell} \in \mathcal{X}^{\ell-k-1}$, so the previous implies that $\mathcal{M}(x_{1:k+1}) \geq f(x_{k+1})$. Since this holds for any $x_{k+1} \in \mathcal{X}$, we obtain that $\mathcal{M}(x_{1:k}\cdot) \geq f$ and thus that $\Delta\mathcal{M}(x_{1:k}) \geq f - \mathcal{M}(x_{1:k})$, or equivalently that $-\Delta\mathcal{M}(x_{1:k}) \leq \mathcal{M}(x_{1:k}) - f$. Since $-\Delta\mathcal{M}(x_{1:k}) \in \mathscr{A}_{x_{1:k}}$ by the fact that $\mathcal{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$, the monotonicity property D5$_{28}$ implies that also $\mathcal{M}(x_{1:k}) - f \in \mathscr{A}_{x_{1:k}}$. But then it follows from Eq. (3.1)$_{50}$ that

$$\mathcal{M}(x_{1:k}) \geq \overline{\mathrm{Q}}_{x_{1:k}}(f).$$

Hence, we have that $\mathcal{M}(x_{1:k}) \geq \overline{\mathrm{Q}}_{x_{1:k}}(f)$ for any $\mathcal{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathcal{M}(X_{1:\ell}) \geq_{x_{1:k}} f(X_{k+1})$ for some $\ell \geq k$, which implies by Eq. (3.11)$_{63}$ that

$$\overline{\mathrm{Q}}_{x_{1:k}}(f) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k}).$$

Together with the previously obtained inequality that $\overline{\mathrm{Q}}_{x_{1:k}}(f) \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k})$, we obtain that $\overline{\mathrm{Q}}_{x_{1:k}}(f) = \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f(X_{k+1})|x_{1:k})$. □

Proposition 3.D.2. *For any upper expectations tree $\mathscr{A}_\bullet$, any $f \in \mathbb{V}$ and $s \in \mathcal{X}^*$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = \inf\left\{\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\}.$$

Proof. We first prove that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \geq \inf\left\{\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\}$. Consider any real $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$. Then there is a supermartingale $\mathcal{M} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ such that $\mathcal{M}(s) \leq \alpha$ and $\mathcal{M}(X_{1:k}) \geq_s f$ for some $k \geq |s|$. Let $g := \mathcal{M}(X_{1:k})$. Then we clearly have that $g \in \mathbb{F}$ and that $g \geq_s f$. Since $g = \mathcal{M}(X_{1:k})$, we surely have that $\mathcal{M}(X_{1:k}) \geq_s g$, which implies that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \leq \mathcal{M}(s) \leq \alpha$. Since we know that $g \in \mathbb{F}$ and that $g \geq_s f$, we obtain that

$$\inf\{\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\} \leq \alpha.$$

Since this holds for any real $\alpha > \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$, we conclude that

$$\inf\left\{\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\} \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s).$$

The remaining inequality follows trivially from the fact that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ is monotone [WC5$_{84}$], which can easily be inferred from the definition of $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ [Eq. (3.11)$_{63}$]. Indeed, for any $g \in \mathbb{F}$ such that $g \geq_s f$, we have by WC5$_{84}$ that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$, and so also that

$$\inf\left\{\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\} \geq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s).$$ □

The following lemma gathers the two lemmas above, and combines them with Proposition 3.2.10$_{67}$ to show that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies NE1$_{88}$–NE4$_{88}$.

Lemma 3.D.3. *For any acceptable gambles tree $\mathscr{A}_\bullet$ and any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agree according to Eq.* (3.1)$_{50}$*, we have that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies* NE1$_{88}$–NE4$_{88}$.

Proof. Lemma 3.D.1$_{114}$, Proposition 3.2.10$_{67}$ and Proposition 3.D.2$_{\curvearrowleft}$ guarantee that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies respectively NE1$_{88}$, NE3$_{88}$ and NE4$_{88}$. That $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ moreover satisfies NE2$_{88}$ follows straightforwardly from its definition [Eq. (3.11)$_{63}$]. □

3.D.2 Proving the existence and uniqueness of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ using Axioms NE1–NE4

We next aim to show that, apart from being internally consistent, Axioms NE1$_{88}$–NE4$_{88}$ (or NE1$_{88}$–NE3$_{88}$) also suffice for a global upper expectation to be equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$). The existence—and uniqueness—of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ will then follow automatically. We start with a rather basic but technical lemma. It uses, for any situation $x_{1:k} \in \mathscr{X}^*$ and any $(k+1)$-measurable gamble $g(X_{1:k+1})$, the notation $g(x_{1:k}\cdot)$ to denote the local gamble on $\mathscr{X}$ that assumes the value $g(x_{1:k+1})$ in $x_{k+1} \in \mathscr{X}$.

Lemma 3.D.4. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F}\times\mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and that satisfies* WC11$_{85}$ *on $\mathbb{F}\times\mathscr{X}^*$. Then, for any $x_{1:k} \in \mathscr{X}^*$ and any $(k+1)$-measurable gamble f,*

$$\overline{\mathrm{E}}(f|x_{1:k}) = \overline{\mathrm{Q}}_{x_{1:k}}(f(x_{1:k}\cdot)).$$

Proof. We have that $f(x_{1:k}X_{k+1})\mathbb{1}_{x_{1:k}} = f\mathbb{1}_{x_{1:k}}$ and hence, because $\overline{\mathrm{E}}$ satisfies WC11$_{85}$ and coincides with $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$,

$$\begin{aligned}\overline{\mathrm{E}}(f|x_{1:k}) = \overline{\mathrm{E}}(f\mathbb{1}_{x_{1:k}}|x_{1:k}) = \overline{\mathrm{E}}(f(x_{1:k}X_{k+1})\mathbb{1}_{x_{1:k}}|x_{1:k}) &= \overline{\mathrm{E}}(f(x_{1:k}X_{k+1})|x_{1:k})\\ &= \overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}(f(x_{1:k}X_{k+1})|x_{1:k})\\ &= \overline{\mathrm{Q}}_{x_{1:k}}(f(x_{1:k}\cdot)),\end{aligned}$$

where the last equality follows from Eq. (3.13)$_{85}$. □

The following lemma shows that Axioms NE1$_{88}$–NE3$_{88}$ fix the values of a global upper expectation completely on the domain $\mathbb{F}\times\mathscr{X}^*$. For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $k \in \mathbb{N}_0$ and any $(k+1)$-measurable gamble $g(X_{1:k+1})$, we write $\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))$ to denote the k-measurable gamble defined by

$$\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))(x_{1:k}) := \overline{\mathrm{Q}}_{x_{1:k}}(g(x_{1:k}\cdot)) \text{ for all } x_{1:k} \in \mathscr{X}^k.$$

Note that $\overline{\mathrm{Q}}_{X_{1:k}}(g(X_{1:k+1}))$ is indeed a (bounded) gamble, because the local upper expectation $\overline{\mathrm{Q}}_{x_{1:k}}(g(x_{1:k}\cdot))$ for all $x_{1:k} \in \mathscr{X}^k$ is real due to coherence [C5$_{33}$].

Lemma 3.D.5. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F}\times\mathscr{X}^*$ that satisfies* NE1$_{88}$–NE3$_{88}$. *Then, for any $(f, x_{1:k}) \in \mathbb{F}\times\mathscr{X}^*$,*

$$\overline{\mathrm{E}}(f|x_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots\overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\cdots\big)\big)(x_{1:k}),$$

with $\ell \geq k$ any natural number such that f is $(\ell+1)$-measurable.

Proof. First note that Lemma 3.D.4$_{\leftarrow}$ can be applied to $\overline{\mathrm{E}}$ here because $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ due to NE1$_{88}$ and the definition of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ [Eq. (3.13)$_{85}$], and because $\overline{\mathrm{E}}$ satisfies WC11$_{85}$ on $\mathbb{F} \times \mathscr{X}^*$ due to NE2$_{88}$. As a result, the variable $\overline{\mathrm{E}}(f|X_{1:\ell})$ is equal to $\overline{\mathrm{Q}}_{X_{1:\ell}}(f)$ due to Lemma 3.D.4$_{\leftarrow}$ and because f is $(\ell+1)$-measurable. Since $\overline{\mathrm{Q}}_{X_{1:\ell}}(f)$ is real-valued due to the coherence [C5$_{33}$] of $\overline{\mathrm{Q}}_{x_{1:\ell}}$ for all $x_{1:\ell} \in \mathscr{X}^{\ell}$, we obtain that $\overline{\mathrm{E}}(f|X_{1:\ell}) = \overline{\mathrm{Q}}_{X_{1:\ell}}(f)$ is real-valued, and more specifically an ℓ-measurable gamble.

Next, consider the term $\overline{\mathrm{E}}(f|X_{1:\ell-1})$. Since $\overline{\mathrm{E}}(f|X_{1:\ell})$ is real-valued, NE3$_{88}$ says that

$$\overline{\mathrm{E}}(f|X_{1:\ell-1}) = \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:\ell})|X_{1:\ell-1}) = \overline{\mathrm{E}}(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)|X_{1:\ell-1}).$$

Then, since $\overline{\mathrm{Q}}_{X_{1:\ell}}(f)$ is an ℓ-measurable gamble, Lemma 3.D.4$_{\leftarrow}$ implies that

$$\overline{\mathrm{E}}(f|X_{1:\ell-1}) = \overline{\mathrm{E}}(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)|X_{1:\ell-1}) = \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big).$$

Again, since $\overline{\mathrm{Q}}_{X_{1:\ell-1}}(\cdot)$ is real-valued due to the coherence of the local upper expectations $\overline{\mathrm{Q}}_{x_{1:\ell-1}}$, we obtain that $\overline{\mathrm{E}}(f|X_{1:\ell-1})$ is real-valued, and more specifically an $(\ell-1)$-measurable gamble.

We can then apply the same reasoning to the next term $\overline{\mathrm{E}}(f|X_{1:\ell-2})$. Since $\overline{\mathrm{E}}(f|X_{1:\ell-1})$ is real-valued, NE3$_{88}$ and the expression for $\overline{\mathrm{E}}(f|X_{1:\ell-1})$ above imply that

$$\overline{\mathrm{E}}(f|X_{1:\ell-2}) = \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:\ell-1})|X_{1:\ell-2}) = \overline{\mathrm{E}}\big(\overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)|X_{1:\ell-2}\big).$$

The fact that $\overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)$ is $\ell-1$-measurable then implies by Lemma 3.D.4$_{\leftarrow}$ that

$$\overline{\mathrm{E}}(f|X_{1:\ell-2}) = \overline{\mathrm{Q}}_{X_{1:\ell-2}}\big(\overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\big).$$

Now, we can continue to repeat the above reasoning, and it is clear that this will eventually yield

$$\overline{\mathrm{E}}(f|X_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\cdots\big)\big),$$

and therefore, in particular, that

$$\overline{\mathrm{E}}(f|x_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\cdots\big)\big)(x_{1:k}).$$ □

Finally, before proving Theorem 3.4.6$_{88}$, we also need to establish that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies WC1$_{82}$–WC4$_{82}$ for any acceptable gambles tree $\mathscr{A}_{\bullet}$.

Lemma 3.D.6. *For any acceptable gambles tree $\mathscr{A}_{\bullet}$, the upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies* WC1$_{82}$–WC4$_{82}$.

Proof. Recall that $\overline{\mathrm{E}}_{\mathscr{A}}$ is deduced from the set of acceptable global gambles $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ according to Eq. (3.10)$_{60}$, and that the set $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ is coherent by Proposition 3.2.3$_{58}$ [and the definition of the natural extension of a set of acceptable gambles; see Eq. (3.7)$_{57}$ and Definition 2.7$_{38}$]. Hence, by Lemma 3.C.1$_{109}$, we have that $\overline{\mathrm{E}}_{\mathscr{A}}$ is coherent, and thus by Theorem 3.4.3$_{84}$, that $\overline{\mathrm{E}}_{\mathscr{A}}$ satisfies WC1$_{82}$–WC4$_{82}$. The desired statement now follows from Theorem 3.2.7$_{65}$. □

Proof of Theorem 3.4.6$_{88}$. We first show that $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$ exists, and that it is the unique global upper expectation on $\mathbb{F}\times\mathscr{X}^*$ that satisfies NE1$_{88}$–NE3$_{88}$. To this end, it suffices to prove that, for any global upper expectation $\overline{\mathrm{E}}^*\colon \mathbb{F}\times\mathscr{X}^* \to \overline{\mathbb{R}}$ that satisfies NE1$_{88}$–NE3$_{88}$, $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$. Indeed, the existence of such an upper expectation $\overline{\mathrm{E}}^*$, and therefore the existence of $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$, then follows from the fact that the restriction of $\overline{\mathrm{E}}_{\mathscr{A},\mathrm{V}}^{\mathrm{f}}$ to $\mathbb{F}\times\mathscr{X}^*$, with $\mathscr{A}_\bullet$ any acceptable gambles tree that agrees with $\overline{\mathrm{Q}}_\bullet$ [such a tree $\mathscr{A}_\bullet$ exists; recall the discussion surrounding Eq. (3.2)$_{51}$], satisfies NE1$_{88}$–NE3$_{88}$ due to Lemma 3.D.3$_{115}$. The uniqueness of $\overline{\mathrm{E}}^*$ then moreover follows from the equality with $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$, and the fact that $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$ is unique as a result of its definition.

So fix any global upper expectation $\overline{\mathrm{E}}^*\colon \mathbb{F}\times\mathscr{X}^* \to \overline{\mathbb{R}}$ that satisfies NE1$_{88}$–NE3$_{88}$. In order to show that $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$, we need to prove three properties of $\overline{\mathrm{E}}^*$; that $\overline{\mathrm{E}}^*$ extends $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$, that $\overline{\mathrm{E}}^*$ satisfies WC1$_{82}$–WC4$_{82}$, and that $\overline{\mathrm{E}}^*$ is larger or equal than any other global upper expectation on $\mathbb{F}\times\mathscr{X}^*$ extending $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ and satisfying WC1$_{82}$–WC4$_{82}$. The first property follows from the fact that $\overline{\mathrm{E}}^*$ satisfies NE1$_{88}$. The second property, that $\overline{\mathrm{E}}^*$ satisfies WC1$_{82}$–WC4$_{82}$, can be deduced as follows. Since $\overline{\mathrm{E}}^*$ satisfies NE1$_{88}$–NE3$_{88}$ by definition, and since, for any acceptable gambles tree $\mathscr{A}_\bullet$ that agrees with $\overline{\mathrm{Q}}_\bullet$ [again, there is at least one such a tree $\mathscr{A}_\bullet$], $\overline{\mathrm{E}}_{\mathscr{A},\mathrm{V}}^{\mathrm{f}}$ satisfies NE1$_{88}$–NE3$_{88}$ by Lemma 3.D.3$_{115}$, it follows from the expression in Lemma 3.D.5$_{116}$ that $\overline{\mathrm{E}}^*$ and $\overline{\mathrm{E}}_{\mathscr{A},\mathrm{V}}^{\mathrm{f}}$ coincide on the domain $\mathbb{F}\times\mathscr{X}^*$. Lemma 3.D.6$_{\curvearrowleft}$ says that $\overline{\mathrm{E}}_{\mathscr{A},\mathrm{V}}^{\mathrm{f}}$ satisfies WC1$_{82}$–WC4$_{82}$ on its entire domain, and thus also on the restricted domain $\mathbb{F}\times\mathscr{X}^*$, which implies that $\overline{\mathrm{E}}^*$ also satisfies WC1$_{82}$–WC4$_{82}$.

So it only remains to prove that $\overline{\mathrm{E}}^*$ is larger or equal than any other global upper expectation on $\mathbb{F}\times\mathscr{X}^*$ extending $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ and satisfying WC1$_{82}$–WC4$_{82}$. Fix any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F}\times\mathscr{X}^*$ extending $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ and satisfying WC1$_{82}$–WC4$_{82}$, and fix any $(f, x_{1:k}) \in \mathbb{F}\times\mathscr{X}^*$. Let $\ell \geq k$ be any natural number such that f is $(\ell+1)$-measurable [there surely exists such an ℓ because f is finitary]. Since $\overline{\mathrm{E}}$ satisfies WC1$_{82}$–WC4$_{82}$ by assumption, it follows from Proposition 3.4.4$_{84}$ that $\overline{\mathrm{E}}$ satisfies WC13$_{85}$ and WC5$_{84}$, and so we have that

$$\begin{aligned}\overline{\mathrm{E}}(f|X_{1:k}) &\leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k})\\ &\leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+2})|X_{1:k+1})|X_{1:k})\\ &\leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\cdots\overline{\mathrm{E}}(f|X_{1:\ell})\cdots|X_{1:k+1})|X_{1:k}).\end{aligned} \tag{3.32}$$

We can then replace each of the global upper expectations on the right-hand side by local upper expectations. Indeed, $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ by assumption and moreover satisfies WC11$_{85}$ due to Proposition 3.4.4$_{84}$ and the fact that it satisfies WC1$_{82}$–WC4$_{82}$ by assumption. So by Lemma 3.D.4$_{116}$ we infer that, for any $x_{1:i} \in \mathscr{X}^*$ and any $(i+1)$-measurable gamble g,

$$\overline{\mathrm{E}}(g|z_{1:i}) = \overline{\mathrm{Q}}_{z_{1:i}}(g(z_{1:i}\cdot)).$$

Equivalently, we can write that, for any $i \in \mathbb{N}_0$ and any $(i+1)$-measurable gamble g,

$$\overline{\mathrm{E}}(g|X_{1:i}) = \overline{\mathrm{Q}}_{X_{1:i}}(g).$$

Applying this equation to each of the (global) upper expectations in the expression on the right-hand side of Eq. (3.32)$_{\leftarrow}$,[12] we obtain that

$$\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\cdots\big)\big). \tag{3.33}$$

On the other hand, since $\overline{\mathrm{E}}^*$ satisfies NE1$_{88}$–NE3$_{88}$, we have by Lemma 3.D.5$_{116}$ that

$$\overline{\mathrm{E}}^*(f|x_{1:k}) = \overline{\mathrm{Q}}_{X_{1:k}}\big(\overline{\mathrm{Q}}_{X_{1:k+1}}\big(\cdots \overline{\mathrm{Q}}_{X_{1:\ell-1}}\big(\overline{\mathrm{Q}}_{X_{1:\ell}}(f)\big)\cdots\big)\big)(x_{1:k}).$$

Combined with Eq. (3.33), this implies that $\overline{\mathrm{E}}(f|x_{1:k}) \leq \overline{\mathrm{E}}^*(f|x_{1:k})$. Since this is the case for any $(f, x_{1:k}) \in \mathbb{F} \times \mathscr{X}^*$ and any global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{F} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfies WC1$_{82}$–WC4$_{82}$, we indeed have that $\overline{\mathrm{E}}^*$ is the natural extension $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ to $\mathbb{F} \times \mathscr{X}^*$ under WC1$_{82}$–WC4$_{82}$.

The second part of this proof is now concerned with showing that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ exists, and that it is the unique global upper expectation on $\mathbb{V} \times \mathscr{X}^*$ that satisfies NE1$_{88}$–NE4$_{88}$. In the same way as before, it suffices to prove that, for any global upper expectation $\overline{\mathrm{E}}^* \colon \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ that satisfies NE1$_{88}$–NE4$_{88}$, $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. Indeed, the existence of $\overline{\mathrm{E}}^*$, and in that case the existence of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, then follows from the fact that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, with $\mathscr{A}_\bullet$ any acceptable gambles tree that agrees with $\overline{\mathrm{Q}}_\bullet$ [where, again, there is at least one such a tree $\mathscr{A}_\bullet$], satisfies NE1$_{88}$–NE4$_{88}$ due to Lemma 3.D.3$_{115}$. The uniqueness of $\overline{\mathrm{E}}^*$ then again follows from the equality with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is unique due to its definition.

So fix any global upper expectation $\overline{\mathrm{E}}^* \colon \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ that satisfies NE1$_{88}$–NE4$_{88}$. Since, as we have just proved above, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ exists and is the unique global upper expectation on $\mathbb{F} \times \mathscr{X}^*$ that satisfies NE1$_{88}$–NE3$_{88}$, we have that $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$. So, because $\overline{\mathrm{E}}^*$ satisfies NE4$_{88}$, we find that, for any $(f, s) \in \mathbb{V} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}^*(f|s) = \inf\Big\{\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\Big\}. \tag{3.34}$$

In order to show that $\overline{\mathrm{E}}^*$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, we need to show that $\overline{\mathrm{E}}^*$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, that $\overline{\mathrm{E}}^*$ satisfies WC1$_{82}$–WC4$_{82}$, and that $\overline{\mathrm{E}}^*$ is larger or equal than any other global upper expectation on $\mathbb{V} \times \mathscr{X}^*$ extending $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfying WC1$_{82}$–WC4$_{82}$. The first property follows from the fact that $\overline{\mathrm{E}}^*$ satisfies NE1$_{88}$. The second property, that $\overline{\mathrm{E}}^*$ satisfies WC1$_{82}$–WC4$_{82}$, can be deduced in a similar way as before. Since, for any acceptable gambles tree $\mathscr{A}_\bullet$ that agrees with $\overline{\mathrm{Q}}_\bullet$, the global upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ satisfies NE1$_{88}$–NE4$_{88}$ by Lemma 3.D.3$_{115}$, we have that, similarly as for $\overline{\mathrm{E}}^*$ [which is simply a general global upper expectation satisfying NE1$_{88}$–NE4$_{88}$],

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = \inf\Big\{\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\Big\}.$$

So, by Eq. (3.34), $\overline{\mathrm{E}}^*$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ coincide. Hence, it follows from Lemma 3.D.6$_{117}$ that $\overline{\mathrm{E}}^*$ satisfies WC1$_{82}$–WC4$_{82}$.

It now only remains to prove that $\overline{\mathrm{E}}^*$ is larger or equal than any other global upper expectation on $\mathbb{V} \times \mathscr{X}^*$ extending $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfying WC1$_{82}$–WC4$_{82}$. Fix any

[12] Note that the arguments of the (global) upper expectations in the expression on the right-hand side of Eq. (3.32)$_{\leftarrow}$ must be (finitary) gambles because $\overline{\mathrm{E}}$ is only defined on $\mathbb{F} \times \mathscr{X}^*$, and so otherwise this expression cannot be valid.

global upper expectation $\overline{\mathrm{E}}$ on $\mathbb{V} \times \mathscr{X}^*$ extending $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ and satisfying WC1$_{82}$–WC4$_{82}$. Then note that, since $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}$ is by definition the pointwise largest extension of $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{pre}}$ to $\mathbb{F} \times \mathscr{X}^*$ under WC1$_{82}$–WC4$_{82}$, we have that $\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}(g|s) \geq \overline{\mathrm{E}}(g|s)$ for all $(g,s) \in \mathbb{F} \times \mathscr{X}^*$. Hence, for any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, due to Eq. (3.34)$_{\frown}$,

$$\overline{\mathrm{E}}^*(f|s) = \inf\left\{\overline{\mathrm{E}}_{\mathrm{Q}}^{\mathrm{fin}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\} \\ \geq \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\} \geq \overline{\mathrm{E}}(f|s),$$

where the last equality follows from the fact that $\overline{\mathrm{E}}$ satisfies WC5$_{84}$, which itself follows from Proposition 3.4.4$_{84}$ and the fact that $\overline{\mathrm{E}}$ satisfies WC1$_{82}$–WC4$_{82}$. □

3.E Proof of Proposition 3.5.5

3.E.1 Topological results for precise probability trees

Recall from Section 2.1$_{18}$ that $\mathbb{P}(\mathscr{X})$ denotes the set of all probability mass functions on $\mathscr{X}$. Let $d(\cdot,\cdot)$ be the total variation distance [24, Section 7.1] defined, for any two mass functions $\pi_1, \pi_2 \in \mathbb{P}(\mathscr{X})$, by

$$d(\pi_1, \pi_2) := \max_{A \subseteq \mathscr{X}} |\pi_1(A) - \pi_2(A)| = \frac{1}{2} \sum_{x \in \mathscr{X}} |\pi_1(x) - \pi_2(x)|, \qquad (3.35)$$

where we allowed ourselves a slight abuse of notation by writing $\pi_i(A)$ to mean $\sum_{x \in A} \pi_i(x)$ for $i \in \{1,2\}$. Let $\mathbb{P}(\mathscr{X})$ be endowed with the topology induced by d, which is equivalent—see [24, Appendix A]—to the topology of pointwise convergence that we have implicitly adopted in the main text. So $\mathbb{P}(\mathscr{X})$ is metrizable and, by [24, Section 7], compact. Also, note that any precise probability tree $p \colon s \in \mathscr{X}^* \mapsto p(\cdot|s) \in \mathbb{P}(\mathscr{X})$ can be regarded as an element of the product space $\times_{s \in \mathscr{X}^*} \mathbb{P}(\mathscr{X})$, and that any imprecise probability tree $\mathscr{P}_\bullet$ can be seen as a subset of $\times_{s \in \mathscr{X}^*} \mathbb{P}(\mathscr{X})$. Saying that a precise probability tree p is compatible with an imprecise probability tree $\mathscr{P}_\bullet$ is then the same as saying that $p \in \mathscr{P}_\bullet$. We will moreover endow the space $\times_{s \in \mathscr{X}^*} \mathbb{P}(\mathscr{X})$ with the product topology or, equivalently, the topology of pointwise convergence. A sequence of precise probability trees $(p_i)_{i \in \mathbb{N}}$ then converges if, for each situation $s \in \mathscr{X}^*$, the mass functions $(p_i(\cdot|s))_{i \in \mathbb{N}}$ converge pointwise.

Lemma 3.E.1. *Any sequence $(p_i)_{i \in \mathbb{N}}$ of precise probability trees has a convergent subsequence.*

Proof. Since by Tychonoff's theorem [111, Theorem 17.8] any product of compact spaces is compact in the product topology, the compactness of $\mathbb{P}(\mathscr{X})$ [and the fact that $\times_{s \in \mathscr{X}^*} \mathbb{P}(\mathscr{X})$ is endowed with the product topology] implies the compactness of $\times_{s \in \mathscr{X}^*} \mathbb{P}(\mathscr{X})$. Moreover, [111, Theorem 22.3] says that any countable product of metrizable spaces [if equipped with the product topology] is itself metrizable, so the metrizability of $\mathbb{P}(\mathscr{X})$ and the countability of $\mathscr{X}^*$ imply the metrizability of $\bigtimes_{s\in\mathscr{X}^*}\mathbb{P}(\mathscr{X})$. [111, 17G.3.] says that for metric spaces—and thus also for metrizable spaces—compactness is equivalent to sequential compactness. Since we know $\bigtimes_{s\in\mathscr{X}^*}\mathbb{P}(\mathscr{X})$ to be compact and metrizable, we infer that it is sequentially compact. Hence, by definition of sequential compactness, each sequence $(p_i)_{i\in\mathbb{N}}$ (of precise probability trees) in $\bigtimes_{s\in\mathscr{X}^*}\mathbb{P}(\mathscr{X})$ has a convergent subsequence. □

Lemma 3.E.2. *Consider any sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees that converges to some limit probability tree p. Then we have that*

$$\lim_{i\to+\infty}\overline{\mathrm{E}}_{p_i}(g|s)=\overline{\mathrm{E}}_p(g|s)\text{ for all }g\in\mathbb{F}\text{ and all }s\in\mathscr{X}^*.$$

Proof. Fix any $g\in\mathbb{F}$ and any $s\in\mathscr{X}^*$. Observe that, by Proposition 3.3.8$_{79}$,

$$\overline{\mathrm{E}}_p(g|s)=\sum_{z_{1:\ell}\in\mathscr{X}^\ell}g(z_{1:\ell})\mathrm{P}_p(z_{1:\ell}|s) \tag{3.36}$$

and, for all $i\in\mathbb{N}$,

$$\overline{\mathrm{E}}_{p_i}(g|s)=\sum_{z_{1:\ell}\in\mathscr{X}^\ell}g(z_{1:\ell})\mathrm{P}_{p_i}(z_{1:\ell}|s), \tag{3.37}$$

where P_p and P_{p_i} on $\langle\mathscr{X}^*\rangle\times\mathscr{X}^*$ are related to respectively p and p_i according to Proposition 3.3.4$_{73}$, and where $\ell>|s|$ is any natural number such that g is ℓ-measurable. Let $x_{1:k}\in\mathscr{X}^*$ be such that $s=x_{1:k}$ and fix any $z_{1:\ell}\in\mathscr{X}^\ell$. Then we have that $\ell>k$. We next show that $\mathrm{P}_{p_i}(z_{1:\ell}|s)$ converges to $\mathrm{P}_p(z_{1:\ell}|s)$ as a function of $i\in\mathbb{N}$.

If $z_{1:k}\neq x_{1:k}$, then $\mathrm{P}_{p_i}(z_{1:\ell}|s)=0$ for all $i\in\mathbb{N}$ and also $\mathrm{P}_p(z_{1:\ell}|s)=0$, so $\mathrm{P}_{p_i}(z_{1:\ell}|s)$ surely converges to $\mathrm{P}_p(z_{1:\ell}|s)$. So it remains to check whether it is true for the case that $z_{1:k}=x_{1:k}$. In that case, $\mathrm{P}_{p_i}(z_{1:\ell}|x_{1:k})=\prod_{n=k}^{\ell-1}p_i(z_{n+1}|z_{1:n})$ converges to $\mathrm{P}_p(z_{1:\ell}|x_{1:k})=\prod_{n=k}^{\ell-1}p\,(z_{n+1}|z_{1:n})$ if, for all $n\in\{k,\dots,\ell-1\}$, $p_i(z_{n+1}|z_{1:n})$ converges to $p\,(z_{n+1}|z_{1:n})$. The latter is implied by the convergence of p_i to p. Indeed, since $\bigtimes_{s\in\mathscr{X}^*}\mathbb{P}(\mathscr{X})$ is equipped with the product topology, the convergence of p_i to p implies that, for any $n\in\{k,\dots,\ell-1\}$, the mass function $p_i(\cdot|z_{1:n})$ converges to $p\,(\cdot|z_{1:n})$. Since the set $\mathbb{P}(\mathscr{X})$ on its turn is equipped with the topology of pointwise convergence, this implies that $p_i(z_{n+1}|z_{1:n})$ converges to $p\,(z_{n+1}|z_{1:n})$.

Now, to conclude the proof, note that the sums in Equations (3.36) and (3.37) are over a finite set $\mathscr{X}^\ell$—because $\mathscr{X}$ is finite—and the coefficients $g(z_{1:\ell})$ are real because g is a gamble. Since we have just shown that, for any $z_{1:\ell}\in\mathscr{X}^\ell$, the probability $\mathrm{P}_{p_i}(z_{1:\ell}|s)$ converges to $\mathrm{P}_p(z_{1:\ell}|s)$, it is therefore clear that the expectation $\overline{\mathrm{E}}_{p_i}(g|s)$ converges to $\overline{\mathrm{E}}_p(g|s)$. □

The following lemma uses the supremum norm $\|\cdot\|_\infty$ on the set of all gambles $\mathscr{L}(\mathscr{Y})$ on a general non-empty set $\mathscr{Y}$; it is defined by $\|f\|_\infty:=\sup_{y\in\mathscr{Y}}|f(y)|$ for all $f\in\mathscr{L}(\mathscr{Y})$.

Lemma 3.E.3. *Consider any two probability mass functions $p,\tilde{p}\in\mathbb{P}(\mathscr{X})$, and let E_p and $\mathrm{E}_{\tilde{p}}$ be the corresponding linear expectations on $\mathscr{L}(\mathscr{X})$ according to Eq. (2.1)$_{21}$. Then, for any $f\in\mathscr{L}(\mathscr{X})$,*

$$|\mathrm{E}_p(f)-\mathrm{E}_{\tilde{p}}(f)|\leq d(p,\tilde{p})2\|f\|_\infty.$$

Proof. Since $\mathscr{X}$ is finite, [92, Proposition 1] says that

$$\max_{g \in \mathscr{F}_1} |\mathrm{E}_p(g) - \mathrm{E}_{\tilde{p}}(g)| = d(p, \tilde{p}),$$

where $\mathscr{F}_1$ is the set of all non-negative gambles in $\mathscr{L}(\mathscr{X})$ such that $\|g\|_\infty \leq 1$. Fix any $f \in \mathscr{L}(\mathscr{X})$ and note that $f - \min f$ is non-negative. Moreover, if $f - \min f \neq 0$, we have that $\frac{f-\min f}{\|f-\min f\|_\infty} \in \mathscr{F}_1$ and therefore that

$$|\mathrm{E}_p\big(\tfrac{f-\min f}{\|f-\min f\|_\infty}\big) - \mathrm{E}_{\tilde{p}}\big(\tfrac{f-\min f}{\|f-\min f\|_\infty}\big)| \leq d(p, \tilde{p}).$$

It then suffices to use the non-negative homogeneity and the constant additivity of E_p and $\mathrm{E}_{\tilde{p}}$ [since they are defined by Eq. $(2.1)_{21}$] to obtain that indeed

$$|\mathrm{E}_p(f) - \mathrm{E}_{\tilde{p}}(f)| \leq d(p, \tilde{p})\|f - \min f\|_\infty \leq d(p, \tilde{p})2\|f\|_\infty.$$

If on the other hand $f - \min f = 0$, then it is clear that $f = c$ for some $c \in \mathbb{R}$, and therefore that

$$|\mathrm{E}_p(f) - \mathrm{E}_{\tilde{p}}(f)| = |\mathrm{E}_p(c) - \mathrm{E}_{\tilde{p}}(c)| = c - c = 0 \leq d(p, \tilde{p})2\|f\|_\infty.$$

□

Lemma 3.E.4. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and any convergent sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees such that $p_i \sim \mathscr{P}_\bullet$ for all $i \in \mathbb{N}$. Let p be the limit of $(p_i)_{i\in\mathbb{N}}$. Then, for any $\epsilon > 0$, there is a precise probability tree $\tilde{p} \sim \mathscr{P}_\bullet$ such that*

$$|\overline{\mathrm{E}}_p(g|s) - \overline{\mathrm{E}}_{\tilde{p}}(g|s)| \leq \epsilon\|g\|_\infty \text{ for all } g \in \mathbb{F} \text{ and all } s \in \mathscr{X}^*.$$

Proof. Recall that $\times_{s\in\mathscr{X}^*} \mathbb{P}(\mathscr{X})$ is equipped with the product topology, so the convergence of p_i to p implies that, for any $s \in \mathscr{X}^*$, the mass functions $p_i(\cdot\,|s)$ converge to $p\,(\cdot\,|s)$. Since $\mathbb{P}(\mathscr{X})$ was endowed with the topology induced by d, this in turn implies that, for any $s \in \mathscr{X}^*$ and any $\xi > 0$, there is an $i(s,\xi) \in \mathbb{N}$ such that $d(p(\cdot|s), p_i(\cdot|s)) \leq \xi$ for all $i \geq i(s,\xi)$. Now fix any $\epsilon > 0$ and let $(\xi_k)_{k\in\mathbb{N}_0}$ be defined by $\xi_k := \epsilon 2^{-k-1}$ for all $k \in \mathbb{N}_0$. Consider the precise probability tree $\tilde{p}$ defined, for all $s \in \mathscr{X}^*$, by $\tilde{p}(\cdot|s) := p_j(\cdot|s)$ with $j := i(s, \xi_{|s|})$. Then we have that $d(p(\cdot|s), \tilde{p}(\cdot|s)) \leq \xi_{|s|} = \epsilon 2^{-|s|-1}$ for all $s \in \mathscr{X}^*$. Moreover, since $p_i \sim \mathscr{P}_\bullet$ for all $i \in \mathbb{N}$, and therefore, for all $s \in \mathscr{X}^*$, $\tilde{p}(\cdot|s) = p_j(\cdot|s) \in \mathscr{P}_s$ with $j := i(s, \xi_{|s|})$, we also have that $\tilde{p} \sim \mathscr{P}_\bullet$.

Next, let $\mathrm{Q}_\bullet := \overline{\mathrm{Q}}_{\bullet,p}$ and $\mathrm{Q}'_\bullet := \overline{\mathrm{Q}}_{\bullet,\tilde{p}}$ be the (upper) expectations trees associated with respectively p and $\tilde{p}$ according to Eq. $(3.4)_{52}$. Fix any $g \in \mathbb{F}$ and any $x_{1:k} \in \mathscr{X}^*$. Since g is finitary, it is surely $(\ell+1)$-measurable for some $\ell \geq k$. Then, by combining Proposition $3.5.4_{92}$ and Lemma $3.D.5_{116}$, we find that

$$\overline{\mathrm{E}}_p(g|x_{1:k}) = \mathrm{Q}_{X_{1:k}}\big(\mathrm{Q}_{X_{1:k+1}}\big(\cdots \mathrm{Q}_{X_{1:\ell-1}}\big(\mathrm{Q}_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}). \tag{3.38}$$

For all $z_{1:\ell} \in \mathscr{X}^\ell$, Lemma 3.E.3↶ implies that

$$\begin{aligned}\mathrm{Q}_{z_{1:\ell}}(g(z_{1:\ell}\cdot)) &\leq \mathrm{Q}'_{z_{1:\ell}}(g(z_{1:\ell}\cdot)) + d(p(\cdot|z_{1:\ell}), \tilde{p}(\cdot|z_{1:\ell}))2\|g(z_{1:\ell}\cdot)\|_\infty \\ &\leq \mathrm{Q}'_{z_{1:\ell}}(g(z_{1:\ell}\cdot)) + d(p(\cdot|z_{1:\ell}), \tilde{p}(\cdot|z_{1:\ell}))2\|g\|_\infty \\ &\leq \mathrm{Q}'_{z_{1:\ell}}(g(z_{1:\ell}\cdot)) + \epsilon 2^{-\ell}\|g\|_\infty.\end{aligned}$$

So we have that

$$\mathrm{Q}_{X_{1:\ell}}(g) \leq \mathrm{Q}'_{X_{1:\ell}}(g) + \epsilon 2^{-\ell}\|g\|_\infty.$$

Plugging this back into Eq. (3.38)$_\leftarrow$, and using the monotonicity [C4$_{33}$] and the constant additivity [C6$_{33}$] of all local (upper) expectations Q_s, we get that

$$\overline{\mathrm{E}}_p(g|x_{1:k}) \leq \mathrm{Q}_{X_{1:k}}\big(\mathrm{Q}_{X_{1:k+1}}\big(\cdots \mathrm{Q}_{X_{1:\ell-1}}\big(\mathrm{Q}'_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}) + \epsilon 2^{-\ell}\|g\|_\infty. \tag{3.39}$$

Next, we apply a similar reasoning to the local (upper) expectation $\mathrm{Q}_{X_{1:\ell-1}}$ and the corresponding ℓ-measurable gamble $\mathrm{Q}'_{X_{1:\ell}}(g)$. We have by coherence [C5$_{33}$] that $\mathrm{Q}'_{X_{1:\ell}}(g) \leq \|g\|_\infty$. Hence, in the same way as before, we find that

$$\mathrm{Q}_{X_{1:\ell-1}}(\mathrm{Q}'_{X_{1:\ell}}(g)) \leq \mathrm{Q}'_{X_{1:\ell-1}}(\mathrm{Q}'_{X_{1:\ell}}(g)) + \epsilon 2^{-(\ell-1)}\|g\|_\infty.$$

Plugging this back into Eq. (3.39), and using the monotonicity [C4$_{33}$] and the constant additivity [C6$_{33}$] of all local (upper) expectations Q_s, we get that

$$\overline{\mathrm{E}}_p(g|x_{1:k}) \leq \mathrm{Q}_{X_{1:k}}\big(\mathrm{Q}_{X_{1:k+1}}\big(\cdots \mathrm{Q}'_{X_{1:\ell-1}}\big(\mathrm{Q}'_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}) + \epsilon 2^{-(\ell-1)}\|g\|_\infty + \epsilon 2^{-\ell}\|g\|_\infty.$$

We can continue to repeat this reasoning until we eventually arrive at

$$\begin{aligned}\overline{\mathrm{E}}_p(g|x_{1:k}) &\leq \mathrm{Q}'_{X_{1:k}}\big(\mathrm{Q}'_{X_{1:k+1}}\big(\cdots \mathrm{Q}'_{X_{1:\ell-1}}\big(\mathrm{Q}'_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}) + \sum_{n=k}^{\ell}\epsilon 2^{-n}\|g\|_\infty \\ &\leq \mathrm{Q}'_{X_{1:k}}\big(\mathrm{Q}'_{X_{1:k+1}}\big(\cdots \mathrm{Q}'_{X_{1:\ell-1}}\big(\mathrm{Q}'_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}) + \epsilon\|g\|_\infty.\end{aligned}$$

Since, again by Proposition 3.5.4$_{92}$ and Lemma 3.D.5$_{116}$, we have that

$$\overline{\mathrm{E}}_{\tilde{p}}(g|x_{1:k}) = \mathrm{Q}'_{X_{1:k}}\big(\mathrm{Q}'_{X_{1:k+1}}\big(\cdots \mathrm{Q}'_{X_{1:\ell-1}}\big(\mathrm{Q}'_{X_{1:\ell}}(g)\big)\cdots\big)\big)(x_{1:k}),$$

it follows that

$$\overline{\mathrm{E}}_p(g|x_{1:k}) \leq \overline{\mathrm{E}}_{\tilde{p}}(g|x_{1:k}) + \epsilon\|g\|_\infty. \tag{3.40}$$

The inequality above holds for any general $g \in \mathbb{F}$ and any $x_{1:k} \in \mathscr{X}^*$. Since $-g$ is a finitary gamble if g is a finitary gamble, we therefore also have that

$$\overline{\mathrm{E}}_p(-g|s) \leq \overline{\mathrm{E}}_{\tilde{p}}(-g|s) + \epsilon\|g\|_\infty \text{ for all } g \in \mathbb{F} \text{ and all } s \in \mathscr{X}^*,$$

or, by the linearity of $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\tilde{p}}$ on $\mathbb{F}$ [Proposition 3.3.8$_{79}$], that

$$\overline{\mathrm{E}}_p(g|s) \geq \overline{\mathrm{E}}_{\tilde{p}}(g|s) - \epsilon\|g\|_\infty \text{ for all } g \in \mathbb{F} \text{ and all } s \in \mathscr{X}^*.$$

Together with Eq. (3.40) [taking into account that this holds for any general $g \in \mathbb{F}$ and any $x_{1:k} \in \mathscr{X}^*$], and the fact that $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\tilde{p}}$ are real-valued on $\mathbb{F} \times \mathscr{X}^*$ [this can easily be seen from Proposition 3.3.8$_{79}$] we obtain that

$$|\overline{\mathrm{E}}_p(g|s) - \overline{\mathrm{E}}_{\tilde{p}}(g|s)| \leq \epsilon\|g\|_\infty \text{ for all } g \in \mathbb{F} \text{ and all } s \in \mathscr{X}^*,$$

as desired. □

3.E.2 Proof of Proposition 3.5.5

Before we prove NE4$_{88}$ for $\overline{\mathrm{E}}_{\mathscr{P}}$, we first establish that $\overline{\mathrm{E}}_p$ satisfies this property for any precise probability tree p, and show that both $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ are monotone.

Lemma 3.E.5. *For any precise probability tree p, the upper expectation $\overline{\mathrm{E}}_p$ satisfies* NE4$_{88}$.

Proof. First note that $\overline{\mathrm{E}}_p$ satisfies WC5$_{84}$ on $\mathbb{F} \times \mathscr{X}^*$; indeed, this can easily be checked taking into account the last expression in Proposition 3.3.8$_{79}$. We will use this property two times in this proof.

Fix any $(f, s) \in \mathbb{V} \times \mathscr{X}^*$. Then we have that

$$\begin{aligned}\overline{\mathrm{E}}_p(f|s) &= \inf\left\{\overline{\mathrm{E}}_p(g|s) \colon g \in \operatorname{span}(\langle \mathscr{X}^* \rangle) \text{ and } g \geq f\right\}\\ &= \inf\left\{\overline{\mathrm{E}}_p(g|s) \colon g \in \mathbb{F} \text{ and } g \geq f\right\},\end{aligned} \tag{3.41}$$

where the first equality follows from Definition 3.5$_{78}$, Proposition 3.3.6(ii)$_{76}$ and Definition 3.3$_{76}$, and where the second equality follows from Lemma 3.3.5$_{75}$. It thus follows that

$$\overline{\mathrm{E}}_p(f|s) \geq \inf\left\{\overline{\mathrm{E}}_p(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\}.$$

To see that the converse inequality holds, consider any $g \in \mathbb{F}$ such that $g \geq_s f$. Let g' be the gamble that is equal to $g(\omega)$ for all $\omega \in \Gamma(s)$ and that is equal to the constant $\sup f$ for all other paths $\omega \in \Omega \setminus \Gamma(s)$. Then it is clear that $g' \geq f$ and that $g' \in \mathbb{F}$. Hence, Eq. (3.41) implies that $\overline{\mathrm{E}}_p(f|s) \leq \overline{\mathrm{E}}_p(g'|s)$. But we have that $g' =_s g$, and therefore in particular that $g' \leq_s g$, which by WC5$_{84}$ of $\overline{\mathrm{E}}_p$ on $\mathbb{F} \times \mathscr{X}^*$ implies that

$$\overline{\mathrm{E}}_p(f|s) \leq \overline{\mathrm{E}}_p(g'|s) \leq \overline{\mathrm{E}}_p(g|s).$$

Since this holds for any $g \in \mathbb{F}$ such that $g \geq_s f$, we infer that

$$\overline{\mathrm{E}}_p(f|s) \leq \inf\left\{\overline{\mathrm{E}}_p(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\right\}.$$

Taken together with the inequality above, we indeed conclude that $\overline{\mathrm{E}}_p$ satisfies NE4$_{88}$. □

Lemma 3.E.6. *For any precise probability tree p and any imprecise probability tree $\mathscr{P}_\bullet$, the upper expectations $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfy* WC5$_{84}$ *on* $\mathbb{V} \times \mathscr{X}^*$.

Proof. Since $\overline{\mathrm{E}}_p$ satisfies NE4$_{88}$ by Lemma 3.E.5, $\overline{\mathrm{E}}_p$ satisfies WC5$_{84}$ on $\mathbb{V}\times\mathscr{X}^*$. To see that WC5$_{84}$ also holds for the upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ corresponding to any imprecise probability tree $\mathscr{P}_\bullet$, it suffices to recall Definition 3.6$_{79}$ and use that, as we have just proved above, for any precise probability tree p, the upper expectation $\overline{\mathrm{E}}_p$ satisfies WC5$_{84}$ on $\mathbb{V} \times \mathscr{X}^*$. □

Most of the mathematical machinery that enables us to prove Proposition 3.5.5$_{93}$ is tucked away inside the following two lemmas. Though fairly technical and abstract, we state them separately—instead of in one single proof—because they will be used later on in Section 5.4_{240} to prove a downward type of continuity for the measure-theoretic upper expectation of Chapter 5_{217}.

Lemma 3.E.7. *For any imprecise probability tree $\mathscr{P}_\bullet$, any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ and any $s \in \mathscr{X}^*$,*

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) \geq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s),$$

where the precise probability tree p is the limit of some convergent sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees, each of which being compatible with the imprecise tree $\mathscr{P}_\bullet$.

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is decreasing and $\overline{\mathrm{E}}_{\mathscr{P}}$ is monotone due to Lemma $3.\mathrm{E}.6_{\leftarrow}$, we have that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s)$ exists. Furthermore, note that $\overline{\mathrm{E}}_{\mathscr{P}}(f_n|s)$ is real for all $n \in \mathbb{N}$. Indeed, by Proposition $3.5.4_{92}$, $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies $\mathrm{NE1}_{88}$–$\mathrm{NE3}_{88}$, and so $\overline{\mathrm{E}}_{\mathscr{P}}$ on $\mathbb{F} \times \mathscr{X}^*$ is given by the expression in Lemma $3.\mathrm{D}.5_{116}$. Since $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{F}$, and since the local upper expectations $\overline{\mathrm{Q}}_\bullet$ are real-valued due to coherence [$\mathrm{C5}_{33}$], this indeed implies that $\overline{\mathrm{E}}_{\mathscr{P}}(f_n|s)$ is real for all $n \in \mathbb{N}$. As a result, for any fixed $\epsilon > 0$, there is a sequence $(p_n)_{n\in\mathbb{N}}$ of precise probability trees such that $p_n \sim \mathscr{P}_\bullet$ and

$$\overline{\mathrm{E}}_{p_n}(f_n|s) + \epsilon/n \geq \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_p(f_n|s) = \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s) \text{ for all } n \in \mathbb{N}. \tag{3.42}$$

By Lemma $3.\mathrm{E}.1_{120}$, the sequence $(p_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(p_{n(i)})_{i\in\mathbb{N}}$. Let $p := \lim_{i\to+\infty} p_{n(i)}$ be the limit of this sequence, and let $(f_{n(i)})_{i\in\mathbb{N}}$ be the associated subsequence of $(f_n)_{n\in\mathbb{N}}$. Since $(f_n)_{n\in\mathbb{N}}$ is decreasing, $(f_{n(i)})_{i\in\mathbb{N}}$ is also decreasing. So, since $\overline{\mathrm{E}}_p$ is monotone by Lemma $3.\mathrm{E}.6_{\leftarrow}$, the limit $\lim_{i\to+\infty} \overline{\mathrm{E}}_p(f_{n(i)}|s)$ exists. Fix any real number $c > \lim_{i\to+\infty} \overline{\mathrm{E}}_p(f_{n(i)}|s)$ and any $j \in \mathbb{N}$ such that $c > \overline{\mathrm{E}}_p(f_{n(j)}|s)$. Since $(p_{n(i)})_{i\in\mathbb{N}}$ converges to p, and since $f_{n(j)}$ is finitary, Lemma $3.\mathrm{E}.2_{121}$ guarantees that $\lim_{i\to+\infty} \overline{\mathrm{E}}_{p_{n(i)}}(f_{n(j)}|s) = \overline{\mathrm{E}}_p(f_{n(j)}|s)$. Taking into account that $c > \overline{\mathrm{E}}_p(f_{n(j)}|s)$, this implies that there are arbitrarily large $k \geq j$ such that $c > \overline{\mathrm{E}}_{p_{n(k)}}(f_{n(j)}|s)$. For each such k, since $(f_{n(i)})_{i\in\mathbb{N}}$ is decreasing and $\overline{\mathrm{E}}_{p_{n(k)}}$ is monotone by Lemma $3.\mathrm{E}.6_{\leftarrow}$, and since $k \geq j$, this in turn implies that $c > \overline{\mathrm{E}}_{p_{n(k)}}(f_{n(k)}|s)$. Using Eq. (3.42), we obtain that $c > \overline{\mathrm{E}}_{\mathscr{P}}(f_{n(k)}|s) - \epsilon/n(k)$. Since this holds for arbitrarily large $k \geq j$ [and thus also for arbitrarily large $n(k)$], we have that $c \geq \liminf_{i\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_{n(i)}|s)$. Once more using the decreasing character of $(f_{n(i)})_{i\in\mathbb{N}}$ and the monotonicity of $\overline{\mathrm{E}}_{\mathscr{P}}$, we infer that $\lim_{i\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_{n(i)}|s)$ exists and that $c \geq \lim_{i\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_{n(i)}|s)$. This inequality holds for any real number $c > \lim_{i\to+\infty} \overline{\mathrm{E}}_p(f_{n(i)}|s)$, so we have that

$$\lim_{i\to+\infty} \overline{\mathrm{E}}_p(f_{n(i)}|s) \geq \lim_{i\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_{n(i)}|s).$$

The sequence $(f_{n(i)})_{i\in\mathbb{N}}$ is a subsequence of the decreasing sequence $(f_n)_{n\in\mathbb{N}}$, so by the monotonicity of $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ the above inequality implies that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) \geq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s). \tag{3.43}$$

The desired statement then follows by moreover recalling that each p_n (or $p_{n(i)}$) is compatible with $\mathscr{P}_\bullet$, and that $p := \lim_{i\to+\infty} p_{n(i)}$. □

Lemma 3.E.8. *For any imprecise probability tree $\mathscr{P}_\bullet$, any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that converges to a gamble $f \in \mathbb{V}$, and any $s \in \mathscr{X}^*$,*

$$\sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s).$$

Proof. Due to Lemma 3.E.7$_{\curvearrowleft}$, we have that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) \geq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s), \tag{3.44}$$

where p is the limit of a convergent sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees, each of which is compatible with the imprecise tree $\mathscr{P}_\bullet$. Fix any $\epsilon > 0$. Then by Lemma 3.E.4$_{122}$, there is a compatible precise tree $\tilde{p} \sim \mathscr{P}_\bullet$ such that

$$|\overline{\mathrm{E}}_p(g|s) - \overline{\mathrm{E}}_{\tilde{p}}(g|s)| \leq \epsilon\|g\|_\infty \text{ for all } g \in \mathbb{F}.$$

Since $(f_n)_{n\in\mathbb{N}}$ is a sequence of finitary gambles, this implies that

$$|\overline{\mathrm{E}}_p(f_n|s) - \overline{\mathrm{E}}_{\tilde{p}}(f_n|s)| \leq \epsilon\|f_n\|_\infty \text{ for all } n \in \mathbb{N}.$$

Moreover, since $(f_n)_{n\in\mathbb{N}}$ converges decreasingly to f, we have, for all $n \in \mathbb{N}$, that $\inf f \leq f_n \leq \sup f_1$ and therefore that $\|f_n\|_\infty \leq B := \max\{\|f\|_\infty, \|f_1\|_\infty\}$. So the inequality above implies that

$$|\overline{\mathrm{E}}_p(f_n|s) - \overline{\mathrm{E}}_{\tilde{p}}(f_n|s)| \leq \epsilon B \text{ for all } n \in \mathbb{N}.$$

In particular, we then have that

$$\overline{\mathrm{E}}_p(f_n|s) \leq \overline{\mathrm{E}}_{\tilde{p}}(f_n|s) + \epsilon B \text{ for all } n \in \mathbb{N},$$

and therefore, that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) \leq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\tilde{p}}(f_n|s) + \epsilon B,$$

where the existence of the limit on the right-hand side follows from $(f_n)_{n\in\mathbb{N}}$ being decreasing and $\overline{\mathrm{E}}_{\tilde{p}}$ being monotone due to Lemma 3.E.6$_{124}$. Combining this inequality with Eq. (3.44), we obtain that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s) \leq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\tilde{p}}(f_n|s) + \epsilon B,$$

which by the fact that $\tilde{p} \sim \mathscr{P}_\bullet$ implies that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s) \leq \sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s) + \epsilon B,$$

where, again, the existence of the limit on the right-hand side follows from $(f_n)_{n\in\mathbb{N}}$ being decreasing and $\overline{\mathrm{E}}_p$ for any $p \sim \mathscr{P}_\bullet$ being monotone [Lemma 3.E.6$_{124}$]. Since this holds for any $\epsilon > 0$, and since $B = \max\{\|f\|_\infty, \|f_1\|_\infty\}$ is real because f and f_1 are gambles, we infer that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(f_n|s) \leq \sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_p(f_n|s).$$

The converse inequality follows from the fact that $\overline{\mathrm{E}}_{\mathscr{P}}(f_n|s) = \sup_{p'\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p'}(f_n|s) \geq \overline{\mathrm{E}}_p(f_n|s)$ for all $n \in \mathbb{N}$ and all $p \sim \mathscr{P}_\bullet$. □

Proof of Proposition 3.5.5$_{93}$. Fix any $f \in \mathbb{V}$ and any $s \in \mathscr{X}^*$. Note that, because $\overline{\mathrm{E}}_{\mathscr{P}}$ satisfies WC5$_{84}$ by Lemma 3.E.6$_{124}$,

$$\overline{\mathrm{E}}_{\mathscr{P}}(f|s) \leq \inf\{\overline{\mathrm{E}}_{\mathscr{P}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\}.$$

So it suffices to prove that $\inf\{\overline{\mathrm{E}}_{\mathscr{P}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\} \leq \overline{\mathrm{E}}_{\mathscr{P}}(f|s)$.

Consider any $p \sim \mathscr{P}_{\bullet}$. By Lemma 3.E.5$_{124}$ and Lemma 3.E.6$_{124}$, we have that $\overline{\mathrm{E}}_p$ satisfies NE4$_{88}$ and WC5$_{84}$. Hence, if $(g_n)_{n\in\mathbb{N}}$ is the decreasing sequence of finitary gambles defined by $g_n(\omega) \coloneqq \sup_{\tilde{\omega}\in\Gamma(\omega^n)} f(\tilde{\omega})$ for all $n \in \mathbb{N}$ and all $\omega \in \Omega$, then by Lemma 3.5.11$_{97}$ we have that $\overline{\mathrm{E}}_p(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_p(g_n|s)$. Since this holds for any $p \sim \mathscr{P}_{\bullet}$ and since $\overline{\mathrm{E}}_{\mathscr{P}}(f|s) = \sup_{p\sim\mathscr{P}_{\bullet}} \overline{\mathrm{E}}_p(f|s)$, we have that

$$\overline{\mathrm{E}}_{\mathscr{P}}(f|s) = \sup_{p\sim\mathscr{P}_{\bullet}} \lim_{n\to+\infty} \overline{\mathrm{E}}_p(g_n|s). \tag{3.45}$$

Since $(g_n)_{n\in\mathbb{N}}$ is a decreasing sequence of finitary gambles that is bounded below by $\inf f$, it converges to a gamble. Therefore, it follows from Lemma 3.E.8$_{\leftarrow}$ that the right-hand side in the equality above is equal to $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(g_n|s)$. Hence, we have that $\overline{\mathrm{E}}_{\mathscr{P}}(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(g_n|s)$. Hence, since each of the g_n is a finitary gamble for which it holds that $g_n \geq f$ [this follows straightforwardly from their definition], we find that

$$\inf\{\overline{\mathrm{E}}_{\mathscr{P}}(g|s) \colon g \in \mathbb{F} \text{ and } g \geq_s f\} \leq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P}}(g_n|s) = \overline{\mathrm{E}}_{\mathscr{P}}(f|s),$$

as required. □

— Chapter IV —

Game-theoretic upper expectations

The first type of global upper expectation that we consider as an alternative to the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, is the game-theoretic upper expectation introduced and, for the most part, studied by Shafer and Vovk [85, 86, 109]. The conceptual ideas that underlie the definition of this operator were essentially already explained in Section $3.2.3_{61}$, where we introduced the finitary game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$. Just as for $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, the game-theoretic upper expectation that we will consider here represents an infimum over the starting capitals that allow a gambler—that is, Skeptic—to play along the rules—set by Forecaster—and superhedge the gamble at hand. The main difference here, however, is that we will not require Skeptic to superhedge at a finite time point, but only in the limit. As we will see, this modification results in a global upper expectation with completely different—and in many cases more desirable—properties.

The chapter is structured as follows. In Section 4.1_{131}, we first introduce this game-theoretic upper expectation, and then argue why it qualifies as a suitable global model for stochastic processes. At first, we limit ourselves to the domain $\mathbb{V} \times \mathscr{X}^*$ of gambles and situations but, as we have discussed in Section 3.6_{98}, we want a global upper expectation to be defined on $\overline{\mathbb{V}} \times \mathscr{X}^*$. This is why in Section 4.2_{139} we will extend the definition of the game-theoretic upper expectation from $\mathbb{V} \times \mathscr{X}^*$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$. Two possible approaches will be suggested here; an extension using continuity with respect to upper and lower cuts, and an extension using extended real-valued supermartingales. Though we believe the former to bear a more direct interpretation, we continue to work with the latter in the remainder of the chapter because it turns out to be mathematically more convenient and because, as we will show later on, it is completely equivalent to the former. Until that point, we will have parametrized our game-theoretic upper expectations in terms of acceptable gambles trees, yet, Section 4.3_{152} then shows how game-theoretic upper expectations can be alternatively defined starting from upper expectations trees. Similarly to what Corollary $3.5.8_{95}$ showed, it will become clear then that the game-theoretic upper expectation corre-

	local model	global upper expectation	
		finitary	continuity-based
behavioural	$\mathscr{A}_\bullet$ sets of acceptable gambles	$\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ from sets of acceptable gambles or martingales	$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ game-theoretic upper expectations
axiomatic	$\overline{\mathrm{Q}}_\bullet$ coherent upper expectations	$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extension under coherence	Chapter 6
probabilistic	$\mathscr{P}_\bullet$ sets of probability mass functions	$\overline{\mathrm{E}}_{\mathscr{P}}$ from finitely additive probabilities	Chapter 5

Figure 4.1 Overview of the global upper expectations treated in this and previous chapters.

sponding to an acceptable gambles tree can be completely characterised in terms of the—less expressive—agreeing upper expectations tree.

Sections 4.4_{162}–4.7_{180} are all devoted to establishing mathematical properties and/or generalising existing results for the game-theoretic upper expectation. We start in Section 4.4_{162} by proving, amongst other properties, an extended version of coherence, a law of iterated upper expectations, and equality with the natural extension $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$. Section 4.5_{171} presents generalised versions of Doob's convergence theorem and Lévy's zero–one law, and shows that the definition of the game-theoretic upper expectation can be modified in several interesting ways without affecting the values of the resulting game-theoretic upper expectation.

Sections 4.6_{175} and 4.7_{180} finally are concerned with continuity properties of the game-theoretic upper expectation: continuity with respect to increasing bounded below sequences and continuity with respect to decreasing sequences of lower cuts and decreasing sequences of finitary gambles are established. We also show that the game-theoretic upper expectation does in general not satisfy continuity with respect to decreasing sequences nor continuity with respect to pointwise convergence of sequences of finitary gambles.

In Section 4.8_{186}, we summarize our findings and use them to argue for the use of the game-theoretic upper expectation as a global upper expectation. We also return to the question of how game-theoretic upper expectations can be extended from gambles to extended real-valued variables, and show that the approach that uses continuity with respect to upper and lower cuts is equivalent to the approach that uses extended real-valued supermartingales. In the last section, Section 4.9_{187}, we compare our results to Shafer and Vovk's. It seems appropriate to devote an entire section to this topic, because many of what we do is closey related to—or strongly inspired by—the work of Shafer and Vovk.

4.1 Game-theoretic upper expectations on gambles

Recall the finitary game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ that we have introduced in Section $3.2.3_{61}$. It was obtained from an acceptable gambles tree $\mathscr{A}_\bullet$ by using the associated set of supermartingales and then using a notion of superhedging. Concretely, Eq. $(3.11)_{63}$ said that, for all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) = \inf\left\{\mathscr{C}(s) \colon \mathscr{C} \in \overline{\mathbb{M}}(\mathscr{A}_\bullet) \text{ and } (\exists k \geq |s|)\mathscr{C}(X_{1:k}) \geq_s f\right\}. \tag{4.1}$$

Given that we interpret the supermartingales $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$ as the possible evolutions of Skeptic's capital as he is betting against Forecaster's beliefs $\mathscr{A}_\bullet$, the expression above tells us that, for any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, $\overline{\mathrm{E}}_{\mathscr{A}}(f|s)$ is equal to the infimum starting capital $\mathscr{C}(s)$ in s such that Skeptic is able to surely end up with more money—or utility—than what the gamble f would give him. There is an important sidenote to this formulation though; Skeptic is required to superhedge f at some **finite** time instant $k \geq |s|$ in the future. Indeed, Eq. (4.1) involves an infimum that is taken only over the supermartingales $\mathscr{C}$ for which there is at least one finite time instant $k \geq |s|$ such that $\mathscr{C}(X_{1:k}) \geq_s f$. One can observe, for instance by recalling Example $3.6.1_{99}$, that this superhedging at a finite time point becomes problematic when considering the upper expectation of gambles that are non-finitary; the resulting upper expectation is often too conservative. But what happens when we now modify the upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ to also include supermartingales that superhedge at a(n) (idealised) time instant that lies infinitely far into the future?

We first introduce some notation that allows us to conveniently deal with the limit values of processes. For any real process $\mathscr{C}$, we write $\liminf \mathscr{C}$ to denote the extended real-valued variable on Ω defined by $\liminf \mathscr{C}(\omega) := \liminf_{k\to+\infty} \mathscr{C}(\omega^k)$ for all $\omega \in \Omega$, and similarly for the variable $\limsup \mathscr{C}$. Furthermore, for any $\omega \in \Omega$ such that $\liminf \mathscr{C}(\omega) = \limsup \mathscr{C}(\omega)$, we use $\lim \mathscr{C}(\omega)$ to denote the common value $\liminf \mathscr{C}(\omega) = \limsup \mathscr{C}(\omega)$. If $\lim \mathscr{C}(\omega)$ exists for all $\omega \in \Omega$, then we moreover let $\lim \mathscr{C} := \liminf \mathscr{C} = \limsup \mathscr{C}$.

Though we have used the terminology differently before, henceforth, we will say that, for any $(f, s) \in \mathbb{V} \times \mathscr{X}^*$, a real process $\mathscr{C}$ superhedges a gamble $f \in \mathbb{V}$ on $\Gamma(s)$ if $\liminf \mathscr{C} \geq_s f$—and no longer if the stronger condition holds that $\mathscr{C}(X_{1:k}) \geq_s f$ for some $k \geq |s|$. So, a process $\mathscr{C}$ superhedging a gamble f on $\Gamma(s)$ means that, for all paths that go through s, the process $\mathscr{C}$ remains larger than or arbitrarily close to f as time converges to infinity. It is clear that this condition of superhedging at infinity is implied by the condition of superhedging at a finite time point used in Eq. (4.1)↶ above (since stopping a supermartingale preserves it supermartingale character; see the proof of Lemma 3.2.8$_{66}$ or Lemma 4.C.5$_{211}$ below), but not the other way around. By replacing the finitary superhedging condition in the expression of Eq. (4.1)↶ by this weaker notion of superhedging at infinity, we arrive at a more informative—less conservative—type of global upper expectation. This type of upper expectation, we call the **game-theoretic upper expectation**. To explicitly define it, for any acceptable gambles tree $\mathscr{A}_\bullet$, we henceforth let $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet) := \overline{\mathbb{M}}(\mathscr{A}_\bullet)$—this notation will be clarified shortly.

Definition 4.1 (The game-theoretic upper expectation on gambles)**.** For any acceptable gambles tree $\mathscr{A}_\bullet$, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}} \colon \mathbb{V} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ is defined, for all $(f, s) \in \mathbb{V} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) = \inf \left\{ \mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f \right\}. \qquad ◎$$

Readers that are familiar with the work of Shafer and Vovk may observe that this game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is similar to the global upper expectation that they have introduced and strongly advocated for [85, 86, 109]. A thorough discussion of this connection, for $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, as well as for other similar definitions further below, will be given in Section 4.9$_{187}$. Nonetheless, we already want to point out the following key differences.

First of all, most of the work in [85, 86, 109] is not necessarily aimed at a setting where state spaces are finite; most often Shafer and Vovk consider general (possibly infinite) state spaces. Another difference lies in the description of the local models; they do not necessarily assume that the local models—the description of which being the 'moves' of Forecaster—are known beforehand, that is, before Skeptic starts playing. In our case, this is always assumed because we require the acceptable gambles tree $\mathscr{A}_\bullet$, which represents Forecaster's commitments, to be specified from the start. So in both respects, we are less general—it will allow us to obtain stronger results though. On the other hand, in their case, Forecaster's moves—the local models—do not take the form of local sets of acceptable gambles, but rather the form of local upper expectations. We however believe this to be somewhat circuitous, interpretationally speaking, since these local upper expectations are then only used to determine which local gambles Forecaster wants to commit to, and therefore wants to make available to Skeptic. Furthermore, as we have seen in Section 3.1.2$_{48}$, local sets of acceptable gambles are more expressive than local upper expectations, thus this assumption could possibly impact generality in the negative—we will nevertheless prove in Section 4.3$_{152}$ that this is not the case.

Though Shafer and Vovk are largely responsible for the theory of game-theoretic probability and upper expectations as it is currently known, a number of their original ideas can be traced back in some form to Jean Ville [107]; see the discussions in [86, Sections 8.5–8.6] and [85, Chapter 9]. The subscript 'V' in $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is intended to refer to him. The reason that we also accompany the operator $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ by a superscript 'r' is because it is defined through the set $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet) = \overline{\mathbb{M}}(\mathscr{A}_\bullet)$ of all **r**eal-valued (possibly unbounded) supermartingales. In the sequel, we will also introduce game-theoretic upper expectations that additionally use extended real-valued supermartingales, or that are limited to using (extended) real-valued supermartingales that are bounded, or bounded below.

4.1.1 The continuity properties of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$

Of course, since $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ was introduced with the goal of obtaining a global upper expectation that has stronger continuity properties and therefore returns more informative upper expected values than $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, it remains to check whether $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ indeed succeeds in doing so. Recalling Example 3.6.1$_{99}$ already hints at a positive answer.

Example 4.1.1. Reconsider the stochastic process from Example 3.6.1$_{99}$, but where the precise probability tree p is replaced by the acceptable gambles tree $\mathscr{A}_\bullet$, defined by $\mathscr{A}_s := \mathscr{L}_{\geq}(\mathscr{X}) \cup \{f \in \mathscr{L}(\mathscr{X}) \colon f(a) > 0\}$ for all $s \in \mathscr{X}^*$. It can be checked easily that $\mathscr{A}_s$ is coherent for all $s \in \mathscr{X}^*$, and therefore that $\mathscr{A}_\bullet$ is indeed an acceptable gambles tree. It can furthermore be easily seen that these two trees p and $\mathscr{A}_\bullet$ lead to the same agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet$, and thus by Theorem 3.5.1$_{90}$ and Theorem 3.5.2$_{91}$ that their finitary global upper expectations $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ (and $\overline{\mathrm{E}}_{\mathscr{A}}$) are equal. We will now show that—in contrast with what we found for $\overline{\mathrm{E}}_p$—the game-theoretic upper probability $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b) := \overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(\mathbb{I}_{H_b})$ of the event of ever hitting the state b is equal to zero.

Fix any $\epsilon > 0$. Let $\mathscr{M}$ be the real process that starts in $\mathscr{M}(\square) := \epsilon$ and for which the process difference in any situation a^k with $k \in \mathbb{N}_0$ takes the value $\Delta\mathscr{M}(a^k)(a) := -\epsilon/2^{k+1}$ in a and the value $\Delta\mathscr{M}(a^k)(b) := 1 - \mathscr{M}(a^k)$ in b, and for which the process difference in any other situation $s \in \mathscr{X}^* \setminus \{a^k \colon k \in \mathbb{N}_0\}$ is equal to the constant $\Delta\mathscr{M}(s) := 0$. Then clearly, $\Delta\mathscr{M}(s) \in -\mathscr{A}_s$ for all $s \in \mathscr{X}^*$, so $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$. Moreover note that, for any situation a^k with

$k \in \mathbb{N}_0$, the value of $\mathscr{M}$ is equal to

$$\mathscr{M}(a^k) = \mathscr{M}(\square) + \sum_{i=1}^{k} \Delta\mathscr{M}(a^{i-1})(a) = \epsilon + \sum_{i=1}^{k}(-\tfrac{\epsilon}{2^i}) = \epsilon\Big(1 - \sum_{i=1}^{k} \tfrac{1}{2^i}\Big) \geq 0.$$

Hence, for the path $\omega' := aaa\cdots$, we have that

$$\liminf \mathscr{M}(\omega') = \liminf_{k\to+\infty} \mathscr{M}(a^k) \geq 0 = \mathbb{I}_{H_b}(\omega').$$

On the other hand, for any situation $a^k b$ with $k \in \mathbb{N}_0$, we have that

$$\mathscr{M}(a^k b) = \mathscr{M}(a^k) + \Delta\mathscr{M}(a^k)(b) = \mathscr{M}(a^k) + 1 - \mathscr{M}(a^k) = 1.$$

Since for any situation $s \in \mathscr{X}^*$ such that $s \sqsupseteq a^k b$ for some $k \in \mathbb{N}_0$, we have that $\mathscr{M}(s) = \mathscr{M}(a^k b) = 1$, it follows that $\liminf \mathscr{M}(\omega) = 1 \geq \mathbb{I}_{H_b}(\omega)$ for all $\omega \in H_b = \Omega \setminus \{\omega'\}$. Since we already deduced that $\liminf \mathscr{M}(\omega') \geq \mathbb{I}_{H_b}(\omega')$, we conclude that $\liminf \mathscr{M}(\omega) \geq \mathbb{I}_{H_b}(\omega)$ for all $\omega \in \Omega$ and therefore that $\mathscr{M}$ superhedges $\mathbb{I}_{H_b}$ on all of Ω. Then, recalling that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$, we have by Definition 4.1_{132} that

$$\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b) = \overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(\mathbb{I}_{H_b}) \leq \mathscr{M}(\square) = \epsilon.$$

Since this holds for all $\epsilon > 0$, we infer that $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b) \leq 0$. That $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b) \geq 0$, and therefore $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b) = 0$, can be deduced from the fact that no supermartingale $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ can ever increase along the path $\omega' = aaa\cdots$, and that any $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ must in particular superhedge $\mathbb{I}_{H_b}(\omega') = 0$ on the path ω' for it to be included in the infimum of Definition 4.1_{132}.

So the game-theoretic upper probability $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$—or the corresponding upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$—indeed returns the desired value 0 for the global gamble $\mathbb{I}_{H_b}$. Furthermore, using the same type of supermartingale as the one above, we can also infer that, for all $k \in \mathbb{N}_0$, $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(H_b^k) := \overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(\mathbb{I}_{H_b^k})$ is equal to 0, for H_b^k the event of hitting b before time $k+1$. Recalling—from Example $3.6.1_{99}$—that $(\mathbb{I}_{H_b^k})_{k\in\mathbb{N}_0}$ is an increasing sequence such that $\lim_{k\to+\infty} \mathbb{I}_{H_b^k} = \mathbb{I}_{H_b}$, we conclude that $\overline{\mathrm{P}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is continuous with respect to the increasing sequence $(H_b^k)_{k\in\mathbb{N}_0}$—or equivalently that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is continuous with respect to the increasing sequence $(\mathbb{I}_{H_b^k})_{k\in\mathbb{N}_0}$. ◇

In the example above, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ exhibits more desirable continuity behaviour than the finitary upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$—and therefore also than the upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$. Though no formal proof will be given at this point, we can already assert that the desirable continuity behaviour of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is not limited to the example above; the upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ satisfies continuity with respect to general increasing sequences of gambles (that converge to a gamble) and with respect to decreasing sequences of finitary gambles (that converge to a gamble). Hence, the upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ seems to be a suitable option when aiming for a global model with decent continuity properties.

4.1.2 Coherence and the relation between $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and the finitary global upper expectations

Of course, there is more to choosing a global model than only strong continuity properties; an additional and very basic criterion is whether the global model is coherent [Definition 3.7_{82}]—or, equivalently, whether it satisfies $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$. We already announce that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is coherent, but postpone a proof to Section 4.8_{186}.

Another element that we might want to take into account is the relation of our global model with the finitary upper expectations $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}_{\mathscr{A}}$, et cetera. As we have discussed in Section 3.6_{98}, the finitary upper expectations are too conservative for dealing with general global variables (and situations) in a satisfactory manner, but there is—or seems to be—no problem with using them on the restricted domain of finitary gambles (and situations). In fact, their simple and intuitive construction—and the fact that they are all equal—makes them more suited to be applied on this restricted domain than any of the more complex continuity-based variants. Hence, from this point of view, it would thus be desirable that our global model is at least as small—or at least as informative—as $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ (or any other finitary upper expectation) for general (possibly non-finitary) global variables, but, if possible, coincides with $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ on finitary gambles. The game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ also seems to score well on this count. The following result shows that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is always smaller than or equal to $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$. Afterwards, with Proposition $4.1.4_{\curvearrowright}$, we establish that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ on the domain $\mathbb{F}\times\mathscr{X}^*$.

Proposition 4.1.2. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \text{ for all } (f,s)\in\mathbb{V}\times\mathscr{X}^*.$$

Proof. Fix any $(f,s)\in\mathbb{V}\times\mathscr{X}^*$, and consider any $\mathscr{M}\in\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:\ell})\geq_s f$ for some $k\geq|s|$ and all $\ell\geq k$. Then, for any $\omega\in\Gamma(s)$, we clearly have that $\liminf\mathscr{M}(\omega)=\liminf_{\ell\to+\infty}\mathscr{M}(\omega^\ell)\geq f(\omega)$, and so $\liminf\mathscr{M}\geq_s f$. Hence, by Definition 4.1_{132}, we have that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s)\leq\mathscr{M}(s)$. Since this holds for any $\mathscr{M}\in\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\mathscr{M}(X_{1:\ell})\geq_s f$ for some $k\geq|s|$ and all $\ell\geq k$, Lemma $3.2.8_{66}$ implies that indeed $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s)\leq\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$. □

To prove that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{F}\times\mathscr{X}^*$, we will use the following lemma, which is a simple consequence of Lemma $3.2.5_{63}$.

Lemma 4.1.3. *For any acceptable gambles tree $\mathscr{A}_\bullet$, any $\mathscr{M}\in\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ and any $s\in\mathscr{X}^*$, we have that*

$$\mathscr{M}(s)\geq\inf_{\omega\in\Gamma(s)}\limsup\mathscr{M}(\omega)\geq\inf_{\omega\in\Gamma(s)}\liminf\mathscr{M}(\omega).$$

Proof. Since $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ [and recalling that we previously used $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$ to denote $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$], Lemma 3.2.5$_{63}$ implies that, for any $t \in \mathscr{X}^*$, there is some $x \in \mathscr{X}$ such that $\mathscr{M}(t) \geq \mathscr{M}(tx)$. In particular, this holds for the situation s, and so there is some $x \in \mathscr{X}$ such that $\mathscr{M}(s) \geq \mathscr{M}(sx)$. Then we can apply the same property to the situation $s' := sx$, therefore implying that there is some $x' \in \mathscr{X}$ such that $\mathscr{M}(s') \geq \mathscr{M}(s'x') = \mathscr{M}(sxx')$. By continuing in this way,[1] we obtain the existence of a path $\omega = sxx' \cdots$ for which it holds that $\mathscr{M}(s) \geq \mathscr{M}(\omega^k)$ for all $k \geq |s|$. As a result, we have that $\mathscr{M}(s) \geq \inf_{\omega \in \Gamma(s)} \limsup \mathscr{M}(\omega)$. The rest of the proof is now trivial. □

Proposition 4.1.4. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \mathbb{F} \times \mathscr{X}^*.$$

Proof. The fact that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s)$ for all $(f,s) \in \mathbb{F} \times \mathscr{X}^*$ follows immediately from Proposition 4.1.2$_\curvearrowleft$. To prove the converse inequality, consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$. Since f is finitary, there is some $k \geq |s|$ and some $g \in \mathscr{L}(\mathscr{X}^k)$ such that $f = g(X_{1:k})$. Since $\liminf \mathscr{M} \geq_s f$, we also have that $\liminf \mathscr{M} \geq_s g(X_{1:k})$. Then, for any $x_{1:k} \in \mathscr{X}^k$ such that $x_{1:k} \sqsupseteq s$, we have that $\liminf \mathscr{M}(\omega) \geq g(x_{1:k})$ for all $\omega \in \Gamma(x_{1:k})$, and therefore also that $\inf_{\omega \in \Gamma(x_{1:k})} \liminf \mathscr{M}(\omega) \geq g(x_{1:k})$. By Lemma 4.1.3$_\curvearrowleft$, this implies that $\mathscr{M}(x_{1:k}) \geq g(x_{1:k})$. Since this holds for any $x_{1:k} \in \mathscr{X}^k$ such that $x_{1:k} \sqsupseteq s$, and recalling that $k \geq |s|$, and therefore that $\Gamma(s) = \bigcup_{x_{1:k} \sqsupseteq s} \Gamma(x_{1:k})$, it follows that $\mathscr{M}(X_{1:k}) \geq_s g(X_{1:k}) = f$. So by Eq. (3.11)$_{63}$, we have $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$, Definition 4.1$_{132}$ implies that $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s)$. □

4.1.3 Game-theoretic upper expectations in terms of bounded below and bounded supermartingales

What makes the upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ particularly attractive compared to, for instance, the measure-theoretic global (upper) expectations in Chapter 5$_{217}$, is the fact that it does not rely on abstract mathematical constructs such as measurability or σ-algebras. That the definition of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ involves superhedging at an infinite time horizon may be considered somewhat of an abstract and operationally meaningless concept, yet, apart from that, $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is entirely built on behavioural arguments. This does not only grant the upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ a relatively clear and direct interpretation, it also favours generality in the sense that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is then naturally defined on all gambles or variables, instead of only the measurable ones.

On top of this, there are two alternative ways of defining $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ that allow for an even more direct interpretation: using bounded below real-valued supermartingales and using bounded (below and above) real-valued supermartingales. The fact that supermartingales should be bounded below seems like a realistic assumption; for if a supermartingale really represents

[1] And by adopting the **Axiom of Dependent Choice (DC)**.

a subject's capital—or Skeptic's capital—as he is gambling on the subsequent state values of a stochastic process, then negative supermartingale values correspond to our subject borrowing money, which he can never do unboundedly. We therefore think that a version of the game-theoretic upper expectation with bounded below supermartingales makes more sense from an interpretational point of view. Such a version was also adopted in the past by both ourselves [8, 94] and by Shafer and Vovk [86, 109]. We define this version below in Definition 4.2.

In the same way as we have argued for the use of bounded below supermartingales, one could also argue for the use bounded (below and above) supermartingales. Indeed, since our subject—Skeptic—should receive his money from someone—Forecaster—and since Forecaster can never borrow an unbounded amount of money himself, Skeptic can never gain an unbounded amount of money neither. So in this sense, it seems justified to restrict ourselves to supermartingales that are not only bounded below, but also bounded above. Let us next define a version of the game-theoretic upper expectation with both bounded below and bounded supermartingales.

We let $\overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet)$ be the set of all supermartingales $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ that are bounded below; i.e. for which there is a $c \in \mathbb{R}$ such that $\mathscr{M}(s) \geq c$ for all $s \in \mathscr{X}^*$. Let $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$ be furthermore the subset of $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ consisting of all supermartingales $\mathscr{M}$ that are bounded; so for which $\mathscr{M}$ and $-\mathscr{M}$ are bounded below.

Definition 4.2. For any acceptable gambles tree $\mathscr{A}_\bullet$, the game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ are defined, for all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(f|s) := \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\};$$
$$\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) := \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\}. \qquad \circledcirc$$

We next prove that these game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ both coincide with $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ on the entire domain $\mathbb{V} \times \mathscr{X}^*$. To do this, we first establish the following lemma which says that bounding a supermartingale from above does not impact the fact that it is a supermartingale. We will use, for any real process $\mathscr{C}$ and any $B \in \mathbb{R}$, the notation $\mathscr{C}^{\wedge B}$ to denote the process defined by $\mathscr{C}^{\wedge B}(s) := \min\{\mathscr{C}(s), B\}$ for all $s \in \mathscr{X}^*$.

Lemma 4.1.5. *For any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ and all $B \in \mathbb{R}$, we have that $\mathscr{M}^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$.*

Proof. Since $\mathscr{M}^{\wedge B}(s) \leq \mathscr{M}(s)$ for all $s \in \mathscr{X}^*$, it follows that $\mathscr{M}^{\wedge B}(s\cdot) \leq \mathscr{M}(s\cdot)$ for all $s \in \mathscr{X}^*$. Fix any $s \in \mathscr{X}^*$. We first show that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$.

If $\mathscr{M}(s) \leq B$, then $\mathscr{M}^{\wedge B}(s) = \mathscr{M}(s)$ and therefore $\Delta\mathscr{M}^{\wedge B}(s) = \mathscr{M}^{\wedge B}(s\cdot) - \mathscr{M}(s) \leq \Delta\mathscr{M}(s)$. Since $-\Delta\mathscr{M}(s) \in \mathscr{A}_s$, we have by the monotonicity property [$\mathrm{D5}_{28}$] that

$-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$. If, on the other hand, $\mathscr{M}(s) > B$, then $\mathscr{M}^{\wedge B}(s) = B$, which by the fact that $\mathscr{M}^{\wedge B}(s\cdot) \leq B$ implies that $\Delta\mathscr{M}^{\wedge B}(s) \leq 0$. Then we once more have by D1$_{27}$ that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$. So for all situations $s \in \mathscr{X}^*$ we find that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$, and therefore that $\mathscr{M}^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$. □

In order to prove the equality between $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{V} \times \mathscr{X}^*$, we will need yet another lemma; it says that bounding the limit inferior of a process from above is the same as first bounding the process from above, and taking the limit inferior.

Lemma 4.1.6. *For any real process $\mathscr{C}$ and any path $\omega \in \Omega$, we have that*

$$\min\left\{B, \liminf_{n\to+\infty} \mathscr{C}(\omega^n)\right\} = \liminf_{n\to+\infty} \mathscr{C}^{\wedge B}(\omega^n) \text{ for all } B \in \mathbb{R}.$$

Proof. This follows as a special case of Lemma 4.2.10$_{151}$ further below. □

Proposition 4.1.7. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

Proof. Fix any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$. Since clearly $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet) \subset \overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet) \subset \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$, we have that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s)$. So it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) \geq \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s)$. Consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$. Let $\mathscr{M}_s$ be the real process defined by $\mathscr{M}_s(t) := \mathscr{M}(t)$ for all $t \sqsupset s$, and $\mathscr{M}_s(t) := \mathscr{M}(s)$ for all $t \not\sqsupset s$. Then note that $\Delta\mathscr{M}_s(t) = \Delta\mathscr{M}(t)$ for all $t \sqsupseteq s$, and $\Delta\mathscr{M}_s(t) = 0$ for all $t \not\sqsupseteq s$. Since, for all $t \in \mathscr{X}^*$, $-\Delta\mathscr{M}(t) \in \mathscr{A}_t$ [because $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$] and $0 \in \mathscr{A}_t$ [due to the fact that $\mathscr{A}_t$ satisfies D1$_{27}$], it follows that $\mathscr{M}_s \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$. Moreover, we clearly also have that $\liminf \mathscr{M}_s =_s \liminf \mathscr{M} \geq_s f$. Let $B := \max\{\sup f, \mathscr{M}_s(s)+1\}$ [which is real because f is a gamble] and note that $\mathscr{M}_s^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ by Lemma 4.1.5$_\curvearrowleft$. We moreover have that $\mathscr{M}_s^{\wedge B}$ is bounded. Indeed, $\mathscr{M}_s^{\wedge B}$ is bounded above by $B \in \mathbb{R}$ and bounded below by

$$\begin{aligned}\inf_{t\in\mathscr{X}^*} \mathscr{M}_s^{\wedge B}(t) = \inf_{t\in\mathscr{X}^*} \min\{\mathscr{M}_s(t), B\} \geq \inf_{t\in\mathscr{X}^*} \min\{\mathscr{M}_s(t), \mathscr{M}_s(s)+1\} &= \inf_{t\in\mathscr{X}^*} \mathscr{M}_s(t)\\ &= \inf_{t\sqsupseteq s} \mathscr{M}(t),\end{aligned}$$

where the last equality follows from the definition of $\mathscr{M}_s$. To see that this lower bound is real, observe that, due to Lemma 4.1.3$_{135}$ and the fact that $\liminf \mathscr{M} \geq_s f$,

$$\begin{aligned}\inf_{t\sqsupseteq s} \mathscr{M}(t) \geq \inf_{t\sqsupseteq s} \inf_{\omega\in\Gamma(t)} \liminf \mathscr{M}(\omega) &\geq \inf_{t\sqsupseteq s} \inf_{\omega\in\Gamma(s)} \liminf \mathscr{M}(\omega)\\ &= \inf_{\omega\in\Gamma(s)} \liminf \mathscr{M}(\omega) \geq \inf_{\omega\in\Gamma(s)} f(\omega).\end{aligned}$$

Since f is a gamble and thus bounded, we indeed have that $\inf_{\omega\in\Gamma(s)} f(\omega) \in \mathbb{R}$, and therefore that $\mathscr{M}_s^{\wedge B}$ is bounded below. Recalling that $\mathscr{M}_s^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ and that $\mathscr{M}_s^{\wedge B}$ is bounded above, we conclude that $\mathscr{M}_s^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$. Moreover, since $\liminf \mathscr{M}_s \geq_s f$ and since $B \geq \sup f$, we have by Lemma 4.1.6 that $\liminf \mathscr{M}_s^{\wedge B} \geq_s f$. Hence, it follows from the definition of $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ that $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}_s^{\wedge B}(s) \leq \mathscr{M}_s(s) = \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$, we obtain from the definition of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ that $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s)$. □

4.2 Towards an appropriate definition for the game-theoretic upper expectation on extended real-valued variables

For all we could tell so far, the game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and—especially—$\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ present themselves as excellent global models. Yet, there is one issue: the domain of these global upper expectations contains only gambles, and no unbounded or extended real(-valued) variables in $\overline{\mathbb{V}}$. This is with good reason though, since extending them to the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$—in a trivial way, by simply applying the same characterising expressions—would yield global upper expectations with undesirable properties.

4.2.1 The problem with $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$

Let us start by pointing out the issue with the version $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ based on real-valued (unbounded) supermartingales. In the sequel, we assume that the domain of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ is extended to $\overline{\mathbb{V}}\times\mathscr{X}^*$ in a trivial way, by using the expression in Definition 4.1_{132}. We will also require the corresponding global game-theoretic lower expectation $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$; similarly as on p.64, it is defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) \coloneqq \sup\left\{\mathscr{M}(s) \colon \mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet) \text{ and } \limsup \mathscr{M} \leq_s f\right\},$$

where $\underline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ is an alternative notation for the set $\underline{\mathbb{M}}(\mathscr{A}_\bullet)$ of all real submartingales according to $\mathscr{A}_\bullet$, which was introduced in Section $3.2.3_{61}$. Recall that $\underline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ is then the set of all real processes $\mathscr{M}$ such that $-\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$, or equivalently, the real processes $\mathscr{M}$ for which there is a betting process $\mathscr{G}$ such that $\mathscr{M}(s) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(s)$ and $\mathscr{G}(s) \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$. Furthermore, one may again check that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ are related by conjugacy; so $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f|s) = -\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(-f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.

An obvious property that we want general upper and lower expectations to have is that lower expectations are always lower than or equal to the corresponding upper expectations. Yet, it is this very basic requirement that is not always satisfied by the game-theoretic upper and lower expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$. This issue was already raised by De Cooman et al. [8, Example 1], and the following example is borrowed from them.

Example 4.2.1. Let $\mathscr{X} \coloneqq \{a,b\}$ and let $\mathscr{A}_\bullet$ be defined by $\mathscr{A}_s \coloneqq \{f \in \mathscr{L}(\mathscr{X}) \colon f(a)+f(b) \geq 0\}$ for all $s \in \mathscr{X}^*$. Clearly, the tree $\mathscr{A}_\bullet$ is an acceptable gambles tree. For any $\alpha \in \mathbb{R}_>$, let $\mathscr{G}_\alpha$ be the betting process defined by

$\mathscr{G}_\alpha(\square) := 2\alpha(\mathbb{I}_a - \mathbb{I}_b)$ and by

$$\mathscr{G}_\alpha(x_{1:k}) := \begin{cases} \alpha 2^{k-1}(\mathbb{I}_a - \mathbb{I}_b) & \text{if } x_{1:k} = a^k; \\ 3\alpha 2^{k-1}(\mathbb{I}_a - \mathbb{I}_b) & \text{if } x_{1:k} = b^k; \\ 0 & \text{otherwise,} \end{cases} \text{ for all } x_{1:k} \in \mathscr{X}^* \setminus \{\square\}.$$

Then, for all $s \in \mathscr{X}^*$, since $\mathscr{G}_\alpha(s)(a) + \mathscr{G}_\alpha(s)(b) = 0$, we have that $-\mathscr{G}_\alpha(s) \in \mathscr{A}_s$. Hence, if we let $\mathscr{M}_\alpha$ be the real process defined by $\mathscr{M}_\alpha(s) := -\alpha + \mathscr{E}^{\mathscr{G}_\alpha}(s)$ for all $s \in \mathscr{X}^*$, then $\mathscr{M}_\alpha \in \overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$. Moreover, note that

$$\mathscr{M}(x_{1:k}) = \begin{cases} \alpha 2^{k-1} & \text{if } x_{1:k} = a^k; \\ -3\alpha 2^{k-1} & \text{if } x_{1:k} = b^k; \\ 0 & \text{otherwise,} \end{cases} \text{ for all } x_{1:k} \in \mathscr{X}^*.$$

Hence, we have that $\lim \mathscr{M}_\alpha = \liminf \mathscr{M}_\alpha$ is equal to—and therefore superhedges—the variable $f \in \overline{\mathbb{V}}$ defined by

$$f(\omega) := \begin{cases} +\infty & \text{if } \omega = aaa\cdots; \\ -\infty & \text{if } \omega = bbb\cdots; \\ 0 & \text{otherwise,} \end{cases} \text{ for all } \omega \in \Omega.$$

As a consequence, by the expression in Definition 4.1$_{132}$, we have that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f) \leq \mathscr{M}_\alpha(\square) = -\alpha$. Since this holds for any $\alpha > 0$, we obtain that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f) = -\infty$. However, we also have that $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f) = -\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(-f) = -(-\infty) = +\infty$ because of symmetry considerations: indeed, if we swap the a's and b's in the reasoning above, then f turns into $-f$ but the local models $\mathscr{A}_s$ remain unaltered. A completely similar derivation therefore yields that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(-f) < \alpha$ for all $\alpha \in \mathbb{R}_>$, and therefore that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(-f) = -\infty$. As a result, we find that $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f) = -\infty < +\infty = \underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}(f)$. ◇

As a consequence of this example, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ with real-valued (unbounded) supermartingales $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$ is unsuitable. We are thus left with the versions $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ as possible game-theoretic upper expectations. One may check that the issue raised in the example above disappears if, instead of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, we were to work with $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$—hence the reason why the former was used in [8, 94]. We do not tread into detail here, because, as we will show next, there are other issues with $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ that make these versions unsuitable as well.

Let us first focus on the more popular version $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$. The domain of $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ is again extended to $\overline{\mathbb{V}} \times \mathscr{X}^*$ in a trivial way, by applying the same expression as in Definition 4.2$_{137}$.

Example 4.2.2. Consider a stochastic process with state space $\mathscr{X} := \{a, b\}$. Let $\mathscr{A}_\bullet$ be the acceptable gambles tree defined by $\mathscr{A}_\square := \mathscr{L}_\geq(\mathscr{X}) \cup \{f \in$

$\mathscr{L}(\mathscr{X})\colon f(a) > 0\}$ and, for all $s \in \mathscr{X}^* \setminus \{\square\}$, by $\mathscr{A}_s := \mathscr{L}_{\geq}(\mathscr{X})$. It can easily be checked that $\mathscr{A}_\bullet$ is indeed an acceptable gambles tree. In particular, $\mathscr{A}_\bullet$ models the case where our subject is willing to commit to any gamble on X_1 as long as the gamble's pay-off for the outcome a is positive (or non-negative if the pay-off for b is also non-negative), and where we are not willing to gamble—in a non-trivial way—on the state value at any other time instant. This can be interpreted as saying that our subject is (practically) certain about the fact that X_1 will take the value a, but that he is completely uncertain or is not willing to take any risks with respect to values of the variables $X_2, X_3, \dots$ In terms of upper expectations, one can easily check that $\overline{\mathrm{Q}}_\square(\mathbb{I}_b) = 0$ for the upper expectations tree $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ that agrees with $\mathscr{A}_\bullet$, and additionally that, due to Proposition $4.1.7_{138}$, Proposition $4.1.4_{136}$ and Lemma $3.\mathrm{D}.1_{114}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\mathbb{I}_b) = 0$.

Let $\tau_a \in \overline{\mathbb{V}}$ be the hitting time of the state a:

$$\tau_a(\omega) := \inf\{k \in \mathbb{N}\colon \omega_k = a\} \text{ for all } \omega \in \Omega.$$

We want to determine the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a)$ of τ_a with respect to the tree $\mathscr{A}_\bullet$. To this end, consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq \tau_a$. Since $\mathscr{A}_s = \mathscr{L}_{\geq}(\mathscr{X})$ for all $s \in \mathscr{X}^* \setminus \{\square\}$, we have that $\Delta\mathscr{M}(s) \leq 0$ for all $s \in \mathscr{X}^* \setminus \{\square\}$, and therefore clearly that $\mathscr{M}(x_1) \geq \mathscr{M}(x_{1:k})$ for all $x_{1:k} \in \mathscr{X}^* \setminus \{\square\}$. This implies that $\mathscr{M}(X_1) \geq \liminf \mathscr{M} \geq \tau_a$. Since $\tau_a(bbb\cdots) = +\infty$, this implies that $\mathscr{M}(b) = +\infty$, which is impossible because $\mathscr{M}$ is assumed to be real-valued. Hence, there are no supermartingales $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet)$ for which it holds that $\liminf \mathscr{M} \geq \tau_a$, and so $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a)$ is equal to the infimum over an empty set—so it is equal to $+\infty$.

The result that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = +\infty$ is again in conflict with our intuition; we would expect that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = 1$ because, according to $\mathscr{A}_\bullet$ or its agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet$, our subject is practically certain about the fact that $X_1 = a$. The fact that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = +\infty$ not only shows that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ is sometimes too conservative; as we will show next, it also implies that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ does not satisfy continuity with respect to general increasing sequences of finitary gambles—which, in fact, could essentially be seen as the cause of its conservative behaviour. Note, by the way, that this is not in contradiction with our claim from Section $4.1.1_{133}$, where we said that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ satisfies continuity with respect to increasing sequences of finitary gambles that converge to a gamble—the latter part of this statement is crucial.

Consider the sequence $(\tau_a^{\wedge k})_{k\in\mathbb{N}}$ of hitting times of a that are stopped at time k; so $\tau_a^{\wedge k}(\omega) := \min\{\tau_a(\omega), k\}$ for all $k \in \mathbb{N}$ and all $\omega \in \Omega$. Each $\tau_a^{\wedge k}$ is a finitary gamble because it only depends on the first k states $X_{1:k}$ and is bounded above by k (and below by 1). The sequence $(\tau_a^{\wedge k})_{k\in\mathbb{N}}$ is moreover clearly increasing and converges pointwise to τ_a. However, one can verify that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a^{\wedge k}) = 1$ for all $k \in \mathbb{N}$ [hint: consider, for any $\epsilon > 0$ and any

$k \in \mathbb{N}$, the real process $\mathscr{M}$ defined by $\mathscr{M}(\square) := 1+\epsilon$, $\mathscr{M}(s) := 1$ for all $s \sqsupseteq a$ and $\mathscr{M}(s) := k$ for all $s \sqsupseteq b$]. Hence, we conclude that

$$\lim_{k\to+\infty} \overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a^{\wedge k}) = 1 \neq +\infty = \overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = \overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\lim_{k\to+\infty} \tau_a^{\wedge k}).$$

◇

It can easily be seen that exactly the same issues would arise if we were to replace $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ with $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ in the example above (when, again, extending the domain of $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$ in a trivial way). Or, alternatively, as an immediate consequence of Definition 4.2$_{137}$ and the definitions of the sets $\overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet)$ and $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$, one may observe that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ is always smaller than or equal to—at least as informative as—$\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$, and thus that also $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(\tau_a) = +\infty$. The fact that $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(\tau_a^{\wedge k}) = 1$ for all $k \in \mathbb{N}$ can moreover be inferred from Proposition 4.1.7$_{138}$.

In summary, the three game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ are all unsuitable when considering the domain $\overline{\mathbb{V}} \times \mathscr{X}^*$. We shall therefore want to further modify the definition(s) of (one of) these global upper expectations in such a way that we obtain a new game-theoretic upper expectation with more desirable properties. We consider two possible—and appropriate—ways of doing so: firstly, using extensions based on upper and lower cuts; secondly, defining extended local models and considering extended real-valued supermartingales (that are bounded below). As we will show in Section 4.8$_{186}$, the two global upper expectations that result from these approaches always coincide.

4.2.2 An extension using continuity with respect to upper and lower cuts

A straightforward approach for obtaining a suitable extended game-theoretic upper expectation is to simply use $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ on the smaller domain $\mathbb{V} \times \mathscr{X}^*$ where they behave nicely, and then subsequently extend them to $\overline{\mathbb{V}} \times \mathscr{X}^*$ by imposing continuity with respect to so-called upper and lower cuts.[2] We continue to work with $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ from here on, because it is the version that was used the most often in the past, but one could equally well use $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$—where the latter is, interpretationally speaking, to be preferred.

We use, for any extended real-valued function $f \in \overline{\mathscr{L}}(\mathscr{Y})$ on a non-empty set $\mathscr{Y}$, and any $c \in \mathbb{R}$, the notation $f^{\wedge c}$ to denote the variable in $\overline{\mathscr{L}}(\mathscr{Y})$ defined by $f^{\wedge c}(y) := \min\{f(y), c\}$ for all $y \in \mathscr{Y}$, and, analogously, $f^{\vee c}$ to denote the pointwise maximum of f and c. For any upper expectation

[2] This is similar to how Troffaes & De Cooman [106, Chapter 15] extend the notion of coherence from gambles to unbounded real-valued variables.

$\overline{\mathrm{E}}\colon \overline{\mathscr{L}}(\mathscr{Y}) \to \overline{\mathbb{R}}$, the following properties are then called continuity with respect to **upper** and **lower cuts**, respectively:[3]

CU1. $\overline{\mathrm{E}}(f) = \lim_{c\to+\infty} \overline{\mathrm{E}}(f^{\wedge c})$ for all $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$;
CU2. $\overline{\mathrm{E}}(f) = \lim_{c\to-\infty} \overline{\mathrm{E}}(f^{\vee c})$ for all $f \in \overline{\mathscr{L}}(\mathscr{Y})$.

Note that Property CU1 only involves extended real-valued functions that are bounded below. Our reason for doing so is purely mathematical: it allows us to use CU1 and CU2 as a tool to extend any monotone upper expectation on the set of all gambles $\mathscr{L}(\mathscr{Y})$ in an unambiguous way to the set $\overline{\mathscr{L}}(\mathscr{Y})$. For the sake of completeness; we say that a general (unconditonal) upper expectation $\overline{\mathrm{E}}\colon \mathscr{K} \to \overline{\mathbb{R}}$ with $\mathscr{K} \subseteq \overline{\mathscr{L}}(\mathscr{Y})$ is monotone if $\overline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(g)$ for any two $f, g \in \mathscr{K}$ such that $f \leq g$.

Lemma 4.2.3. *For any upper expectation* $\overline{\mathrm{E}}$ *on* $\mathscr{L}(\mathscr{Y})$ *that is monotone, there is a unique extension* $\overline{\mathrm{E}}^{\uparrow}$ *to* $\overline{\mathscr{L}}(\mathscr{Y})$ *that satisfies* CU1 *and* CU2.

Proof. Let $\overline{\mathrm{E}}'$ be the upper expectation on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ that is equal to $\overline{\mathrm{E}}$ on $\mathscr{L}(\mathscr{Y})$ and that is defined, for all $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y}) \setminus \mathscr{L}(\mathscr{Y})$, by $\overline{\mathrm{E}}'(f) := \lim_{c\to+\infty} \overline{\mathrm{E}}(f^{\wedge c})$. Note that, since $\overline{\mathrm{E}}$ is monotone, the limit on the right-hand side indeed exists. Next, let $\overline{\mathrm{E}}^{\uparrow}$ be the upper expectation on $\overline{\mathscr{L}}(\mathscr{Y})$ that is equal to $\overline{\mathrm{E}}'$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ and that is defined, for all $\overline{\mathscr{L}}(\mathscr{Y}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$, by $\overline{\mathrm{E}}^{\uparrow}(f) := \lim_{c\to-\infty} \overline{\mathrm{E}}'(f^{\vee c})$. Again, since $\overline{\mathrm{E}}'$ is monotone due to its definition and the fact that $\overline{\mathrm{E}}$ is monotone, the limit on the right-hand side exists.

Since $\overline{\mathrm{E}}^{\uparrow}$ extends $\overline{\mathrm{E}}'$, and $\overline{\mathrm{E}}'$ extends $\overline{\mathrm{E}}$, we also have that $\overline{\mathrm{E}}^{\uparrow}$ extends $\overline{\mathrm{E}}$. Moreover, $\overline{\mathrm{E}}^{\uparrow}$ satisfies CU2 on $\overline{\mathscr{L}}(\mathscr{Y}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ by definition. That it also satisfies CU2 on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ is trivial; any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ is bounded below, so there is a real number c' such that $f^{\vee c} = f$ for all $c \leq c'$. Furthermore, since $\overline{\mathrm{E}}'$ satisfies CU1 on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y}) \setminus \mathscr{L}(\mathscr{Y})$ by definition, and since $\overline{\mathrm{E}}^{\uparrow}$ extends $\overline{\mathrm{E}}'$, $\overline{\mathrm{E}}^{\uparrow}$ also satisfies CU1 on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y}) \setminus \mathscr{L}(\mathscr{Y})$. To see that it also satisfies CU1 on $\mathscr{L}(\mathscr{Y})$ is again trivial; any $f \in \mathscr{L}(\mathscr{Y})$ is bounded, so there is surely a real number c' such that $f^{\wedge c} = f$ for all $c \geq c'$. Hence, in summary, $\overline{\mathrm{E}}^{\uparrow}$ is an extension of $\overline{\mathrm{E}}$ to $\overline{\mathscr{L}}(\mathscr{Y})$ that satisfies CU1 and CU2. To see that $\overline{\mathrm{E}}^{\uparrow}$ is moreover the only upper expectation on $\overline{\mathscr{L}}(\mathscr{Y})$ that extends $\overline{\mathrm{E}}$ and that satisfies CU1 and CU2, can then again be reasoned in a step-wise manner, first checking that CU1 uniquely determines the values on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y}) \setminus \mathscr{L}(\mathscr{Y})$ (in terms of the values of $\overline{\mathrm{E}}$), and then checking that CU2 uniquely determines the values on $\overline{\mathscr{L}}(\mathscr{Y}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$. □

We next apply this extension method to the upper expectation $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\cdot|s)$ for all $s \in \mathscr{X}^*$ to arrive at our definition of the global game-theoretic upper expectation $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$.

Definition 4.3. For any acceptable gambles tree $\mathscr{A}_\bullet$, we let $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ be the unique global upper expectation on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and is such that, for all $s \in \mathscr{X}^*$, $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}(\cdot|s)$ satisfies CU1 and CU2. ◎

[3] Shafer and Vovk call CU2 'bounded-below support'; see [85, Exercise 6.9]

Proof. Let us show that $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ exists (and is unique). Consider any $s \in \mathscr{X}^*$. It can easily be inferred from Definition 4.2_{137} that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\cdot|s)$ on $\mathbb{V}$ is monotone, and thus by Lemma $4.2.3_{\curvearrowleft}$ that there is a unique extension $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}(\cdot|s)$ to $\overline{\mathbb{V}}$ that satisfies $\mathrm{CU1}_{\curvearrowleft}$ and $\mathrm{CU2}_{\curvearrowleft}$. Hence, the upper expectations $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}(\cdot|s)$ for all $s \in \mathscr{X}^*$ form a global upper expectation $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that satisfies the desired properties. □

Why do we believe this way of defining an extended game-theoretic upper expectation $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ is appropriate? Our answer is based on our interpretation of unbounded and extended real-valued global variables; a variable $f \in \overline{\mathbb{V}}$ that is not bounded is, at least in this game-theoretic chapter, regarded as an abstract idealisation of the gamble $(f^{\wedge c_1})^{\vee c_2}$—that is bounded above by c_1 and bounded below by c_2—for arbitrarily large positive $c_1 \in \mathbb{R}_>$ and arbitrarily large negative $c_2 \in \mathbb{R}_<$. This makes sense, to us, because game-theoretic upper expectations have a behavioural justification and so variables—bounded, unbounded or extended real-valued—are typically interpreted as uncertain pay-offs. But what does it mean for a pay-off to be infinite? Or even, what does it mean for a bet to be unbounded in its possible values? In reality, there is always only a finite amount of money, and so the variables in $\overline{\mathbb{V}} \setminus \mathbb{V}$ can be given only an indirect interpretation.[4] The approach described above then seems one of the most intuitive ways to do so. Then, given this indirect interpretation, extending $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ with $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ seems like a logical thing to do; $\mathrm{CU1}_{\curvearrowleft}$ and $\mathrm{CU2}_{\curvearrowleft}$ guarantee that the values of $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ on unbounded, possibly extended real-valued variables are simply idealised, limit values of $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ on gambles obtained from cutting variables at sufficiently large values.

Another reason is that Axioms $\mathrm{CU1}_{\curvearrowleft}$ and $\mathrm{CU2}_{\curvearrowleft}$ are rather weak. Axiom $\mathrm{CU2}_{\curvearrowleft}$ can be justified by a conservativity argument; imposing it on top of $\mathrm{CU1}_{\curvearrowleft}$ is equivalent to taking the largest—the most conservative—monotone global upper expectation that coincides with $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and satisfies $\mathrm{CU1}_{\curvearrowleft}$. Axiom $\mathrm{CU1}_{\curvearrowleft}$ on the other hand can be seen as a weakened version of the continuity with respect to increasing sequences—or continuity from below—that is often adopted, either directly, e.g. in [85, Part II], or indirectly as a consequence of the continuity of the underlying probability measure or capacity; see e.g. [5, 29, 31].

In spite of all this, extending the upper expectation $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ (or $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$) through Axioms $\mathrm{CU1}_{\curvearrowleft}$ and $\mathrm{CU2}_{\curvearrowleft}$ is not that common. A technique that is used more often consists in directly applying the expression for the upper expectation $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ in Definition 4.2_{137} to the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$, but with the real-valued supermartingales replaced by extended real-valued ones [85, 97, 98]. This of course first requires us to extend the local un-

[4] A similar observation can actually be made about gambles in $\mathbb{V}$ that are non-finitary; see Section 6.1_{285}.

certainty models $\mathscr{A}_s$—or $\overline{\mathrm{Q}}_s$—to allow for such extended real-valued supermartingales. Such an approach will be the topic of the next section.

Remarkably enough, as we will see in Section 4.8186, the extended game-theoretic upper expectation that results from this 'extended supermartingale'-approach is identical to the operator $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ we have introduced here. It therefore does not matter which extension procedure is chosen. We will state and derive most of our future results in terms of the game-theoretic upper expectation with extended real supermartingales, because it is mathematically more convenient and because this type of upper expectation is more widely used [85, 88]. In principle, however, we favour the use of $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$, because relying on extended real-valued supermartingales undermines what we think is a key strength of the game-theoretic approach: that supermartingales—and hence the resulting game-theoretic upper expectations—can be given a clear behavioural meaning in terms of betting.

4.2.3 An extension using extended local sets of acceptable gambles

In order to allow for extended real-valued supermartingales, we first need to extend the local sets of acceptable gambles $\mathscr{A}_\bullet$ such that they can possibly also contain extended real variables on $\mathscr{X}$. However, our notion of acceptability hinged on a behavioural interpretation, which we cannot directly apply to extended real variables—a point that we have already raised in the previous section. We therefore first need to extend the notion of acceptability itself. We propose the following approach for finite state spaces.

Definition 4.4 (Acceptability for extended real variables)**.** We say that an extended real variable $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})$ is acceptable if there is some $c_1 \in \mathbb{R}_>$ such that the gamble $(f^{\wedge c_1})^{\vee c_2}$ is acceptable for all $c_2 \in \mathbb{R}_<$. ◎

Definition 4.4 can be seen to make most sense when reasoning in a stepwise manner as follows. Suppose that for some $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})$, there is a $c_1 \in \mathbb{R}_>$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_<$. Then the gamble $(f^{\wedge c_1})^{\vee c_2}$ is acceptable no matter how far we bound $f^{\wedge c_1}$ from below by $c_2 < 0$. As we have discussed in the previous section, we regard $f^{\wedge c_1}$ to be an abstraction of $(f^{\wedge c_1})^{\vee c_2}$ for arbitrarily large negative $c_2 \in \mathbb{R}_<$, so it is sensible to call $f^{\wedge c_1}$ itself acceptable—note that this is indeed in accordance with our definition above, because $((f^{\wedge c_1})^{\wedge c_1})^{\vee c_2} = (f^{\wedge c_1})^{\vee c_2}$ is acceptable for all $c_2 \in \mathbb{R}_<$. The fact that we then also call f acceptable, follows from a monotonicity argument: f is deemed acceptable because $f^{\wedge c_1}$ is acceptable and $f^{\wedge c_1} \leq f$.

Unlike our original notion of acceptability, which was purely interpretational and had no direct mathematical consequences—coherence did the work there—our extended definition of acceptability inevitably imposes structure on a set of acceptable variables. This structure immediately allows us to associate with a general coherent set of acceptable gambles $\mathscr{A}$ on $\mathscr{X}$ a unique extended set of acceptable variables $\mathscr{A}^{\uparrow}$.

Definition 4.5. For any coherent set of acceptable gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{X})$, the corresponding extended set of acceptable variables $\mathscr{A}^{\uparrow} \subseteq \overline{\mathscr{L}}(\mathscr{X})$ is the set that includes $\mathscr{A}$ and additionally contains any variable $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})$ for which there is some $c_1 \in \mathbb{R}_{>}$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. ◎

The following corollary provides a characterisation for the extended sets $\mathscr{A}^{\uparrow}$ that is more practical to work with.

Corollary 4.2.4. *For any coherent set of acceptable gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{X})$, the corresponding extended set of acceptable variables $\mathscr{A}^{\uparrow} \subseteq \overline{\mathscr{L}}(\mathscr{X})$ is the set of all variables $f \in \overline{\mathscr{L}}(\mathscr{X})$ for which there is some $c_1 \in \mathbb{R}_{>}$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$.*

Proof. Let $\mathscr{A}'$ be the set of all variables $f \in \overline{\mathscr{L}}(\mathscr{X})$ for which there is some $c_1 \in \mathbb{R}_{>}$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. Then it is clear by Definition 4.5 that the intersection of $\mathscr{A}'$ with $\overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})$ is equal to the intersection of $\mathscr{A}^{\uparrow}$ with $\overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})$. Hence, since

$$\mathscr{A}' = \big(\mathscr{A}' \cap \big(\overline{\mathscr{L}}(\mathscr{X}) \setminus \mathscr{L}(\mathscr{X})\big)\big) \cup \big(\mathscr{A}' \cap \mathscr{L}(\mathscr{X})\big),$$

and similarly for $\mathscr{A}^{\uparrow}$, it suffices to prove that $\mathscr{A}' \cap \mathscr{L}(\mathscr{X})$ is equal to $\mathscr{A}^{\uparrow} \cap \mathscr{L}(\mathscr{X})$. The latter is equal to $\mathscr{A}$ due to Definition 4.5, so we need to show that $\mathscr{A}' \cap \mathscr{L}(\mathscr{X}) = \mathscr{A}$.

Consider any gamble in $f \in \mathscr{A}$ and let $c_1 \in \mathbb{R}_{>}$ be any positive real number such that $c_1 \geq \sup f \in \mathbb{R}$. Then we have that $f^{\wedge c_1} = f \in \mathscr{A}$. Moreover, for any $c_2 \in \mathbb{R}_{<}$, we have that $(f^{\wedge c_1})^{\vee c_2} \geq f^{\wedge c_1}$. Then since $f^{\wedge c_1} \in \mathscr{A}$ and since $\mathscr{A}$ satisfies D5$_{28}$, we also find that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$. Hence, we have that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$, and therefore by the definition of $\mathscr{A}'$ that $f \in \mathscr{A}'$. Since this is true for any gamble $f \in \mathscr{A}$, we obtain that $\mathscr{A}' \supseteq \mathscr{A}$ and therefore that $\mathscr{A}' \cap \mathscr{L}(\mathscr{X}) \supseteq \mathscr{A}$. To prove the converse inclusion, consider any gamble $f \in \mathscr{A}' \cap \mathscr{L}(\mathscr{X})$. Then, according to the definition of $\mathscr{A}'$, there is a $c_1 \in \mathbb{R}_{>}$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. In particular, for any $c_2 \in \mathbb{R}_{<}$ such that $c_2 \leq \inf f^{\wedge c_1} \in \mathbb{R}$, we have that $f^{\wedge c_1} = (f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$. Since $f \geq f^{\wedge c_1}$ and since f is a gamble, the coherence [D5$_{28}$] of $\mathscr{A}$ implies that $f \in \mathscr{A}$. Hence, we have that $f \in \mathscr{A}$ for all $f \in \mathscr{A}' \cap \mathscr{L}(\mathscr{X})$, and therefore that $\mathscr{A}' \cap \mathscr{L}(\mathscr{X}) \subseteq \mathscr{A}$ as desired. □

The following result moreover shows that an extended version of the monotonicity property [D5$_{28}$] for coherent sets of acceptable gambles holds for the extended sets $\mathscr{A}^{\uparrow}$. We will require it later on, to prove Proposition 4.3.1$_{153}$.

Lemma 4.2.5. *Consider any local set of acceptable gambles $\mathscr{A}$ and let $\mathscr{A}^{\uparrow}$ be its extension according to Definition 4.5$_{\leftarrow}$. Then, for any $f, g \in \overline{\mathscr{L}}(\mathscr{X})$ such that $f \leq g$, we have that $g \in \mathscr{A}^{\uparrow}$ if $f \in \mathscr{A}^{\uparrow}$.*

Proof. First note that, for any two gambles $f, g \in \mathscr{L}(\mathscr{X})$ such that $f \leq g$, the monotonicity property holds because $\mathscr{A}$ satisfies D5$_{28}$ due to its coherence, and because $\mathscr{A}^{\uparrow} \cap \mathscr{L}_{\geq}(\mathscr{X}) = \mathscr{A}$ due to Definition 4.5$_{\leftarrow}$. Now consider any two general $f, g \in \overline{\mathscr{L}}(\mathscr{X})$ such that $f \leq g$, and assume that $f \in \mathscr{A}^{\uparrow}$. Then Corollary 4.2.4$_{\leftarrow}$ implies that there is a $c_1 \in \mathbb{R}_{>}$ such that $(f^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. Since $f \leq g$, we also have that $(f^{\wedge c_1})^{\vee c_2} \leq (g^{\wedge c_1})^{\vee c_2}$ for all $c_2 \in \mathbb{R}_{<}$. Moreover, since both $(f^{\wedge c_1})^{\vee c_2}$ and $(g^{\wedge c_1})^{\vee c_2}$ are gambles for all $c_2 \in \mathbb{R}_{<}$, and since we already established the monotonicity property for gambles in $\mathscr{A}$, we obtain that $(g^{\wedge c_1})^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. Hence, by Corollary 4.2.4$_{\leftarrow}$, we have that $g \in \mathscr{A}^{\uparrow}$. □

Extended real supermartingales

The previously introduced definition of an extended set of acceptable gambles can in particular be applied to the individual components of an acceptable gambles tree $\mathscr{A}_{\bullet}$ to form the corresponding extended tree $\mathscr{A}_{\bullet}^{\uparrow}$; for any $s \in \mathscr{X}^*$, $\mathscr{A}_s^{\uparrow}$ is then the extended set of acceptable variables that corresponds to $\mathscr{A}_s$ as described in Definition 4.5$_{\leftarrow}$. Such an extended tree $\mathscr{A}_{\bullet}^{\uparrow}$ can then subsequently be used to define extended real(-valued) supermartingales. Before we do so, let us reconsider the relation between processes and their (process) differences in case they are extended real-valued.

We adopt similar definitions as in Chapter 3$_{45}$; an **extended real(-valued) process** $\mathscr{C}$ is an extended real-valued map on $\mathscr{X}^*$, while an **extended betting process** $\mathscr{G}$ is a map that associates with each $s \in \mathscr{X}^*$ an extended real variable $\mathscr{G}(s) \in \overline{\mathscr{L}}(\mathscr{X})$. The process difference $\Delta\mathscr{C}$ can also be defined similarly as before, as the extended betting process which is, for any $s \in \mathscr{X}^*$, equal to

$$\Delta\mathscr{C}(s) := \mathscr{C}(s\cdot) - \mathscr{C}(s) \text{ with } \mathscr{C}(s\cdot)(x) := \mathscr{C}(sx) \text{ for all } x \in \mathscr{X}.$$

Conversely, with an extended betting process $\mathscr{G}$, we associate an extended real process $\mathscr{C}^{\mathscr{G}}$ defined by

$$\mathscr{C}^{\mathscr{G}}(x_{1:k}) := \sum_{\ell=0}^{k-1} \mathscr{G}(x_{1:\ell})(x_{\ell+1}) \text{ for all } x_{1:k} \in \mathscr{X}^*.$$

However, note that, since we are summing and subtracting extended real numbers—recall Section 1.6$_{14}$ for the associated conventions—these relations between processes and process differences are not one-to-one any more (even if we fixed the initial value of the process). That is, there may be multiple different extended real processes—with the same initial value—that have the same process difference and, vice versa, there may be multiple different extended betting processes that result in the same extended real process. As a result, given an extended real process $\mathscr{C}$, we do not necessarily have, for all $x_{1:k} \in \mathscr{X}^*$, that $\mathscr{C}(x_{1:k})$ is equal to $\mathscr{C}(\square) + \sum_{\ell=0}^{k-1} \Delta\mathscr{C}(x_{1:\ell})(x_{\ell+1})$. In fact, there does not even necessarily exist an extended betting process $\mathscr{G}$ such that $\mathscr{C}$ is equal to the extended real process $\mathscr{C}(\square) + \mathscr{C}^{\mathscr{G}}$. This is for instance the case if $\mathscr{C}$ is such that $\mathscr{C}(s) = +\infty$ and $\mathscr{C}(t) \in \mathbb{R}$ for some $s, t \in \mathscr{X}^*$ and $t \sqsupset s$.

Similarly as in Section 3.2.3$_{61}$, we say that an extended real process $\mathscr{M}$ is a(n) **(extended real) supermartingale** according to an acceptable gambles tree $\mathscr{A}_{\bullet}$ if there is an extended betting process $\mathscr{G}$ such that $\mathscr{M}(s) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(s)$ and $-\mathscr{G}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$. Note that, since the sets $\mathscr{A}_s^{\uparrow}$ include the sets $\mathscr{A}_s$ by definition, we have that any—real—supermartingale according to our earlier definition from Section 3.2.3$_{61}$ is still an extended real supermartingale according to our current definition, hence why we simply adopted the same terminology. A subtlety worth pointing out, though, is that, in contrast with the real supermartingales introduced in Section 3.2.3$_{61}$, the extended real supermartingales defined here cannot be alternatively characterised by means of process differences; that is, an extended real process $\mathscr{C}$ for which $-\Delta\mathscr{C}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$ is not necessarily a supermartingale, and conversely, for an extended real supermartingale $\mathscr{M}$, we do not necessarily have that $-\Delta\mathscr{M}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$. This is due to the fact that, as just mentioned, the relations between extended real processes, process differences and betting processes are more tedious than if they were to take values in the reals. We choose to define extended real supermartingales as above, with acceptable betting processes, and not with process differences, because the latter would preclude a supermartingale to remain (in all the following situations) in $+\infty$ once it has attained $+\infty$; indeed, if $\mathscr{M}(s) = +\infty$ and $\mathscr{M}(s\cdot) = +\infty$, then we have that $-\Delta\mathscr{M}(s) = -(+\infty) = -\infty$, which can never be an element of $\mathscr{A}_s^{\uparrow}$ (whatever the acceptable gambles tree) due to D2$_{27}$. A similar observation can be made for a process attaining the constant value $-\infty$. It can be shown that forcing a supermartingale to change its value (in at least one of the following situations) after it has reached $+\infty$ or $-\infty$ would yield some rather undesirable effects; and intuitively too, it would plainly seem unnatural if we were to preclude a supermartingale from remaining constant on some values.

Finally, similarly as before and without going into the details, we say that an extended real process $\mathscr{M}$ is a(n) **(extended real) submartingale** according to $\mathscr{A}_{\bullet}$ if $-\mathscr{M}$ is an extended real supermartingale according to $\mathscr{A}_{\bullet}$—it can be checked that any real submartingale as defined before, in Section 3.2.3$_{61}$, is an extended real submartingale as defined here.

A definitive version of the game-theoretic upper expectation

We next apply the previously introduced concepts about extended real-valued supermartingales and submartingales to define a modified version of the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$. We let $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ be the set of all **e**xtended real **b**ounded below supermartingales according to $\mathscr{A}_\bullet$, and we let $\underline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ be the set of all extended real bounded **above** submartingales according to $\mathscr{A}_\bullet$. So $\mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ if and only if $-\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$.

Definition 4.6 (The game-theoretic upper/lower expectation with **e**xtended real supermartingales that are **b**ounded below/above)**.** For any acceptable gambles tree $\mathscr{A}_\bullet$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ are defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \coloneqq \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\};$$
$$\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \coloneqq \sup\left\{\mathscr{M}(s) \colon \mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \text{ and } \limsup \mathscr{M} \leq_s f\right\}. \qquad \circledcirc$$

Due to the following conjugacy relation, we are again allowed to focus on upper expectations.

Corollary 4.2.6 (Conjugacy)**.** *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) = -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(-f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.*

Proof. This can easily be derived from the definitions of $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$, and the fact that $\mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ if and only if $-\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$. □

Similar modifications of the upper expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ with extended real supermartingales can also be proposed, but it is easy to see, based on our earlier considerations, that these modifications will turn out to be unsuitable. Indeed, in the case of $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$, one cannot really speak of a modification because working with bounded extended real supermartingales is the same as working with the set $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$ of all bounded real supermartingales, which was already used in Definition 4.2_{137}. The modification of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, then, would consist in working with all extended real (unbounded) supermartingales, which, as mentioned before, includes all real (unbounded) supermartingales $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A})$. The resulting global upper expectation would thus surely be smaller than or equal to $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$—and the corresponding global lower expectation would be larger than or equal to $\underline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$—and so the same issue as in Example $4.2.1_{139}$ would then arise.

To the contrary, the global game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ fixes all the issues raised previously. Let us check that it does so for the issue in Example $4.2.2_{140}$.

Example 4.2.7. Let $\mathscr{A}_\bullet$ be the same acceptable gambles tree as in Example $4.2.2_{140}$; so $\mathscr{A}_\square$ is equal to $\mathscr{L}_{\geq}(\mathscr{X}) \cup \{f \in \mathscr{L}(\mathscr{X}) \colon f(a) > 0\}$, and

$\mathscr{A}_s = \mathscr{L}_{\geq}(\mathscr{X})$ for all $s \in \mathscr{X}^* \setminus \{\square\}$. Then it can be checked using Definition 4.5146 that

$$\mathscr{A}_\square^\uparrow = \overline{\mathscr{L}}_{\geq}(\mathscr{X}) \cup \{f \in \overline{\mathscr{L}}(\mathscr{X}) : f(a) > 0\},$$

where $\overline{\mathscr{L}}_{\geq}(\mathscr{X})$ is the set of all non-negative variables in $\overline{\mathscr{L}}(\mathscr{X})$. Hence, an extended real process $\mathscr{M}$ that is obtained from an extended betting process $\mathscr{G}$ for which $\mathscr{G}(\square)(a) < 0$ and, for all $s \in \mathscr{X}^* \setminus \{\square\}$, $-\mathscr{G}(s) = 0 \in \mathscr{A}_s = \mathscr{L}_{\geq}(\mathscr{X})$, is a supermartingale according to $\mathscr{A}_\bullet$. In particular, for any $\epsilon > 0$, the extended real process $\mathscr{M}$ defined by

$$\mathscr{M}(s) := \begin{cases} 1+\epsilon & \text{if } s = \square; \\ 1 & \text{if } s \sqsupseteq a; \\ +\infty & \text{if } s \sqsupseteq b, \end{cases} \quad \text{for all } s \in \mathscr{X}^*,$$

is a supermartingale according to $\mathscr{A}_\bullet$. Since it is also clearly bounded below, we have that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$. Moreover, $\mathscr{M}$ superhedges the variable τ_a on all paths; even on $\omega = bbb\cdots$ where $\tau_a(\omega) = +\infty$. So it follows from Definition 4.6↶ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(\tau_a) \leq \mathscr{M}(\square) = 1+\epsilon$. Since this is true for all $\epsilon > 0$, we find that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(\tau_a) \leq 1$. To see that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(\tau_a) \geq 1$, note that, for any $s \in \mathscr{X}^* \setminus \{\square\}$, since $\mathscr{A}_s = \mathscr{L}_{\geq}(\mathscr{X})$, the extended set of acceptable variables $\mathscr{A}_s^\uparrow$ is equal to $\overline{\mathscr{L}}_{\geq}(\mathscr{X})$. Hence, together with the form of $\mathscr{A}_\square^\uparrow$, we infer that any supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ must always remain equal or decrease on all paths $\omega \in \Gamma(a)$. So, for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq \tau_a \geq 1$, we have that $\mathscr{M}(\square) \geq 1$, and thus by Definition 4.6↶ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(\tau_a) \geq 1$. Hence, we find that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = 1$, which indeed corresponds to our intuition. This is contrast with the result in Example 4.2.2140, where we obtained that $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}(\tau_a) = +\infty$. ◇

We encourage the reader to moreover check that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ does also not suffer from the same issue as the one raised for $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ in Example 4.2.1139. Furthermore, as we will show next, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ is an extension of (the restriction of) $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{V} \times \mathscr{X}^*$—or, by Proposition 4.1.7138, $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{V} \times \mathscr{X}^*$—and therefore $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{V} \times \mathscr{X}^*$ can be interpreted in the same intuitive way as $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$. Hence, due to this equality, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ behaves in the same desirable way in Example 4.1.1133 as $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$.

Proposition 4.2.8. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \textit{ for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

This equality remains to hold if we replace $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ by $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$.

The proof of this result is based on the following two lemmas, which are similar to Lemma 4.1.5137 and Lemma 4.1.6138, but deal with extended real processes. Just as before, for any extended real process $\mathscr{C}$ and any real number B, $\mathscr{C}^{\wedge B}$ denotes the extended real process defined by $\mathscr{C}^{\wedge B}(s) := \min\{\mathscr{C}(s), B\}$ for all $s \in \mathscr{X}^*$.

Lemma 4.2.9. *For any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ and all $B \in \mathbb{R}$, we have that $\mathscr{M}^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$.*

Proof. Let $\mathscr{G}$ be the extended betting process such that $\mathscr{M}(s) = \mathscr{M}(\Box) + \mathscr{C}^{\mathscr{G}}(s)$ and $-\mathscr{G}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$. Note that the process $\mathscr{M}^{\wedge B}$ is bounded (and thus real-valued) because $\mathscr{M}$ is bounded below. We moreover prove that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$.

Fix any $s \in \mathscr{X}^*$. First suppose that $\mathscr{M}(s) \leq B$ and therefore that $\mathscr{M}(s) = \mathscr{M}^{\wedge B}(s)$. Since $\mathscr{M}^{\wedge B}$ is real-valued, $\mathscr{M}(s)$ is real. We moreover have that $\mathscr{M}(s\cdot) \geq \mathscr{M}^{\wedge B}(s\cdot)$ by the definition of $\mathscr{M}^{\wedge B}$, so we can infer that

$$\begin{aligned}\mathscr{G}(s) = \mathscr{G}(s) + \mathscr{M}(s) - \mathscr{M}(s) = \mathscr{M}(s\cdot) - \mathscr{M}(s) &\geq \mathscr{M}^{\wedge B}(s\cdot) - \mathscr{M}(s) \\ &= \mathscr{M}^{\wedge B}(s\cdot) - \mathscr{M}^{\wedge B}(s) \\ &= \Delta\mathscr{M}^{\wedge B}(s)\end{aligned}$$

Hence, since $-\mathscr{G}(s) \in \mathscr{A}_s^{\uparrow}$ by assumption, we infer by Lemma 4.2.5$_{146}$ that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s^{\uparrow}$. Then also $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$ because $\Delta\mathscr{M}^{\wedge B}(s)$ is a gamble [since $\mathscr{M}^{\wedge B}$ is real-valued] and $\mathscr{A}_s^{\uparrow} \cap \mathscr{L}(\mathscr{X}) = \mathscr{A}_s$ [due to Definition 4.5$_{146}$].

On the other hand, suppose that $\mathscr{M}(s) > B$ and therefore that $\mathscr{M}^{\wedge B}(s) = B$. Then since $\mathscr{M}^{\wedge B}(s\cdot) \leq B$, we know that $\Delta\mathscr{M}^{\wedge B}(s) \leq 0$. Since $\Delta\mathscr{M}^{\wedge B}(s)$ is moreover a gamble [since $\mathscr{M}^{\wedge B}$ is real-valued], we have by D1$_{27}$ that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$. So we conclude that $-\Delta\mathscr{M}^{\wedge B}(s) \in \mathscr{A}_s$ for all $s \in \mathscr{X}^*$, and therefore that $\mathscr{M}^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$ as desired. □

Lemma 4.2.10. *For any extended real process $\mathscr{C}$ and any path $\omega \in \Omega$, we have that*

$$\min\left\{B, \liminf_{n\to+\infty} \mathscr{C}(\omega^n)\right\} = \liminf_{n\to+\infty} \mathscr{C}^{\wedge B}(\omega^n) \text{ for all } B \in \mathbb{R}.$$

Proof. Fix any $B \in \mathbb{R}$. It is easy to check that

$$\begin{aligned}\liminf_{n\to+\infty} \mathscr{C}^{\wedge B}(\omega^n) = \sup_{m\in\mathbb{N}} \inf_{n\geq m} \mathscr{C}^{\wedge B}(\omega^n) &= \sup_{m\in\mathbb{N}} \inf_{n\geq m} \min\{\mathscr{C}(\omega^n), B\} \\ &= \sup_{m\in\mathbb{N}} \min\{\inf_{n\geq m} \mathscr{C}(\omega^n), B\} \\ &= \min\left\{\sup_{m\in\mathbb{N}} \inf_{n\geq m} \mathscr{C}(\omega^n), B\right\} \\ &= \min\left\{B, \liminf_{n\to+\infty} \mathscr{C}(\omega^n)\right\}.\end{aligned}$$ □

Proof of Proposition 4.2.8$_{\leftarrow}$. Fix any $(f, s) \in \mathbb{V} \times \mathscr{X}^*$. That $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s)$ follows immediately from the fact that $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ [because each $\mathscr{A}_s^{\uparrow}$ is an extension of $\mathscr{A}_s$] and the definitions of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ [Definition 4.6_{149} and Definition 4.2_{137}]. To prove the converse inequality, consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$, and let B be any real number such that $B > \sup(f|s)$—which exists because f is a gamble, and therefore $\sup(f|s) \in \mathbb{R}$. Then, by Lemma $4.2.9_{\curvearrowleft}$, we have that $\mathscr{M}^{\wedge B} \in \overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$. Moreover, it follows from Lemma $4.2.10_{\curvearrowleft}$ that, for any $\omega \in \Gamma(s)$,

$$\begin{aligned}\liminf \mathscr{M}^{\wedge B}(\omega) = \min\{B, \liminf \mathscr{M}(\omega)\} &\geq \min\{B, f(\omega)\}\\ &\geq \min\{\sup(f|s), f(\omega)\} = f(\omega),\end{aligned}$$

which implies that $\liminf \mathscr{M}^{\wedge B} \geq_s f$. Hence, by Definition 4.2_{137}, we find that $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}^{\wedge B}(s) \leq \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$, we conclude that by Definition 4.6_{149} that $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s)$. The remaining statement for $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ follows from Proposition $4.1.7_{138}$. □

From what we know so far, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ seems to be the best choice among all the game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ (and the hypothetical one $\overline{\mathrm{E}}^{\mathrm{e}}_{\mathscr{A},\mathrm{V}}$). Furthermore, as we will show in further sections of this chapter, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ satisfies a broad variety of desirable properties. On top of this, the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ does not only coincide with $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ on $\mathbb{V} \times \mathscr{X}^*$, but, as we have claimed in Section $4.2.2_{142}$, and as we will prove in Section 4.8_{186}, it also coincides—on all of $\overline{\mathbb{V}} \times \mathscr{X}^*$—with the version $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ that results from the more direct, and perhaps more intuitive approach using upper and lower cuts. As a result of these considerations, we will adopt the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ as our game-theoretic upper expectation of choice.

4.3 Game-theoretic upper expectations in terms of upper expectations trees

As a first step in our mathematical analysis of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$, we develop an alternative expression for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ in terms of upper expectations trees $\overline{\mathrm{Q}}_\bullet$ and their corresponding extensions $\overline{\mathrm{Q}}^{\uparrow}_\bullet$. The reason for doing this is that it leads to a conclusion that is similar to Corollary $3.5.8_{95}$: as far as the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ is concerned, the boundary structure of the sets $\mathscr{A}_s$ is irrelevant—the agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet$ defined according to Eq. $(3.1)_{50}$ completely characterises $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$. As a consequence, it makes sense to prove such a property in the beginning, before we derive further mathematical properties, because it will then allow us to reduce the degrees of freedom along which the initial local models can vary, therefore simplifying matters considerably. Moreover, the representation of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ in terms of upper expectations trees lies closer to Shafer and Vovk's approach, and will therefore, in Section 4.9_{187}, allow us to relate our work to theirs in a clearer fashion.

In order to establish this result, we first need to know how we extend an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ to a tree $\overline{\mathrm{Q}}^\uparrow_\bullet$ consisting of extended upper expectations $\overline{\mathrm{Q}}^\uparrow_s$ on $\overline{\mathscr{L}}(\mathscr{X})$. These can then be used to define the notion of an extended real supermartingale corresponding to a tree $\overline{\mathrm{Q}}_\bullet$ and to subsequently define corresponding game-theoretic upper expectations.

4.3.1 Extended local upper expectations

Just as we did in Section 4.2.2$_{142}$ for global upper expectations, we can uniquely extend a local upper expectation's domain from $\mathscr{L}(\mathscr{X})$ to $\overline{\mathscr{L}}(\mathscr{X})$ by imposing continuity with respect to **upper** and **lower cuts**. Indeed, any local upper expectation $\overline{\mathrm{Q}}$ (corresponding to any general situation) is coherent and therefore monotone [C4$_{33}$], so Lemma 4.2.3$_{143}$ ensures that there is a unique extension $\overline{\mathrm{Q}}^\uparrow$ to $\overline{\mathscr{L}}(\mathscr{X})$ that satisfies CU1$_{143}$ and CU2$_{143}$. We refer to $\overline{\mathrm{Q}}^\uparrow$ as the **extended local upper expectation** corresponding to $\overline{\mathrm{Q}}$. We will also call any map $\overline{\mathrm{Q}}^\uparrow\colon \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$ an extended local upper expectation (without further ado) if $\overline{\mathrm{Q}}^\uparrow$ satisfies CU1$_{143}$ and CU2$_{143}$, and if $\overline{\mathrm{Q}}^\uparrow$ is the extension of a (coherent) local upper expectation $\overline{\mathrm{Q}}$ on $\mathscr{L}(\mathscr{X})$—or, equivalently, if the restriction of $\overline{\mathrm{Q}}^\uparrow$ to $\mathscr{L}(\mathscr{X})$ is coherent. Care should be taken here, because the unconditional notion of coherence introduced in Definition 2.6$_{32}$ only applies to (unconditional) upper expectations that are real-valued. For general (unconditional) upper expectations on $\mathscr{L}(\mathscr{X})$, being real-valued is henceforth implicitly adopted as part of—in addition to either of the conditions (i)$_{32}$–(iii)$_{32}$ in Definition 2.6$_{32}$—the definition of coherence.

Notably, this extension procedure for local upper expectations can be additionally motivated by the fact that it is in accordance with how we have extended local sets of acceptable gambles in Section 4.2.3$_{145}$. For, consider any (coherent) local set of acceptable gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{X})$, its extension $\mathscr{A}\uparrow$ according to Definition 4.5$_{146}$, and let $\overline{\mathrm{Q}}^\uparrow_{\mathscr{A}} : \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$ be defined similarly as in Eq. (3.1)$_{50}$, by

$$\overline{\mathrm{Q}}^\uparrow_{\mathscr{A}}(f) := \inf\{\alpha \in \mathbb{R} : \alpha - f \in \mathscr{A}\uparrow\}. \tag{4.2}$$

Then, as we next show, $\overline{\mathrm{Q}}^\uparrow_{\mathscr{A}}$ is an extended local upper expectation according to our definition above. The converse is also true; for any extended local upper expectation $\overline{\mathrm{Q}}^\uparrow$ according to our definition above, there is always a (coherent) local set of acceptable gambles $\mathscr{A} \subseteq \mathscr{L}(\mathscr{X})$ such that $\overline{\mathrm{Q}}^\uparrow = \overline{\mathrm{Q}}^\uparrow_{\mathscr{A}}$.

Proposition 4.3.1. *For any local set of acceptable gambles $\mathscr{A}$, $\overline{\mathrm{Q}}^\uparrow_{\mathscr{A}}$ is equal to the extended local upper expectation $\overline{\mathrm{Q}}^\uparrow$ corresponding to the local upper expectation $\overline{\mathrm{Q}} := \overline{\mathrm{Q}}_{\mathscr{A}}$ that agrees with $\mathscr{A}$ according to Eq.* (3.1)$_{50}$*. Furthermore,* *for any local upper expectation $\overline{Q}$, the extension $\overline{Q}^{\uparrow}$ is equal to $\overline{Q}^{\uparrow}_{\mathscr{A}}$, with $\mathscr{A}$ any local set of acceptable gambles that agrees with $\overline{Q}$ according to Eq.* (3.1)$_{50}$.

This result essentially states that, for any local set of acceptable gambles $\mathscr{A}$ and any local upper expectation $\overline{Q}$ that agree in the sense of Eq. (3.1)$_{50}$, the corresponding extensions $\mathscr{A}^{\uparrow}$ and $\overline{Q}^{\uparrow}$ also 'agree', in the sense that $\overline{Q}^{\uparrow}_{\mathscr{A}} = \overline{Q}^{\uparrow}$. Then also note that, since $\overline{Q}$ is equal to $\overline{Q}_{\mathscr{A}}$ by Eq. (3.1)$_{50}$, this equality between $\overline{Q}^{\uparrow}_{\mathscr{A}}$ and $\overline{Q}^{\uparrow} = (\overline{Q}_{\mathscr{A}})^{\uparrow}$ makes sure that no possible confusion can arise about the meaning of $\overline{Q}^{\uparrow}_{\mathscr{A}}$.

Our proof of Proposition 4.3.1$_{\frown}$ relies on the following lemma, which says that $\overline{Q}^{\uparrow}_{\mathscr{A}}$ for any local set of acceptable gambles $\mathscr{A}$ is an extension of the agreeing upper expectation $\overline{Q}_{\mathscr{A}}$.

Lemma 4.3.2. *For any local set of acceptable gambles $\mathscr{A}$, $\overline{Q}^{\uparrow}_{\mathscr{A}}$ extends the upper expectation $\overline{Q}_{\mathscr{A}}$ deduced from $\mathscr{A}$ according to Eq.* (3.1)$_{50}$.

Proof. Consider any $f \in \mathscr{L}(\mathscr{X})$ and observe that also $\alpha - f \in \mathscr{L}(\mathscr{X})$ for all $\alpha \in \mathbb{R}$. So, since $\mathscr{A}^{\uparrow} \cap \mathscr{L}(\mathscr{X}) = \mathscr{A}$ according to Definition 4.5$_{146}$, we obtain from Eq. (4.2)$_{\frown}$ and Eq. (3.1)$_{50}$ that

$$\begin{aligned}\overline{Q}^{\uparrow}_{\mathscr{A}}(f) = \inf\{\alpha \in \mathbb{R} : \alpha - f \in \mathscr{A}^{\uparrow}\} &= \inf\{\alpha \in \mathbb{R} : \alpha - f \in \mathscr{A}^{\uparrow} \cap \mathscr{L}(\mathscr{X})\} \\ &= \inf\{\alpha \in \mathbb{R} : \alpha - f \in \mathscr{A}\} = \overline{Q}_{\mathscr{A}}(f). \qquad \square\end{aligned}$$

We will also need the following monotonicity property for the extended models $\overline{Q}^{\uparrow}_{\mathscr{A}}$.

Lemma 4.3.3. *For any local set of acceptable gambles $\mathscr{A}$, and any two $f, g \in \mathscr{L}(\mathscr{X})$ such that $f \leq g$, we have that $\overline{Q}^{\uparrow}_{\mathscr{A}}(f) \leq \overline{Q}^{\uparrow}_{\mathscr{A}}(g)$.*

Proof. Observe from Lemma 4.2.5$_{146}$ that, for any $\alpha \in \mathbb{R}$ such that $(\alpha - g) \in \mathscr{A}^{\uparrow}$, $(\alpha - f) \in \mathscr{A}^{\uparrow}$. The desired inequality then follows from Eq. (3.1)$_{50}$. $\square$

Proof of Proposition 4.3.1$_{\frown}$. Consider any (coherent) local set of acceptable gambles $\mathscr{A}$, let $\mathscr{A}^{\uparrow}$ be the corresponding extended set [according to Definition 4.5$_{146}$], and let $\overline{Q}^{\uparrow}_{\mathscr{A}}$ be defined by Eq. (4.2)$_{\frown}$. First note that, due to Lemma 4.3.2, the upper expectation $\overline{Q}^{\uparrow}_{\mathscr{A}}$ extends $\overline{Q} = \overline{Q}_{\mathscr{A}}$. We next show that $\overline{Q}^{\uparrow}_{\mathscr{A}}$ satisfies CU1$_{143}$. Fix any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Due to Lemma 4.3.3, we have that

$$\overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c_1}) \leq \overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c_2}) \leq \overline{Q}^{\uparrow}_{\mathscr{A}}(f) \text{ for any two reals } c_1 \leq c_2.$$

On the one hand, this implies that the limit $\lim_{c\to+\infty} \overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c})$ exists, and on the other hand, this implies that $\lim_{c\to+\infty} \overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c}) \leq \overline{Q}^{\uparrow}_{\mathscr{A}}(f)$. So it remains to prove that $\lim_{c\to+\infty} \overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c}) \geq \overline{Q}^{\uparrow}_{\mathscr{A}}(f)$. Consider any $\alpha \in \mathbb{R}$ such that $\alpha > \lim_{c\to+\infty} \overline{Q}^{\uparrow}_{\mathscr{A}}(f^{\wedge c})$. Choose any $c_1 \in \mathbb{R}_{>}$ such that $c_1 > \alpha - \inf f = \sup(\alpha - f)$ [which is possible because f is bounded below] and any $c_2 \in \mathbb{R}_{<}$. Consider also any $c' \in \mathbb{R}_{>}$ such that $c' > \alpha - c_2$. Then by Lemma 4.3.3 and since $f^{\wedge c}$ is increasing for increasing c, we have that

$\alpha > \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\wedge c}) \geq \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\wedge c'})$. Then it follows from Eq. (4.2)$_{153}$ that there is a real $\alpha_{c'} \leq \alpha$ such that $\alpha_{c'} - f^{\wedge c'} \in \mathscr{A}\!\uparrow$, which by Lemma 4.2.5$_{146}$ implies that $\alpha - f^{\wedge c'} \in \mathscr{A}\!\uparrow$. Some basic manipulations of the latter variable then gives us that

$$(\alpha - f)^{\vee \alpha - c'} = (\alpha + (-f)^{\vee -c'}) = (\alpha - f^{\wedge c'}) \in \mathscr{A}\!\uparrow.$$

Since $c' > \alpha - c_2$ and thus $c_2 > \alpha - c'$, it follows once again from Lemma 4.2.5$_{146}$ that $(\alpha - f)^{\vee c_2} \in \mathscr{A}\!\uparrow$. We also have that $c_1 > \sup(\alpha - f)$ and thus that $\big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2} = (\alpha - f)^{\vee c_2} \in \mathscr{A}\!\uparrow$. Hence, since $\big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2}$ is a gamble, we have by Definition 4.5$_{146}$ that $\big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2} \in \mathscr{A}$. Since this holds for any $c_2 \in \mathbb{R}_{<}$, and since c_1 is a positive real number, it follows from Corollary 4.2.4$_{146}$ that $(\alpha - f) \in \mathscr{A}\!\uparrow$. Hence, by Eq. (4.2)$_{153}$, we obtain that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f) \leq \alpha$. Since this holds for any $\alpha \in \mathbb{R}$ such that $\alpha > \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\wedge c})$, we indeed have that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f) \leq \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\wedge c})$.

Next, we prove that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}$ also satisfies CU2$_{143}$. Consider any $f \in \overline{\mathscr{L}}(\mathscr{X})$. That $\lim_{c\to-\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c})$ exists and that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f) \leq \lim_{c\to-\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c})$ follows in a similar way as before from Lemma 4.3.3$_{\leftarrow}$. To prove that $\lim_{c\to-\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c}) \leq \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f)$, consider any $\alpha \in \mathbb{R}$ such that $(\alpha - f) \in \mathscr{A}\!\uparrow$. Then Corollary 4.2.4$_{146}$ guarantees that there is a $c_1 \in \mathbb{R}_{>}$ such that $\big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2} \in \mathscr{A}$ for all $c_2 \in \mathbb{R}_{<}$. Consider any $c \in \mathbb{R}_{<}$ such that $c < \alpha - c_1$. We show that $\alpha - f^{\vee c} \in \mathscr{A}\!\uparrow$. To this end, start by noting that

$$\alpha - f^{\vee c} = \alpha + (-f)^{\wedge -c} = (\alpha - f)^{\wedge \alpha - c}.$$

So, for any $c_2 \in \mathbb{R}_{<}$, we have that

$$\big((\alpha - f^{\vee c})^{\wedge c_1}\big)^{\vee c_2} = \big(((\alpha - f)^{\wedge \alpha - c})^{\wedge c_1}\big)^{\vee c_2},$$

which by the fact that $c < \alpha - c_1$, and thus $c_1 < \alpha - c$, implies that

$$\big((\alpha - f^{\vee c})^{\wedge c_1}\big)^{\vee c_2} = \big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2}.$$

Since $\big((\alpha - f)^{\wedge c_1}\big)^{\vee c_2} \in \mathscr{A}$, we thus have that $\big((\alpha - f^{\vee c})^{\wedge c_1}\big)^{\vee c_2} \in \mathscr{A}$. Since this holds for any $c_2 \in \mathbb{R}_{<}$, we obtain by Corollary 4.2.4$_{146}$ that $\alpha - f^{\vee c} \in \mathscr{A}\!\uparrow$ and therefore by Eq. (4.2)$_{153}$ that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c}) \leq \alpha$. By Lemma 4.3.3$_{\leftarrow}$, this then also implies that $\lim_{c\to-\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c}) \leq \alpha$. This holds for any $\alpha \in \mathbb{R}$ such that $(\alpha - f) \in \mathscr{A}\!\uparrow$, so by Eq. (4.2)$_{153}$ this implies that $\lim_{c\to-\infty} \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f^{\vee c}) \leq \overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}(f)$ as desired.

Hence, we have shown that the upper expectation $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}$ satisfies CU1$_{143}$ and CU2$_{143}$, and that it extends $\overline{\mathrm{Q}}$. As a consequence, $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}$ is equal to the extended local upper expectation $\overline{\mathrm{Q}}^{\uparrow}$ corresponding to $\overline{\mathrm{Q}}$.

The remaining implication now follows straightforwardly from the first. Indeed, consider any local upper expectation $\overline{\mathrm{Q}}$, let $\overline{\mathrm{Q}}^{\uparrow}$ be the corresponding extension, and let $\mathscr{A}$ be any (coherent) local set of acceptable gambles such that $\overline{\mathrm{Q}} = \overline{\mathrm{Q}}_{\mathscr{A}}$ with $\overline{\mathrm{Q}}_{\mathscr{A}}$ defined by Eq. (3.1)$_{50}$ [it is clear from our considerations in Section 3.1.2$_{48}$ that there is always such a set; e.g. the one described in Eq. (3.2)$_{51}$]. It then follows from the first implication that $\overline{\mathrm{Q}}^{\uparrow}_{\mathscr{A}}$ is equal to the extended local upper expectation $\overline{\mathrm{Q}}^{\uparrow}$. □

4.3.2 Basic properties for extended local upper expectations

We next prove that extended local upper expectations satisfy some basic properties that are similar to the coherence properties [C1$_{32}$–C6$_{33}$], but extended to involve extended real variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Moreover, we also show that extended local upper expectations satisfy a monotone convergence and a countable subadditivity property for variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$; see LE6 and LE7 below. We focus on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ instead of $\overline{\mathscr{L}}(\mathscr{X})$, because these properties will mainly be used later on to say something about the local behaviour of supermartingales, which we always consider to be bounded below. Moreover, because we are only interested in variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, Axiom CU2$_{143}$ is irrelevant and so, to remain as general as possible, we do not require CU2$_{143}$ to be satisfied for the upper expectations in the following proposition.

Proposition 4.3.4. *Consider any upper expectation $\overline{\mathrm{Q}}^{\uparrow}\colon \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$ that is coherent on $\mathscr{L}(\mathscr{X})$ and that satisfies* CU1$_{143}$. *Then we have that*

LE1. $-\infty < \inf f \leq \overline{\mathrm{Q}}^{\uparrow}(f) \leq \sup f$ *for all* $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE2. $\overline{\mathrm{Q}}^{\uparrow}(f+g) \leq \overline{\mathrm{Q}}^{\uparrow}(f) + \overline{\mathrm{Q}}^{\uparrow}(g)$ *for all* $f, g \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE3. $\overline{\mathrm{Q}}^{\uparrow}(\lambda f) = \lambda\overline{\mathrm{Q}}^{\uparrow}(f)$ *for all* $\lambda \in \mathbb{R}_{\geq}$ *and all* $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE4. $f \leq g \Rightarrow \overline{\mathrm{Q}}^{\uparrow}(f) \leq \overline{\mathrm{Q}}^{\uparrow}(g)$ *for all* $f, g \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE5. $\overline{\mathrm{Q}}^{\uparrow}(f+\mu) = \overline{\mathrm{Q}}^{\uparrow}(f) + \mu$ *for all* $\mu \in \mathbb{R} \cup \{+\infty\}$ *and all* $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE6. $\lim_{n\to+\infty} \overline{\mathrm{Q}}^{\uparrow}(f_n) = \overline{\mathrm{Q}}^{\uparrow}\left(\lim_{n\to+\infty} f_n\right)$ *for any increasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$*;*

LE7. $\overline{\mathrm{Q}}^{\uparrow}\left(\sum_{n\in\mathbb{N}} f_n\right) \leq \sum_{n\in\mathbb{N}} \overline{\mathrm{Q}}^{\uparrow}(f_n)$ *for any sequence* $(f_n)_{n\in\mathbb{N}}$ *of non-negative variables in* $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$*;*

LE8. $\overline{\mathrm{Q}}^{\uparrow}((+\infty)f) = (+\infty)\overline{\mathrm{Q}}^{\uparrow}(f)$ *for all non-negative* $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$.

Proof. We first prove LE4. Fix any $f, g \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$ such that $f \leq g$. Then, for any $c \in \mathbb{R}$, we also have that $f^{\wedge c} \leq g^{\wedge c}$. Both $f^{\wedge c}$ and $g^{\wedge c}$ are gambles because f and g are bounded below, so it follows from the coherence [C4$_{33}$] of $\overline{\mathrm{Q}}^{\uparrow}$ on $\mathscr{L}(\mathscr{X})$ that $\overline{\mathrm{Q}}^{\uparrow}(f^{\wedge c}) \leq \overline{\mathrm{Q}}^{\uparrow}(g^{\wedge c})$. Since this holds for any $c \in \mathbb{R}$, we have by CU1$_{143}$ that

$$\overline{\mathrm{Q}}^{\uparrow}(f) = \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}(f^{\wedge c}) \leq \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}(g^{\wedge c}) = \overline{\mathrm{Q}}^{\uparrow}(g).$$

LE1. Fix any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. If $\sup f = +\infty$, we trivially have that $\overline{\mathrm{Q}}^{\uparrow}(f) \leq \sup f$. If $\sup f$ is real, it follows immediately from LE4 that $\overline{\mathrm{Q}}^{\uparrow}(f) \leq \overline{\mathrm{Q}}^{\uparrow}(\sup f)$. We have that $\overline{\mathrm{Q}}^{\uparrow}(\sup f) = \sup f$ because $\sup f$ is real and $\overline{\mathrm{Q}}^{\uparrow}$ is coherent [C5$_{33}$] on $\mathscr{L}(\mathscr{X})$. Hence, we then also have that $\overline{\mathrm{Q}}^{\uparrow}(f) \leq \sup f$. That $\sup f = -\infty$, is impossible because f is bounded below. To see that $-\infty < \inf f \leq \overline{\mathrm{Q}}^{\uparrow}(f)$, note that $\inf f$ is real or equal to $+\infty$ [because f is bounded below] and therefore that $-\infty < \inf f$ is automatically satisfied. Moreover, for any real $\alpha < \inf f$ we clearly have that $\alpha < f$, implying by LE4 and the coherence [C5$_{33}$] of $\overline{\mathrm{Q}}^{\uparrow}$ on $\mathscr{L}(\mathscr{X})$ that $\alpha = \overline{\mathrm{Q}}^{\uparrow}(\alpha) \leq \overline{\mathrm{Q}}^{\uparrow}(f)$. Since this holds for any $\alpha < \inf f$ we indeed have that $\inf f \leq \overline{\mathrm{Q}}^{\uparrow}(f)$.

LE5. Fix any $\mu \in \mathbb{R} \cup \{+\infty\}$ and any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$. If $\mu = +\infty$, then $f + \mu = +\infty$ because f is bounded below. The fact that f is bounded below also implies by LE1 that $\overline{\mathrm{Q}}^{\uparrow}(f) \in \mathbb{R} \cup \{+\infty\}$, and thus that $\overline{\mathrm{Q}}^{\uparrow}(f) + \mu = +\infty$. So it suffices to prove that $\overline{\mathrm{Q}}^{\uparrow}(+\infty) = +\infty$; this holds due to LE1$_{\leftarrow}$. So assume that $\mu \in \mathbb{R}$. Consider any $c \in \mathbb{R}$ and note that $(f+\mu)^{\wedge c} = f^{\wedge c-\mu} + \mu$. Since $f^{\wedge c-\mu}$ is a gamble because f is bounded below, it then follows from the coherence [C6$_{33}$] of $\overline{\mathrm{Q}}^{\uparrow}$ on $\mathscr{L}(\mathscr{X})$ that $\overline{\mathrm{Q}}^{\uparrow}((f+\mu)^{\wedge c}) = \overline{\mathrm{Q}}^{\uparrow}(f^{\wedge c-\mu}) + \mu$. Since this holds for all $c \in \mathbb{R}$, it follows from CU1$_{143}$ that

$$\overline{\mathrm{Q}}^{\uparrow}(f+\mu) = \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((f+\mu)^{\wedge c}) = \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}(f^{\wedge c-\mu}) + \mu = \overline{\mathrm{Q}}^{\uparrow}(f) + \mu.$$

LE2$_{\leftarrow}$. Fix any $f, g \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Let $a, b \in \mathbb{R}$ be such that $a \leq \inf f$ and $b \leq \inf g$ [which is possible because f and g are bounded below], and let $f' := f - a$ and $g' := g - b$; then f' and g' are both non-negative. Consider any positive $c \in \mathbb{R}_{>}$ and note that the non-negativity of f' and g' implies that $(f'+g')^{\wedge c} \leq (f')^{\wedge c} + (g')^{\wedge c}$. Indeed, for any $x \in \mathscr{X}$, we either have that $f'(x) \geq c$, that $g'(x) \geq c$ or that both $f'(x) < c$ and $g'(x) < c$. If $f'(x) \geq c$, then $(f'+g')^{\wedge c}(x) \leq c = (f')^{\wedge c}(x)$ and so by the non-negativity of g' [and thus also $(g')^{\wedge c}$ because $c > 0$] that $(f'+g')^{\wedge c}(x) \leq (f')^{\wedge c}(x) + (g')^{\wedge c}(x)$. In a completely analogous way, we can establish that the same is true if $g'(x) \geq c$. Finally, if both $f'(x) < c$ and $g'(x) < c$, then we infer that $(f')^{\wedge c}(x) + (g')^{\wedge c}(x) = f'(x) + g'(x) = (f'+g')(x) \geq (f'+g')^{\wedge c}(x)$. Since this holds for all $x \in \mathscr{X}$, we indeed have that $(f'+g')^{\wedge c} \leq (f')^{\wedge c} + (g')^{\wedge c}$. Hence, by LE4$_{\leftarrow}$ and since $\overline{\mathrm{Q}}^{\uparrow}$ is coherent [C2$_{32}$] on $\mathscr{L}(\mathscr{X})$ and $(f')^{\wedge c}$ and $(g')^{\wedge c}$ are gambles, we have that

$$\overline{\mathrm{Q}}^{\uparrow}((f'+g')^{\wedge c}) \leq \overline{\mathrm{Q}}^{\uparrow}((f')^{\wedge c} + (g')^{\wedge c}) \leq \overline{\mathrm{Q}}^{\uparrow}((f')^{\wedge c}) + \overline{\mathrm{Q}}^{\uparrow}((g')^{\wedge c}).$$

Since this holds for any $c \in \mathbb{R}_{>}$, and since $\overline{\mathrm{Q}}^{\uparrow}((f')^{\wedge c})$ and $\overline{\mathrm{Q}}^{\uparrow}((g')^{\wedge c})$ are increasing in c because of LE4$_{\leftarrow}$, we then infer that

$$\begin{aligned}\overline{\mathrm{Q}}^{\uparrow}(f'+g') \overset{\mathrm{CU1}_{143}}{=} \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((f'+g')^{\wedge c}) &\leq \lim_{c\to+\infty} [\overline{\mathrm{Q}}^{\uparrow}((f')^{\wedge c}) + \overline{\mathrm{Q}}^{\uparrow}((g')^{\wedge c})] \\ &= \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((f')^{\wedge c}) + \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((g')^{\wedge c}) \\ &\overset{\mathrm{CU1}_{143}}{=} \overline{\mathrm{Q}}^{\uparrow}(f') + \overline{\mathrm{Q}}^{\uparrow}(g').\end{aligned}$$

It now suffices to apply LE5$_{\leftarrow}$ to both sides, to arrive at the fact that $\overline{\mathrm{Q}}^{\uparrow}(f+g) \leq \overline{\mathrm{Q}}^{\uparrow}(f) + \overline{\mathrm{Q}}^{\uparrow}(g)$.

LE3$_{\leftarrow}$. Fix any $\lambda \in \mathbb{R}_{\geq}$ and any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. If $\lambda = 0$, it suffices, because of the convention $0 \cdot (+\infty) = 0 \cdot (-\infty) = 0$, to prove that $\overline{\mathrm{Q}}^{\uparrow}(0) = 0$. This follows directly from LE1$_{\leftarrow}$. If $\lambda \in \mathbb{R}_{>}$, note that for any $c \in \mathbb{R}_{>}$ we have that $(\lambda f)^{\wedge c} = \lambda (f)^{\wedge c/\lambda}$. Hence, we have that

$$\begin{aligned}\overline{\mathrm{Q}}^{\uparrow}(\lambda f) \overset{\mathrm{CU1}_{143}}{=} \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((\lambda f)^{\wedge c}) = \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}(\lambda (f)^{\wedge c/\lambda}) &= \lambda \lim_{c\to+\infty} \overline{\mathrm{Q}}^{\uparrow}((f)^{\wedge c/\lambda}) \\ &\overset{\mathrm{CU1}_{143}}{=} \lambda \overline{\mathrm{Q}}^{\uparrow}(f),\end{aligned}$$

where the penultimate step follows from the coherence [C3$_{32}$] of $\overline{\mathrm{Q}}^{\uparrow}$ on $\mathscr{L}(\mathscr{X})$ and the fact that each $(f)^{\wedge c/\lambda}$ is a gamble.

LE6$_{\leftarrow}$. Fix any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Let $f := \lim_{n\to+\infty} f_n \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Then, for any $c \in \mathbb{R}$, $(f_n^{\wedge c})_{n\in\mathbb{N}}$ is an increasing sequence in $\mathscr{L}(\mathscr{X})$ that clearly converges pointwise to $f^{\wedge c} \in \mathscr{L}(\mathscr{X})$. Moreover, since $f^{\wedge c}$ is a real-valued function on a finite set $\mathscr{X}$, the sequence $(f_n^{\wedge c})_{n\in\mathbb{N}}$ converges uniformly to $f^{\wedge c}$. Hence, we have that

$$\begin{aligned}\overline{\mathrm{Q}}^{\uparrow}(f) \overset{\mathrm{CU1}_{143}}{=} \lim_{c\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(f^{\wedge c}) \overset{\mathrm{C8}_{33}}{=} \lim_{c\to+\infty}\lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(f_n^{\wedge c}) \overset{\mathrm{LE4}_{156}}{=} \sup_{c\in\mathbb{R}}\sup_{n\in\mathbb{N}}\overline{\mathrm{Q}}^{\uparrow}(f_n^{\wedge c})\\ = \sup_{n\in\mathbb{N}}\sup_{c\in\mathbb{R}}\overline{\mathrm{Q}}^{\uparrow}(f_n^{\wedge c}) \overset{\mathrm{LE4}_{156}}{=} \lim_{n\to+\infty}\sup_{c\in\mathbb{R}}\overline{\mathrm{Q}}^{\uparrow}(f_n^{\wedge c}) \overset{\mathrm{LE4}_{156}}{=} \lim_{n\to+\infty}\lim_{c\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(f_n^{\wedge c})\\ \overset{\mathrm{CU1}_{143}}{=} \lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(f_n).\end{aligned}$$

LE7$_{156}$. Fix any sequence $(f_n)_{n\in\mathbb{N}}$ of non-negative variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Let $(g_n)_{n\in\mathbb{N}}$ be the sequence of non-negative variables defined by $g_n := \sum_{i=1}^{n} f_i$ for all $n\in\mathbb{N}$. Then, $(g_n)_{n\in\mathbb{N}}$ is increasing because $(f_n)_{n\in\mathbb{N}}$ is non-negative. Moreover, it is clear that $(g_n)_{n\in\mathbb{N}}$ converges pointwise to $\sum_{n\in\mathbb{N}} f_n$. Hence, we can apply LE6$_{156}$ to find that

$$\overline{\mathrm{Q}}^{\uparrow}\left(\sum_{n\in\mathbb{N}} f_n\right) = \lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(g_n) = \lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}\left(\sum_{i=1}^{n} f_i\right) \overset{\mathrm{LE2}_{156}}{\leq} \lim_{n\to+\infty}\sum_{i=1}^{n}\overline{\mathrm{Q}}^{\uparrow}(f_i) = \sum_{n\in\mathbb{N}}\overline{\mathrm{Q}}^{\uparrow}(f_n),$$

where the limit on the right-hand side of the inequality exists because all $\overline{\mathrm{Q}}^{\uparrow}(f_i)$ are non-negative as a consequence of LE1$_{156}$.

LE8$_{156}$. Fix any non-negative $f\in\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ and observe that $(nf)_{n\in\mathbb{N}}$ is an increasing sequence in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ that converges pointwise to $(+\infty)f$ [because of the convention that $(+\infty)\,0 = 0$]. Hence,

$$\overline{\mathrm{Q}}^{\uparrow}\left((+\infty)f\right) = \overline{\mathrm{Q}}^{\uparrow}\left(\lim_{n\to+\infty} nf\right) \overset{\mathrm{LE6}_{156}}{=} \lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}(nf) \overset{\mathrm{LE3}_{156}}{=} \lim_{n\to+\infty} n\,\overline{\mathrm{Q}}^{\uparrow}(f) = (+\infty)\overline{\mathrm{Q}}^{\uparrow}(f),$$

where we once more used the convention that $(+\infty)\,0 = 0$ for the last step, together with the fact that $\overline{\mathrm{Q}}^{\uparrow}(f)\geq 0$ because of LE1$_{156}$. □

4.3.3 Supermartingales and game-theoretic upper expectations in terms of extended upper expectations trees

For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, let $\overline{\mathrm{Q}}_{\bullet}^{\uparrow}$ be the corresponding **extended upper expectations tree**; it consists, for all $s\in\mathscr{X}^*$, of the extended local upper expectation $\overline{\mathrm{Q}}_s^{\uparrow}$ corresponding to $\overline{\mathrm{Q}}_s$. All previously stated results for extended local upper expectations thus apply in particular to the components of an extended upper expectations tree. Most importantly, it follows from Proposition 4.3.1$_{153}$ that, for any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ and acceptable gambles tree $\mathscr{A}_{\bullet}$ that agree according to Eq. (3.1)$_{50}$, the extended upper expectations tree $\overline{\mathrm{Q}}_{\bullet}^{\uparrow}$ corresponding to $\overline{\mathrm{Q}}_{\bullet}$ is equal to the map $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}^{\uparrow}\colon s\in\mathscr{X}^*\mapsto\overline{\mathrm{Q}}_{s,\mathscr{A}}^{\uparrow}$, where $\overline{\mathrm{Q}}_{s,\mathscr{A}}^{\uparrow}$ for all $s\in\mathscr{X}^*$ is deduced from $\mathscr{A}_s^{\uparrow}$ according to Eq. (4.2)$_{153}$.

Extended real supermartingales based on extended upper expectations trees

Given this correlation between extended upper expectations trees and extended acceptable gambles trees, and taking into account the definition of the supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$, it is now natural to define an extended real supermartingale corresponding to an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ as any extended real process $\mathscr{M}$ for which there is some extended betting process $\mathscr{G}$ such that

$$\mathscr{M}(s) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(s) \text{ and } \overline{\mathrm{Q}}_s^{\uparrow}(\mathscr{G}(s)) \leq 0 \text{ for all } s \in \mathscr{X}^*.$$

We use $\overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet)$ to denote the set of all extended real supermartingales that are defined in this way and that are bounded below.

Though the above definition of an extended real supermartingale is intuitive when acquainted with sets of acceptable gambles and/or acceptable gambles tree, most of the time, we will work with the following slightly different—and more direct—definition: any extended real process $\mathscr{M}$ is called a(n) **(extended) real supermartingale** according to $\overline{\mathrm{Q}}_\bullet$ if

$$\overline{\mathrm{Q}}_s^{\uparrow}(\mathscr{M}(s\cdot)) \leq \mathscr{M}(s) \text{ for all } s \in \mathscr{X}^*. \tag{4.3}$$

We let $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ be the set of all supermartingales that are defined in this way and that are bounded below.

Let us first show that the class $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ of supermartingales is at least as large as $\overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet)$, and that, for any tree $\mathscr{A}_\bullet$ that agrees with $\overline{\mathrm{Q}}_\bullet$, both $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and $\overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet)$ are supersets of the class $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$ of (extended real) supermartingales according to $\mathscr{A}_\bullet$.

Proposition 4.3.5. *Consider any acceptable gambles tree $\mathscr{A}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ be the agreeing upper expectations tree according to Eq.* (3.1)$_{50}$. *Then we have that*

$$\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet).$$

Proof. Let us first show that $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet)$. Consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$. Then there is an extended betting process $\mathscr{G}$ such that $\mathscr{M}(s) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(s)$ and $-\mathscr{G}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$. Due to Proposition 4.3.1$_{153}$, we know that $\overline{\mathrm{Q}}_\bullet^{\uparrow}$ coincides with $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}^{\uparrow}$, so it suffices to show that $\overline{\mathrm{Q}}_{s,\mathscr{A}}^{\uparrow}(\mathscr{G}(s)) \leq 0$ for all $s \in \mathscr{X}^*$. This follows trivially from Eq. (4.2)$_{153}$ and the fact that $-\mathscr{G}(s) \in \mathscr{A}_s^{\uparrow}$ for all $s \in \mathscr{X}^*$.

We next prove that $\overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}^{\mathscr{G}}(\overline{\mathrm{Q}}_\bullet)$. Then there is an extended betting process $\mathscr{G}$ such that $\mathscr{M}(s) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(s)$ and $\overline{\mathrm{Q}}_s^{\uparrow}(\mathscr{G}(s)) \leq 0$ for all $s \in \mathscr{X}^*$. Fix any $x_{1:k} \in \mathscr{X}^*$. Then, for any $x_{k+1} \in \mathscr{X}$, we have that

$$\begin{aligned}\mathscr{M}(x_{1:k+1}) = \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(x_{1:k+1}) &= \mathscr{M}(\square) + \sum_{i=0}^{k} \mathscr{G}(x_{1:i})(x_{i+1}) \\ &= \mathscr{M}(\square) + \mathscr{C}^{\mathscr{G}}(x_{1:k}) + \mathscr{G}(x_{1:k})(x_{k+1}) \\ &= \mathscr{M}(x_{1:k}) + \mathscr{G}(x_{1:k})(x_{k+1}).\end{aligned}$$

Since this holds for any $x_{k+1} \in \mathscr{X}$, we obtain that $\mathscr{M}(x_{1:k}\cdot) = \mathscr{M}(x_{1:k}) + \mathscr{G}(x_{1:k})$. Recall that $\mathscr{M}$ is bounded below, so $\mathscr{M}(x_{1:k}) \in \mathbb{R} \cup \{+\infty\}$, and therefore, since the extended local upper expectation $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}$ satisfies LE5$_{156}$ by Proposition 4.3.4$_{156}$, we have that

$$\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}) + \mathscr{G}(x_{1:k})) = \mathscr{M}(x_{1:k}) + \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{G}(x_{1:k})).$$

Since we know that $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{G}(x_{1:k})) \leq 0$, it follows that $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot)) \leq \mathscr{M}(x_{1:k})$. Since this holds for any any $x_{1:k} \in \mathscr{X}^*$ [and since $\mathscr{M}$ is bounded below], we obtain that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ as desired. □

It can actually be shown that the inclusion $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ in the result above is strict; for instance, a supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ can attain the value $+\infty$ and afterwards become real-valued; one may observe that this is not possible for a supermartingale in $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. The inclusion $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \subseteq \overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ too, can sometimes become strict, depending on the form of the local sets of acceptable gambles $\mathscr{A}_s$.

Finally, before introducing global game-theoretic upper expectations corresponding to $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, note that the definitions of the supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ never relied on process differences. The reason is similar to why we did not use process differences in Section 4.2.3$_{145}$ to define the supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$; if an extended real supermartingale $\mathscr{M}$ were to be characterised by the condition that $\overline{\mathrm{Q}}^{\uparrow}_{s}(\Delta\mathscr{M}(s)) \leq 0$ for all $s \in \mathscr{X}^*$, then it can be checked using LE1$_{156}$ that a supermartingale cannot remain in $+\infty$ (for all following situations) once it has attained $+\infty$, which we consider to be undesirable.

Game-theoretic upper expectations based on extended upper expectations trees

Using the sets $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, we can define the corresponding game-theoretic upper expectations as follows.

Definition 4.7. For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) := \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\};$$

The upper expectation $\overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\mathrm{Q,V}}$ is defined similarly, with $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ replaced by $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. ◎

The following theorem states that these two versions $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\mathrm{Q,V}}$ of the game-theoretic upper expectation based on an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ coincide, and that they moreover (both) coincide with the version $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ based directly on an acceptable gambles tree $\mathscr{A}_\bullet$, given that the trees $\overline{\mathrm{Q}}_\bullet$ and $\mathscr{A}_\bullet$ agree in the sense of Eq. (3.1)$_{50}$. The proof of it can be found in Appendix 4.A$_{197}$.

Theorem 4.3.6. *Consider any acceptable gambles tree* $\mathscr{A}_\bullet$ *and let* $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ *be the agreeing upper expectations tree according to Eq.* $(3.1)_{50}$. *Then we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Theorem 4.3.6 shows that the subtle difference between the supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ pointed out earlier, is irrelevant when concerned with the values of the associated global game-theoretic upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\overline{\mathrm{Q}},\mathrm{V}}$. This allows us to use the mathematically more convenient set $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and its corresponding game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$, rather than the set $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ which is—from a behavioural point of view—actually more natural to adopt.

Theorem 4.3.6 also shows that, as far as the resulting game-theoretic upper expectations are concerned, acceptable gambles trees again have an unnecessary rich and complex structure: for any two acceptable gambles trees $\mathscr{A}^1_\bullet$ and $\mathscr{A}^2_\bullet$ for which it holds that $\overline{\mathrm{Q}}_{\bullet,\mathscr{A}^1} = \overline{\mathrm{Q}}_{\bullet,\mathscr{A}^2}$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A}^1,\mathrm{V}} = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A}^2,\mathrm{V}}$. This is a similar conclusion to the one we have drawn in Section $3.5.3_{93}$ for the finitary global upper expectations $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{A}}$. As a result, here too, it seems sensible—at least from a mathematical point of view—to work with upper expectations trees instead of acceptable gambles trees when parametrizing a stochastic process. We will henceforth do so and therefore typically write $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ to denote a generic game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$.

Moreover, note that, as far as the resulting values of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ are concerned, **it does not matter whether we do, or do not impose $\mathbf{CU2_{143}}$** onto the local upper expectations $\overline{\mathrm{Q}}^{\uparrow}_s$. Indeed, the supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ are required to be bounded below, so the values of the local models $\overline{\mathrm{Q}}^{\uparrow}_s$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ are all that matters for the values of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. Since $\mathrm{CU2}_{143}$ is obviously always satisfied on the restricted domain $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ by any local upper expectation, imposing $\mathrm{CU2}_{143}$ does not impact the values that can be taken by $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. One could therefore choose to not adopt $\mathrm{CU2}_{143}$, and therefore remain slightly more general. We will nevertheless choose to adopt $\mathrm{CU2}_{143}$ because (i) as pointed out in Section $4.2.2_{142}$, it results from a conservativity assumption, (ii) the extended upper expectations trees satisfying $\mathrm{CU2}_{143}$ can be seen as to result from extended acceptable gambles tree [Proposition $4.3.1_{153}$], and, most importantly, (iii) because, as we will show in Section $4.6.2_{178}$, Axiom $\mathrm{CU2}_{143}$ is required to guarantee compatibility with the global game-theoretic upper expectation on the entire domain $\overline{\mathscr{L}}(\mathscr{X})$ of local variables.

Finally, let us establish that the equality in Theorem 4.3.6 also holds for the lower expectations $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. The latter is defined as follows. Similarly as we did in Section $4.2.3_{145}$, an **(extended real) submartingale** $\mathscr{M}$ according to an upper expectations tree $\overline{\mathrm{Q}}_\bullet$ is an extended real process such that $-\mathscr{M}$ is a supermartingale according to $\overline{\mathrm{Q}}_\bullet$. It can be checked that $\mathscr{M}$ is then a submartingale according to $\overline{\mathrm{Q}}_\bullet$ if and only if

$$\underline{\mathrm{Q}}_s^{\uparrow}(\mathscr{M}(s\cdot)) \geq \mathscr{M}(s) \text{ for all } s \in \mathscr{X}^*, \tag{4.4}$$

where the (extended) lower expectations $\underline{\mathrm{Q}}_s^{\uparrow}$ are obtained from the upper expectations $\overline{\mathrm{Q}}_s^{\uparrow}$ through conjugacy; $\underline{\mathrm{Q}}_s^{\uparrow}(f) := -\overline{\mathrm{Q}}_s^{\uparrow}(-f)$ for all $f \in \overline{\mathscr{L}}(\mathscr{X})$ and $s \in \mathscr{X}^*$. Let us denote the set of all bounded above submartingales corresponding to a tree $\overline{\mathrm{Q}}_\bullet$ by $\underline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. The lower expectation $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is now defined, for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) := \sup\left\{\mathscr{M}(s) \colon \mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \limsup \mathscr{M} \leq_s f\right\}. \tag{4.5}$$

Once more, one may easily check that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is related to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ by conjugacy.

Corollary 4.3.7 (Conjugacy). *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, we have that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.*

Proof. This follows immediately from the definitions of $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$, and the fact that $\mathscr{M} \in \underline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ if and only if $-\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ for any extended real process $\mathscr{M}$. □

Since $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ are also related by conjugacy [Corollary 4.2.6$_{149}$], it follows that the equality in Theorem 4.3.6↶ also holds for lower expectations.

Corollary 4.3.8. *Consider any acceptable gambles tree $\mathscr{A}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$ be the agreeing upper expectations tree according to Eq.* (3.1)$_{50}$. *Then we have that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.*

4.4 Basic properties of game-theoretic upper expectations

Our decision to pick $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$—or, equivalently, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$—as our game-theoretic upper expectation of choice was so far only backed by a series of examples where $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ behaved nicely, and where the alternatives $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ (and the hypothetical one $\overline{\mathrm{E}}^{\mathrm{e}}_{\mathscr{A},\mathrm{V}}$) did not. From what is yet to come in this chapter, however, it will become clear that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is also in general a global upper expectation with desirable properties: e.g. it is coherent on $\mathbb{V} \times \mathscr{X}^*$, it extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$, it satisfies continuity with respect to increasing sequences of bounded below variables, it satisfies continuity with respect to decreasing sequences of finitary bounded above variables, …

In this section, we start by establishing some basic, yet essential properties for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$. Most importantly, we establish that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is coherent on $\mathbb{V} \times \mathscr{X}^*$, that it satisfies some 'extended' coherence properties on $\overline{\mathbb{V}} \times \mathscr{X}^*$, that it satisfies a general law of iterated upper expectations, and that it coincides with the finitary global upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ (or any other type of finitary global upper expectation) on $\mathbb{F} \times \mathscr{X}^*$.

4.4.1 Extended coherence properties, the law of iterated upper expectations and conditional coherence

We start by proving that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies extended versions of the coherence properties WC1$_{82}$–WC3$_{82}$, WC5$_{84}$–WC7$_{84}$ and WC11$_{85}$ for extended real-valued global variables. They are given, for any global upper expectation $\overline{\mathrm{E}} \colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ and the conjugate lower expectation $\underline{\mathrm{E}}$, as follows: for all $f, g \in \overline{\mathbb{V}}$, all $\lambda \in \mathbb{R}_{\geq}$, all $\mu \in \mathbb{R} \cup \{+\infty\}$ and all situations $s \in \mathscr{X}^*$,

EC1. $\inf(f|s) \leq \underline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(f|s) \leq \sup(f|s)$ [bounds];

EC2. $\overline{\mathrm{E}}(f+g|s) \leq \overline{\mathrm{E}}(f|s) + \overline{\mathrm{E}}(g|s)$ [sub-additivity];

EC3. $\overline{\mathrm{E}}(\lambda f|s) = \lambda \overline{\mathrm{E}}(f|s)$ [non-negative homogeneity];

EC4. $f \leq_s g \Rightarrow \overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(g|s)$ [monotonicity];

EC5. $\overline{\mathrm{E}}(f+\mu|s) = \overline{\mathrm{E}}(f|s) + \mu$ [constant additivity];

EC6. $\overline{\mathrm{E}}(f|s) = \overline{\mathrm{E}}(f \mathbb{1}_s|s)$ [conditioning invariance].

To prove this, we need the following two, rather abstract lemmas about supermartingales. The first simply says that Lemma 4.1.3$_{135}$ also holds for the extended real supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Its proof is similar to that of [8, Lemma 1], where instead real-valued supermartingales were used.

Lemma 4.4.1. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ and any situation $s \in \mathscr{X}^*$. Then*

$$\mathscr{M}(s) \geq \inf_{\omega \in \Gamma(s)} \limsup \mathscr{M}(\omega) \geq \inf_{\omega \in \Gamma(s)} \liminf \mathscr{M}(\omega).$$

Proof. Since $\mathscr{M}$ is a supermartingale, we have that $\overline{\mathrm{Q}}^{\uparrow}_s(\mathscr{M}(s\,\cdot)) \leq \mathscr{M}(s)$, which by LE1$_{156}$ and the fact that $\mathscr{M}$ is bounded below implies that $\inf_{x\in\mathscr{X}} \mathscr{M}(sx) \leq \mathscr{M}(s)$. Hence, since $\mathscr{X}$ is finite, there is at least one $x \in \mathscr{X}$ such that $\mathscr{M}(sx) \leq \mathscr{M}(s)$. Repeating this argument over and over again, leads us to the conclusion that there is some $\omega \in \Gamma(s)$ such that $\limsup_{n\to+\infty} \mathscr{M}(\omega^n) \leq \mathscr{M}(s)$ and therefore also $\inf_{\omega\in\Gamma(s)} \limsup \mathscr{M}(\omega) \leq \mathscr{M}(s)$. The remaining inequality follows now trivially. □

Lemma 4.4.2. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any countable collection $(\mathscr{M}_n)_{n\in\mathbb{N}_0}$ of supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, and any countable collection of non-negative extended real numbers $(\lambda_n)_{n\in\mathbb{N}_0}$. If all $\mathscr{M}_n$ are non-negative, then $\mathscr{M} := \sum_{n\in\mathbb{N}_0} \lambda_n \mathscr{M}_n$ is also a non-negative supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. On the other hand, if the supermartingales $(\mathscr{M}_n)_{n\in\mathbb{N}_0}$ have a common lower bound (but are not necessarily non-negative), and if $\sum_{n\in\mathbb{N}_0} \lambda_n$ is a real number λ, then $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.*

Proof. We start by proving the first statement. Since all $\mathscr{M}_n$ and λ_n are non-negative, the extended real process $\mathscr{M} = \sum_{n\in\mathbb{N}_0} \lambda_n \mathscr{M}_n$ exists and is non-negative. To see that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, observe that, for any $s \in \mathscr{X}^*$,

$$\begin{aligned}\overline{\mathrm{Q}}_s^{\uparrow}(\mathscr{M}(s\cdot)) = \overline{\mathrm{Q}}_s^{\uparrow}\Big(\sum_{n\in\mathbb{N}_0} \lambda_n \mathscr{M}_n(s\cdot)\Big) &\overset{\mathrm{LE7}_{156}}{\leq} \sum_{n\in\mathbb{N}_0} \overline{\mathrm{Q}}_s^{\uparrow}\Big(\lambda_n \mathscr{M}_n(s\,\cdot)\Big) \\ &\overset{\mathrm{LE3}_{156},\,\mathrm{LE8}_{156}}{=} \sum_{n\in\mathbb{N}_0} \lambda_n \overline{\mathrm{Q}}_s^{\uparrow}(\mathscr{M}_n(s\,\cdot)) \\ &\leq \sum_{n\in\mathbb{N}_0} \lambda_n \mathscr{M}_n(s) = \mathscr{M}(s),\end{aligned}$$

where we were allowed to apply $\mathrm{LE7}_{156}$, $\mathrm{LE3}_{156}$ and $\mathrm{LE8}_{156}$ because all $\mathscr{M}_n(s\cdot)$ are non-negative [and thus bounded below], and where the last inequality followed from the non-negativity of all λ_n and the fact that all $\mathscr{M}_n$ are non-negative supermartingales.

So it remains to prove the second statement. Suppose that the supermartingales $(\mathscr{M}_n)_{n\in\mathbb{N}_0}$ have a common lower bound, say $B \in \mathbb{R}$, and that the sum $\sum_{n\in\mathbb{N}_0} \lambda_n$ is a real number λ, which in particular implies that all λ_n are real. Since all $\mathscr{M}_n$ are bounded below by B, the processes $\mathscr{M}_n - B$ will be non-negative. Moreover, it can be easily checked using Property $\mathrm{LE5}_{156}$ of the local models $\overline{\mathrm{Q}}_\bullet^{\uparrow}$, that the supermartingale character of all $\mathscr{M}_n$ implies the supermartingale character of all $\mathscr{M}_n - B$. So all $\mathscr{M}_n - B$ are non-negative supermartingales in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Hence, by the first part of our proof, we have that $\sum_{n\in\mathbb{N}_0} \lambda_n[\mathscr{M}_n - B]$ is a non-negative supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Since B and all λ_n are real, we furthermore have that, for all $s \in \mathscr{X}^*$,

$$\begin{aligned}\sum_{n\in\mathbb{N}_0} \lambda_n[\mathscr{M}_n(s) - B] = \sum_{n\in\mathbb{N}_0} [\lambda_n \mathscr{M}_n(s) - \lambda_n B] &= \lim_{n\to+\infty}\Big(\sum_{i=1}^{n} \lambda_i \mathscr{M}_i(s) - \sum_{i=1}^{n} \lambda_i B\Big) \\ &= \lim_{n\to+\infty} \sum_{i=1}^{n} \lambda_i \mathscr{M}_i(s) - \lambda B \\ &= \sum_{n\in\mathbb{N}} \lambda_n \mathscr{M}_n(s) - \lambda B = \mathscr{M}(s) - \lambda B,\end{aligned}$$

where the third equality follows from the fact that $\lim_{n\to+\infty} \sum_{i=1}^{n} \lambda_i B = \lambda B$ is real [because λ and B are real]. As a result, since $\sum_{n\in\mathbb{N}_0} \lambda_n[\mathscr{M}_n - B]$ is a non-negative supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, $\mathscr{M} - \lambda B$ is also a non-negative supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Clearly, $\mathscr{M}$ is then bounded below [by λB], and it can easily be deduced from Property $\mathrm{LE5}_{156}$ of the local models $\overline{\mathrm{Q}}_\bullet^{\uparrow}$ that then moreover $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. ☐

The two lemmas above now allow us to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies $\mathrm{EC1}_{\curvearrowleft}$–$\mathrm{EC6}_{\curvearrowleft}$. A first result that established similar such properties was stated in [86, Chapter 8], yet, our proof of the following result bears a closer resemblance to that of [8, Proposition 14]; we adapt that proof to our present setting which involves dealing with extended real-valued supermartingales instead of real-valued ones.

Proposition 4.4.3. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the global upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies* $\mathrm{EC1}_{\curvearrowleft}$–$\mathrm{EC6}_{\curvearrowleft}$.

Proof. Let us first prove the third inequality in EC1$_{163}$; that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \sup(f|s)$. If $\sup(f|s) = +\infty$, then the inequality is trivially satisfied. If not, consider any real $M \geq \sup(f|s)$ and the real process $\mathscr{M}$ that assumes the constant value M. Then clearly $\mathscr{M}$ is a bounded below supermartingale and moreover $\liminf \mathscr{M}(\omega) = M \geq f(\omega)$ for all $\omega \in \Gamma(s)$. Hence, Definition 4.7$_{160}$ implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \mathscr{M}(s) = M$. Since this is true for all real $M \geq \sup(f|s)$, the inequality $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \sup(f|s)$ follows.

EC2$_{163}$. If either $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$ is equal to $+\infty$, then the inequality is trivially true. So suppose that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) < +\infty$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) < +\infty$ and consider any real $c_1 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ and any real $c_2 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$. Then there are two bounded below supermartingales $\mathscr{M}_1$ and $\mathscr{M}_2$ such that $\mathscr{M}_1(s) \leq c_1$ and $\mathscr{M}_2(s) \leq c_2$ and moreover $\liminf \mathscr{M}_1 \geq_s f$ and $\liminf \mathscr{M}_2 \geq_s g$. Now consider the extended real process $\mathscr{M} := \mathscr{M}_1 + \mathscr{M}_2$. Then $\mathscr{M}$ is a bounded below supermartingale because of Lemma 4.4.2$_{163}$, which we can apply because $\mathscr{M}_1$ and $\mathscr{M}_2$ are both bounded below and hence have a common lower bound [note that the countable sum in Lemma 4.4.2$_{163}$ can be turned into a finite sum by setting all remaining supermartingales equal to zero]. Moreover, for any $\omega \in \Omega$, we have that

$$\begin{aligned}
\liminf(\mathscr{M}_1 + \mathscr{M}_2)(\omega) &= \liminf_{n\to+\infty}(\mathscr{M}_1(\omega^n) + \mathscr{M}_2(\omega^n)) \\
&= \lim_{m\to+\infty} \inf_{n\geq m}(\mathscr{M}_1(\omega^n) + \mathscr{M}_2(\omega^n)) \\
&\geq \lim_{m\to+\infty} (\inf_{n\geq m} \mathscr{M}_1(\omega^n) + \inf_{n\geq m} \mathscr{M}_2(\omega^n)) \\
&= \lim_{m\to+\infty} \inf_{n\geq m} \mathscr{M}_1(\omega^n) + \lim_{m\to+\infty} \inf_{n\geq m} \mathscr{M}_2(\omega^n) \\
&= \liminf_{n\to+\infty} \mathscr{M}_1(\omega^n) + \liminf_{n\to+\infty} \mathscr{M}_2(\omega^n),
\end{aligned}$$

where the limit in the fourth term (after the inequality) exists because both $\inf_{n\geq m} \mathscr{M}_1(\omega^n)$ and $\inf_{n\geq m} \mathscr{M}_2(\omega^n)$ are increasing in m, and where the third equality follows from the fact that, again, $\inf_{n\geq m} \mathscr{M}_1(\omega^n)$ and $\inf_{n\geq m} \mathscr{M}_2(\omega^n)$ are increasing in m, and that these terms take values in $\mathbb{R} \cup \{+\infty\}$ for all m—and thus also converge in $\mathbb{R} \cup \{+\infty\}$ for increasing m. Since this holds for all $\omega \in \Omega$, we have that $\liminf(\mathscr{M}_1 + \mathscr{M}_2) \geq \liminf \mathscr{M}_1 + \liminf \mathscr{M}_2$ and therefore, since $\liminf \mathscr{M}_1 \geq_s f$ and $\liminf \mathscr{M}_2 \geq_s g$, that $\liminf \mathscr{M} \geq_s f+g$. Combined with the fact that $\mathscr{M}$ is a bounded below supermartingale, it follows from Definition 4.7$_{160}$ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f+g|s) \leq \mathscr{M}(s) = \mathscr{M}_1(s) + \mathscr{M}_2(s) \leq c_1 + c_2$. Since this holds for any real $c_1 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ and any real $c_2 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$, it follows that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f+g|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) + \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$.

EC3$_{163}$. For $\lambda \in \mathbb{R}_>$, it suffices to note that $\mathscr{M}$ is a bounded below supermartingale such that $\liminf \mathscr{M} \geq_s f$ if and only if $\lambda\mathscr{M}$ is a bounded below supermartingale such that $\liminf \lambda\mathscr{M} \geq_s \lambda f$. If $\lambda = 0$, then $\lambda\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = 0$ because $(+\infty)\cdot 0 = (-\infty)\cdot 0 = 0$. To see that also $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lambda f|s) = 0$, start by noting that $\lambda f = 0$ and hence, because of the third inequality in EC1$_{163}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lambda f|s) \leq 0$. That $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lambda f|s) < 0$ is impossible, follows from Lemma 4.4.1$_{163}$ and Definition 4.7$_{160}$. Hence, we indeed have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lambda f|s) = 0$.

EC4$_{163}$. Consider any two $f, g \in \overline{\mathbb{V}}$ such that $f \leq_s g$. Then for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ such that $\liminf \mathscr{M} \geq_s g$, we also have that $\liminf \mathscr{M} \geq_s f$, and hence, by Definition 4.7$_{160}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$.

EC1$_{163}$. We have already proved the third inequality. The first inequality then follows from the fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ and $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ are related by conjugacy [Corollary 4.3.7$_{162}$].

To prove the second inequality, assume **ex absurdo** that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. Then $0 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) - \underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ which by conjugacy and EC2$_{163}$ implies that $0 > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f + (-f)|s)$. Since, according to our convention, the extended real variable $f + (-f)$ only assumes values in $\{0, +\infty\}$, we have that $f + (-f) \geq 0$ and therefore, by EC4$_{163}$ and EC3$_{163}$, that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f+(-f)|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(0|s) = 0$. This is a contradiction.

EC5$_{163}$. If $\mu = +\infty$, then it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(+\infty|s) = +\infty$, which follows from EC1$_{163}$. On the other hand, if $\mu \in \mathbb{R}$, then for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ such that $\liminf \mathscr{M} \geq_s f + \mu$, we have by LE5$_{156}$ that $\mathscr{M} - \mu \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$, and moreover that $\liminf(\mathscr{M} - \mu) \geq_s f$. Hence, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \mathscr{M}(s) - \mu$ and therefore also $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) + \mu \leq \mathscr{M}(s) - \mu + \mu = \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ such that $\liminf \mathscr{M} \geq_s f + \mu$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) + \mu \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f + \mu|s)$. By applying this inequality to $f' = f + \mu$ and $\mu' = -\mu$, we also find that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f + \mu|s) - \mu \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

EC6$_{163}$. This follows immediately from Definition 4.7$_{160}$. □

It follows from Proposition 4.4.3 [EC1$_{163}$–EC3$_{163}$] that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies the coherence axioms WC1$_{82}$–WC3$_{82}$. The fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ also satisfies WC4$_{82}$, and thus by Theorem 3.4.3$_{84}$ that it is coherent on $\mathbb{V} \times \mathscr{X}^*$, will follow straightforwardly from the following general law of iterated upper expectations for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. The idea of the proof for this theorem goes back to [86, Proposition 8.7], yet, our proof is more similar to that of [8, Theorem 16].

Theorem 4.4.4 (Law of iterated upper expectations)**.** *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, any $f \in \overline{\mathbb{V}}$ and any $k \in \mathbb{N}_0$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|X_{1:k}) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|X_{1:k+1})\,\big|X_{1:k}\big).$$

Proof. We need to show that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|X_{1:k+1})\,|x_{1:k})$ for any $x_{1:k} \in \mathscr{X}^k$. To this end, due to Proposition 4.4.3 [EC6$_{163}$], it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}X_{k+1})\,|x_{1:k})$. Let us first show that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}X_{k+1})\big|x_{1:k}\big) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}).$$

If $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}) = +\infty$, this is trivially satisfied. If not, then for any fixed real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k})$ there is a bounded below supermartingale $\mathscr{M}$ such that $\mathscr{M}(x_{1:k}) \leq \alpha$ and $\liminf \mathscr{M} \geq_{x_{1:k}} f$. Then it is clear that, for all $x_{k+1} \in \mathscr{X}$, $\liminf \mathscr{M} \geq_{x_{1:k+1}} f$, and hence $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1})$ by Definition 4.7$_{160}$. Let $\mathscr{M}'$ be the process that is equal to $\mathscr{M}$ for all situations s such that $s \not\sqsupset x_{1:k}$, and that is equal to the constant $\mathscr{M}(x_{1:k+1})$ for all situations s such that $s \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Clearly, $\mathscr{M}'$ is again a bounded below supermartingale and, because of the reasoning above, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}X_{k+1}) \leq \mathscr{M}(x_{1:k}X_{k+1}) =_{x_{1:k}} \liminf \mathscr{M}'$. Hence, it follows from Definition 4.7$_{160}$ that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}X_{k+1})|x_{1:k}) \leq \mathscr{M}'(x_{1:k}) = \mathscr{M}(x_{1:k}) \leq \alpha.$$

Since this holds for any real number $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k})$, we indeed have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}X_{k+1})|x_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k})$.

We now prove the other inequality. Again, if $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}X_{k+1})|x_{1:k}) = +\infty$ it trivially holds, so we can assume it to be real or equal to $-\infty$. Fix any real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}X_{k+1})|x_{1:k})$ and any $\epsilon \in \mathbb{R}_{>}$. Then there must be a bounded below supermartingale $\mathscr{M}$ such that $\mathscr{M}(x_{1:k}) \leq \alpha$ and $\liminf \mathscr{M} \geq_{x_{1:k}} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}X_{k+1})$. Consider any such bounded below supermartingale. Then for any $x_{k+1} \in \mathscr{X}$, we have that $\liminf \mathscr{M} \geq_{x_{1:k+1}} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k+1})$, which by Lemma 4.4.1$_{163}$ implies that $\mathscr{M}(x_{1:k+1}) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k+1})$. Fix any $x_{k+1} \in \mathscr{X}$. Then $\mathscr{M}(x_{1:k+1})$ is either real or equal to $+\infty$ because $\mathscr{M}$ is bounded below. If $\mathscr{M}(x_{1:k+1})$ is real, then since $\mathscr{M}(x_{1:k+1}) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k+1})$, it follows from Definition 4.7$_{160}$ that there is a bounded below supermartingale $\mathscr{M}_{x_{1:k+1}}$ such that $\mathscr{M}_{x_{1:k+1}}(x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1}) + \epsilon$ and $\liminf \mathscr{M}_{x_{1:k+1}} \geq_{x_{1:k+1}} f$. If $\mathscr{M}(x_{1:k+1})$ is $+\infty$, let $\mathscr{M}_{x_{1:k+1}}$ be the constant supermartingale that is equal to $+\infty$ everywhere. So, for all $x_{k+1} \in \mathscr{X}$, we have found a bounded below supermartingale $\mathscr{M}_{x_{1:k+1}}$ such that $\mathscr{M}_{x_{1:k+1}}(x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1})+\epsilon$ and $\liminf \mathscr{M}_{x_{1:k+1}} \geq_{x_{1:k+1}} f$. Let $\mathscr{M}^*$ be the process that is equal to $\mathscr{M}+\epsilon$ for all situations s such that $s \not\sqsupset x_{1:k}$, and that is equal to $\mathscr{M}_{x_{1:k+1}}$ for all situations s such that $s \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Note that $\liminf \mathscr{M}^* \geq_{x_{1:k}} f$ because, for each $x_{k+1} \in \mathscr{X}$, we have that $\liminf \mathscr{M}^* =_{x_{1:k+1}} \liminf \mathscr{M}_{x_{1:k+1}} \geq_{x_{1:k+1}} f$. We moreover show that $\mathscr{M}^*$ is a bounded below supermartingale.

The process $\mathscr{M}^*$ is clearly bounded below because $\mathscr{M}$ and all $\mathscr{M}_{x_{1:k+1}}$ are bounded below and $\mathscr{X}$ is finite. Furthermore, for any $x_{k+1} \in \mathscr{X}$, we have that $\mathscr{M}^*(x_{1:k+1}) = \mathscr{M}_{x_{1:k+1}}(x_{1:k+1}) \leq \mathscr{M}(x_{1:k+1})+\epsilon$, implying that $\mathscr{M}^*(x_{1:k}\cdot) \leq \mathscr{M}(x_{1:k}\cdot)+\epsilon$ and therefore, by LE4$_{156}$ and LE5$_{156}$, that

$$\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}^*(x_{1:k}\cdot)) \leq \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot)+\epsilon) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot))+\epsilon \leq \mathscr{M}(x_{1:k})+\epsilon = \mathscr{M}^*(x_{1:k}).$$

Moreover, for all situations $s \not\sqsupset x_{1:k}$, we have by LE5$_{156}$ that $\overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}^*(s\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}(s\cdot)+\epsilon) = \overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}(s\cdot))+\epsilon \leq \mathscr{M}(s)+\epsilon = \mathscr{M}^*(s)$, and for all $s \in \mathscr{X}^*$ such that $s \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$, we have that $\overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}^*(s\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}_{x_{1:k+1}}(s\cdot)) \leq \mathscr{M}_{x_{1:k+1}}(s) = \mathscr{M}^*(s)$. All together, we have that $\overline{\mathrm{Q}}^{\uparrow}_{s}(\mathscr{M}^*(s\cdot)) \leq \mathscr{M}^*(s)$ for all $s \in \mathscr{X}^*$, implying that $\mathscr{M}^*$ is a supermartingale.

Since $\liminf \mathscr{M}^* \geq_{x_{1:k}} f$ and $\mathscr{M}^*(x_{1:k}) = \mathscr{M}(x_{1:k})+\epsilon \leq \alpha+\epsilon$, Definition 4.7$_{160}$ now implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}) \leq \alpha+\epsilon$. This holds for any $\epsilon \in \mathbb{R}_{>}$ and any real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}X_{k+1})|x_{1:k})$, so we indeed conclude that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|x_{1:k}X_{k+1})|x_{1:k})$. □

It now follows in a trivial way from the previous results that the restriction of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ to $\mathbb{V}\times\mathscr{X}^*$ is coherent [Definition 3.7$_{82}$].

Corollary 4.4.5 (Conditional coherence). *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the restriction of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ to $\mathbb{V}\times\mathscr{X}^*$ satisfies* WC1$_{82}$–WC4$_{82}$*, and is therefore coherent.*

Proof. Proposition 4.4.3$_{164}$ guarantees that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ satisfies WC1$_{82}$–WC3$_{82}$ on $\mathbb{V}\times\mathscr{X}^*$. So it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ satisfies WC4$_{82}$ on $\mathbb{V}\times\mathscr{X}^*$, because the coherence will then follow from Theorem 3.4.3$_{84}$. Consider any $f \in \mathbb{V}$ and any $s,t \in \mathscr{X}^*$ such that $s \sqsubseteq t$. Let $k := |s|$ and $\ell := |t|$; so we have that $k \leq \ell$. By iteratively applying

Theorem 4.4.4$_{166}$, we find that

$$\begin{aligned}
&\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:k}\big)\\
&\qquad=\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:k+1}\big)|X_{1:k}\big)\\
&\qquad\ \vdots\\
&\qquad=\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\cdots\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:\ell}\big)|X_{1:\ell-1}\big)\cdots|X_{1:k+1}\big)|X_{1:k}\big).
\end{aligned}\tag{4.6}$$

Observe that the inner most upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:\ell}\big)$ is identically zero. Indeed, for any $x_{1:\ell}\in\mathscr{X}^\ell\setminus\{t\}$ [recall that $\ell=|t|$] we have that $(f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t=_{x_{1:\ell}}0$ and therefore by Proposition 4.4.3 [EC1$_{163}$] that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|x_{1:\ell}\big)=0$. On the other hand, for the situation t itself, since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies EC5$_{163}$ and EC6$_{163}$ by Proposition 4.4.3, we infer that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|t\big)\overset{\mathrm{EC6}_{163}}{=}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t)|t\big)\overset{\mathrm{EC5}_{163}}{=}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t)-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t)=0,$$

where we were allowed to use EC5$_{163}$ because, by Proposition 4.4.3 [EC1$_{163}$] and the fact that $f\in\mathbb{V}$, we know that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t)\in\mathbb{R}$. So we indeed have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:\ell}\big)=0$. Plugging this back into Eq. (4.6), and then using the fact that $\overline{\mathrm{E}}(0|t')=0$ for all $t'\in\mathscr{X}^*$ [due to EC1$_{163}$], we obtain that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|X_{1:k}\big)=0$. Recalling that $k=|s|$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big((f-\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t))\mathbb{1}_t|s\big)=0$ as desired. □

Another interesting consequence of Theorem 4.4.4$_{166}$ is that, for any fixed $f\in\overline{\mathscr{L}}_{\mathrm{b}}(\Omega)$, the upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|\cdot)\colon s\in\mathscr{X}^*\mapsto\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ itself form a supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. We henceforth use $\overline{\mathbb{V}}_{\mathrm{b}}$ as a shorthand notation for the set $\overline{\mathscr{L}}_{\mathrm{b}}(\Omega)$ of all (extended real) global variables that are bounded below.

Corollary 4.4.6. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any $f\in\overline{\mathbb{V}}_{\mathrm{b}}$, the extended real process $\mathscr{C}$, defined by $\mathscr{C}(s):=\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for all $s\in\mathscr{X}^*$, is a supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.*

Proof. The process $\mathscr{C}$ is bounded below because f is bounded below and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies EC1$_{163}$. Moreover, if for any $s\in\mathscr{X}^*$ we let $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s\,\cdot)$ be the (bounded below) local variable that assumes the value $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|sx)$ for all $x\in\mathscr{X}$, then it follows from Proposition 4.4.7$_{\rightarrow}$ and Theorem 4.4.4$_{166}$ that

$$\begin{aligned}
\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{C}(x_{1:k}\cdot))=\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k}\cdot)\big)&=\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|X_{1:k+1})|x_{1:k}\big)\\
&=\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|x_{1:k})=\mathscr{C}(x_{1:k})\text{ for all }x_{1:k}\in\mathscr{X}^*.
\end{aligned}$$

Hence, $\mathscr{C}$ is indeed a supermartingale. □

4.4.2 Relation with the local upper expectations and the finitary global upper expectations

For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is deduced from $\overline{\mathrm{Q}}_\bullet$ as a generalisation, we will want $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ to be 'compatible' with $\overline{\mathrm{Q}}_\bullet$, in the sense that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ should satisfy NE1$_{88}$. The following proposition shows that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies such a type of compatibility with $\overline{\mathrm{Q}}^{\uparrow}_\bullet$ on the domain $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Since $\overline{\mathrm{Q}}^{\uparrow}_s$ extends $\overline{\mathrm{Q}}_s$ for all $s \in \mathscr{X}^*$, this indeed implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies NE1$_{88}$. We will moreover show later on, in Section 4.6.2$_{178}$, that the compatibility with $\overline{\mathrm{Q}}^{\uparrow}_\bullet$ can be extended to the entire domain $\overline{\mathscr{L}}(\mathscr{X})$ of all local extended real variables.

Proposition 4.4.7 (Partial compatibility with local models)**.** *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $x_{1:k} \in \mathscr{X}^*$ (with $k \in \mathbb{N}_0$) and any $(k+1)$-measurable extended real variable f that is bounded below. Then,*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|x_{1:k}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f(x_{1:k}\cdot)).$$

In specific, the global upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies NE1$_{88}$.

Proof. Our proof is similar to that of [8, Corollary 3]. Consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\liminf \mathscr{M} \geq_{x_{1:k}} f$. Then it follows from Lemma 4.4.1$_{163}$ that, for all $x_{k+1} \in \mathscr{X}$,

$$\mathscr{M}(x_{1:k+1}) \geq \inf_{\omega \in \Gamma(x_{1:k+1})} \liminf \mathscr{M}(\omega) \geq \inf_{\omega \in \Gamma(x_{1:k+1})} f(\omega) = f(x_{1:k+1}).$$

Hence, we have that $\mathscr{M}(x_{1:k}\cdot) \geq f(x_{1:k}\cdot)$, which implies by LE4$_{156}$ and the supermartingale character of $\mathscr{M}$ that

$$\mathscr{M}(x_{1:k}) \geq \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot)) \geq \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f(x_{1:k}\cdot)).$$

Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\liminf \mathscr{M} \geq_{x_{1:k}} f$, it follows from Definition 4.7$_{160}$ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|x_{1:k}) \geq \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f(x_{1:k}\cdot))$. To see that the inequality is an equality, consider the extended real process $\mathscr{M}$ defined by $\mathscr{M}(s) := \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f(x_{1:k}\cdot))$ for all $s \not\sqsupset x_{1:k}$, and by $\mathscr{M}(s) := f(x_{1:k+1})$ for any $s \in \mathscr{X}^*$ such that $s \sqsupseteq x_{1:k+1}$ for some $x_{k+1} \in \mathscr{X}$. Then $\mathscr{M}$ is bounded below because f is bounded below and $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}$ satisfies LE1$_{156}$. It is also a supermartingale because $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}(x_{1:k}\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f(x_{1:k}\cdot)) = \mathscr{M}(x_{1:k})$ and, for all $s \neq x_{1:k}$, $\overline{\mathrm{Q}}^{\uparrow}_s(\mathscr{M}(s\,\cdot)) = \mathscr{M}(s)$ because of LE1$_{156}$ and the fact that $\mathscr{M}(s\cdot)$ is constant and equal to $\mathscr{M}(s)$. It is moreover easy to see that $\liminf \mathscr{M} \geq_{x_{1:k}} f$ is guaranteed because f is $(k+1)$-measurable.

The final statement, that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies NE1$_{88}$, follows easily from the first statement that we have just proved, and the fact that $\overline{\mathrm{Q}}^{\uparrow}_s$ extends $\overline{\mathrm{Q}}_s$ by definition for all $s \in \mathscr{X}^*$. □

As argued in Section 4.1.2$_{135}$, we want a global upper expectation—and thus in specific $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$—to be at least as informative as the finitary global upper expectations from Chapter 3$_{45}$ on the domain $\mathbb{V}\times\mathscr{X}^*$, and preferably to coincide with (any of) these finitary global upper expectations on $\mathbb{F}\times\mathscr{X}^*$. These conditions too are satisfied by $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$, and can easily be inferred from our earlier considerations. We state these results for the finitary global upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, but due to Theorem 3.5.1$_{90}$ and Theorem 3.5.2$_{91}$ they can just as well be stated for the other types of finitary global upper expectations (as long as we consider agreeing trees).

Corollary 4.4.8. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) \text{ for all } (f,s)\in\mathbb{V}\times\mathscr{X}^*.$$

Proof. Note that Proposition 4.4.7$_\curvearrowleft$ implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, where the latter was defined by Eq. (3.13)$_{85}$. Indeed, for any $f\in\mathscr{L}(\mathscr{X})$ and any $x_{1:k}\in\mathscr{X}^*$, we have by Proposition 4.4.7$_\curvearrowleft$ that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(f) = \overline{\mathrm{Q}}_{x_{1:k}}(f) \overset{(3.13)}{=} \overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}(f(X_{k+1})|x_{1:k}).$$

where the second equality follows from the fact that $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}$ coincides by definition with $\overline{\mathrm{Q}}_{x_{1:k}}$ on local gambles. So, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, and since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ moreover satisfies WC1$_{82}$–WC4$_{82}$ according to Corollary 4.4.5$_{167}$, we obtain from the definition of the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) \text{ for all } (f,s)\in\mathbb{V}\times\mathscr{X}^*. \qquad \square$$

Corollary 4.4.9. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|s) \text{ for all } (f,s)\in\mathbb{F}\times\mathscr{X}^*.$$

Proof. According to Theorem 3.4.6$_{88}$, it suffices to show that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—or its restriction to $\mathbb{F}\times\mathscr{X}^*$—satisfies NE1$_{88}$–NE3$_{88}$. To this end, observe that NE1$_{88}$ follows from Proposition 4.4.7$_\curvearrowleft$, NE2$_{88}$ follows from Proposition 4.4.3 [EC6$_{163}$], and NE3$_{88}$ follows from Theorem 4.4.4$_{166}$. $\square$

That $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is sometimes strictly more informative than $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ on $\mathbb{V}\times\mathscr{X}^*$ can easily be seen by recalling Examples 3.6.1$_{99}$ and 4.1.1$_{133}$, where in the latter $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ will give the same result as $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ due to Theorem 4.3.6$_{161}$—for $\overline{\mathrm{Q}}$ the appropriate agreeing upper expectations tree. The fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ will yield desirable—informative—values in the case of Example 4.1.1$_{133}$ can alternatively be inferred from the fact that, as we will show later in Section 4.6$_{175}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous with respect to increasing bounded below sequences.

4.5 Doob's convergence theorem, Lévy's zero–one law, and their implications for the definition of the game-theoretic upper expectation

The current section is devoted to two technical results that have proved essential in the theory of game-theoretic probabilities and upper expectations: Doob's convergence theorem and Lévy's Zero–one law. Both of these results are also well-known to hold in a—more traditional—measure-theoretic context [5, 33, 61, 90]; however, the versions that we will state here do not require any measurability conditions, nor do they require the local models to be precise. The game-theoretic versions of these two results will be instrumental for us as well, for instance, in order to establish that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous from below [Theorem 4.6.1$_{175}$]. Though both results are entirely due to Shafer, Vovk and Takemura [85, 88, 109], we nonetheless present independent, yet very similar proofs for them because our framework slightly differs from theirs; see Section 4.9$_{187}$. As some of the involved arguments are rather lengthy and technical, we have chosen to relegate these proofs to Appendix 4.B$_{199}$.

To state the results, we require some new terminology. For any $s \in \mathscr{X}^*$, we say that a supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ is an **s-test supermartingale** (for $\overline{\mathrm{Q}}_\bullet$) if it is non-negative and $\mathscr{M}(s) = 1$. If $s = \square$, we simply say it is a test supermartingale. For any $s \in \mathscr{X}^*$, we say that an event $A \subseteq \Omega$ is **almost sure (a.s.)** in $\Gamma(s)$ if there is an s-test supermartingale that converges to $+\infty$ on $\Gamma(s) \setminus A$. In that case, we call the event A^c **null** in $\Gamma(s)$. If $s = \square$, we drop the 'in' and simply speak of 'almost sure' and 'null'. For any two $f, g \in \overline{\mathbb{V}}$, note that $f \geq_s g$ a.s. in $\Gamma(s)$ if and only if $f \geq g$ a.s. in $\Gamma(s)$—and similarly for $\leq_s$, $>_s$ and $<_s$.

Recall from Section 3.1.3$_{52}$ that the (game-theoretic) upper probability $\overline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ corresponding to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is obtained by restricting $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ to the domain of indicators (and situations), and that the (game-theoretic) lower probability $\underline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is obtained in a similar way from $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. Then it can be shown easily that an event $A \subseteq \Omega$ is almost sure in $\Gamma(s)$ if and only $\overline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A^c|s) = 0$ or, equivalently,[5] if and only if $\underline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A|s) = 1$; we refer to [85, Proposition 8.4] for an illustration of how this can be deduced in the case where $s = \square$. This is similar to the traditional measure-theoretic definition of an almost sure event; that is, a measurable event with (measure-theoretic) probability one; see

[5] This follows from the fact that

$$\overline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A^c|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{I}_{A^c}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(1-\mathbb{I}_A|s) = 1+\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(-\mathbb{I}_A|s) = 1-\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{I}_A|s) = 1-\underline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A|s),$$

where we used Proposition 4.4.3$_{164}$ [EC5$_{163}$] for the third equality and conjugacy [Corollary 4.3.7$_{162}$] for the fourth equality.

Appendix 5.A263. In contrast with the measure-theoretic definition however, the game-theoretic approach provides a clear behavioural interpretation for strictly null events $A \subseteq \Omega$: it says that Forecaster allows Skeptic to play in such a way that he can start with capital equal to one (in the situation s) and become infinitely rich on all paths $\omega \in A$ (that moreover go through s) without ever borrowing money.

Using this terminology, Theorem 4.5.2 below establishes a version of Doob's convergence theorem. It states that a bounded below supermartingale converges to a real number almost surely. This is somewhat intuitive (yet, not at all trivial): since a supermartingale is expected to decrease, one would expect a bounded below supermartingale to converge to a real number. We precede Theorem 4.5.2 with a technical result that is very similar to Doob's convergence theorem—and from which Doob's convergence theorem can easily be derived; see Appendix 4.B199. We state it separately because it will be used later on to prove Proposition 4.5.4→.

Proposition 4.5.1. *Consider any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$ *and any supermartingale* $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. *If* $\mathscr{M}(s)$ *is real for some* $s \in \mathscr{X}^*$, *then there is an s-test supermartingale* $\mathscr{M}^*$ *that converges to* $+\infty$ *on all paths* $\omega \in \Gamma(s)$ *where* $\mathscr{M}$ *does not converge to an extended real number, and that converges to an extended real number on all paths* $\omega \in \Gamma(s)$ *where* $\mathscr{M}$ *converges to a real number.*

Theorem 4.5.2 (Doob's convergence theorem)**.** *Consider any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$ *and any supermartingale* $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. *If* $\mathscr{M}(s)$ *is real for some* $s \in \mathscr{X}^*$, *then* $\mathscr{M}$ *converges to a real number almost surely in* $\Gamma(s)$.

Lévy's zero–one law captures yet another intuitive idea: in particular, it says that the upper probability of an event $C \subseteq \Omega$ conditional on a situation ω^n should—or, is expected to—converge to 1 as $n \to +\infty$ if $\omega \in C$. The law as stated below is more general though, as it applies to bounded below variables $f \in \overline{\mathbb{V}}_{\mathrm{b}}$ instead of merely events $C \subseteq \Omega$; the version for events corresponds to choosing $f = \mathbb{I}_C$.

Theorem 4.5.3 (Lévy's zero–one law)**.** *For any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$, *any* $f \in \overline{\mathbb{V}}_{\mathrm{b}}$ *and any* $s \in \mathscr{X}^*$, *the event*

$$A := \left\{\omega \in \Omega\colon \liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|\omega^n) \geq f(\omega)\right\} \text{ is a.s. in } \Gamma(s).$$

One of the major consequences of Doob's convergence theorem and Lévy's zero–one law is that they allow us to draw some interesting conclusions about the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$. In particular, we can use Doob's convergence theorem—or, rather, the technical Proposition 4.5.1 underlying

Doob's convergence theorem—to show that this operator is not impacted much by changes in Definition 4.7_{160} that concern the limit behaviour of supermartingales; more specifically, the following result shows that the limit inferior in Definition 4.7_{160} can be replaced by a limit, without affecting the values of the resulting operator. As was the case for the earlier results in this section [Section 4.5_{171}], and as will also be the case for the future results in this section, the ideas underlying the proof of this result are also due to Shafer, Vovk and Takemura [85, 88].

Proposition 4.5.4. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \inf\Big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \lim \mathscr{M} \geq_s f\Big\},$$

where the condition $\lim \mathscr{M} \geq_s f$ is taken to imply that $\lim \mathscr{M}$ exists within $\Gamma(s)$.

Proof. The inequality '$\leq$' is trivially satisfied since $\liminf \mathscr{M} =_s \lim \mathscr{M}$ for any bounded below supermartingale $\mathscr{M}$ whose limit $\lim \mathscr{M}$ exists within $\Gamma(s)$. It remains to prove the converse inequality. If $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = +\infty$, it is trivially satisfied. If not, fix any real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. Then, due to Definition 4.7_{160}, there is some supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\mathscr{M}(s) \leq \alpha$ and $\liminf \mathscr{M} \geq_s f$. Because $\mathscr{M}$ is bounded below and α is real, $\mathscr{M}(s)$ is also real. So, by Proposition $4.5.1_{\leftarrow}$, there is an s-test supermartingale $\mathscr{M}^*$ that converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ where $\mathscr{M}$ does not converge in $\overline{\mathbb{R}}$ and converges in $\overline{\mathbb{R}}$ on all paths $\omega \in \Gamma(s)$ where $\mathscr{M}$ converges in $\mathbb{R}$.

Fix any $\epsilon \in \mathbb{R}_>$ and consider the process $\mathscr{M}'$ defined by $\mathscr{M}'(t) := \mathscr{M}(t) + \epsilon\mathscr{M}^*(t)$ for all situations $t \in \mathscr{X}^*$. Then $\mathscr{M}'$ is again a bounded below supermartingale because of Lemma $4.4.2_{163}$ [which we can apply because $\mathscr{M}$ and $\mathscr{M}^*$ are both bounded below and hence have a common lower bound]. We moreover have that $\liminf \mathscr{M}' \geq_s f$ because $\epsilon\mathscr{M}^*$ is non-negative and $\liminf \mathscr{M} \geq_s f$. We will now show that, on top of this, for all $\omega \in \Gamma(s)$, this process $\mathscr{M}'$ converges in $\overline{\mathbb{R}}$.

For any $\omega \in \Gamma(s)$, if $\mathscr{M}$ does not converge in $\overline{\mathbb{R}}$, $\mathscr{M}^*$ converges to $+\infty$, and therefore so does $\mathscr{M}'$ because $\mathscr{M}$ is bounded below and ϵ is positive. If $\mathscr{M}$ does converge in $\overline{\mathbb{R}}$, it converges either to a real number or to $+\infty$ (convergence to $-\infty$ is impossible because $\mathscr{M}$ is bounded below). If $\mathscr{M}$ converges to a real number, $\mathscr{M}^*$ converges in $\overline{\mathbb{R}}$ and therefore $\mathscr{M}'$ also converges in $\overline{\mathbb{R}}$. If $\mathscr{M}$ converges to $+\infty$, then so does $\mathscr{M}'$ because $\epsilon\mathscr{M}^*$ is non-negative. Hence, for all $\omega \in \Gamma(s)$, $\mathscr{M}'$ converges in $\overline{\mathbb{R}}$ and the limit $\lim \mathscr{M}'(\omega)$ therefore exists.

Now, recall that $\lim \mathscr{M}' = \liminf \mathscr{M}' \geq_s f$ and that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Hence, we have that

$$\inf\Big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \lim \mathscr{M} \geq_s f\Big\} \leq \mathscr{M}'(s) = \mathscr{M}(s) + \epsilon\mathscr{M}^*(s) \leq \alpha + \epsilon.$$

This holds for any $\epsilon \in \mathbb{R}_>$ and any $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$, which implies that indeed

$$\inf\Big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \lim \mathscr{M} \geq_s f\Big\} \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s). \qquad \square$$

The following result shows that Definition 4.7_{160} can be modified in yet another way; it says that the condition $\liminf \mathscr{M} \geq_s f$ in Definition 4.7_{160} need in fact only hold almost surely in $\Gamma(s)$.

Proposition 4.5.5. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$. Then*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = \inf\Big\{\mathscr{M}(s) : \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \textit{ and } \liminf \mathscr{M} \geq f \textit{ a.s. in } \Gamma(s)\Big\}. \tag{4.7}$$

Proof. Since for every supermartingale $\mathscr{M}$ that satisfies $\liminf \mathscr{M} \geq_s f$ it holds that $\liminf \mathscr{M} \geq f$ a.s. in $\Gamma(s)$, we clearly have that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \geq \inf\Big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq f \text{ a.s. in } \Gamma(s)\Big\}.$$

So it remains to prove the converse inequality. If the right-hand side of Eq. (4.7) is equal to $+\infty$, then this inequality is trivially satisfied. So consider the case where it is not. Fix any $\alpha \in \mathbb{R}$ such that $\alpha > \inf\big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq f \text{ a.s. in } \Gamma(s)\big\}$ and any $\epsilon \in \mathbb{R}_>$. Then there is some bounded below supermartingale $\mathscr{M}_\alpha$ such that $\liminf \mathscr{M}_\alpha \geq f$ a.s. in $\Gamma(s)$ and

$$\mathscr{M}_\alpha(s) \leq \alpha. \tag{4.8}$$

Since $\liminf \mathscr{M}_\alpha \geq f$ a.s. in $\Gamma(s)$, there is some s-test supermartingale $\mathscr{M}^*_\alpha$ that converges to $+\infty$ on $A := \{\omega \in \Gamma(s) \colon \liminf \mathscr{M}_\alpha(\omega) < f(\omega)\}$. Consider the extended real process $\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha$. This process is again a bounded below supermartingale because of Lemma $4.4.2_{163}$ [which we can apply because $\mathscr{M}_\alpha$ and $\mathscr{M}^*_\alpha$ are both bounded below and hence have a common lower bound]. Since $\mathscr{M}^*_\alpha$ converges to $+\infty$ on A and because $\mathscr{M}_\alpha$ is bounded below, we have that $\liminf(\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha)(\omega) = +\infty \geq f(\omega)$ for all $\omega \in A$. Moreover, for all $\omega \in \Gamma(s) \setminus A$, we also have that $\liminf(\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha)(\omega) \geq f(\omega)$, because $\liminf \mathscr{M}_\alpha(\omega) \geq f(\omega)$ and because $\epsilon\mathscr{M}^*_\alpha$ is non-negative. Hence, $\liminf(\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha) \geq_s f$ and consequently, also $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq (\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha)(s)$. It therefore follows from Eq. (4.8) that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq (\mathscr{M}_\alpha + \epsilon\mathscr{M}^*_\alpha)(s) = \mathscr{M}_\alpha(s) + \epsilon \leq \alpha + \epsilon.$$

Since this holds for any $\epsilon \in \mathbb{R}_>$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \alpha$, and since this is true for all $\alpha \in \mathbb{R}$ such that $\alpha > \inf\big\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq f \text{ a.s. in } \Gamma(s)\big\}$, it follows that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \inf\Big\{\mathscr{M}(s) : \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq f \text{ a.s. in } \Gamma(s)\Big\}. \qquad \square$$

Clearly, the infimum in Eq. (4.7) is taken over a larger set compared to the infimum in Definition 4.7_{160}. Though Proposition 4.5.5 shows that the resulting game-theoretic upper expectation is not impacted by this difference, it does make sure that the infimum in Eq. (4.7) is actually attained for bounded below variables $f \in \overline{\mathbb{V}}_{\mathrm{b}}$—this follows from Lévy's zero–one law.

Proposition 4.5.6. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \overline{\mathbb{V}}_\mathrm{b}$ and any $s \in \mathscr{X}^*$, the infimum in Eq. $(4.7)_{\leftarrow}$ is attained.*

Proof. Let $\mathscr{C}$ be the extended real process defined by $\mathscr{C}(t) := \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t)$ for all $t \in \mathscr{X}^*$. Then $\mathscr{C}$ is a bounded below supermartingale because of Corollary $4.4.6_{168}$. Moreover, because of Theorem $4.5.3_{172}$, we have that $\liminf \mathscr{C} \geq f$ almost surely in $\Gamma(s)$. Since $\mathscr{C}(s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$, this concludes the proof. □

4.6 Continuity of the game-theoretic upper expectation with respect to monotone sequences

We now turn to the final part in our analysis of game-theoretic upper expectations: their continuity properties. As was illustrated for instance in Sections 3.6_{98} and 4.1_{131}, these properties are crucial when aiming to develop a mathematical theory that is sufficiently elegant and powerful to work with.

4.6.1 Continuity from below

Our first continuity result establishes that, similarly to the local models $\overline{\mathrm{Q}}_s^{\uparrow}$, the global upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies continuity with respect to increasing sequences that are bounded below. This type of result, although usually with measurability conditions, is known under the name of 'the monotone convergence theorem' [5, 31, 89]. The idea behind our result goes back to Vovk & Shafer [109, Theorem 6.6], but an updated version can now also be found in their latest book [85, Proposition 8.3]. Once more, the setting for which [85, Proposition 8.3] is stated slightly differs from ours; more specifically, the authors do not necessarily consider a finite state space, and their local models are assumed to satisfy different axioms compared to ours; see Section 4.9_{187} for a more elaborate discussion. Moreover, they only give an explicit proof for the case that there is a single, fixed local model $\overline{\mathrm{Q}}$ in all situations. For these reasons, we provide an independent proof.

Theorem 4.6.1 (Continuity from below). *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $s \in \mathscr{X}^*$ and any increasing sequence $(f_n)_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_\mathrm{b}$,*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s), \textit{ with } f := \sup_{n\in\mathbb{N}_0} f_n = \lim_{n\to+\infty} f_n.$$

Proof. Throughout the proof, we will use Properties $\mathrm{EC1}_{163}$–$\mathrm{EC6}_{163}$ for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$; these follow from Proposition $4.4.3_{164}$, yet we will not explicitly refer to Proposition $4.4.3_{164}$ each time one of them is used. As $f_0 \in \overline{\mathbb{V}}_\mathrm{b}$ is bounded below and the sequence $(f_n)_{n\in\mathbb{N}_0}$ is increasing, there is an $M \in \mathbb{R}$ such that $f_n \geq M$ for all $n \in \mathbb{N}_0$

and therefore, f is also bounded below by M. Hence, since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is constant additive [EC5$_{163}$], we can assume without loss of generality that f and all f_n are non-negative.

That $\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s)$ exists, follows from the increasing character of $(f_n)_{n\in\mathbb{N}_0}$ and EC4$_{163}$. Moreover, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s)$ because $f \geq f_n$ [since $(f_n)_{n\in\mathbb{N}_0}$ is increasing] and because $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies EC4$_{163}$. It remains to prove the converse inequality.

For any $n \in \mathbb{N}_0$, consider the extended real process $\mathscr{C}_n$, defined by $\mathscr{C}_n(t) := \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|t)$ for all $t \in \mathscr{X}^*$ and the extended real process $\mathscr{C}$ defined by the limit $\mathscr{C}(t) := \lim_{n\to+\infty}\mathscr{C}_n(t)$ for all $t \in \mathscr{X}^*$. This limit exists because $(\mathscr{C}_n(t))_{n\in\mathbb{N}_0}$ is an increasing sequence for all $t \in \mathscr{X}^*$, due to the monotonicity [EC4$_{163}$] of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. As f_n is non-negative for all $n \in \mathbb{N}_0$, $\mathscr{C}_n$ is non-negative for all $n \in \mathbb{N}_0$ because of EC1$_{163}$ and therefore $\mathscr{C}$ is also non-negative. As a result, $\mathscr{C}$ and all $\mathscr{C}_n$ are non-negative extended real processes.

It now suffices to prove that $\mathscr{C}$ is a bounded below supermartingale such that $\liminf \mathscr{C} \geq f$ a.s. in $\Gamma(s)$, because it will then follow from Proposition 4.5.5$_{174}$ that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) &= \inf\left\{\mathscr{M}(s)\colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq f \text{ a.s. in } \Gamma(s)\right\}\\ &\leq \mathscr{C}(s) = \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s).\end{aligned}$$

This is what we now set out to do.

We first show that $\mathscr{C}$ is a supermartingale; that it is bounded below follows trivially from its non-negativity. For all situations $t \in \mathscr{X}^*$, we already know that $(\mathscr{C}_n(t\,\cdot))_{n\in\mathbb{N}_0}$ is an increasing sequence that converges to $\mathscr{C}(t\,\cdot)$. Since $\mathscr{C}_n$ and $\mathscr{C}$ are non-negative, we also have that $\mathscr{C}_n(t\,\cdot), \mathscr{C}(t\,\cdot) \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Then, due to LE6$_{156}$, we have that

$$\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{C}(t\,\cdot)) = \lim_{n\to+\infty}\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{C}_n(t\,\cdot)) \text{ for all } t \in \mathscr{X}^*. \tag{4.9}$$

$\mathscr{C}_n$ is a supermartingale for all $n \in \mathbb{N}_0$ because of Corollary 4.4.6$_{168}$, so it follows that $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{C}_n(t\,\cdot)) \leq \mathscr{C}_n(t)$ for all $n \in \mathbb{N}_0$ and all $t \in \mathscr{X}^*$. This implies, together with Eq. (4.9), that

$$\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{C}(t\,\cdot)) \leq \lim_{n\to+\infty}\mathscr{C}_n(t) = \mathscr{C}(t) \text{ for all } t \in \mathscr{X}^*.$$

Hence, $\mathscr{C}$ is a supermartingale.

To prove that $\liminf \mathscr{C} \geq f$ a.s. in $\Gamma(s)$, we will use Lévy's zero–one law. It follows from Theorem 4.5.3$_{172}$ that, for any $n \in \mathbb{N}_0$, there is an s-test supermartingale $\mathscr{M}_n$ that converges to $+\infty$ on the event

$$A_n := \left\{\omega \in \Gamma(s)\colon \liminf_{m\to+\infty}\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|\omega^m) < f_n(\omega)\right\}.$$

Now, consider the extended real process $\mathscr{M}$, defined by

$$\mathscr{M}(t) := \sum_{n\in\mathbb{N}_0}\lambda_n\mathscr{M}_n(t) \text{ for all } t \in \mathscr{X}^*,$$

where the coefficients $\lambda_n > 0$ sum to 1. Then it follows from Lemma 4.4.2$_{163}$ that $\mathscr{M}$ is again a non-negative supermartingale. Moreover, it is clear that $\mathscr{M}(s) = 1$ and hence, $\mathscr{M}$ is an s-test supermartingale.

We show that $\mathscr{M}$ converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ for which $\liminf_{m\to+\infty} \mathscr{C}(\omega^m) < f(\omega)$. Clearly, $\mathscr{M}$ converges to $+\infty$ on $\cup_{n\in\mathbb{N}_0} A_n =: A$. Consider now any path $\omega \in \Gamma(s)$ for which $\liminf_{m\to+\infty} \mathscr{C}(\omega^m) < f(\omega)$. As explained before, $\mathscr{C}_n(t)$ is increasing in n for all $t \in \mathscr{X}^*$, so we have that $\sup_{n\in\mathbb{N}_0} \mathscr{C}_n(\omega^m) = \lim_{n\to+\infty} \mathscr{C}_n(\omega^m) = \mathscr{C}(\omega^m)$ for all $m \in \mathbb{N}_0$. Since $\liminf_{m\to+\infty} \mathscr{C}(\omega^m) < f(\omega)$, this implies that

$$\liminf_{m\to+\infty} \sup_{n\in\mathbb{N}_0} \mathscr{C}_n(\omega^m) < \sup_{n\in\mathbb{N}_0} f_n(\omega).$$

Since $\sup_{n\in\mathbb{N}_0} \liminf_{m\to+\infty} \mathscr{C}_n(\omega^m) \leq \liminf_{m\to+\infty} \sup_{n\in\mathbb{N}_0} \mathscr{C}_n(\omega^m)$ [because we obviously have that $\mathscr{C}_n(\omega^m) \leq \sup_{n\in\mathbb{N}_0} \mathscr{C}_n(\omega^m)$ for all $n, m \in \mathbb{N}_0$], this implies that

$$\sup_{n\in\mathbb{N}_0} \liminf_{m\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|\omega^m) = \sup_{n\in\mathbb{N}_0} \liminf_{m\to+\infty} \mathscr{C}_n(\omega^m) \leq \liminf_{m\to+\infty} \sup_{n\in\mathbb{N}_0} \mathscr{C}_n(\omega^m) < \sup_{n\in\mathbb{N}_0} f_n(\omega). \tag{4.10}$$

Hence, there is some $n_\omega \in \mathbb{N}_0$ such that

$$\sup_{n\in\mathbb{N}_0} \liminf_{m\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|\omega^m) < f_{n_\omega}(\omega),$$

and therefore, we see that in particular

$$\liminf_{m\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_{n_\omega}|\omega^m) < f_{n_\omega}(\omega).$$

So $\omega \in A_{n_\omega} \subseteq A$ and, as a consequence, $\mathscr{M}$ converges to $+\infty$ on ω. Hence, the s-test supermartingale $\mathscr{M}$ converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ such that $\liminf_{m\to+\infty} \mathscr{C}(\omega^m) < f(\omega)$, and therefore $\liminf \mathscr{C} \geq f$ almost surely in $\Gamma(s)$. □

A fairly immediate consequence of Theorem 4.6.1$_{175}$ is that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies a version of Fatou's lemma [5, 89]. Here, and also further on in this dissertation, we will say that a sequence $(f_n)_{n\in\mathbb{N}}$ of variables in $\overline{\mathbb{V}}$ is **uniformly bounded below** if there is some $c \in \mathbb{R}$ such that $c \leq \inf f_n$ for all $n \in \mathbb{N}$.

Corollary 4.6.2 (Fatou's Lemma). *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any situation $s \in \mathscr{X}^*$ and any sequence $(f_n)_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_\mathrm{b}$ that is uniformly bounded below, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s)$ where $f := \liminf_{n\to+\infty} f_n$.*

Proof. Consider any $s \in \mathscr{X}^*$ and any sequence $(f_n)_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_\mathrm{b}$ that is uniformly bounded below. For all $k \in \mathbb{N}_0$, let g_k be the global variable defined by $g_k(\omega) := \inf_{n\geq k} f_n(\omega)$ for all $\omega \in \Omega$. Then $\lim_{k\to+\infty} g_k = \liminf_{n\to+\infty} f_n = f$. Furthermore, $(g_k)_{k\in\mathbb{N}_0}$ is clearly increasing and it is a sequence in $\overline{\mathbb{V}}_\mathrm{b}$ because $(f_n)_{n\in\mathbb{N}_0}$ is uniformly bounded below. Hence, we can use Theorem 4.6.1$_{175}$ to find that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \lim_{k\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_k|s) = \liminf_{k\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_k|s) \leq \liminf_{k\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_k|s),$$

where the inequality holds because, for all $k \in \mathbb{N}_0$, $g_k \leq f_k$ and therefore, because of Proposition 4.4.3 [EC4$_{163}$], also $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_k|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_k|s)$. □

4.6.2 Continuity with respect to lower cuts and compatibility with the local models

The following result states that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ is continuous with respect to decreasing sequences of lower cuts. In other words, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\cdot|s)$ satisfies Axiom CU2$_{143}$, which was used in Section 4.2.2$_{142}$ as a part of our approach to extend game-theoretic upper expectations with real-valued supermartingales from $\mathbb{V} \times \mathscr{X}^*$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$.

Proposition 4.6.3 (Continuity w.r.t. lower cuts)**.** *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$,*

$$\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s).$$

Proof. $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s)$ is increasing in c because $f^{\vee c}$ is increasing in c and because the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ is monotone [EC4$_{163}$ in Proposition 4.4.3$_{164}$], and therefore $\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s)$ exists. Moreover, $f^{\vee c} \geq f$ for all $c \in \mathbb{R}$, implying, by the monotonicity [EC4$_{163}$ in Proposition 4.4.3$_{164}$] of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$, that $\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$. It therefore only remains to prove the converse inequality.

If $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s) = +\infty$, then $\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$ holds trivially. If $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s) < +\infty$, fix any real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$. Then it follows from the definition of the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$ that there is some supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\mathscr{M}(s) \leq \alpha$ and $\liminf \mathscr{M} \geq_s f$. Since $\mathscr{M}$ is bounded below, it immediately follows that there is some $B \in \mathbb{R}$ such that $\liminf \mathscr{M} \geq c$ for all $c \leq B$. For any such $c \leq B$, we have that $\liminf \mathscr{M} \geq_s f^{\vee c}$, which by Definition 4.7$_{160}$ implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) \leq \mathscr{M}(s) \leq \alpha$. This holds for all $c \leq B$, so we infer that $\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) \leq \alpha$, and since this holds for any $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$, we conclude that indeed $\lim_{c \to -\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f^{\vee c}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(f|s)$. ☐

Proposition 4.6.3 implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ is uniquely determined by its values on $\overline{\mathbb{V}}_{\mathrm{b}} \times \mathscr{X}^*$. Moreover, due to Theorem 4.6.1$_{175}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\cdot|s)$ for any $s \in \mathscr{X}^*$ also satisfies CU1$_{143}$, so the values of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ on $\overline{\mathbb{V}}_{\mathrm{b}} \times \mathscr{X}^*$ are on their turn uniquely determined by its values on $\mathbb{V} \times \mathscr{X}^*$. Together, these observations imply that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ is uniquely determined by its values on $\mathbb{V} \times \mathscr{X}^*$. We will moreover show in Section 4.8$_{186}$ that these values coincide with those of $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathbb{V}}$ on $\mathbb{V} \times \mathscr{X}^*$—if $\mathscr{A}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ agree—and therefore, since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}(\cdot|s)$ for any $s \in \mathscr{X}^*$ satisfies CU1$_{143}$ and CU2$_{143}$, that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ coincides with the extended upper expectation $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$.

Proposition 4.6.3 also immediately confirms part of our claim at the end of Section 4.3.3$_{158}$, where we said that, although imposing CU2$_{143}$ onto the local models does not affect the values of the corresponding global game-theoretic upper expectation, the axiom is crucial when we desire full compatibility of local and global upper expectations, rather than only the partial compatibility that was established by Proposition 4.4.7$_{169}$. We only show here that CU2$_{143}$—together with CU1$_{143}$—is sufficient for full compatibility; later on, in Section 4.9$_{187}$, we will show that it is also necessary.

Corollary 4.6.4 (Compatibility with local models). *Consider any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ and let $\overline{\mathrm{Q}}^{\uparrow}_{\bullet}$ be the corresponding extended upper expectations tree. Then $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|x_{1:n}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}(f(x_{1:n}\cdot))$ for all $x_{1:n} \in \mathscr{X}^*$ and all $(n+1)$-measurable variables $f \in \overline{\mathbb{V}}$.*

Proof. Consider any $x_{1:n} \in \mathscr{X}^*$ and any $(n+1)$-measurable extended real variable $f \in \overline{\mathbb{V}}$. Clearly, $f^{\vee c}$ is bounded below and remains to be $(n+1)$-measurable for any $c \in \mathbb{R}$. Due to Proposition 4.4.7$_{169}$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\vee c}|x_{1:n}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}(f^{\vee c}(x_{1:n}\cdot))$ for any $c \in \mathbb{R}$. Then, because $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\cdot|x_{1:n})$ satisfies CU2$_{143}$ due to Proposition 4.6.3$_{\leftarrow}$, and $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}$ satisfies CU2$_{143}$ by definition, we clearly also have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|x_{1:n}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}(f(x_{1:n}\cdot))$. □

4.6.3 $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ may fail continuity from above

Even though the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is continuous with respect to increasing sequences and with respect to decreasing sequences of lower cuts, it is not necessarily continuous with respect to general decreasing sequences—and therefore certainly not with respect to general pointwise convergence. This is shown by the following example.

Example 4.6.5. Let $\mathscr{X} := \{a, b\}$ and consider the upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ defined by $\overline{\mathrm{Q}}_s(f) := \sup f$ for all $s \in \mathscr{X}^*$ and all $f \in \mathscr{L}(\mathscr{X})$. Then it can be checked easily that each $\overline{\mathrm{Q}}_s$ is coherent, and that the extended tree $\overline{\mathrm{Q}}^{\uparrow}_{\bullet}$ satisfies $\overline{\mathrm{Q}}^{\uparrow}_s(f) = \sup f$ for all $s \in \mathscr{X}^*$ and all $f \in \overline{\mathscr{L}}(\mathscr{X})$. Moreover, since no supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ is ever able to increase, it can be inferred that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) = \sup(g|s) \text{ for all } (g, s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Now consider the decreasing sequence $(A_n)_{n\in\mathbb{N}}$ of events defined by $A_n := \Gamma(a^n) \setminus \{aaa\cdots\}$ for all $n \in \mathbb{N}$; so for any $n \in \mathbb{N}$ and $\omega \in \Omega$, we have that $\omega \in A_n$ if (and only if) at least the first n components of ω are a, but not all of them. Then we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\mathbb{I}_{A_n}|\square) = \sup(\mathbb{I}_{A_n}) = 1$ for all $n \in \mathbb{N}$. On the other hand, it can easily be checked that $\lim_{n\to+\infty} A_n = \emptyset$, and therefore that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lim_{n\to+\infty} \mathbb{I}_{A_n}|\square) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(0|\square) = \sup(0) = 0$. Hence, we find that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\mathbb{I}_{A_n}|\square) = 1 \neq 0 = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lim_{n\to+\infty} \mathbb{I}_{A_n}|\square),$$

which shows that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ does not satisfy continuity with respect to general decreasing sequences. ◇

4.7 Behaviour of the game-theoretic upper expectation with respect to sequences of finitary variables

In the previous section, we saw that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ does not necessarily satisfy continuity with respect to pointwise convergence, not even if the considered sequence is downward monotone. Luckily enough though, sequences of interest will in many cases be composed of variables that are finitary (note that this is not the case for Example 4.6.5); such sequences tend to be more well-behaved and therefore allow us to establish stronger continuity properties for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$. In the present section, we set out to do so; one of the most important continuity properties is continuity with respect to decreasing sequences of bounded above finitary variables.

Sequences of finitary variables or, more specifically, finitary gambles are also interesting from a practical point of view, because their associated global upper expectations can be computed fairly efficiently; see e.g. [100]. If these computational methods are combined with the appropriate continuity properties—which tend to be stronger for sequences of finitary variables—we also immediately have a method for computing (or approximating) upper expectations for many non-finitary variables.

4.7.1 Some notes about finitary variables and their pointwise limits

Because of their importance in this section, we first want to establish some basic, yet convenient properties for sequences of finitary variables. First is the fact that any sequence of finitary variables can be equivalently considered as a sequence of n-measurable variables; the latter is a sequence $(f_n)_{n\in\mathbb{N}}$ of global variables where, for any $n \in \mathbb{N}$, the variable $f_n \in \overline{\mathbb{V}}$ is n-measurable. It is clear that this is not the case for all sequences of finitary variables, yet we can always modify it, arriving at a sequence of n-measurable variables, while not affecting most of the other sequence characteristics, including its pointwise limit (should it exist). This can be done using the following construction.

Consider any sequence $(g_n)_{n\in\mathbb{N}_0}$ of finitary variables, and let $(g^{\xi}_n)_{n\in\mathbb{N}_0}$ and $\xi\colon \mathbb{N} \to \mathbb{N}_0$ be defined by the following recursive expressions, where $g^{\xi}_0 := c \in \overline{\mathbb{R}}$ is a freely chosen extended real number and $\xi(1) := 0$:

$$g^{\xi}_n := \begin{cases} g_{\xi(n)} & \text{if } g_{\xi(n)} \text{ is } n\text{-measurable;} \\ g^{\xi}_{n-1} & \text{otherwise,} \end{cases}$$
$$\text{and,} \tag{4.11}$$
$$\xi(n+1) := \begin{cases} \xi(n)+1 & \text{if } g_{\xi(n)} \text{ is } n\text{-measurable;} \\ \xi(n) & \text{otherwise,} \end{cases}$$

for all $n \in \mathbb{N}$. The construction above seems more technical than it actually is; informally speaking, the sequence $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is simply created from $(g_n)_{n\in\mathbb{N}_0}$ by keeping the sequence $(g_n)_{n\in\mathbb{N}_0}$ constant for a number of steps, then switching to the next variable in the sequence $(g_n)_{n\in\mathbb{N}_0}$ at the appropriate moment. For instance, consider the sequence $(g_n)_{n\in\mathbb{N}_0}$ such that $g_0 = g_1 = g_2 = \mathbb{1}_{aa}$, $g_3 = \mathbb{1}_{a^6}$ and $g_n = \mathbb{1}_{a^n}$ for all $n \geq 4$, with a some state in the state space $\mathscr{X}$ for which it holds that $|\mathscr{X}| > 1$. Then $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is given by $c, c, g_0, g_1, g_2, g_2, g_3, g_4, g_5, g_6, \ldots$.

The following lemma establishes our claim that, amongst other things, the newly created sequence $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is a sequence of n-measurable variables.

Lemma 4.7.1. *Consider any sequence $(g_n)_{n\in\mathbb{N}_0}$ of finitary variables and let $(g_n^{\xi})_{n\in\mathbb{N}_0}$ be defined by Eq. (4.11)$_{\leftarrow}$. Then we have that*

(i) *$(g_n^{\xi})_{n\in\mathbb{N}_0}$ is a sequence of n-measurable variables.*

(ii) *If $(g_n)_{n\in\mathbb{N}_0}$ is increasing and $g_0^{\xi} \leq \inf g_0$, then $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is increasing too.*

(iii) *If $(g_n)_{n\in\mathbb{N}_0}$ is decreasing and $g_0^{\xi} \geq \sup g_0$, then $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is decreasing too.*

(iv) *If $(g_n)_{n\in\mathbb{N}_0}$ is uniformly bounded below, then so is $(g_n^{\xi})_{n\in\mathbb{N}_0}$.*

(v) *If $(g_n)_{n\in\mathbb{N}_0}$ is a sequence of gambles and $c \in \mathbb{R}$, then so is $(g_n^{\xi})_{n\in\mathbb{N}_0}$.*

(vi) $\liminf_{n\to+\infty} g_n = \liminf_{n\to+\infty} g_n^{\xi}$ *and* $\limsup_{n\to+\infty} g_n = \limsup_{n\to+\infty} g_n^{\xi}$.

(vii) $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n|s) = \liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n^{\xi}|s)$ *and* $\limsup_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n|s) = \limsup_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n^{\xi}|s)$.

Proof. We prove (i) by induction. $g_0^{\xi} = c$ is clearly 0-measurable. To prove the induction step, suppose that g_{k-1}^{ξ} is $(k-1)$-measurable for some $k \in \mathbb{N}$. Then if $g_{\xi(k)}$ is k-measurable, so is $g_k^{\xi} = g_{\xi(k)}$. Otherwise, g_k^{ξ} is equal to g_{k-1}^{ξ} implying that g_k^{ξ} is $(k-1)$-measurable and therefore also k-measurable. This concludes the induction step.

In order to prove (ii) and (iii), observe that $(g_n)_{n\in\mathbb{N}_0}$ is a subsequence of $(g_n^{\xi})_{n\in\mathbb{N}_0}$, and more specifically that there is a function $N\colon \mathbb{N}_0 \to \mathbb{N}$ such that $(g_n^{\xi})_{n\in\mathbb{N}_0}$ consists of $N(0)$ times c, then $N(1)$ times g_0, then $N(2)$ times g_1, and so on; indeed, the fact that each g_n only appears a finite number of times in $(g_n^{\xi})_{n\in\mathbb{N}_0}$ is due to the fact that g_n is finitary, and thus m-measurable for some $m \in \mathbb{N}_0$. It is then obvious, by a suitable choice of $c \in \overline{\mathbb{R}}$—that is, any $c \leq \inf g_0$ if $(g_n)_{n\in\mathbb{N}_0}$ is increasing, or any $c \geq \sup g_0$ if $(g_n)_{n\in\mathbb{N}_0}$ is decreasing—that $(g_n^{\xi})_{n\in\mathbb{N}_0}$ has the same monotone character as the original sequence $(g_n)_{n\in\mathbb{N}_0}$. Properties (iv)–(vii) follow from similar observations. □

Due to Lemma 4.7.1 (vi), the pointwise limits of sequences of n-measurable variables constitute the same subset of $\overline{\mathbb{V}}$ as the pointwise limits of sequences of finitary variables. The following result additionally shows that when such limits are bounded below, we can restrict our attention to sequences of n-measurable **gambles**. We use $\overline{\mathbb{L}}_{\mathrm{b}}$ to denote the set of all bounded below variables $f \in \overline{\mathbb{V}}_{\mathrm{b}}$ such that $f = \lim_{n\to+\infty} f_n$ for some sequence $(f_n)_{n\in\mathbb{N}_0}$ of finitary variables.

Proposition 4.7.2. *Any $f \in \overline{\mathbb{L}}_{\mathrm{b}}$ is the pointwise limit of a sequence $(f_n)_{n\in\mathbb{N}_0}$ of n-measurable gambles. Furthermore, we can guarantee that $B \leq f_n \leq \sup f$ for all $n \in \mathbb{N}_0$, where B is any real number if $\inf f = +\infty$, and $B = \inf f$ otherwise.*

Proof. Fix any $f \in \overline{\mathbb{L}}_{\mathrm{b}}$. Then, according to the definition of $\overline{\mathbb{L}}_{\mathrm{b}}$, f is the pointwise limit of a sequence $(g_n)_{n\in\mathbb{N}_0}$ of finitary variables. Let $(g_n^\xi)_{n\in\mathbb{N}_0}$ be the sequence defined by Eq. $(4.11)_{180}$, with $c = 0$, which by Lemma 4.7.1(i)$_{\curvearrowleft}$ is a sequence of n-measurable variables. By Lemma 4.7.1(vi)$_{\curvearrowleft}$, the sequences $(g_n)_{n\in\mathbb{N}_0}$ and $(g_n^\xi)_{n\in\mathbb{N}_0}$ have the same limit behaviour, so $(g_n^\xi)_{n\in\mathbb{N}_0}$ converges pointwise to f. Let B be any real if $\inf f = +\infty$ and let $B := \inf f$ if $\inf f \in \mathbb{R}$ [the case where $\inf f = -\infty$ is impossible because f is bounded below]. Let $(f_n)_{n\in\mathbb{N}_0}$ be the sequence defined by bounding each g_n^ξ above by $\min\{n, \sup f\}$ and below by B; so $f_n(\omega) := \max\{\min\{g_n^\xi(\omega), n, \sup f\}, B\}$ for all $\omega \in \Omega$ and all $n \in \mathbb{N}_0$. Then it is clear that $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles because $(g_n^\xi)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable (possibly extended real) variables. It also converges pointwise to f because

$$\begin{aligned} f(\omega) = \max\Big\{\min\{f(\omega), \sup f\}, B\Big\} &= \max\Big\{\min\big\{\lim_{n\to+\infty} g_n^\xi(\omega), \lim_{n\to+\infty} n, \sup f\big\}, B\Big\} \\ &= \lim_{n\to+\infty} \max\Big\{\min\{g_n^\xi(\omega), n, \sup f\}, B\Big\} \\ &= \lim_{n\to+\infty} f_n(\omega), \end{aligned}$$

for all $\omega \in \Omega$, where the first equality follows from the fact that $B \leq \inf f \leq f$. Moreover, for all $n \in \mathbb{N}_0$, we clearly have that $B \leq f_n$, and also $f_n \leq \sup f$ because $\min\{g_n^\xi(\omega), n, \sup f\} \leq \sup f$ for all $\omega \in \Omega$ and $B \leq \inf f \leq \sup f$. Hence, $(f_n)_{n\in\mathbb{N}_0}$ satisfies all of the conditions in the proposition. □

4.7.2 Continuity with respect to sequences of finitary variables

We now present two important results concerning the behaviour of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ with respect to sequences of finitary variables. The first one guarantees that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous with respect to decreasing sequences of finitary bounded above variables. The second one states that, for any $f \in \overline{\mathbb{L}}_{\mathrm{b}}$, there is always a sequence of n-measurable gambles—and therefore also a sequence of finitary gambles—that converges pointwise to f and for which $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous. The proofs of these results can be found in Appendix 4.C$_{208}$.

Theorem 4.7.3 (Continuity w.r.t. decreasing finitary variables)**.** *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}_0}$ of finitary, bounded above variables that converges pointwise to a variable $f \in \overline{\mathbb{V}}$, we have that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.*

Theorem 4.7.4. *For any upper expectations tree* $\overline{\mathrm{Q}}_{\bullet}$, *any* $s \in \mathscr{X}^*$ *and* $f \in \overline{\mathbb{L}}_{\mathrm{b}}$, *there is a sequence* $(f_n)_{n\in\mathbb{N}_0}$ *of n-measurable gambles that is uniformly bounded below and that converges pointwise to* f *such that* $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

Observe that Theorem 4.7.3$_{\leftarrow}$ is especially strong if it is combined with Theorem 4.6.1$_{175}$: together, they imply that both the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and the game-theoretic lower expectation $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ are continuous with respect to increasing sequences of bounded below finitary variables, and continuous with respect to decreasing sequences of bounded above finitary variables.[6] In practice, this comes down to being continuous with respect to almost all **monotone** sequences of **finitary gambles**. This property has already been used by Krak et al. [58] in order to obtain an equivalence result about hitting times and hitting probabilities in imprecise Markov chains. Theorem 4.7.4, on the other hand, further establishes the importance of finitary variables and their limits when it comes to characterising $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. In fact, Theorem 4.7.4 will be a key result for obtaining our alternative axiomatic characterisation of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ in Section 6.2$_{290}$.

In light of Theorems 4.7.3$_{\leftarrow}$ and 4.7.4, one might now wonder how far the continuity of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ with respect to sequences of finitary gambles stretches. We know that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is not necessarily continuous with respect to general pointwise convergence, but perhaps it could still be continuous if we restrict ourselves to converging sequences of finitary gambles. Unfortunately, as the following example shows, this is not the case, not even for sequences of indicators of cylinder events.

Example 4.7.5. Let $\mathscr{X} := \{a, b\}$ and consider the same upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ as in Example 4.6.5$_{179}$; as explained in that example, we then have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) = \sup(g|s)$ for all $(g, s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. Observe that for the sequence $(a^n b)_{n\in\mathbb{N}}$ of situations, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{1}_{a^n b}|\square) = \sup(\mathbb{1}_{a^n b}) = 1$ for all $n \in \mathbb{N}$. Yet, it can also be checked that $\lim_{n\to+\infty} \mathbb{1}_{a^n b} = 0$, which implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\lim_{n\to+\infty} \mathbb{1}_{a^n b}|\square) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(0|\square) = 0$, where the last equality follows from Proposition 4.4.3 [EC1$_{163}$]. So we find that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{1}_{a^n b}|\square) = 1 \neq 0 = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\lim_{n\to+\infty} \mathbb{1}_{a^n b}|\square).$$

Hence, since $(\mathbb{1}_{a^n b})_{n\in\mathbb{N}}$ is a sequence of finitary gambles, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ does in general not satisfy continuity with respect to pointwise convergence of finitary gambles. ◇

[6] Indeed, by conjugacy [Corollary 4.3.7$_{162}$], Theorem 4.6.1$_{175}$ implies that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous with respect to decreasing sequences that are bounded above, and Theorem 4.7.3$_{\leftarrow}$ implies that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is continuous with respect to increasing sequences that are bounded below.

4.7.3 Expressions for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ in terms of (limits of) finitary gambles

Our next two results further emphasize the central role of the variables in $\mathbb{F}$ and $\overline{\mathbb{L}}_{\mathrm{b}}$ for the characterisation of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. The first one states that the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ of any variable $f \in \overline{\mathbb{V}}$, conditional on any $s \in \mathscr{X}^*$, is the lower envelope of the upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s)$ of variables $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ that are (pointwise) equal to or larger than f on $\Gamma(s)$. The second one further parses this expression and fully characterises $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ using only its values on the finitary gambles. Together with Theorem 4.7.4↶, both of these results will be crucial in Chapter 6_{283}, where $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ will be given an alternative characterisation in terms of some fairly simple axioms. It is also interesting to compare these results to Proposition $3.5.10_{97}$, which fully characterises $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ using only its values on $\mathbb{F} \times \mathscr{X}^*$, as it sheds some light on the differences between $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$.

Proposition 4.7.6. *Consider any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $f \in \overline{\mathbb{V}}$ and any $s \in \mathscr{X}^*$. Then*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq_s f\right\}$$
$$= \inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\}. \quad (4.12)$$

Proof. Because $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is monotone [$\mathrm{EC4}_{163}$ in Proposition $4.4.3_{164}$], we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s)$ for any $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ such that $f \leq_s g$. It therefore follows immediately that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq_s f\right\} \leq \inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\},$$

where the last inequality follows from the fact that $g \geq f$ implies $g \geq_s f$ for any $g \in \overline{\mathbb{V}}$. It remains to prove that $\inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

Consider any $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\lim \mathscr{M}'$ exists within $\Gamma(s)$ and such that $\lim \mathscr{M}' \geq_s f$. Let $\mathscr{M}$ be the extended real process defined by $\mathscr{M}(t) := \mathscr{M}'(t)$ for all $t \sqsupseteq s$, and by $\mathscr{M}(t) := +\infty$ for all $t \not\sqsupseteq s$. We show that $\mathscr{M}$ is a bounded below supermartingale such that $\lim \mathscr{M} \geq f$. The process $\mathscr{M}$ is bounded below because $\mathscr{M}'$ is. Moreover, we have, for all $t \sqsupseteq s$, that $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}(t\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}'(t\cdot)) \leq \mathscr{M}'(t) = \mathscr{M}(t)$ because $\mathscr{M}'$ is a supermartingale, and, for all $t \not\sqsupseteq s$, we also have that $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}(t\cdot)) \leq \mathscr{M}(t)$ because then $\mathscr{M}(t) = +\infty$. Hence, $\mathscr{M}$ is also a supermartingale. Furthermore, note that $\lim \mathscr{M} =_s \lim \mathscr{M}' \geq_s f$ and, for any path ω not going through s, that $\lim \mathscr{M}(\omega) = +\infty \geq f(\omega)$, which all together implies that $\lim \mathscr{M} \geq f$.

Now, let $(g_n)_{n\in\mathbb{N}}$ be the sequence defined by $g_n(\omega) := \mathscr{M}(\omega^n)$ for all $n \in \mathbb{N}$ and all $\omega \in \Omega$. Then it is clear that $(g_n)_{n\in\mathbb{N}}$ is a sequence of n-measurable, and therefore finitary, extended real variables that is uniformly bounded below. Moreover, since $\lim \mathscr{M}$ exists everywhere, we have that $g(\omega) := \lim_{n\to+\infty} g_n(\omega) = \lim_{n\to+\infty} \mathscr{M}(\omega^n)$ exists for all $\omega \in \Omega$. Hence, $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ and because $\lim \mathscr{M} \geq f$ also $g \geq f$. It furthermore follows from Definition 4.7_{160} that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \leq \mathscr{M}(s)$ because $\lim \mathscr{M} \geq_s g$ (since, in fact, $\lim \mathscr{M} = g$). This implies that

$$\inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \leq \mathscr{M}(s) = \mathscr{M}'(s).$$

Since this holds for any $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\lim \mathscr{M}'$ exists within $\Gamma(s)$ and $\lim \mathscr{M}' \geq_s f$, it follows from Proposition 4.5.4₁₇₃ that, indeed,

$$\inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s). \qquad \square$$

As already mentioned, the following result fully characterises $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ in terms of its values on $\mathbb{F} \times \mathscr{X}^*$.

Proposition 4.7.7. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) \colon g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}.$$

Proof. Fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. We first prove that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) \colon g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}. \tag{4.13}$$

Fix any sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below—so there is a $B \in \mathbb{R}$ such that $g_n \geq B$ for all $n \in \mathbb{N}$—and such that $\lim_{n\to+\infty} g_n \geq_s f$. Then, by Corollary 4.6.2₁₇₇,

$$\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\lim_{n\to+\infty} g_n|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s),$$

where the last inequality follows from Proposition 4.4.3₁₆₄ [EC4₁₆₃]. Since the inequality above holds for all sequences $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that are uniformly bounded below and for which $\lim_{n\to+\infty} g_n \geq_s f$, we conclude that Eq. (4.13) holds.

To prove the converse inequality, fix any $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ such that $g \geq_s f$. According to Theorem 4.7.4₁₈₃, there is a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below and for which $\lim_{n\to+\infty} g_n = g$ and

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) = \liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s).$$

As a result, since $(g_n)_{n\in\mathbb{N}}$ is moreover uniformly bounded below and is such that $\lim_{n\to+\infty} g_n = g \geq_s f$, we obtain that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \geq \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) \colon g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}.$$

Since the inequality above holds for all $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ such that $g \geq_s f$, we infer from Proposition 4.7.6_← that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g_n|s) \colon g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\},$$

which together with Eq. (4.13) concludes the proof. $\square$

4.8 Concluding notes on the definition of a game-theoretic upper expectation

An important contribution of this chapter to the theory of game-theoretic probability is, besides that it establishes a multitude of fundamental properties for game-theoretic upper expectations, that it provides an overview of the possible definitions and an argumentation for why one specific version stands out. Such an overview and argumentation is particularly relevant because many slightly different definitions for a global game-theoretic upper expectation have appeared in the literature [8, 86, 88, 109], and it is not always clear what these differences entail. Most versions only differ in how the supermartingales are allowed to behave. Propositions 4.5.4$_{173}$ and 4.5.5$_{174}$, as well as [85, Proposition 7.7], show that the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is fairly robust with respect to changes that concern the limit behaviour of supermartingales and, more specifically, how this limit behaviour relates to the variable f in consideration. A choice that does have a large impact is whether to use real-valued or extended real-valued supermartingales, and whether we require them to be bounded below or not. We have chosen to adopt a version of the game-theoretic upper expectation with bounded below extended real-valued supermartingales $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$, mainly because of our findings in Sections 4.1$_{131}$–4.3$_{152}$, but also because of some claims about the desirable features of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$—recall the end of Section 4.2.3$_{145}$. Using the results from Sections 4.4$_{162}$–Section 4.7$_{180}$ we can now confirm these claims. Let us first briefly recall the following considerations from Sections 4.1$_{131}$–4.3$_{152}$.

Given a general acceptable gambles tree $\mathscr{A}_\bullet$, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ is the version with the most direct and intuitive interpretation because it solely involves the use of **bounded** supermartingales; we regard this to be a practically sensible assumption because we interpret supermartingales as capital processes and because, in a realistic, practical context, one can never borrow or gain an unbounded or infinite amount of money. The version $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ furthermore coincides with the versions $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$—which are interpretationally less direct because they use real-valued supermartingales that are not bounded (above)—on the domain $\mathbb{V} \times \mathscr{X}^*$. On the domain $\overline{\mathbb{V}} \times \mathscr{X}^*$, all three the upper expectations $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ turn out to be unsuitable; $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$ has the undesirable feature that it sometimes becomes lower than its corresponding game-theoretic lower expectation [Example 4.2.1$_{139}$]; $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ sometimes return excessively large—conservative—values [Example 4.2.2$_{140}$]. An appropriate solution to these issues was found by adopting the use of extended real supermartingales. This led us to the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$, which on its turn is equivalent to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$ as long as the upper expectations tree $\overline{\mathrm{Q}}_\bullet$ agrees with $\mathscr{A}_\bullet$ through Eq. (3.1)$_{50}$; recall Theorem 4.3.6$_{161}$.[7]

Now, apart from solving the issues raised in Examples 4.2.1$_{139}$ and 4.2.2$_{140}$—and also the one in Example 3.6.1$_{99}$ due to Proposition 4.2.8$_{150}$ and Example 4.1.1$_{133}$—$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ was claimed to have numerous desirable properties; this is now confirmed by the results in Sections 4.4$_{162}$–Section 4.7$_{180}$. Among others, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ satisfies extended coherence properties, a general law of iterated upper expectations, continuity with respect to increasing (bounded below) sequences and continuity with respect to decreasing (bounded above) sequences of finitary variables. These results were established for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$—because the parametrisation in terms of upper expectations trees is more convenient—but, by Theorem 4.3.6$_{161}$, they also hold for any acceptable gambles tree $\mathscr{A}_\bullet$ and the corresponding upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$.

Moreover, a final argument for the use of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathbb{V}}$—or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$—is that, as was claimed at the end of Section 4.2.3$_{145}$, and as we will show next, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ coincides with $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$. Recall from Section 4.2.2$_{142}$ that $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ was by definition equal to $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathbb{V}}$—or, better, $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathbb{V}}$—on $\mathbb{V} \times \mathscr{X}^*$, and was further defined on $\overline{\mathbb{V}}\times\mathscr{X}^*$ by imposing continuity with respect to upper and lower cuts [CU1$_{143}$, CU2$_{143}$]. As we have argued there, we believe the definition of $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ to make more sense, interpretationally speaking, than the one of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ because the former still fundamentally relies on bounded real-valued supermartingales instead of extended real-valued ones. Since the upper expectations $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ coincide, $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ can thus serve as an alternative—and more intuitive—characterisation for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$.

Proposition 4.8.1. *For any acceptable gambles tree $\mathscr{A}_\bullet$, we have that*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}(f|s) = \overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}(f|s) \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Proof. First note that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ extends (the restriction to $\mathbb{V}\times\mathscr{X}^*$ of) $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathbb{V}}$ due to Proposition 4.2.8$_{150}$. Since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}(\cdot|s)$ for any $s \in \mathscr{X}^*$ moreover satisfies CU1$_{143}$ and CU2$_{143}$ by Theorem 4.6.1$_{175}$, Proposition 4.6.3$_{178}$ and Theorem 4.3.6$_{161}$, and since $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}(\cdot|s)$ is by Definition 4.3$_{143}$ the unique global upper expectation that extends $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathbb{V}}$ (on $\mathbb{V} \times \mathscr{X}^*$) and is such that, for all $s \in \mathscr{X}^*$, $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}(\cdot|s)$ satisfies CU1$_{143}$ and CU2$_{143}$, we obtain that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathbb{V}}$ coincides with $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathbb{V}}$ on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$. □

4.9 Relationship to Shafer and Vovk's work

We conclude this chapter on game-theoretic upper expectations with a brief study on the relation between our work and that of Shafer and Vovk. Of

[7] Results similar to Theorem 4.3.6$_{161}$ could also be deduced for the upper expectations $\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathbb{V}}$, $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathbb{V}}$ and $\overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathbb{V}}$.

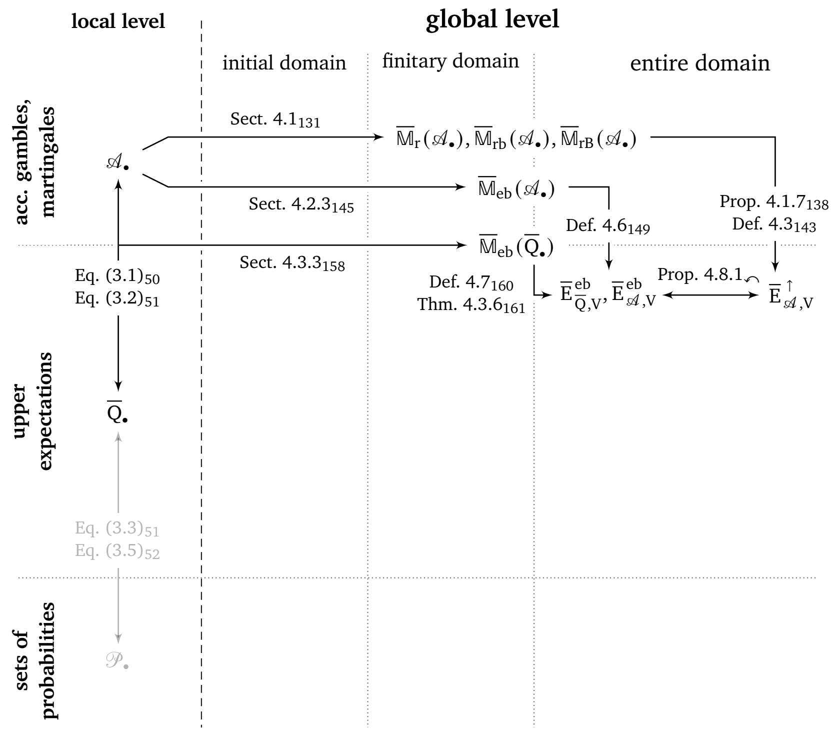

Figure 4.2 Schematic overview of the most important game-theoretic approaches and their connections.

course, many others have contributed significantly to the field as well [8, 9, 21, 74, 88], yet we regard Shafer and Vovk's new book [85] to be our main point of reference because (i) it proposes a full-fledged and self-contained theory of game-theoretic probability that covers a broad range of results and topics; (ii) it is recent and therefore takes into account and/or includes most of the novel contributions to the field—contrary to [86], which is in some aspects already outdated; and (iii) Part II in [85] concerns material that is closely related to what we have presented here.

4.9.1 A brief overview

The starting point in Shafer and Vovk's framework—not only in [85] but also in [86, 88, 109]—is not necessarily an acceptable gambles tree $\mathscr{A}_\bullet$ or an upper expectations tree $\overline{\mathrm{Q}}_\bullet$, but rather a sequential game where three players—Forecaster, Skeptic and Reality—or sometimes two players—Skeptic and World—play according to a so-called testing protocol. Testing protocols lay down the rules of the game. One such testing protocol was already more or less introduced in Section 3.2.3$_{61}$ where we introduced the finitary game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$. Shafer and Vovk, however, also consider a broad range of other testing protocols, varying from very simple ones—see e.g. [85, Protocol 1.1]—to rather abstract and general ones—see e.g. [85, Protocol 7.12]. Pointing out the differences with our approach, and certainly understanding what these differences entail, can however be a rather challenging task. We now sketch an overview of what the most important aspects are one should take into account when comparing our work with theirs. Our results are mainly related to the results situated in [85, Part II], so we focus only on the material presented therein.

(i) In [85, Part II], a player called 'Reality'—or 'World' in absence of Forecaster—decides what the outcome of each round is. This is only a matter of interpretation; one could just as well regard Reality's moves to be the subsequent observations of the state of a stochastic process.

(ii) The local state space—the move space for Reality or World—is in [85, Part II] not necessarily finite, nor fixed; see e.g. [85, Protocol 7.10].

(iii) The local models in [85, Part II]—which are specified by Forecaster—always take the form of a particular type of upper expectation; see [85, Section 6.1] and Definition 4.8$_{191}$ below. Moreover, note that, though this turned out to have no effect on the resulting game-theoretic upper expectation, we actually took coherent sets of acceptable gambles to be starting point rather than local upper expectations.

(iv) The local models $\overline{\mathrm{Q}}_{\bullet}$ or $\mathscr{A}_{\bullet}$ are in our case assumed to be known beforehand; that is, we assume that Skeptic knows what Forecaster's moves—the specification of the models $\overline{\mathrm{Q}}^{\uparrow}_{s}$ or $\mathscr{A}^{\uparrow}_{s}$—for each situation $s \in \mathscr{X}^*$ are going to be, and thus what options Skeptic is going to have in each situation, before he starts playing. In [85, Protocol 7.12], for instance, this is not the case as Forecaster is there only required to reveal his moves in each round, after he has observed previous moves by Skeptic and Reality.[8] His forecasts or moves are in that case called 'prequential' [21]. Mathematically speaking, this comes down to allowing each local upper expectation to depend on the situation **and** on the previous moves by Skeptic. Though this seems to always impact the generality of their approach in the positive, it only effectively does so when we are considering finite or countable state spaces. As Shafer and Vovk argue themselves, for general state spaces, one can

[8] In contrast to Section 3.2.3$_{61}$ where Skeptic's moves are the (process) differences $\Delta\mathscr{M}$ of the corresponding real supermartingale $\mathscr{M}$, Skeptic's moves in the 'extended real' context are the local variables $\mathscr{M}(s\,\cdot)$; see e.g. [85, Protocol 7.1].

always turn a prequential protocol—e.g. [85, Protocol 7.12]—into an equivalent non-prequential one—e.g. [85, Protocol 7.1].

It is clear that the differences described in point (ii)$_{\curvearrowleft}$ and (iv)$_{\curvearrowleft}$ impact the generality of Shafer and Vovk's approach in the positive. To assess how the difference in (iii)$_{\curvearrowleft}$ impacts generality, on the other hand, a little bit more care is required. We will study the relation between Shafer and Vovk's type of upper expectation [85, Section 6.1] and our local upper expectations in Section 4.9.2$_{\rightarrow}$ below. As we will see, their type of (local) upper expectation is more general than ours. Nevertheless, we do not consider this additional expressive power to be a positive feature, because—as we will also show below—on the one hand, the resulting global game-theoretic upper expectations are not affected by it, and on the other hand, compatibility of local and global upper expectations cannot be guaranteed if we were to work with their more general type of (local) upper expectation.

Besides, though our local upper expectations are less general than those of Shafer and Vovk, we have also set forward a game-theoretic approach based entirely on local (coherent) sets of acceptable gambles. By Theorem 4.3.6$_{161}$, we know that this does not affect the values of the resulting global game-theoretic upper expectation, but still, as we have discussed in Section 2.5$_{33}$ and Section 3.1.2$_{48}$, sets of acceptable gambles are more general than upper expectations, and so it was a priori not given, neither trivial, that these two types of local models would lead to equivalent global upper expectations. Additionally, our acceptability-based approach sheds light on the connection between Shafer and Vovk's theory and the traditional field of behavioural imprecise probabilities [3, 106, 110, 113]. Shafer and Vovk have also used a type of local model similar to coherent sets of acceptable gambles in their first book [86, Section 8.3], but the setting there involves only gambles and real-valued supermartingales; we refer to De Cooman & Hermans [9] for an in-depth overview on how the setting in [86, Section 8.3] compares to the behavioural coherence approach of Walley [110], for stochastic processes with a finite time horizon.

Finally, we also want to nuance our earlier statements about the increased generality of Shafer and Vovk's approach in aspects (ii)$_{\curvearrowleft}$ and (iv)$_{\curvearrowleft}$: by restricting ourselves to finite state spaces, and to non-prequential forecasts, we were able to establish some crucial results that are absent or different in Shafer and Vovk's book [85]. Most notably, Theorem 4.7.3$_{182}$ is similar to [85, Lemma 9.12] due to Lemma 5.5.5$_{251}$ below, yet [85, Lemma 9.12] requires strong topological conditions on how the local models are allows to vary, only involves non-negative variables, and is only stated for unconditional global upper expectations. There appears to be no analogon of Proposition 4.7.7$_{185}$ in [85]; neither does there seem to be ones for the more technical results Theorem 4.7.4$_{183}$ or Proposition 4.7.6$_{184}$, which will be crucial in Chapter 6$_{283}$ to establish an axiomatic characterisation of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$.

4.9.2 Upper expectations according to Shafer and Vovk

The following definition specifies what Shafer, Vovk and Takemura call a 'superexpectation' in [88] and what Shafer and Vovk call an 'upper expectation' in [85]. In order to differentiate with our notion of an upper expectation, we will refer to it as an SV-upper expectation.[9] They define this type of upper expectation for general possibility spaces $\mathscr{Y}$, but we immediately apply it here to the case where $\mathscr{Y}$ is the finite state space $\mathscr{X}$.

Definition 4.8 (SV-upper expectations)**.** An SV-upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathscr{L}}(\mathscr{X})$ is an extended real-valued map on $\overline{\mathscr{L}}(\mathscr{X})$ that satisfies the following axioms:

SV1. $\overline{\mathrm{E}}(c) = c$ for all $c \in \mathbb{R}$;

SV2. $\overline{\mathrm{E}}(f+g) \leq \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(g)$ for all $f, g \in \overline{\mathscr{L}}(\mathscr{Y})$;

SV3. $\overline{\mathrm{E}}(\lambda f) = \lambda \overline{\mathrm{E}}(f)$ for all $\lambda \in \mathbb{R}_{>}$ and all $f \in \overline{\mathscr{L}}(\mathscr{Y})$;

SV4. $f \leq g \Rightarrow \overline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}(g)$ for all $f, g \in \overline{\mathscr{L}}(\mathscr{Y})$.

SV5. $\lim_{n\to+\infty} \overline{\mathrm{E}}(f_n) = \overline{\mathrm{E}}\left(\lim_{n\to+\infty} f_n\right)$ for any increasing sequence $(f_n)_{n\in\mathbb{N}_0}$ of non-negative variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$. ◎

We immediately have the following constant additivity property for an SV-upper expectation.

Corollary 4.9.1. *For any SV-upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathscr{L}}(\mathscr{X})$, we have that*

SV6. $\overline{\mathrm{E}}(f+\mu) = \overline{\mathrm{E}}(f) + \mu$ *for all* $f \in \overline{\mathscr{L}}(\mathscr{X})$ *and all* $\mu \in \mathbb{R}$.

Proof. Fix any $f \in \overline{\mathscr{L}}(\mathscr{X})$ and any $\mu \in \mathbb{R}$. By SV2 and SV1, we have that

$$\overline{\mathrm{E}}(f) = \overline{\mathrm{E}}(f) + \mu - \mu = \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(\mu) - \mu \geq \overline{\mathrm{E}}(f+\mu) - \mu = \overline{\mathrm{E}}(f+\mu) + \overline{\mathrm{E}}(-\mu) \geq \overline{\mathrm{E}}(f).$$

So we obtain that $\overline{\mathrm{E}}(f+\mu) - \mu = \overline{\mathrm{E}}(f)$, and thus that $\overline{\mathrm{E}}(f+\mu) = \overline{\mathrm{E}}(f) + \mu$. □

Since Shafer and Vovk use these SV-upper expectations as local models—or moves by Forecaster—in their testing protocols, we are interested in how these SV-upper expectations are related to our notion of an extended local upper expectation, which is characterised by coherence on $\mathscr{L}(\mathscr{X})$, and

[9]Their definition is, as far as we know, not based on a single specific interpretation. Rather, they draw inspiration from various subfields in probability theory to obtain these axioms.

by the continuity properties CU1$_{143}$ and CU2$_{143}$ on $\overline{\mathscr{L}}(\mathscr{X})$. We gather our findings in the following proposition.

Proposition 4.9.2. *For any (unconditional) upper expectation* $\overline{\mathrm{E}}\colon \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$*, the following statements hold:*

(i) *if* $\overline{\mathrm{E}}$ *is coherent on* $\mathscr{L}(\mathscr{X})$ *and satisfies* CU1$_{143}$ *and* CU2$_{143}$*, then* $\overline{\mathrm{E}}$ *is an SV-upper expectation;*

(ii) *if* $\overline{\mathrm{E}}$ *is an SV-upper expectation, then* $\overline{\mathrm{E}}$ *is coherent on* $\mathscr{L}(\mathscr{X})$ *and satisfies* CU1$_{143}$.

Proof. To prove (i), assume that $\overline{\mathrm{E}}$ is coherent on $\mathscr{L}(\mathscr{X})$ and satisfies CU1$_{143}$ and CU2$_{143}$. Then Properties LE1$_{156}$ and LE6$_{156}$ in Proposition 4.3.4$_{156}$ guarantee that $\overline{\mathrm{E}}$ satisfies SV1$_{\frown}$ and SV5$_{\frown}$. We next prove that $\overline{\mathrm{E}}$ also satisfies SV2$_{\frown}$, SV3$_{\frown}$ and SV4$_{\frown}$, and hence, that it is an SV-upper expectation.

SV2$_{\frown}$: Consider any two $f, g \in \overline{\mathscr{L}}(\mathscr{X})$ and any $c \in \mathbb{R}$. Then, since $f^{\vee c} \geq c$ and $g^{\vee c} \geq c$, we have that $f^{\vee c}+g^{\vee c} \geq 2c$. In a similar way, we deduce that $f^{\vee c}+g^{\vee c} \geq f+g$. Hence, combining both inequalities, we obtain that $f^{\vee c} + g^{\vee c} \geq \max\{f+g, 2c\} = (f+g)^{\vee 2c}$. Moreover note that $f^{\vee c}$, $g^{\vee c}$ and $(f+g)^{\vee 2c}$ are all variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, so we can apply LE4$_{156}$ and subsequently LE2$_{156}$ to infer that

$$\overline{\mathrm{E}}\left((f+g)^{\vee 2c}\right) \leq \overline{\mathrm{E}}(f^{\vee c}+g^{\vee c}) \leq \overline{\mathrm{E}}(f^{\vee c}) + \overline{\mathrm{E}}(g^{\vee c}).$$

The inequality above holds for any $c \in \mathbb{R}$, so we have that

$$\begin{aligned}\overline{\mathrm{E}}(f+g) &\overset{\text{CU2}_{143}}{=} \lim_{c\to-\infty} \overline{\mathrm{E}}\left((f+g)^{\vee c}\right) = \lim_{c\to-\infty} \overline{\mathrm{E}}\left((f+g)^{\vee 2c}\right) \leq \lim_{c\to-\infty} \left[\overline{\mathrm{E}}(f^{\vee c}) + \overline{\mathrm{E}}(g^{\vee c})\right] \\ &= \lim_{c\to-\infty} \overline{\mathrm{E}}(f^{\vee c}) + \lim_{c\to-\infty} \overline{\mathrm{E}}(g^{\vee c}) \overset{\text{CU2}_{143}}{=} \overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(g),\end{aligned}$$

where the existence of the limits after the inequality follows from the monotonicity [LE4$_{156}$] of $\overline{\mathrm{E}}$, and where the second to last equality follows from the fact that $\overline{\mathrm{E}}(f^{\vee c})$ and $\overline{\mathrm{E}}(g^{\vee c})$ are increasing in c and our convention that $+\infty - \infty = +\infty$.

SV3$_{\frown}$: Consider any $f \in \overline{\mathscr{L}}(\mathscr{X})$. First note that, since multiplication with a positive constant $\lambda \in \mathbb{R}_{>}$ is order preserving on $\overline{\mathbb{R}}$, we have that

$$\max\{\lambda f(y), c\} = \max\{\lambda f(x), \lambda \tfrac{c}{\lambda}\} = \lambda \max\{f(x), \tfrac{c}{\lambda}\} \text{ for all } x \in \mathscr{X} \text{ and all } c \in \mathbb{R}.$$

Hence, $(\lambda f)^{\vee c} = \lambda f^{\vee \frac{c}{\lambda}}$ for all $c \in \mathbb{R}$ and all $\lambda \in \mathbb{R}_{>}$. Since moreover $f^{\vee \frac{c}{\lambda}} \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, we can apply LE3$_{156}$ to infer that $\overline{\mathrm{E}}\left((\lambda f)^{\vee c}\right) = \overline{\mathrm{E}}\left(\lambda f^{\vee \frac{c}{\lambda}}\right) = \lambda \overline{\mathrm{E}}\left(f^{\vee \frac{c}{\lambda}}\right)$ for all $c \in \mathbb{R}$ and all $\lambda \in \mathbb{R}_{>}$. This then implies that, for any $\lambda \in \mathbb{R}_{>}$,

$$\begin{aligned}\overline{\mathrm{E}}\left(\lambda f\right) &\overset{\text{CU2}_{143}}{=} \lim_{c\to-\infty} \overline{\mathrm{E}}\left((\lambda f)^{\vee c}\right) = \lim_{c\to-\infty} \lambda \overline{\mathrm{E}}\left(f^{\vee \frac{c}{\lambda}}\right) = \lambda \lim_{c\to-\infty} \overline{\mathrm{E}}\left(f^{\vee \frac{c}{\lambda}}\right) \\ &= \lambda \lim_{c\to-\infty} \overline{\mathrm{E}}\left(f^{\vee c}\right) \overset{\text{CU2}_{143}}{=} \lambda \overline{\mathrm{E}}\left(f\right).\end{aligned}$$

Finally, that $\overline{\mathrm{E}}$ satisfies SV4$_{\frown}$ follows trivially from the monotonicity [LE4$_{156}$] of $\overline{\mathrm{E}}$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ in combination with CU2$_{143}$. Hence, $\overline{\mathrm{E}}$ satisfies SV1$_{\frown}$–SV5$_{\frown}$ and is therefore an SV-upper expectation.

To prove (ii)$_{\leftarrow}$, assume that $\overline{\mathrm{E}}$ is an SV-upper expectation. We first show that $\overline{\mathrm{E}}$ is coherent on $\mathscr{L}(\mathscr{X})$. To this end, note that $\overline{\mathrm{E}}$ is real-valued on $\mathscr{L}(\mathscr{X})$ due to SV4$_{191}$ and SV1$_{191}$.[10] Hence, according to Definition 2.6$_{32}$, it suffices to prove that $\overline{\mathrm{E}}$ satisfies C1$_{32}$–C3$_{32}$. This is trivial: C1$_{32}$ follows from SV1$_{191}$ and SV4$_{191}$; C2$_{32}$ follows from SV2$_{191}$; and C3$_{32}$ follows from SV3$_{191}$ and SV1$_{191}$. Now, to see that $\overline{\mathrm{E}}$ satisfies CU1$_{143}$, fix any $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ and any increasing sequence $(c_n)_{n\in\mathbb{N}}$ of non-negative reals such that $\lim_{n\to+\infty} c_n = +\infty$. Then clearly $(f^{\wedge c_n} - \inf f^{\wedge c_1})_{n\in\mathbb{N}}$ is an increasing sequence of non-negative variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ that moreover converges to $f - \inf f^{\wedge c_1}$. Hence, SV5$_{191}$ implies that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}(f^{\wedge c_n} - \inf f^{\wedge c_1}) = \overline{\mathrm{E}}(f - \inf f^{\wedge c_1}). \tag{4.14}$$

Since f is bounded below and c_1 is a non-negative real number, we have that $-\inf f^{\wedge c_1} \in \mathbb{R}$, and therefore by SV6$_{191}$ that $\lim_{n\to+\infty} \overline{\mathrm{E}}(f^{\wedge c_n}) = \overline{\mathrm{E}}(f)$. Furthermore, for any $n, m \in \mathbb{N}$ such that $m > n$, we clearly have that $f^{\wedge c_n} \leq f^{\wedge c} \leq f^{\wedge c_m}$ for all $c \in \mathbb{R}$ such that $c_n \leq c \leq c_m$. Due to SV4$_{191}$, this also implies that $\overline{\mathrm{E}}(f^{\wedge c_n}) \leq \overline{\mathrm{E}}(f^{\wedge c}) \leq \overline{\mathrm{E}}(f^{\wedge c_m})$ for all $c \in \mathbb{R}$ such that $c_n \leq c \leq c_m$. Since this holds for any $n, m \in \mathbb{N}$ such that $m > n$, and since $\lim_{n\to+\infty} c_n = +\infty$, it follows that $\lim_{c\to+\infty} \overline{\mathrm{E}}(f^{\wedge c}) = \lim_{n\to+\infty} \overline{\mathrm{E}}(f^{\wedge c_n}) = \overline{\mathrm{E}}(f)$, where the last equality follows from our earlier considerations. So we indeed conclude that $\overline{\mathrm{E}}$ is coherent on $\mathscr{L}(\mathscr{X})$ and satisfies CU1$_{143}$. ☐

So we see that, as a consequence of (i)$_{\leftarrow}$, each of our local extended upper expectations $\overline{\mathrm{Q}}_s^{\uparrow}$ defined through coherence and CU1$_{143}$ and CU2$_{143}$, can always be seen as an SV-upper expectation. Hence, SV-upper expectations are at least as general as our extended local upper expectations. The following example shows that this class of local models is in fact strictly more general.

Example 4.9.3. Consider any finite state space $\mathscr{X}$ such that $|\mathscr{X}| > 1$ and the upper expectation $\overline{\mathrm{E}} : \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$ defined by

$$\overline{\mathrm{E}}(f) := \begin{cases} -\infty & \text{if } f < +\infty \text{ pointwise and } f(x) = -\infty \text{ for some } x \in \mathscr{X}; \\ \sup f & \text{otherwise,} \end{cases}$$

for all $f \in \overline{\mathscr{L}}(\mathscr{X})$. We show that $\overline{\mathrm{E}}$ satisfies SV1$_{191}$–SV5$_{191}$, but not CU2$_{143}$.

SV1$_{191}$: This follows trivially from the definition of $\overline{\mathrm{E}}$.

SV2$_{191}$: Consider any two $f, g \in \overline{\mathscr{L}}(\mathscr{X})$. If there is some $x \in \mathscr{X}$ such that $f(x) = +\infty$, then we have that $\overline{\mathrm{E}}(f) = +\infty$ and therefore also that $\overline{\mathrm{E}}(f) + \overline{\mathrm{E}}(g) = +\infty$, which implies the desired inequality. Due to symmetry, the inequality is also satisfied if $g(x) = +\infty$ for some $x \in \mathscr{X}$. Hence, consider the case where both $f < +\infty$ and $g < +\infty$ pointwise. Then we clearly also

[10] Recall from the beginning of Section 4.3.1$_{153}$ that being real-valued on $\mathscr{L}(\mathscr{X})$ is a necessity for being coherent on $\mathscr{L}(\mathscr{X})$.

have that $f+g<+\infty$ pointwise. If moreover $f(x)=-\infty$ for some $x\in\mathscr{X}$, then also $f(x)+g(x)=-\infty$ (because $g(x)<+\infty$) which, together with the fact that $f+g<+\infty$ pointwise, implies that $\overline{\mathrm{E}}(f+g)=-\infty$ and thus the desired inequality. Once more, the same can be concluded if $g(x)=-\infty$ for some $x\in\mathscr{X}$ because of symmetry. Hence, we are left with the situation where both f and g—and therefore also $f+g$—are real-valued. Then we can immediately infer that $\overline{\mathrm{E}}(f+g)=\sup(f+g)\leq\sup f+\sup g=\overline{\mathrm{E}}(f)+\overline{\mathrm{E}}(g)$.

SV3$_{191}$: Consider any $\lambda\in\mathbb{R}_{>}$ and any $f\in\overline{\mathscr{L}}(\mathscr{X})$. If $f<+\infty$ pointwise and $f(x)=-\infty$ for some $x\in\mathscr{X}$, then also $\lambda f<+\infty$ pointwise and $\lambda f(x)=-\infty$, which implies that $\lambda\overline{\mathrm{E}}(f)=\lambda(-\infty)=-\infty=\overline{\mathrm{E}}(\lambda f)$. Otherwise, if $f>-\infty$ pointwise or $f(x)=+\infty$ for some $x\in\mathscr{X}$, then also $\lambda f>-\infty$ pointwise or $\lambda f(x)=+\infty$, which implies that $\lambda\overline{\mathrm{E}}(f)=\lambda\sup f=\sup\lambda f=\overline{\mathrm{E}}(\lambda f)$.

SV4$_{191}$: Consider any $f,g\in\overline{\mathscr{L}}(\mathscr{X})$ such that $f\leq g$. If $f<+\infty$ pointwise and $f(x)=-\infty$ for some $x\in\mathscr{X}$, then $\overline{\mathrm{E}}(f)=-\infty$ and therefore automatically $\overline{\mathrm{E}}(f)\leq\overline{\mathrm{E}}(g)$. Otherwise, if $f>-\infty$ pointwise or $f(x)=+\infty$ for some $x\in\mathscr{X}$, then also $g>-\infty$ pointwise or $g(x)=+\infty$ for some $x\in\mathscr{X}$. Then it follows from the definition of $\overline{\mathrm{E}}$ that $\overline{\mathrm{E}}(f)=\sup f\leq\sup g=\overline{\mathrm{E}}(g)$.

SV5$_{191}$: Consider any increasing sequence $(f_n)_{n\in\mathbb{N}_0}$ of non-negative variables in $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Since $f_n\geq 0>-\infty$ pointwise, we have that $\overline{\mathrm{E}}(f_n)=\sup f_n$ for all $n\in\mathbb{N}_0$. Clearly, $f:=\lim_{n\to+\infty}f_n$ is non-negative too, so we also have that $\overline{\mathrm{E}}(f)=\sup f$. Hence, we infer that

$$\begin{aligned}\lim_{n\to+\infty}\overline{\mathrm{E}}(f_n)=\lim_{n\to+\infty}\sup_{x\in\mathscr{X}}f_n(x)=\sup_{n\in\mathbb{N}_0}\sup_{x\in\mathscr{X}}f_n(x)&=\sup_{x\in\mathscr{X}}\sup_{n\in\mathbb{N}_0}f_n(x)\\&=\sup_{x\in\mathscr{X}}f(x)=\overline{\mathrm{E}}(f),\end{aligned}$$

where the second and the fourth equality follows from the increasing character of $(f_n)_{n\in\mathbb{N}_0}$.

As a conclusion, $\overline{\mathrm{E}}$ is an SV-upper expectation on $\overline{\mathscr{L}}(\mathscr{X})$. However, it is easy to see that it does not satisfy CU2$_{143}$. Indeed, consider the extended real variable $-\infty\mathbb{I}_x$, where $x\in\mathscr{X}$. Then we have that $\overline{\mathrm{E}}(-\infty\mathbb{I}_x)=-\infty$. On the other hand, $\overline{\mathrm{E}}((-\infty\mathbb{I}_x)^{\vee c})=\overline{\mathrm{E}}(c\mathbb{I}_x)=0$ for all non-positive $c\in\mathbb{R}$ (indeed, note that $\sup c\mathbb{I}_x=0$ because $|\mathscr{X}|>1$). So $\overline{\mathrm{E}}(-\infty\mathbb{I}_x)=+\infty\neq 0=\lim_{c\to-\infty}\overline{\mathrm{E}}((-\infty\mathbb{I}_x)^{\vee c})$, which implies that $\overline{\mathrm{E}}$ does not satisfy CU2$_{143}$. ◇

Though SV-upper expectations are strictly more general than our extended local upper expectations, it follows from Proposition 4.9.2(ii)$_{192}$ that the increased generality solely concerns the domain $\overline{\mathscr{L}}(\mathscr{X})\setminus\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$; see the corollary below. Furthermore, if an extended local upper expectation—in our sense—and an SV-upper expectation are such that they coincide on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, then the former will always provide conservative bounds for the latter.

Corollary 4.9.4. *For any SV-upper expectation* $\overline{\mathrm{E}}$ *on* $\overline{\mathscr{L}}(\mathscr{X})$*, there is an upper expectation* $\overline{\mathrm{E}}'$ *on* $\overline{\mathscr{L}}(\mathscr{X})$ *that satisfies* CU1$_{143}$ *and* CU2$_{143}$*, whose restriction to* $\mathscr{L}(\mathscr{X})$ *is coherent, and is such that* $\overline{\mathrm{E}}(f) = \overline{\mathrm{E}}'(f)$ *for all* $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$*. For any such* $\overline{\mathrm{E}}'$*, we additionally have that* $\overline{\mathrm{E}}(f) \leq \overline{\mathrm{E}}'(f)$ *for all* $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$*.*

Proof. Let $\overline{\mathrm{E}}'$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ simply be defined as to be equal to $\overline{\mathrm{E}}$, and on $\overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ by CU2$_{143}$. Then, by Proposition 4.9.2(ii)$_{192}$, we have that $\overline{\mathrm{E}}'$ satisfies CU1$_{143}$ and that it is coherent on $\mathscr{L}(\mathscr{X})$. By definition, $\overline{\mathrm{E}}'$ coincides with $\overline{\mathrm{E}}$ on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ and satisfies CU2$_{143}$ on $\overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Since CU2$_{143}$ is moreover trivially satisfied on $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ (by any upper expectation), this establishes the first statement. To see that the second statement holds, fix any $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, and note that

$$\overline{\mathrm{E}}(f) \leq \lim_{c \to -\infty} \overline{\mathrm{E}}(f^{\vee c}) = \lim_{c \to -\infty} \overline{\mathrm{E}}'(f^{\vee c}) \overset{\mathrm{CU2}_{143}}{=} \overline{\mathrm{E}}'(f),$$

where the first inequality and existence of the first limit follows from the monotonicity [SV4$_{191}$] of $\overline{\mathrm{E}}$. □

Hence, if we restrict ourselves to the domain $\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$, our local models are as general as these of Shafer and Vovk. Since game-theoretic upper expectations—in our framework as well as that of Shafer and Vovk—are defined through supermartingales that are bounded below, and since the local models characterise supermartingales through Eq. (4.3)$_{159}$, this implies that, as far as the resulting game-theoretic upper expectations are concerned, it does not matter which type of local upper expectation is being used.

In fact, for this reason, if one wishes to remain as general as possible, one may just as well choose to not impose **any** conditions on the domain $\overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. In our case, this simply corresponds to excluding CU2$_{143}$ from the definition of our local upper expectations. Due to Proposition 4.9.2(ii)$_{192}$ and the example below, this defines a set of upper expectations on $\overline{\mathscr{L}}(\mathscr{X})$ that is strictly larger—and thus more general—than the set of SV-upper expectations.

Example 4.9.5. Let $\overline{\mathrm{E}} \colon \overline{\mathscr{L}}(\mathscr{X}) \to \overline{\mathbb{R}}$ be defined by $\overline{\mathrm{E}}(f) := \sup f$ for all $f \in \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$ and by $\overline{\mathrm{E}}(f) := 0$ for all $f \in \overline{\mathscr{L}}(\mathscr{X}) \setminus \overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{X})$. Then it can easily be checked that $\overline{\mathrm{E}}$ is coherent on $\mathscr{L}(\mathscr{X})$ and that it satisfies CU1$_{143}$. Yet, it clearly does not satisfy SV2$_{191}$ and SV4$_{191}$. Indeed, observe that

$$\overline{\mathrm{E}}(-\infty + 1) = \overline{\mathrm{E}}(-\infty) = 0 \not\geq 1 = \overline{\mathrm{E}}(-\infty) + \overline{\mathrm{E}}(1),$$

so $\overline{\mathrm{E}}$ violates SV2$_{191}$. $\overline{\mathrm{E}}$ also violates SV4$_{191}$ because $\overline{\mathrm{E}}(-\infty) = 0 \not\leq -1 = \overline{\mathrm{E}}(-1)$. ◇

One major reason why we nevertheless impose CU2$_{143}$ onto the local upper expectations is because, without it, we do not necessarily have compatibility with the global upper expectation on the entire domain $\overline{\mathscr{L}}(\mathscr{X})$.

This can be seen as follows. For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any extended upper expectations tree $\overline{\mathrm{Q}}^1_\bullet : s \in \mathscr{X}^* \mapsto \overline{\mathrm{Q}}^1_s$ for which the individual local models $\overline{\mathrm{Q}}^1_s$ extend $\overline{\mathrm{Q}}_s$ and only satisfy CU1$_{143}$ (and not necessarily CU2$_{143}$), the local upper expectations $\overline{\mathrm{Q}}^1_s$ will coincide with the local upper expectations $\overline{\mathrm{Q}}^\uparrow_s$ on $\overline{\mathscr{L}}_\mathrm{b}(\mathscr{X})$, and so the global game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}}^1,\mathrm{V}}$ corresponding to $\overline{\mathrm{Q}}^1_\bullet$—defined in an analogous way as $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—will coincide with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. Hence, by Corollary 4.6.4$_{179}$ and the fact that $\overline{\mathrm{Q}}^\uparrow_s$ satisfies CU2$_{143}$ for all $s \in \mathscr{X}^*$, we have that, for all $(f, x_{1:k}) \in \overline{\mathscr{L}}(\mathscr{X}) \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}}^1,\mathrm{V}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}^\uparrow_{x_{1:k}}(f) = \lim_{c\to-\infty} \overline{\mathrm{Q}}^\uparrow_{x_{1:k}}(f^{\vee c}) = \lim_{c\to-\infty} \overline{\mathrm{Q}}^1_{x_{1:k}}(f^{\vee c})$$

As a result, $\overline{\mathrm{Q}}^1_{x_{1:k}}$ should indeed satisfy CU2$_{143}$ if one wishes to have that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}}^1,\mathrm{V}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}^1_{x_{1:k}}(f).$$

Now let us summarize our considerations above; if we are solely interested in the properties of the global game-theoretic upper expectation, and care little about compatibility with the local models, then the best—the most general—thing to do is to impose coherence and CU1$_{143}$ onto the local upper expectations. This approach is strictly more general than using SV-upper expectations; see Proposition 4.9.2(ii)$_{192}$ and Example 4.9.5$_\curvearrowleft$. If, on the other hand, we find it desirable to have (full) compatibility with the local models, then we should additionally impose CU2$_{143}$ onto the local models. This approach is strictly less general than using SV-upper expectations; see Proposition 4.9.2(i)$_{192}$ and Example 4.9.3$_{193}$. All things considered, we do not see what advantage could be gained from using SV-upper expectations compared to the other two options—that is, adopting coherence and CU1$_{143}$–CU2$_{143}$ (if one desires compatibility) or only adopting coherence and CU1$_{143}$ (if one wishes to remain as general as possible).

On top of this, the definition of an SV-upper expectation is, as far as we can tell, not based on a clear interpretation or argumentation. Our characterisation, on the other hand, starts from the widely encountered notion of coherence—which can be given a clear interpretation in terms of betting behaviour or sets of probabilities [recall Definition 2.6$_{32}$]—and uses a basic and intuitive continuity axiom [CU1$_{143}$] to go from $\mathscr{L}(\mathscr{X})$ to $\overline{\mathscr{L}}_\mathrm{b}(\mathscr{X})$. Our extension from $\overline{\mathscr{L}}_\mathrm{b}(\mathscr{X})$ to $\overline{\mathscr{L}}(\mathscr{X})$ using CU2$_{143}$ subsequently follows from a conservativity argument. Moreover, recall from Proposition 4.3.1$_{153}$ that these extended local upper expectations can also be seen as to result from extended local sets of acceptable gambles.

— Appendices —

4.A Proof of Theorem 4.3.6

Proof of Theorem 4.3.6$_{161}$. Fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. That

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) &= \inf\left\{\mathscr{M}(s) : \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\} \\ &\geq \inf\left\{\mathscr{M}(s) : \mathscr{M} \in \overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\} = \overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \\ &\geq \inf\left\{\mathscr{M}(s) : \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\} = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s),\end{aligned}$$

follows immediately from Proposition 4.3.5$_{159}$. So it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. Consider any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$. Let $\mathscr{M}'$ be the extended real process defined by $\mathscr{M}'(t) := \mathscr{M}(s)$ for all $t \not\sqsupset s$ and, for all situations that follow s, by the recursive relation

$$\mathscr{M}'(tx) := \begin{cases}\mathscr{M}(tx) & \text{if } \mathscr{M}'(t) < +\infty; \\ +\infty & \text{if } \mathscr{M}'(t) = +\infty,\end{cases} \quad \text{for all } t \sqsupseteq s \text{ and all } x \in \mathscr{X}.$$

Then it is clear that $\mathscr{M}'(t) \geq \mathscr{M}(t)$ for all $t \sqsupseteq s$, and therefore that $\liminf \mathscr{M}' \geq_s \liminf \mathscr{M} \geq_s f$. $\mathscr{M}'$ is also bounded below because $\mathscr{M}$ is bounded below. Moreover, to see that $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, note that $\mathscr{M}'(t\cdot) = \mathscr{M}'(t)$ for all $t \not\sqsupseteq s$ and thus by LE1$_{156}$ [which we can apply because $\mathscr{M}'$ is bounded below] that $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}'(t\cdot)) = \mathscr{M}'(t)$. On the other hand, for any $t \sqsupseteq s$, if $\mathscr{M}'(t) < +\infty$, then by the supermartingale character of $\mathscr{M}$ we infer that

$$\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}'(t\cdot)) = \overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}(t\cdot)) \leq \mathscr{M}(t) \leq \mathscr{M}'(t).$$

If $\mathscr{M}'(t) = +\infty$, then also $\mathscr{M}'(t\cdot) = +\infty$ and thus by LE1$_{156}$ $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}'(t\cdot)) = +\infty \leq \mathscr{M}'(t)$. Hence, $\overline{\mathrm{Q}}^{\uparrow}_t(\mathscr{M}'(t\cdot)) \leq \mathscr{M}'(t)$ for all $t \in \mathscr{X}^*$, and since $\mathscr{M}'$ is moreover bounded below, we conclude that indeed $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.

Now fix any $\epsilon > 0$ and let $\mathscr{G}$ be the extended betting process defined by

$$\mathscr{G}(x_{1:k}) := \begin{cases}\Delta\mathscr{M}'(x_{1:k}) - \epsilon 2^{-k} & \text{if } \mathscr{M}'(x_{1:k}) < +\infty; \\ -\epsilon 2^{-k} & \text{if } \mathscr{M}'(x_{1:k}) = +\infty,\end{cases} \quad \text{for all } x_{1:k} \in \mathscr{X}^*.$$

We first prove that

$$\mathscr{M}'(\Box) + \sum_{i=0}^{k-1} \mathscr{G}(x_{1:i})(x_{i+1}) = \mathscr{M}'(x_{1:k}) - \sum_{i=0}^{k-1} \epsilon 2^{-i} \text{ for all } x_{1:k} \in \mathscr{X}^*, \tag{4.15}$$

by using an induction argument on the length $k \in \mathbb{N}_0$ of the situations $x_{1:k}$. That this equality is true for the situation $x_{1:k} = \Box$ is trivial. Then suppose that it holds for all situations $x_{1:k} \in \mathscr{X}^k$ of a certain length $k \in \mathbb{N}_0$. Consider any situation $x_{1:k+1} \in \mathscr{X}^{k+1}$

of length $k+1$. Then we have that

$$\begin{aligned}\mathscr{M}'(\square) + \sum_{i=0}^{k} \mathscr{G}(x_{1:i})(x_{i+1}) &= \mathscr{M}'(\square) + \sum_{i=0}^{k-1} \mathscr{G}(x_{1:i})(x_{i+1}) + \mathscr{G}(x_{1:k})(x_{k+1}) \\ &= \mathscr{M}'(x_{1:k}) - \sum_{i=0}^{k-1} \epsilon 2^{-i} + \mathscr{G}(x_{1:k})(x_{k+1}), \end{aligned} \tag{4.16}$$

where the last equality follows from the induction hypothesis. If $\mathscr{M}'(x_{1:k}) < +\infty$, then we have that $\mathscr{G}(x_{1:k})(x_{k+1}) = \mathscr{M}'(x_{1:k+1}) - \mathscr{M}'(x_{1:k}) - \epsilon 2^{-k}$, and therefore that

$$\begin{aligned}\mathscr{M}'(\square) + \sum_{i=0}^{k} \mathscr{G}(x_{1:i})(x_{i+1}) &= \mathscr{M}'(x_{1:k}) - \sum_{i=0}^{k-1} \epsilon 2^{-i} + \mathscr{M}'(x_{1:k+1}) - \mathscr{M}'(x_{1:k}) - \epsilon 2^{-k} \\ &= \mathscr{M}'(x_{1:k+1}) - \sum_{i=0}^{k} \epsilon 2^{-i},\end{aligned}$$

where we used the fact that $\mathscr{M}'(x_{1:k}) \in \mathbb{R}$ because $\mathscr{M}'(x_{1:k}) < +\infty$ and $\mathscr{M}'$ is bounded below. On the other hand, if $\mathscr{M}'(x_{1:k}) = +\infty$, then we also have that $\mathscr{M}'(x_{1:k+1}) = +\infty$ in the case that $x_{1:k} \sqsupseteq s$ simply due to the definition of $\mathscr{M}'$. The definition of $\mathscr{M}'$ also implies that $\mathscr{M}'(x_{1:k+1}) = +\infty$ in the case that $x_{1:k} \not\sqsupseteq s$, because then $x_{1:k+1} \not\sqsupseteq s$ and so $\mathscr{M}'(x_{1:k+1}) = \mathscr{M}'(s) = \mathscr{M}'(x_{1:k}) = +\infty$. So, in all cases, we have that $\mathscr{M}'(x_{1:k+1}) = +\infty$ if $\mathscr{M}'(x_{1:k}) = +\infty$. Since $\mathscr{G}(x_{1:k})(x_{k+1}) = -\epsilon 2^{-k}$ by the definition of $\mathscr{G}$, Eq. (4.16) implies that

$$\begin{aligned}\mathscr{M}'(\square) + \sum_{i=0}^{k} \mathscr{G}(x_{1:i})(x_{i+1}) &= \mathscr{M}'(x_{1:k}) - \sum_{i=0}^{k-1} \epsilon 2^{-i} + \mathscr{M}'(x_{1:k+1}) - \epsilon 2^{-k} \\ &= \mathscr{M}'(x_{1:k+1}) - \sum_{i=0}^{k} \epsilon 2^{-i}.\end{aligned}$$

So, for any $x_{1:k+1} \in \mathscr{X}^{k+1}$, we have that $\mathscr{M}'(\square) + \sum_{i=0}^{k} \mathscr{G}(x_{1:i})(x_{i+1}) = \mathscr{M}'(x_{1:k+1}) - \sum_{i=0}^{k} \epsilon 2^{-i}$, which proves the induction step and therefore establishes Eq. (4.15)$_{\curvearrowleft}$.

As a result of Eq. (4.15)$_{\curvearrowleft}$ and the fact that $\mathscr{M}'$ superhedges f on $\Gamma(s)$, we find that, for any $\omega \in \Gamma(s)$,

$$\begin{aligned}\liminf_{k\to+\infty} \Big[\mathscr{M}'(\square) + \mathscr{C}^{\mathscr{G}}(\omega^k)\Big] &= \liminf_{k\to+\infty} \Big[\mathscr{M}'(\square) + \sum_{i=0}^{k-1} \mathscr{G}(\omega_{1:i})(\omega_{i+1})\Big] \\ &= \liminf_{k\to+\infty} \Big[\mathscr{M}'(\omega_{1:k}) - \sum_{i=0}^{k-1} \epsilon 2^{-i}\Big] \\ &\geq \liminf \mathscr{M}'(\omega) - 2\epsilon \geq f(\omega) - 2\epsilon.\end{aligned}$$

So the extended real process $\mathscr{M}^*$ defined by $\mathscr{M}^*(t) := \mathscr{M}'(\square) + 2\epsilon + \mathscr{C}^{\mathscr{G}}(t)$ for all $t \in \mathscr{X}^*$ superhedges f on $\Gamma(s)$.

We furthermore show that $-\mathscr{G}(x_{1:k}) \in \mathscr{A}^{\uparrow}_{x_{1:k}}$ for all $x_{1:k} \in \mathscr{X}^*$, and therefore that $\mathscr{M}^*$ is an extended real supermartingale according to $\mathscr{A}_{\bullet}$. For any $x_{1:k} \in \mathscr{X}^*$, we either have that $\mathscr{M}'(x_{1:k}) = +\infty$ or that $\mathscr{M}'(x_{1:k}) < +\infty$. If $\mathscr{M}'(x_{1:k}) = +\infty$, then $\mathscr{G}(x_{1:k}) = -\epsilon 2^{-k}$ by definition. Since $\mathscr{L}_{\geq}(\mathscr{X}) \subseteq \mathscr{A}_{x_{1:k}} \subseteq \mathscr{A}^{\uparrow}_{x_{1:k}}$ due to the coherence [D1$_{27}$] of $\mathscr{A}_{x_{1:k}} = \mathscr{A}^{\uparrow}_{x_{1:k}} \cap \mathscr{L}(\mathscr{X})$, this implies that $-\mathscr{G}(x_{1:k}) \in \mathscr{A}^{\uparrow}_{x_{1:k}}$. On the other hand, if $\mathscr{M}'(x_{1:k}) < +\infty$, then $\mathscr{G}(x_{1:k}) = \Delta\mathscr{M}'(x_{1:k}) - \epsilon 2^{-k}$ by definition. Since $\mathscr{M}'$ is bounded below, we also have that $-\mathscr{M}'(x_{1:k})$ is real and $\mathscr{M}'(x_{1:k}\cdot)$ is bounded below, and so we infer that

$$\begin{aligned}\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}\big(\mathscr{G}(x_{1:k})\big) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}\big(\Delta\mathscr{M}'(x_{1:k}) - \epsilon 2^{-k}\big) &= \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}\big(\mathscr{M}'(x_{1:k}\cdot) - \mathscr{M}'(x_{1:k}) - \epsilon 2^{-k}\big)\\ &\overset{\text{LE5}_{156}}{=} \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{M}'(x_{1:k}\cdot)) - \mathscr{M}'(x_{1:k}) - \epsilon 2^{-k}\\ &\leq \mathscr{M}'(x_{1:k}) - \mathscr{M}'(x_{1:k}) - \epsilon 2^{-k} = -\epsilon 2^{-k},\end{aligned}$$

where the inequality follows from the supermartingale character of $\mathscr{M}'$. Since $\mathscr{A}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ agree, Proposition 4.3.1$_{153}$ implies that $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}} = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:k},\mathscr{A}}$. So, since $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k}}(\mathscr{G}(x_{1:k})) \leq -\epsilon 2^{-k}$, the definition of $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:k},\mathscr{A}}$ implies that there is an $\alpha \in \mathbb{R}_{<}$ such that $(\alpha - \mathscr{G}(x_{1:k})) \in \mathscr{A}^{\uparrow}_{x_{1:k}}$. Using Lemma 4.2.5$_{146}$, we arrive at the fact that $-\mathscr{G}(x_{1:k}) \in \mathscr{A}^{\uparrow}_{x_{1:k}}$. Hence, for any $x_{1:k} \in \mathscr{X}^*$, we have that $-\mathscr{G}(x_{1:k}) \in \mathscr{A}^{\uparrow}_{x_{1:k}}$, which implies that the extended real process $\mathscr{M}^* = \mathscr{M}'(\square) + 2\epsilon + \mathscr{C}^{\mathscr{G}}$ is an extended real supermartingale according to $\mathscr{A}_\bullet$.

It remains to check that $\mathscr{M}^*$ is bounded below. This can be easily deduced by recalling Eq. (4.15)$_{197}$ which implies that, for any $x_{1:k} \in \mathscr{X}^*$,

$$\mathscr{M}^*(x_{1:k}) = \mathscr{M}'(\square) + 2\epsilon + \mathscr{C}^{\mathscr{G}}(x_{1:k}) = \mathscr{M}'(\square) + 2\epsilon + \sum_{i=0}^{k-1}\mathscr{G}(x_{1:i})(x_{i+1}) \tag{4.17}$$

$$= \mathscr{M}'(x_{1:k}) + 2\epsilon - \sum_{i=0}^{k-1}\epsilon 2^{-i}, \tag{4.18}$$

and therefore $\mathscr{M}^*(x_{1:k}) \geq \mathscr{M}'(x_{1:k})$. Then, since $\mathscr{M}'$ is bounded below, we find that $\mathscr{M}^*$ is also bounded below. Together with the fact that it is an extended real supermartingale according to $\mathscr{A}_\bullet$, we obtain that $\mathscr{M}^* \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$. Moreover recalling that $\liminf \mathscr{M}^* \geq_s f$, we have by the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and Eq. (4.17) that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}^*(s) = \mathscr{M}'(s) + 2\epsilon - \sum_{i=0}^{|s|-1}\epsilon 2^{-i} \leq \mathscr{M}'(s) + 2\epsilon.$$

Since this holds for any $\epsilon > 0$, we infer that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}'(s)$. Furthermore, recall that $\mathscr{M}'(s) = \mathscr{M}(s)$ by definition of $\mathscr{M}'$, so we arrive at the fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \mathscr{M}(s)$. Since this holds for any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\liminf \mathscr{M} \geq_s f$, we infer that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}(f|s) \leq \inf\left\{\mathscr{M}(s) \colon \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq_s f\right\} = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s),$$

which concludes the proof. □

4.B Proofs of the results in Section 4.5

In the following proofs, we will frequently use the notion of a (tree) **cut**; a collection $U \subset \mathscr{X}^*$ of pairwise incomparable situations. We call a cut U **complete** if for all $\omega \in \Omega$ there is some $u \in U$ such that $\omega \in \Gamma(u)$. Otherwise, we call U **partial**. For any two cuts U and V, we will write $U \sqsubset V$

if $(\forall v \in V)(\exists u \in U)\, u \sqsubset v$, and similarly for $U \sqsubseteq V$. Furthermore, we write that $U \sqsupset V$ and $U \sqsupseteq V$ if, respectively, $V \sqsubset U$ and $V \sqsubseteq U$. Analogously to what we did before for situations, we say that a path $\omega \in \Omega$ goes through a cut U when there is some $n \in \mathbb{N}_0$ such that $\omega^n \in U$. We will also use the following sets:

$$
\begin{aligned}
[U,V] &:= \{s \in \mathscr{X}^* : (\exists u \in U)(\exists v \in V) u \sqsubseteq s \sqsubseteq v\},\\
[U,V) &:= \{s \in \mathscr{X}^* : (\exists u \in U)(\exists v \in V) u \sqsubseteq s \sqsubset v\},\\
(U,V] &:= \{s \in \mathscr{X}^* : (\exists u \in U)(\exists v \in V) u \sqsubset s \sqsubseteq v\},\\
(U,V) &:= \{s \in \mathscr{X}^* : (\exists u \in U)(\exists v \in V) u \sqsubset s \sqsubset v\}.
\end{aligned}
$$

We will also use the simpler notation s to denote the cut $\{s\}$ that consists of the single situation $s \in \mathscr{X}^*$. In specific, all the previous notations are meaningful if we replace the cuts U or V by a situation.

Proof of Proposition 4.5.1$_{172}$. Let $s \in \mathscr{X}^*$ be any fixed situation where $\mathscr{M}(s)$ is real. We can assume that $\mathscr{M}$ is non-negative and that $\mathscr{M}(s) = 1$ without loss of generality. Indeed, because the original supermartingale is bounded below and real in s, we can obtain such a process by translating and scaling—by adding a positive constant and then multiplying the supermartingale by a positive real—the originally considered supermartingale in an appropriate way. This process will then again be a (bounded below) supermartingale because the local models $\overline{\mathrm{Q}}_t^{\uparrow}$ satisfy LE5$_{156}$ and LE3$_{156}$. Moreover, the new supermartingale will have the same convergence character as the original one.

To start, fix any couple of rational numbers $0 < a < b$ and consider the following recursively constructed sequences of cuts $(U_k^{a,b})_{k\in\mathbb{N}}$ and $(V_k^{a,b})_{k\in\mathbb{N}}$. Let

$$V_1^{a,b} := \{t \sqsupseteq s\colon\ \mathscr{M}(t) < a \text{ and } (\forall t' \in [s,t))\ \mathscr{M}(t') \geq a\},$$

and, for $k \in \mathbb{N}$,

1. let $U_k^{a,b} := \{t \in \mathscr{X}^*\colon V_k^{a,b} \sqsubset t : \mathscr{M}(t) > b \text{ and } (\forall t' \in (V_k^{a,b}, t))\ \mathscr{M}(t') \leq b\}$;
2. let $V_{k+1}^{a,b} := \{t \in \mathscr{X}^*\colon U_k^{a,b} \sqsubset t,\ \mathscr{M}(t) < a \text{ and } (\forall t' \in (U_k^{a,b}, t))\ \mathscr{M}(t') \geq a\}$.

Note that all $U_k^{a,b}$ and all $V_k^{a,b}$ are indeed (partial or complete) cuts.

Next, consider the extended real process $\mathscr{M}^{a,b}$ defined by $\mathscr{M}^{a,b}(t) := \mathscr{M}(s)$ for all $t \not\sqsupseteq s$ and by

$$\mathscr{M}^{a,b}(tx) := \begin{cases} \mathscr{M}^{a,b}(t) + [\mathscr{M}(tx) - \mathscr{M}(t)] & \text{if } V_k^{a,b} \sqsubseteq t \text{ and } U_k^{a,b} \not\sqsubseteq t \text{ for some } k \in \mathbb{N};\\ \mathscr{M}^{a,b}(t) & \text{otherwise,} \end{cases} \tag{4.19}$$

for all $tx \sqsupset s$ with $x \in \mathscr{X}$. We prove that $\mathscr{M}^{a,b}$ is a non-negative supermartingale that converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ such that

$$\liminf \mathscr{M}(\omega) < a < b < \limsup \mathscr{M}(\omega). \tag{4.20}$$

For any situation t and for any $k \in \mathbb{N}$, when $U_k^{a,b} \sqsubset t$, we denote by u_k^t the (necessarily unique) situation in $U_k^{a,b}$ such that $u_k^t \sqsubset t$. Similarly, for any $k \in \mathbb{N}$, when $V_k^{a,b} \sqsubset t$, we denote by v_k^t the (necessarily unique) situation in $V_k^{a,b}$ such that $v_k^t \sqsubset t$. Note that $V_1^{a,b} \sqsubset U_1^{a,b} \sqsubset V_2^{a,b} \sqsubset \cdots \sqsubset V_n^{a,b} \sqsubset U_n^{a,b} \sqsubset \cdots$. Hence, for any situation t we can distinguish the following three cases:

- The first case is that $V_1^{a,b} \not\sqsubset t$. Then we have that

$$\mathscr{M}^{a,b}(t) = \mathscr{M}^{a,b}(s) = \mathscr{M}(s) = 1. \tag{4.21}$$

- The second case is that $V_k^{a,b} \sqsubset t$ and $U_k^{a,b} \not\sqsubset t$ for some $k \in \mathbb{N}$. Then by applying Eq. (4.19)$_\leftarrow$ for each subsequent step, observing that $\mathscr{M}^{a,b}(s)$ is real [because it is equal to 1], and cancelling out the intermediate terms which is possible because $\mathscr{M}$ is real for any situation $t' \in \mathscr{X}^*$ such that $V_{k'}^{a,b} \sqsubseteq t'$ and $U_{k'}^{a,b} \not\sqsubseteq t'$ for some $k' \in \mathbb{N}$ [this follows readily from the definition of the cuts $V_{k'}^{a,b}$ and $U_{k'}^{a,b}$], we have that

$$\mathscr{M}^{a,b}(t) - \mathscr{M}^{a,b}(s) = \sum_{\ell=1}^{k-1} \left[\mathscr{M}(u_\ell^t) - \mathscr{M}(v_\ell^t)\right] + \mathscr{M}(t) - \mathscr{M}(v_k^t). \tag{4.22}$$

- The third case is that $U_k^{a,b} \sqsubset t$ and $V_{k+1}^{a,b} \not\sqsubset t$ for some $k \in \mathbb{N}$. Then we have that

$$\mathscr{M}^{a,b}(t) - \mathscr{M}^{a,b}(s) = \sum_{\ell=1}^{k} [\mathscr{M}(u_\ell^t) - \mathscr{M}(v_\ell^t)], \tag{4.23}$$

  where, again, we used the fact that $\mathscr{M}^{a,b}(s)$ is real, and that $\mathscr{M}$ is real for any situation $t' \in \mathscr{X}^*$ such that $V_{k'}^{a,b} \sqsubseteq t'$ and $U_{k'}^{a,b} \not\sqsubseteq t'$ for some $k' \in \mathbb{N}$.

That $\mathscr{M}^{a,b}(t)$ is non-negative, is trivially satisfied in the first case. To see that this is also true for the third case, observe that $0 < b < \mathscr{M}(u_\ell^t)$ and $0 \leq \mathscr{M}(v_\ell^t) < a$ for all $\ell \in \{1, \ldots, k\}$. This implies that $\mathscr{M}(u_\ell^t) - \mathscr{M}(v_\ell^t) > b - a > 0$ for all $\ell \in \{1, \ldots, k\}$ and therefore directly that $\mathscr{M}^{a,b}(t)$ is non-negative because of Eq. (4.23) and the fact that $\mathscr{M}^{a,b}(s) = \mathscr{M}(s) = 1$. In the second case, it follows from Eqs. (4.22) and (4.23) and the fact that $\mathscr{M}^{a,b}(s) = \mathscr{M}(s) = 1$, that

$$\mathscr{M}^{a,b}(t) = \mathscr{M}^{a,b}(v_k^t) + \mathscr{M}(t) - \mathscr{M}(v_k^t). \tag{4.24}$$

We prove by induction that $\mathscr{M}^{a,b}(v_\ell^t) \geq \mathscr{M}(v_\ell^t)$ for all $\ell \in \{1, \ldots, k\}$, and therefore, by Eq. (4.24) and because $\mathscr{M}$ is non-negative, that $\mathscr{M}^{a,b}(t)$ is non-negative.

If $\ell = 1$, then either $v_1^t = s$ or $v_1^t \neq s$. If $v_1^t = s$, then $\mathscr{M}^{a,b}(v_1^t) = \mathscr{M}^{a,b}(s) = \mathscr{M}(s) = \mathscr{M}(v_1^t)$. If $v_1^t \neq s$, we have, by the definition of $V_1^{a,b}$, that $\mathscr{M}(v_1^t) < a$ and $a \leq \mathscr{M}(s) = \mathscr{M}^{a,b}(s) = \mathscr{M}^{a,b}(v_1^t)$. Hence, in both cases, we have that $\mathscr{M}^{a,b}(v_1^t) \geq \mathscr{M}(v_1^t)$. Now suppose that $\mathscr{M}^{a,b}(v_\ell^t) \geq \mathscr{M}(v_\ell^t)$ for some $\ell \in \{1, \ldots, k-1\}$. Then, again using the fact that $\mathscr{M}$ is real for any situation $t' \in \mathscr{X}^*$ such that $V_{k'}^{a,b} \sqsubseteq t'$ and $U_{k'}^{a,b} \not\sqsubseteq t'$ for some $k' \in \mathbb{N}$, it follows from Eq. (4.19)$_\leftarrow$ that

$$\mathscr{M}^{a,b}(v_{\ell+1}^t) = \mathscr{M}^{a,b}(v_\ell^t) + [\mathscr{M}(u_\ell^t) - \mathscr{M}(v_\ell^t)] \geq \mathscr{M}(u_\ell^t) > b > a > \mathscr{M}(v_{\ell+1}^t),$$

which concludes our induction step. So indeed $\mathscr{M}^{a,b}(v_\ell^t) \geq \mathscr{M}(v_\ell^t)$ for all $\ell \in \{1, \ldots, k\}$.

Next, we show that $\overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t\,\cdot)) \leq \mathscr{M}^{a,b}(t)$ for all $t \in \mathscr{X}^*$, and hence, that $\mathscr{M}$ is a non-negative supermartingale. Consider any $t \in \mathscr{X}^*$. If $V_k^{a,b} \sqsubseteq t$ and $U_k^{a,b} \not\sqsubseteq t$ for some $k \in \mathbb{N}$, it follows from Eq. (4.19)$_{200}$ that

$$\overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t\,\cdot)) = \overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t) + \mathscr{M}(t\,\cdot) - \mathscr{M}(t)) \overset{\text{LE5}_{156}}{=} \overline{Q}_t^{\uparrow}(\mathscr{M}(t\,\cdot)) + \mathscr{M}^{a,b}(t) - \mathscr{M}(t) \leq \mathscr{M}^{a,b}(t),$$

where we were allowed to use LE5$_{156}$ because $\mathscr{M}(t) < +\infty$ as a consequence of the fact that $V_k^{a,b} \sqsubseteq t$ and $U_k^{a,b} \not\sqsubseteq t$ and the definitions of $V_k^{a,b}$ and $U_k^{a,b}$, and where the last step follows from $\mathscr{M}$ being a supermartingale and $0 \leq \mathscr{M}(t) < +\infty$. Otherwise, if, for all $k \in \mathbb{N}$, we have that $V_k^{a,b} \not\sqsubseteq t$ or $U_k^{a,b} \sqsubseteq t$, then $\overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t\,\cdot)) = \overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t)) = \mathscr{M}^{a,b}(t)$, where we have used LE1$_{156}$ for the last inequality. Hence, we have that $\overline{Q}_t^{\uparrow}(\mathscr{M}^{a,b}(t\,\cdot)) \leq \mathscr{M}^{a,b}(t)$ for all $t \in \mathscr{X}^*$, and we can therefore infer that $\mathscr{M}^{a,b}$ is indeed a non-negative supermartingale in $\overline{\mathbb{M}}_{\text{eb}}(\overline{Q}_\bullet)$.

Let us now show that $\mathscr{M}^{a,b}$ converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ for which Eq. (4.20)$_{200}$ holds. Consider such a path ω. First, it follows from $\liminf \mathscr{M}(\omega) < a$ that there exists some $n_1 \in \mathbb{N}_0$ such that $\omega^{n_1} \sqsupseteq s$ and $\mathscr{M}(\omega^{n_1}) < a$. Take the first such n_1. Then it follows from the definition of $V_1^{a,b}$ that $\omega^{n_1} \in V_1^{a,b}$. Next, it follows from $\limsup \mathscr{M}(\omega) > b$ that there exists some $m_1 \in \mathbb{N}_0$ for which $m_1 > n_1$ and $\mathscr{M}(\omega^{m_1}) > b$. Take the first such m_1. Then it follows from the definition of $U_1^{a,b}$ that $\omega^{m_1} \in U_1^{a,b}$. Repeating similar arguments over and over again allows us to conclude that ω goes through all the cuts $V_1^{a,b} \sqsubset U_1^{a,b} \sqsubset V_2^{a,b} \sqsubset \cdots \sqsubset V_k^{a,b} \sqsubset U_k^{a,b} \sqsubset \cdots$. For all $n > n_1$, let $k_n \in \mathbb{N}$ be the index such that $V_{k_n}^{a,b} \sqsubset \omega^n$ and $V_{k_n+1}^{a,b} \not\sqsubset \omega^n$. Note that $k_n \to +\infty$ for $n \to +\infty$. Now, if $V_{k_n}^{a,b} \sqsubset \omega^n$ and $U_{k_n}^{a,b} \not\sqsubset \omega^n$ for some $n > n_1$, then we use Eq. (4.22)$_\curvearrowleft$ to see that $\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)$ is bounded below by $(k_n - 1)(b - a) + \mathscr{M}(\omega^n) - a \geq (k_n - 1)(b - a) - a$ [$\mathscr{M}$ is non-negative]. If on the other hand $U_{k_n}^{a,b} \sqsubset \omega^n$ and $V_{k_n+1}^{a,b} \not\sqsubset \omega^n$ for some $n > n_1$, then Eq. (4.23)$_\curvearrowleft$ implies that $\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)$ is bounded below by $k_n(b - a) \geq (k_n - 1)(b - a) - a$. All together, $\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)$ is bounded below by $(k_n - 1)(b - a) - a$ for all $n > n_1$, which implies that

$$\lim_{n\to+\infty} \big(\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)\big) \geq \lim_{n\to+\infty} (k_n - 1)(b - a) - a = +\infty,$$

because $\lim_{n\to+\infty} k_n = +\infty$ and $(b-a) > 0$. This also implies that $\lim_{n\to+\infty} \mathscr{M}^{a,b}(\omega^n) = +\infty$ because $\mathscr{M}^{a,b}(s) = \mathscr{M}(s) = 1$.

We now use the countable set of rational couples $K := \{(a, b) \in \mathbb{Q}^2 : 0 < a < b\}$ to define the process $\mathscr{M}^*$:

$$\mathscr{M}^* := \sum_{(a,b)\in K} w^{a,b} \mathscr{M}^{a,b},$$

with coefficients $w^{a,b} > 0$ that sum to 1. Hence, $\mathscr{M}^*$ is a countable convex combination of the non-negative supermartingales $\mathscr{M}^{a,b} \in \overline{\mathbb{M}}_{\text{eb}}(\overline{Q}_\bullet)$. By Lemma 4.4.2$_{163}$, $\mathscr{M}^*$ is then a non-negative supermartingale in $\overline{\mathbb{M}}_{\text{eb}}(\overline{Q}_\bullet)$. It is moreover clear that $\mathscr{M}^*(s) = \mathscr{M}(s) = 1$, implying, together with its non-negativity, that $\mathscr{M}^*$ is a s-test supermartingale. We now show that $\mathscr{M}^*$ converges in the desired way as described by the proposition.

If $\mathscr{M}$ does not converge to an extended real number on some path $\omega \in \Gamma(s)$, then $\liminf \mathscr{M}(\omega) < \limsup \mathscr{M}(\omega)$. Since $\liminf \mathscr{M}(\omega) \geq \inf_{t\in\mathscr{X}^*} \mathscr{M}(t) \geq 0$, there is at least one couple $(a', b') \in K$ such that $\liminf \mathscr{M}(\omega) < a' < b' < \limsup \mathscr{M}(\omega)$, and as a consequence $\mathscr{M}^{a',b'}$ converges to $+\infty$ on ω. Then also $\lim w^{a',b'} \mathscr{M}^{a',b'}(\omega) = +\infty$ since $w^{a',b'} > 0$. For all other couples $(a, b) \in K \setminus \{(a', b')\}$, we have that $w^{a,b}\mathscr{M}^{a,b}$ is non-negative, so $\mathscr{M}^*$ indeed converges to $+\infty$ on ω.

Finally, we show that $\mathscr{M}^*$ converges in $\overline{\mathbb{R}}$ on every path $\omega \in \Gamma(s)$ where $\mathscr{M}$ converges to a real number. Fix any such $\omega \in \Gamma(s)$. Then for any $\epsilon \in \mathbb{R}_>$, there is an $n^* \in \mathbb{N}_0$ such that, for all $\ell \geq n \geq n^*$, $|\mathscr{M}(\omega^\ell) - \mathscr{M}(\omega^n)| \leq \epsilon$ and therefore $\mathscr{M}(\omega^\ell) - \mathscr{M}(\omega^n) \geq -\epsilon$. Now fix any couple of rational numbers $0 < a < b$ and, for any $i \in \mathbb{N}$, let v_i^ω and u_i^ω be the situations in respectively $V_i^{a,b}$ and $U_i^{a,b}$ where ω passes through [if it passes through these cuts]. We prove that $\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) \geq -2\epsilon$ for any $\ell \geq n \geq n^*$. To do so, let us distinguish the following four cases:

- $V_1^{a,b} \not\sqsubset \omega^n$ or $U_k^{a,b} \sqsubset \omega^n \not\sqsupset V_{k+1}^{a,b}$ for some $k \in \mathbb{N}$ and moreover, $V_1^{a,b} \not\sqsubset \omega^\ell$ or $U_{k'}^{a,b} \sqsubset \omega^\ell \not\sqsupset V_{k'+1}^{a,b}$ for some $k' \in \mathbb{N}$. Using Eqs. (4.21) and (4.23) for both ω^n and ω^ℓ, we get that

$$\begin{aligned}\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) &= [\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(s)] - [\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)] \\ &= \sum_{i=1}^{k'} [\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega)] - \sum_{i=1}^{k} [\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega)],\end{aligned}$$

  where we assume $k' = 0$ if $V_1^{a,b} \not\sqsubset \omega^\ell$ and $k = 0$ if $V_1^{a,b} \not\sqsubset \omega^n$. Since $k' \geq k$ [because $n \leq \ell$ and therefore $\omega^n \sqsubseteq \omega^\ell$] and $\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega) > b - a > 0$ for all $i \in \mathbb{N}$, we have that $\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) \geq 0 > -2\epsilon$ [where we also implicitly use the convention that $+\infty - \infty = +\infty$].

- $V_1^{a,b} \not\sqsubset \omega^n$ or $U_k^{a,b} \sqsubset \omega^n \not\sqsupset V_{k+1}^{a,b}$ for some $k \in \mathbb{N}$ and moreover, $V_{k'}^{a,b} \sqsubset \omega^\ell \not\sqsupset U_{k'}^{a,b}$ for some $k' \in \mathbb{N}$. Using Eqs. (4.21) and (4.23) for ω^n and Eq. (4.22)$_{201}$ for ω^ℓ, we find that

$$\begin{aligned}\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) &= [\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(s)] - [\mathscr{M}^{a,b}(\omega^n) - \mathscr{M}^{a,b}(s)] \\ &= \Big[\sum_{i=1}^{k'-1} [\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega)] + \mathscr{M}(\omega^\ell) - \mathscr{M}(v_{k'}^\omega) \Big] \\ &\qquad - \sum_{i=1}^{k} [\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega)], \qquad (4.25)\end{aligned}$$

  where we assume $k = 0$ if $V_1^{a,b} \not\sqsubset \omega^n$. Note that $k' \geq k + 1$, because $k' \geq k$ [since $\omega^n \sqsubseteq \omega^\ell$] and $k' = k$ is impossible. Indeed, if $k = 0$, $k' = k$ is impossible because $k' \in \mathbb{N}$. Otherwise, if $k > 0$, $k' = k$ would imply that $U_{k'}^{a,b} = U_k^{a,b} \sqsubset \omega^n \sqsubseteq \omega^\ell$, contradicting the assumption that $\omega^\ell \not\sqsupset U_{k'}^{a,b}$. Hence, taking into account that $\mathscr{M}(u_i^\omega) - \mathscr{M}(v_i^\omega) > b - a > 0$ for all $i \in \mathbb{N}$, we infer from Eq. (4.25) that $\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) \geq \mathscr{M}(\omega^\ell) - \mathscr{M}(v_{k'}^\omega)$ [again, also using the convention that $+\infty - \infty = +\infty$]. Finally, observe that $\omega^{n^*} \sqsubseteq \omega^n \sqsubseteq v_{k+1}^\omega \sqsubseteq v_{k'}^\omega \sqsubset \omega^\ell$—the situation v_{k+1}^ω exists because $V_{k+1}^{a,b} \sqsubseteq V_{k'}^{a,b} \sqsubset \omega^\ell$—and therefore, recalling how n^* was chosen,

$$\mathscr{M}^{a,b}(\omega^\ell) - \mathscr{M}^{a,b}(\omega^n) \geq \mathscr{M}(\omega^\ell) - \mathscr{M}(v_{k'}^\omega) \geq -\epsilon \geq -2\epsilon.$$

- $V_k^{a,b} \sqsubset \omega^n \not\sqsupset U_k^{a,b}$ for some $k \in \mathbb{N}$ and $U_{k'}^{a,b} \sqsubset \omega^\ell \not\sqsupset V_{k'+1}^{a,b}$ for some $k' \in \mathbb{N}$ [we automatically have that $V_1^{a,b} \sqsubset \omega^\ell$ because $V_1^{a,b} \sqsubseteq V_k^{a,b} \sqsubset \omega^n \sqsubseteq \omega^\ell$]. Using Eq. $(4.22)_{201}$ for ω^n and Eq. $(4.23)_{201}$ for ω^ℓ, we get that

$$\begin{aligned}
&\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \\
&\quad= \sum_{i=1}^{k'}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \Big[\sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^n) - \mathcal{M}(v_k^\omega)\Big] \\
&\quad= \sum_{i=1}^{k'}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \Big[\sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^n)\Big] + \mathcal{M}(v_k^\omega) \\
&\quad= \sum_{i=1}^{k'}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \mathcal{M}(\omega^n) + \mathcal{M}(v_k^\omega),
\end{aligned}$$

where the second step follows because $\mathcal{M}(v_k^\omega)$ is real [as a consequence of the definition of $V_k^{a,b}$] and the third step follows because $\sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] \geq 0$ [since all $\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)$ are positive] and $\mathcal{M}(\omega^n) \geq 0$ [because $\mathcal{M}$ is non-negative]. Using the fact that $k' \geq k$ and that all $\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)$ are positive, the equation above implies that

$$\begin{aligned}
\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) &\geq \sum_{i=k}^{k'}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \mathcal{M}(\omega^n) + \mathcal{M}(v_k^\omega) \\
&= \sum_{i=k+1}^{k'}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(u_k^\omega) - \mathcal{M}(\omega^n) \\
&\geq \mathcal{M}(u_k^\omega) - \mathcal{M}(\omega^n),
\end{aligned}$$

where the equality follows from the fact that $\mathcal{M}(v_k^\omega)$ is real-valued and the last inequality follows once more from the positivity of all $\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)$. Then since $\omega^{n^*} \sqsubseteq \omega^n \sqsubseteq u_k^\omega$—the situation u_k^ω exists because $U_k^{a,b} \sqsubseteq U_{k'}^{a,b} \sqsubset \omega^\ell$—we infer from our assumptions about n^* that

$$\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \geq \mathcal{M}(u_k^\omega) - \mathcal{M}(\omega^n) \geq -\epsilon \geq -2\epsilon.$$

- $V_k^{a,b} \sqsubset \omega^n \not\sqsupset U_k^{a,b}$ for some $k \in \mathbb{N}$ and $V_{k'}^{a,b} \sqsubset \omega^\ell \not\sqsupset U_{k'}^{a,b}$ for some $k' \in \mathbb{N}$. Using Eq. $(4.22)_{201}$ for both ω^n and ω^ℓ, we find that

$$\begin{aligned}
&\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \\
&\quad= \Big[\sum_{i=1}^{k'-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^\ell) - \mathcal{M}(v_{k'}^\omega)\Big] \\
&\qquad\qquad - \Big[\sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^n) - \mathcal{M}(v_k^\omega)\Big] \\
&\quad= \sum_{i=1}^{k'-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^\ell) - \mathcal{M}(v_{k'}^\omega) \\
&\qquad\qquad - \sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] - \mathcal{M}(\omega^n) + \mathcal{M}(v_k^\omega) \\
&\quad\geq \sum_{i=k}^{k'-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^\ell) - \mathcal{M}(v_{k'}^\omega) - \mathcal{M}(\omega^n) + \mathcal{M}(v_k^\omega)
\end{aligned}$$

where the two last steps follow in a similar way as before; first using the real-valuedness of $\mathcal{M}(v_k^\omega)$ and the non-negativity of both $\mathcal{M}(\omega^n)$ and $\sum_{i=1}^{k-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)]$, and then using the fact that $k' \geq k$ and that all $\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)$ are positive. If $k' = k$, and therefore $\mathcal{M}(v_{k'}^\omega) = \mathcal{M}(v_k^\omega) \in \mathbb{R}$, it follows from the expression above that $\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \geq \mathcal{M}(\omega^\ell) - \mathcal{M}(\omega^n) \geq -\epsilon \geq -2\epsilon$. Otherwise, if $k' > k$, then we use the real-valuedness of $\mathcal{M}(v_k^\omega)$ to deduce from the expression above that

$$\begin{aligned}\mathcal{M}^{a,b}(\omega^\ell)-&\mathcal{M}^{a,b}(\omega^n)\\ &\geq \sum_{i=k+1}^{k'-1}[\mathcal{M}(u_i^\omega) - \mathcal{M}(v_i^\omega)] + \mathcal{M}(\omega^\ell) - \mathcal{M}(v_{k'}^\omega) + \mathcal{M}(u_k^\omega) - \mathcal{M}(\omega^n)\\ &\geq \mathcal{M}(\omega^\ell) - \mathcal{M}(v_{k'}^\omega) + \mathcal{M}(u_k^\omega) - \mathcal{M}(\omega^n) \geq -2\epsilon,\end{aligned}$$

where the last inequality follows from our assumptions about n^* and the fact that $\omega^{n^*} \sqsubseteq \omega^n \sqsubseteq u_k^\omega \sqsubset v_{k'}^\omega \sqsubset \omega^\ell$.

Hence, we conclude that for any $\epsilon \in \mathbb{R}_>$, there is an $n^* \in \mathbb{N}_0$ such that $\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \geq -2\epsilon$ for all $\ell \geq n \geq n^*$ and any couple of rational numbers $0 < a < b$.

To see that this implies that $\mathcal{M}^*$ converges to an extended real number on ω, assume **ex absurdo** that it does not. Then there is some $\epsilon \in \mathbb{R}_>$ such that $\liminf \mathcal{M}^*(\omega) < \limsup \mathcal{M}^*(\omega) - 2\epsilon$. As proved above, there is an n^* such that, for all $\ell \geq n \geq n^*$ and any couple of rational numbers $0 < a < b$, $\mathcal{M}^{a,b}(\omega^\ell) - \mathcal{M}^{a,b}(\omega^n) \geq -2\epsilon$ and therefore also $\mathcal{M}^{a,b}(\omega^\ell) \geq \mathcal{M}^{a,b}(\omega^n) - 2\epsilon$. Then it follows directly from the definition of $\mathcal{M}^*$ that also $\mathcal{M}^*(\omega^\ell) \geq \mathcal{M}^*(\omega^n) - 2\epsilon$ for all $\ell \geq n \geq n^*$. However, this is in contradiction with $\liminf \mathcal{M}^*(\omega) < \limsup \mathcal{M}^*(\omega) - 2\epsilon$, because the latter would require that there is some couple $\ell \geq n \geq n^*$ such that $\mathcal{M}^*(\omega^\ell) < \mathcal{M}^*(\omega^n) - 2\epsilon$. Hence, $\mathcal{M}^*$ converges to an extended real number on ω. □

Proof of Theorem 4.5.2$_{172}$. Due to Proposition 4.5.1$_{172}$, there is an s-test supermartingale $\mathcal{M}^*$ that converges to $+\infty$ on every path $\omega \in \Gamma(s)$ where $\mathcal{M}$ does not converge to an extended real number. Let $B \in \mathbb{R}$ be a lower bound for $\mathcal{M}$ and let $\mathcal{M}' := \frac{1}{\mathcal{M}(s)-B+1}(\mathcal{M} - B + \mathcal{M}^*)$. Since both $\mathcal{M}^*$ and $\mathcal{M} - B$ are supermartingales [because of LE5$_{156}$], Lemma 4.4.2$_{163}$ implies that $(\mathcal{M} - B + \mathcal{M}^*)$ is a supermartingale too and therefore, since $1 \leq \mathcal{M}(s) - B + 1 < +\infty$ [$\mathcal{M}(s)$ is real], LE3$_{156}$ implies that $\mathcal{M}'$ is a supermartingale. Moreover, $\mathcal{M}'$ is non-negative because both $\mathcal{M} - B$ and $\mathcal{M}^*$ are non-negative and $\mathcal{M}(s) - B + 1 \geq 1$ which, together with $\mathcal{M}'(s) = 1$, allows us to conclude that $\mathcal{M}'$ is an s-test supermartingale. Furthermore, consider any path $\omega \in \Gamma(s)$ such that $\mathcal{M}(\omega^n)$ does not converge to a real number. Then either it converges to $+\infty$ or it does not converge in $\overline{\mathbb{R}}$ [because $\mathcal{M}$ is bounded below]. In the first case, it follows from the non-negativity of $\mathcal{M}^*$ and the positivity of $\frac{1}{\mathcal{M}(s)-B+1}$ that $\mathcal{M}'$ also converges to $+\infty$ on ω. If $\mathcal{M}(\omega^n)$ does not converge in $\overline{\mathbb{R}}$, then $\mathcal{M}^*$ converges to $+\infty$ on ω and therefore, because $\mathcal{M} - B$ is non-negative and $\frac{1}{\mathcal{M}(s)-B+1}$ is positive, $\mathcal{M}'$ also converges to $+\infty$ on ω. All together, we have that $\mathcal{M}'$ is an s-test supermartingale that converges to $+\infty$ on every path $\omega \in \Gamma(s)$ where $\mathcal{M}$ does not converge to a real number. □

Proof of Theorem 4.5.3$_{172}$. Since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|t)$ satisfies EC5$_{163}$ for any $t \in \mathscr{X}^*$, we have, for any $c \in \mathbb{R}$ and any $\omega \in \Omega$, that $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|\omega^n) \geq f(\omega)$ if and only if $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f+c|\omega^n) \geq f(\omega)+c$. Therefore, and because f is bounded below, we can assume without loss of generality that f is a global extended real variable such that $\inf f > 0$.

We now associate with any couple of rational numbers $0 < a < b$ the following recursively constructed sequences of cuts $(U_k^{a,b})_{k\in\mathbb{N}_0}$ and $(V_k^{a,b})_{k\in\mathbb{N}}$. Let $U_0^{a,b} := \{s\}$ and, for $k \in \mathbb{N}$,

1. let $V_k^{a,b} := \{v \in \mathscr{X}^* : U_{k-1}^{a,b} \sqsubset v,\ \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|v) < a \text{ and } (\forall t \in (U_{k-1}^{a,b}, v))\ \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|t) \geq a\}$;
2. if $V_k^{a,b}$ is non-empty, choose a positive supermartingale $\mathscr{M}_k^{a,b} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\mathscr{M}_k^{a,b}(v) < a$ and $\liminf \mathscr{M}_k^{a,b} \geq_v f$ for all $v \in V_k^{a,b}$, and let
$$U_k^{a,b} := \{u \in \mathscr{X}^* : V_k^{a,b} \sqsubset u : \mathscr{M}_k^{a,b}(u) > b \text{ and } (\forall t \in (V_k^{a,b}, u))\ \mathscr{M}_k^{a,b}(t) \leq b\};$$
3. if $V_k^{a,b}$ is empty, let $U_k^{a,b} := \emptyset$.

Note that all $U_k^{a,b}$ and all $V_k^{a,b}$ are indeed (partial or complete) cuts. We now first show that, if $V_k^{a,b}$ is non-empty, there always is a supermartingale $\mathscr{M}_k^{a,b}$ that satisfies the conditions above. We infer from the definition of the cut $V_k^{a,b}$ that

$$\inf\left\{\mathscr{M}(v) : \mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \text{ and } \liminf \mathscr{M} \geq_v f\right\} = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|v) < a \text{ for all } v \in V_k^{a,b}.$$

So, for all $v \in V_k^{a,b}$, we can choose a supermartingale $\mathscr{M}_{k,v}^{a,b} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\mathscr{M}_{k,v}^{a,b}(v) < a$ and $\liminf \mathscr{M}_{k,v}^{a,b} \geq_v f$. Consider now the extended real process $\mathscr{M}_k^{a,b}$ defined, for all $t \in \mathscr{X}^*$, by

$$\mathscr{M}_k^{a,b}(t) := \begin{cases} \mathscr{M}_{k,v}^{a,b}(t) & \text{if } v \sqsubseteq t \text{ for some } v \in V_k^{a,b};\\ a & \text{otherwise.} \end{cases}$$

It is clear that $\mathscr{M}_k^{a,b}(v) < a$ and $\liminf \mathscr{M}_k^{a,b} \geq_v f$ for all $v \in V_k^{a,b}$. We furthermore show that $\mathscr{M}_k^{a,b}$ is a positive supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.

It follows from Lemma 4.4.1$_{163}$ that, for all $v \in V_k^{a,b}$,

$$\mathscr{M}_{k,v}^{a,b}(t) \geq \inf_{\omega\in\Gamma(t)} \liminf \mathscr{M}_{k,v}^{a,b}(\omega) \geq \inf_{\omega\in\Gamma(t)} f(\omega) \geq \inf f > 0 \text{ for all } t \sqsupseteq v. \tag{4.26}$$

Since also $a > 0$, it follows that $\mathscr{M}_k^{a,b}$ is positive. To show that $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_k^{a,b}(t\,\cdot)) \leq \mathscr{M}_k^{a,b}(t)$ for all $t \in \mathscr{X}^*$, fix any $t \in \mathscr{X}^*$ and consider two cases. If $V_k^{a,b} \sqsubseteq t$, then $\mathscr{M}_k^{a,b}(t) = \mathscr{M}_{k,v}^{a,b}(t)$ and $\mathscr{M}_k^{a,b}(t\,\cdot) = \mathscr{M}_{k,v}^{a,b}(t\,\cdot)$ for some $v \in V_k^{a,b}$, and therefore $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_k^{a,b}(t\,\cdot)) = \overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_{k,v}^{a,b}(t\,\cdot)) \leq \mathscr{M}_{k,v}^{a,b}(t) = \mathscr{M}_k^{a,b}(t)$. If $V_k^{a,b} \not\sqsubseteq t$, then for any $x \in \mathscr{X}$ we either have that $tx \in V_k^{a,b}$, which implies that $\mathscr{M}_k^{a,b}(tx) < a$, or $V_k^{a,b} \not\sqsubseteq tx$ and therefore $\mathscr{M}_k^{a,b}(tx) = a$. Hence, we infer that $\mathscr{M}_k^{a,b}(t\,\cdot) \leq a$, and therefore, by LE4$_{156}$ and LE1$_{156}$, that $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_k^{a,b}(t\,\cdot)) \leq a = \mathscr{M}_k^{a,b}(t)$. So we can conclude that $\mathscr{M}_k^{a,b}$ is a positive supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.

Next, consider the extended real process $\mathscr{T}^{a,b}$ defined by $\mathscr{T}^{a,b}(t) := 1$ for all $t \not\sqsupset s$, and

$$\mathscr{T}^{a,b}(tx) := \begin{cases} \mathscr{M}_k^{a,b}(tx)\mathscr{T}^{a,b}(t)/\mathscr{M}_k^{a,b}(t) & \text{if } V_k^{a,b} \sqsubseteq t \text{ and } U_k^{a,b} \not\sqsubseteq t \text{ for some } k \in \mathbb{N}; \\ \mathscr{T}^{a,b}(t) & \text{otherwise,} \end{cases} \tag{4.27}$$

for all $t \sqsupseteq s$ and all $x \in \mathscr{X}$. We prove that this process is a positive s-test supermartingale that converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ such that

$$\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|\omega^n) < a < b < f(\omega). \tag{4.28}$$

That $\mathscr{T}^{a,b}$ is well-defined follows from the fact that, for any $k \in \mathbb{N}$ and any situation t such that $V_k^{a,b} \sqsubseteq t$ and $U_k^{a,b} \not\sqsubseteq t$, $\mathscr{M}_k^{a,b}(t)$ is positive and moreover real because of the definition of $U_k^{a,b}$. The process $\mathscr{T}^{a,b}$ is also positive because, for any $k \in \mathbb{N}$ and any situation t such that $V_k^{a,b} \sqsubseteq t$ and $U_k^{a,b} \not\sqsubseteq t$, $\mathscr{M}_k^{a,b}(t)$ is real and positive and $\mathscr{M}_k^{a,b}(t\,\cdot)$ is positive, and therefore $\mathscr{M}_k^{a,b}(t\,\cdot)/\mathscr{M}_k^{a,b}(t)$ is positive. Furthermore, if $t \in \mathscr{X}^*$ is such that $V_k^{a,b} \sqsubseteq t$ and $U_k^{a,b} \not\sqsubseteq t$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} \overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{T}^{a,b}(t\,\cdot)) &= \overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_k^{a,b}(t\,\cdot)\mathscr{T}^{a,b}(t)/\mathscr{M}_k^{a,b}(t)) \\ &\overset{\mathrm{LE3}_{156},\mathrm{LE8}_{156}}{=} \overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_k^{a,b}(t\,\cdot))\mathscr{T}^{a,b}(t)/\mathscr{M}_k^{a,b}(t) \leq \mathscr{T}^{a,b}(t), \end{aligned}$$

where the second step also uses the fact that $\mathscr{M}_k^{a,b}(t\,\cdot)$ and $\mathscr{T}^{a,b}(t)/\mathscr{M}_k^{a,b}(t)$ are positive [since, as we have shown above, $\mathscr{M}_k^{a,b}(t)$ is real and positive, and $\mathscr{T}^{a,b}(t)$ is positive], and where the last step uses the supermartingale character of $\mathscr{M}_k^{a,b}$ together with the fact that $\mathscr{T}^{a,b}(t)/\mathscr{M}_k^{a,b}(t)$ is positive. Otherwise, if t is such that, for all $k \in \mathbb{N}$, $V_k^{a,b} \not\sqsubseteq t$ or $U_k^{a,b} \sqsubseteq t$, then $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{T}^{a,b}(t\,\cdot)) = \overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{T}^{a,b}(t)) = \mathscr{T}^{a,b}(t)$ because of $\mathrm{LE1}_{156}$. Hence, we have that $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{T}^{a,b}(t\,\cdot)) \leq \mathscr{T}^{a,b}(t)$ for all $t \in \mathscr{X}^*$, which together with the fact that $\mathscr{T}^{a,b}(s) = 1$, allows us to conclude that $\mathscr{T}^{a,b}$ is indeed a positive s-test supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.

Next, we show that $\mathscr{T}^{a,b}$ converges to $+\infty$ on all paths $\omega \in \Gamma(s)$ for which Eq. (4.28) holds. Consider such a path ω. Then ω goes through all the cuts $U_0^{a,b} \sqsubset V_1^{a,b} \sqsubset U_1^{a,b} \sqsubset \dots \sqsubset V_k^{a,b} \sqsubset U_k^{a,b} \sqsubset \dots$. Indeed, it is trivial that ω goes through $U_0^{a,b} = \{s\}$. Furthermore, it follows from $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|\omega^n) < a$ that there is an $n_1 \in \mathbb{N}$ such that $\omega^{n_1} \sqsupset s$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|\omega^{n_1}) < a$. Take the first such $n_1 \in \mathbb{N}$. Then it follows from the definition of $V_1^{a,b}$ that $\omega^{n_1} \in V_1^{a,b}$. Next, it follows from $\liminf_{n\to+\infty} \mathscr{M}_1^{a,b}(\omega^n) \geq f(\omega) > b$ that there exists some $m_1 \in \mathbb{N}$ for which $m_1 > n_1$ and $\mathscr{M}_1^{a,b}(\omega^{m_1}) > b$. Take the first such m_1. Then it follows from the definition of $U_1^{a,b}$ that $\omega^{m_1} \in U_1^{a,b}$. Repeating similar arguments over and over again allows us to conclude that ω indeed goes through all the cuts $U_0^{a,b} \sqsubset V_1^{a,b} \sqsubset U_1^{a,b} \sqsubset \dots \sqsubset V_k^{a,b} \sqsubset U_k^{a,b} \sqsubset \dots$.

In what follows, we use the following notation. For any $k \in \mathbb{N}_0$, let u_k^ω be the (necessarily unique) situation in $U_k^{a,b}$ where ω goes through. Similarly, for any $k \in \mathbb{N}$, let v_k^ω be the (necessarily unique) situation in $V_k^{a,b}$ where ω goes through. For all $n \in \mathbb{N}_0$, let $k_n \in \mathbb{N}_0$ be defined by $k_n := 0$ if $V_1^{a,b} \not\sqsubset \omega^n$ and otherwise, let k_n be such that $V_{k_n}^{a,b} \sqsubset \omega^n$ and $V_{k_n+1}^{a,b} \not\sqsubset \omega^n$. Note that $k_n \to +\infty$ for $n \to +\infty$ because ω goes through all the cuts $V_1^{a,b} \sqsubset V_2^{a,b} \sqsubset \dots \sqsubset V_k^{a,b} \sqsubset \dots$. For any $n \in \mathbb{N}_0$ such that $k_n \geq 1$, we now have one of the following two cases:

1. The first case is that $\omega^n \in (V^{a,b}_{k_n}, U^{a,b}_{k_n}]$. Then by applying Eq. (4.27)$_{\curvearrowleft}$ for each subsequent step, recalling that $\mathscr{T}^{a,b}(s) = 1$, and cancelling out the intermediate terms which is possible because $\mathscr{M}^{a,b}_\ell(s')$ is real for any $s' \in [V^{a,b}_\ell, U^{a,b}_\ell)$ and any $\ell \in \mathbb{N}$ [this follows readily from the definition of the cuts $V^{a,b}_\ell$ and $U^{a,b}_\ell$], we find that
$$\mathscr{T}^{a,b}(\omega^n) = \left(\prod_{\ell=1}^{k_n-1} \frac{\mathscr{M}^{a,b}_\ell(u^\omega_\ell)}{\mathscr{M}^{a,b}_\ell(v^\omega_\ell)}\right)\frac{\mathscr{M}^{a,b}_{k_n}(\omega^n)}{\mathscr{M}^{a,b}_{k_n}(v^\omega_{k_n})}.$$
Since $\mathscr{M}^{a,b}_{k_n}(\omega^n) \geq \inf f > 0$ [due to Eq. (4.26)$_{206}$], $\mathscr{M}^{a,b}_\ell(u^\omega_\ell) > b > 0$ for all $\ell \in \{1, \ldots, k_n - 1\}$ and $0 < \mathscr{M}^{a,b}_\ell(v^\omega_\ell) < a$ for all $\ell \in \{1, \ldots, k_n\}$, we get that
$$\mathscr{T}^{a,b}(\omega^n) \geq \left(\frac{b}{a}\right)^{k_n-1}\frac{\mathscr{M}^{a,b}_{k_n}(\omega^n)}{a} \geq \left(\frac{b}{a}\right)^{k_n-1}\left(\frac{\inf f}{a}\right).$$
2. The second case is that $\omega^n \in (U^{a,b}_{k_n}, V^{a,b}_{k_n+1}]$. Then, by repeatedly applying Eq. (4.27)$_{\curvearrowleft}$, and since $\mathscr{T}^{a,b}(s) = 1$, we have that
$$\mathscr{T}^{a,b}(\omega^n) = \prod_{\ell=1}^{k_n} \frac{\mathscr{M}^{a,b}_\ell(u^\omega_\ell)}{\mathscr{M}^{a,b}_\ell(v^\omega_\ell)}.$$
Since $\mathscr{M}^{a,b}_\ell(u^\omega_\ell) > b > 0$ and $0 < \mathscr{M}^{a,b}_\ell(v^\omega_\ell) < a$ for all $\ell \in \{1, \ldots, k_n\}$, we find that
$$\mathscr{T}^{a,b}(\omega^n) > \left(\frac{b}{a}\right)^{k_n}.$$

Since $\inf f > 0$, $a > 0$ and $\frac{b}{a} > 1$, and since $\lim_{n\to+\infty} k_n = +\infty$, it follows from the two expressions above that indeed $\lim_{n\to+\infty} \mathscr{T}^{a,b}(\omega^n) = +\infty$.

To finish, we use the countable set of rational couples $K := \{(a, b) \in \mathbb{Q}^2 : 0 < a < b\}$ to define the process $\mathscr{T}$:
$$\mathscr{T} := \sum_{(a,b)\in K} w^{a,b}\mathscr{T}^{a,b},$$
with coefficients $w^{a,b} > 0$ that sum to 1. Hence, $\mathscr{T}$ is a countable convex combination of the positive s-test supermartingales $\mathscr{T}^{a,b}$ in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. By Lemma 4.4.2$_{163}$, $\mathscr{T}$ is then also a supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. It is also positive, because all $w^{a,b}\mathscr{T}^{a,b}$ are positive. Since it is moreover clear that $\mathscr{T}(s) = 1$, the process $\mathscr{T}$ is a positive s-test supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Furthermore, $\mathscr{T}$ converges to $+\infty$ on the paths $\omega \in \Gamma(s)$ where $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|\omega^n) < f(\omega)$. Indeed, consider such a path ω. Then since $f(\omega) \geq \inf f > 0$, there is at least one couple $(a', b') \in K$ such that $\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|\omega^n) < a' < b' < f(\omega)$, and as a consequence $\lim_{n\to+\infty} \mathscr{T}^{a',b'}(\omega^n) = +\infty$. Then also $\lim_{n\to+\infty} w^{a',b'}\mathscr{T}^{a',b'}(\omega^n) = +\infty$, and since $w^{a,b}\mathscr{T}^{a,b}$ is positive for all other couples $(a, b) \in K \setminus (a', b')$, the positive s-test supermartingale $\mathscr{T}$ indeed converges to $+\infty$ on ω. □

4.C Stopping times, and proofs of Theorems 4.7.3 and 4.7.4

To prove Theorems 4.7.3$_{182}$ and 4.7.4$_{183}$, we start with some technical results concerning the nature of stopping times in discrete-time stochastic processes with a finite state space.

4.C.1 Stopping times for discrete-time stochastic processes with finite state space

A **stopping time** $\sigma \in \overline{\mathbb{V}}$ is a variable taking values in $\mathbb{N}_0$ and is such that, for any $\omega \in \Omega$ and with $n = \sigma(\omega)$, we have that $\sigma(\tilde{\omega}) = n$ for all $\tilde{\omega} \in \Gamma(\omega^n)$. Or, alternatively, we could say that σ is a stopping time if it takes values in $\mathbb{N}_0$ and if, for any $n \in \mathbb{N}_0$ and any $\omega \in \sigma^{-1}(n) := \{\tilde{\omega} \in \Omega : \sigma(\tilde{\omega}) = n\}$, we have that $\Gamma(\omega^n) \subseteq \sigma^{-1}(n)$. The following lemma shows that our definition of a stopping time agrees with the more traditional one that can for instance be found in [5].

Lemma 4.C.1. *A variable $\sigma \in \overline{\mathbb{V}}$ taking values in $\mathbb{N}_0$ is a stopping time if and only if the event $\sigma^{-1}(n)$ for all $n \in \mathbb{N}_0$ is a union $\bigcup_{x_{1:n} \in \mathscr{S}} \Gamma(x_{1:n})$ of cylinder events of situations $x_{1:n}$ with length n.*

Proof. Let σ be a stopping time and consider any $n \in \mathbb{N}_0$. For any $\omega \in \sigma^{-1}(n)$, since σ is a stopping time, we have that $\Gamma(\omega^n) \subseteq \sigma^{-1}(n)$. Hence, we also have that the union $\bigcup_{\omega \in \sigma^{-1}(n)} \Gamma(\omega^n)$ of the sets $\Gamma(\omega^n)$ over all $\omega \in \sigma^{-1}(n)$ is a subset of $\sigma^{-1}(n)$. That $\bigcup_{\omega \in \sigma^{-1}(n)} \Gamma(\omega^n)$ is also a superset of $\sigma^{-1}(n)$, is trivial, so we find that $\bigcup_{\omega \in \sigma^{-1}(n)} \Gamma(\omega^n)$ is equal to $\sigma^{-1}(n)$.

Conversely, suppose that σ is a variable taking values in $\mathbb{N}_0$ such that the event $\sigma^{-1}(n)$ for all $n \in \mathbb{N}_0$ is a union $\bigcup_{x_{1:n} \in \mathscr{S}} \Gamma(x_{1:n})$ of cylinder events of situations $x_{1:n}$ with length n. Then consider any $\omega \in \Omega$ and any $\tilde{\omega} \in \Gamma(\omega^n)$ with $n = \sigma(\omega)$. Since $\sigma^{-1}(n)$ is a union of cylinder events of situations $x_{1:n}$ with length n, and since obviously $\omega \in \sigma^{-1}(n)$ because $n = \sigma(\omega)$, we obtain that $\Gamma(\omega^n) \subseteq \sigma^{-1}(n)$ and therefore that $\tilde{\omega} \in \sigma^{-1}(n)$. As a result, we conclude that $\sigma(\tilde{\omega}) = n$. □

For discrete-time stochastic processes with general—not necessarily finite—state spaces, stopping times are not necessarily bounded above. If the state space is assumed finite however, as is the case for our setting here, then stopping times are automatically bounded above, and therefore always belong to the space $\mathbb{F}$ of finitary gambles. This results from the following fundamental lemma, which we will also use further on to prove some of our crucial results.

Lemma 4.C.2. *For any decreasing sequence $(A_n)_{n \in \mathbb{N}}$ consisting of sets A_n that are each non-empty and a finite union of cylinder events of situations of the same length, we have that $\lim_{n \to +\infty} A_n$ is non-empty.*[11]

Proof. This follows from the lemma on [5, p. 29]; again, as already mentioned in the proof of Lemma 3.3.3$_{72}$, a 'cylinder event' according to [5] is in our language a finite union of cylinder events of situations of the same length, and that the notion of a 'thin

[11] Note that the finiteness (or better, compactness) of the local state space $\mathscr{X}$ is a necessary assumption for Lemma 4.C.2 to hold.

cylinder event' in [5] corresponds to our notion of a cylinder event. Alternatively, this result is also a special case of [111, Theorem 17.4 (a)–(b)] since $\mathscr{X}$ is finite and therefore $\Omega = \mathscr{X}^{\mathbb{N}}$—if equipped with the product topology—is compact; we refer to Section 5.5.2250 and Appendix 5.C274 for some basic topological facts about Ω. □

Lemma 4.C.3. *Any stopping time σ is a finitary gamble that is $(\sup\sigma)$-measurable, with $\sup\sigma \in \mathbb{N}_0$.*

Proof. We first show that σ is bounded, and therefore a gamble. σ is clearly bounded below because it takes values in $\mathbb{N}_0$. To prove that it is bounded above, consider the sequence of events $(A_n)_{n\in\mathbb{N}}$ defined by $A_n \coloneqq \{\omega \in \Omega \colon \sigma(\omega) \geq n\}$ for all $n \in \mathbb{N}$. Then, for any $n \in \mathbb{N}$, the event A_n is a finite union of cylinder events of situations with length n. Indeed, for any $k \in \mathbb{N}_0$ such that $k < n$, we have by Lemma 4.C.1↶ that $\sigma^{-1}(k) = \{\omega \in \Omega \colon \sigma(\omega) = k\}$ is a union $\bigcup_{x_{1:k}\in\mathscr{S}'}\Gamma(x_{1:k})$ of cylinder events of situations of length k, and thus because $k < n$ also the union $\bigcup_{x_{1:k}\in\mathscr{S}'}\bigcup_{x_{k+1:n}\in\mathscr{X}^{n-k}}\Gamma(x_{1:n})$ of cylinder events of situations of length n. The event $A_n^c = \{\omega \in \Omega \colon \sigma(\omega) < n\}$ is equal to the union $\bigcup_{k=0}^{n-1}\sigma^{-1}(k)$, so A_n^c is also a union of cylinder events of situations of length n. Let us denote this union by $\bigcup_{x_{1:n}\in\mathscr{S}}\Gamma(x_{1:n})$ with $\mathscr{S} \subseteq \mathscr{X}^n$. Since $\Omega = \bigcup_{x_{1:n}\in\mathscr{X}^n}\Gamma(x_{1:n})$, we have that $A_n = \Omega \setminus A_n^c = \bigcup_{x_{1:n}\in\mathscr{X}^n\setminus\mathscr{S}}\Gamma(x_{1:n})$. Hence, since $\mathscr{X}$ is finite, A_n is indeed the finite union of cylinder events of situations of length n. Furthermore, by its definition, $(A_n)_{n\in\mathbb{N}}$ is clearly decreasing. Hence, if A_n is non-empty for all $n \in \mathbb{N}$, then Lemma 4.C.2↶ implies that $\lim_{n\to+\infty} A_n$ is also non-empty. This would mean that there is a path $\omega \in \Omega$ such that $\omega \in A_n$ for all $n \in \mathbb{N}$, and so by definition of A_n, that $\sigma(\omega) \geq n$ for all $n \in \mathbb{N}$. But this is in contradiction with the fact that σ takes values in $\mathbb{N}_0$, so we must have that A_n is empty for some $n \in \mathbb{N}$ and therefore that $\sigma(\omega) < n$ for all $\omega \in \Omega$. So σ is bounded above, which together with the fact that σ is bounded below, implies that σ is a gamble. The fact that σ is bounded above and that it takes values in $\mathbb{N}_0$, also clearly implies that $\sup\sigma \in \mathbb{N}_0$. To see that σ is $(\sup\sigma)$-measurable, consider any $\omega \in \Omega$ and any $\tilde{\omega} \in \Gamma(\omega^{\sup\sigma})$. Then $\tilde{\omega} \in \Gamma(\omega^{\sigma(\omega)})$ because $\sigma(\omega) \leq \sup\sigma$ and therefore, since σ is a stopping time, we have that $\sigma(\omega) = \sigma(\tilde{\omega})$. □

With any stopping time σ, we can also naturally associate a (tree) cut $U_\sigma \coloneqq \{t \in \mathscr{X}^* \colon (\exists\omega \in \Omega)\, t = \omega^{\sigma(\omega)}\} = \{\omega^{\sigma(\omega)} \colon \omega \in \Omega\}$ (see Appendix 4.B199 for definitions and notations concerning cuts). The following lemma shows that U_σ is indeed a cut, and that this cut is moreover complete.

Lemma 4.C.4. *For any stopping time σ, the set $U_\sigma \subset \mathscr{X}^*$ is a complete cut.*

Proof. To see that U_σ is a cut, suppose **ex absurdo** that there are two (different) situations $\omega_1^{\sigma(\omega_1)}$ and $\omega_2^{\sigma(\omega_2)}$ in U_σ such that $\omega_1^{\sigma(\omega_1)} \sqsubseteq \omega_2^{\sigma(\omega_2)}$. Then we have that $\sigma(\omega_1) \leq \sigma(\omega_2)$ and that $\omega_2 \in \Gamma(\omega_1^{\sigma(\omega_1)})$. The latter implies, by the fact that σ is a stopping time, that $\sigma(\omega_2) = \sigma(\omega_1)$. So since $\omega_1^{\sigma(\omega_1)} \sqsubseteq \omega_2^{\sigma(\omega_2)}$, it must be that $\omega_1^{\sigma(\omega_1)} = \omega_2^{\sigma(\omega_2)}$. This is in contradiction with our assumption that $\omega_1^{\sigma(\omega_1)}$ and $\omega_2^{\sigma(\omega_2)}$ are different, so we conclude that all situations in U_σ are pairwise incomparable and therefore, that U_σ is a cut. To see why U_σ is moreover a complete cut, observe that since σ takes values in $\mathbb{N}_0$, it follows that, for any $\omega \in \Omega$, $\omega^{\sigma(\omega)}$ is a situation, which by definition is an element of U_σ; it is moreover clear that for this situation $\omega^{\sigma(\omega)}$, we have that $\omega \in \Gamma(\omega^{\sigma(\omega)})$. Hence, U_σ is a complete cut, because for all $\omega \in \Omega$, there is a situation $u \in U$ such that $\omega \in \Gamma(u)$. □

4.C.2 Proofs of Theorems 4.7.3 and 4.7.4

For the following lemmas, we will associate with any sequence $(f_n)_{n\in\mathbb{N}_0}$ of n-measurable variables and any stopping time σ, the global variable $f_\sigma \in \overline{\mathbb{V}}$ defined by $f_\sigma(\omega) := f_{\sigma(\omega)}(\omega)$. For any extended real process $\mathscr{C}$ and any (tree) cut U, we will furthermore use $\mathscr{C}_{\dashv U}$ to denote the extended real process stopped at U:

$$\mathscr{C}_{\dashv U}(s) := \begin{cases} \mathscr{C}(s) & \text{if } s \not\sqsupseteq U; \\ \mathscr{C}(u(s)) & \text{if } s \sqsupseteq U, \end{cases} \quad \text{for all } s \in \mathscr{X}^*,$$

where, for any $s \sqsupseteq U$, $u(s)$ denotes the unique situation in the cut U such that $s \sqsupseteq u(s)$. That $u(s)$ is unique follows from the fact that U is a cut, and thus the situations in U are incomparable; indeed, for any second $u'(s) \in U$ such that $s \sqsupseteq u'(s)$, and for $n := |u(s)|$ and $m := |u'(s)|$, we will have that $u(s) = s_{1:n} \sqsupseteq s_{1:m} = u'(s)$ if $n \le m$, or $u'(s) = s_{1:m} \sqsupseteq s_{1:n} = u(s)$ if $m \le n$, which contradicts the incomparability of $u(s)$ and $u'(s)$. For any stopping time σ, we also use the notation $\mathscr{C}_{\dashv\sigma}$ to denote the extended real process stopped at the cut U_σ associated with σ.

The following basic lemma shows that stopping a supermartingale does not impact the fact that it is a supermartingale.

Lemma 4.C.5. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, and any (complete or partial) cut U, we have that the stopped process $\mathscr{M}_{\dashv U}$ is a supermartingale in $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. In particular, for any stopping time σ, we have that $\mathscr{M}_{\dashv\sigma} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.*

Proof. The process $\mathscr{M}_{\dashv U}$ is bounded below because $\mathscr{M}$ is. To see that $\mathscr{M}_{\dashv U}$ is a supermartingale, note that

$$\mathscr{M}_{\dashv U}(t\cdot) := \begin{cases} \mathscr{M}(u(t)) & \text{if } t \sqsupseteq U; \\ \mathscr{M}(t\cdot) & \text{if } t \not\sqsupseteq U, \end{cases} \quad \text{for all } t \in \mathscr{X}^*.$$

So for any situation $t \sqsupseteq U$, we have that $\mathscr{M}_{\dashv U}(t) = \mathscr{M}(u(t))$ and that $\mathscr{M}_{\dashv U}(t\cdot) = \mathscr{M}(u(t))$, which implies that $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_{\dashv U}(t\cdot)) = \mathscr{M}_{\dashv U}(t)$ because of LE1$_{156}$. On the other hand, for any situation $t \not\sqsupseteq U$, we have that $\mathscr{M}_{\dashv U}(t) = \mathscr{M}(t)$ and that $\mathscr{M}_{\dashv U}(t\cdot) = \mathscr{M}(t\cdot)$. So here too, because $\mathscr{M}$ is a supermartingale and thus $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}(t\cdot)) \le \mathscr{M}(t)$, we have that $\overline{\mathrm{Q}}_t^{\uparrow}(\mathscr{M}_{\dashv U}(t\cdot)) \le \mathscr{M}_{\dashv U}(t)$. As a consequence, $\mathscr{M}_{\dashv U} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$.

The last statement, that $\mathscr{M}_{\dashv\sigma} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ for any stopping time σ, then follows from the fact that U_σ is a cut, due to Lemma 4.C.4$_{210}$, and the fact that $\mathscr{M}_{\dashv\sigma} = \mathscr{M}_{\dashv U_\sigma}$. □

Lemma 4.C.6. *Consider any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, any $s \in \mathscr{X}^*$ and any sequence $(f_n)_{n\in\mathbb{N}_0}$ of n-measurable gambles that converges pointwise to a variable $f \in \overline{\mathbb{V}}$ that is bounded above. Then, for any $m \in \mathbb{N}_0$ and any $\alpha \in \mathbb{R}$ such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) < \alpha$, there is a stopping time σ such that $m \leq \sigma$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_\sigma|s) \leq \alpha$.*

Proof. Fix any $m \in \mathbb{N}_0$, any $\alpha \in \mathbb{R}$ such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) < \alpha$, and any $\epsilon \in \mathbb{R}_{>}$. According to the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$, there is a supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ such that $\mathscr{M}(s) \leq \alpha$ and $\liminf \mathscr{M} \geq_s f$. We start by showing that, for any $\omega \in \Gamma(s)$ and any $n^* \in \mathbb{N}_0$, there is some natural number $n \geq n^*$ such that $\mathscr{M}(\omega^n) + \epsilon \geq f_n(\omega)$.

So consider any $\omega \in \Gamma(s)$. First note that $\liminf \mathscr{M}(\omega) + \epsilon > f(\omega)$ because $\liminf \mathscr{M} \geq_s f$, $\liminf \mathscr{M}(\omega) > -\infty$ [$\mathscr{M}$ is bounded below] and $f(\omega) < +\infty$ [f is bounded above]. This implies that there is a real number β such that $\liminf \mathscr{M}(\omega) + \epsilon > \beta > f(\omega)$. Then, since $(f_n(\omega))_{n\in\mathbb{N}_0}$ converges to $f(\omega)$ and β is a real such that $\beta > f(\omega)$, there is some index $N(\omega) \in \mathbb{N}_0$ such that $\beta > f_n(\omega)$ for all $n \geq N(\omega)$. Furthermore, by the definition of the limit inferior and the fact that β is a real such that $\liminf \mathscr{M}(\omega) + \epsilon > \beta$, there is a second index $M(\omega) \in \mathbb{N}_0$ such that $\mathscr{M}(\omega^n) + \epsilon > \beta$ for all $n \geq M(\omega)$. This tells us that $\mathscr{M}(\omega^n) + \epsilon > \beta > f_n(\omega)$ for all $n \geq \max\{N(\omega), M(\omega)\}$. This indeed implies that, for any $n^* \in \mathbb{N}_0$, there some $n \geq n^*$ such that $\mathscr{M}(\omega^n) + \epsilon \geq f_n(\omega)$.

Let ℓ be the length of the string s and consider the variable $\sigma \in \overline{\mathbb{V}}$ defined by

$$\sigma(\omega) := \begin{cases} \inf\left\{n \geq \max\{\ell, m\} : \mathscr{M}(\omega^n) + \epsilon \geq f_n(\omega)\right\} & \text{if } \omega \in \Gamma(s); \\ \max\{\ell, m\} & \text{otherwise,} \end{cases} \quad \text{for all } \omega \in \Omega.$$

It clearly follows from the argument above that σ takes values in $\mathbb{N}_0$. We will now also show that $\sigma(\omega) = \sigma(\tilde{\omega})$ for any $\omega \in \Omega$ and any $\tilde{\omega} \in \Gamma(\omega^{\sigma(\omega)})$, implying that σ is a stopping time, and therefore, by Lemma 4.C.3$_{210}$, that $\sup\sigma \in \mathbb{N}_0$ and that σ is a $(\sup\sigma)$-measurable gamble. Furthermore, we then trivially have that $\sigma \geq \ell$ and $\sigma \geq m$.

To this end, consider any $\omega \in \Omega$ and any $\tilde{\omega} \in \Gamma(\omega^{\sigma(\omega)})$. We distinguish two cases: $\omega \in \Gamma(s)$ and $\omega \notin \Gamma(s)$. If $\omega \in \Gamma(s)$, then it follows from the definition of σ that $\mathscr{M}(\omega^{\sigma(\omega)}) + \epsilon \geq f_{\sigma(\omega)}(\omega)$. Since $\omega^{\sigma(\omega)} = \tilde{\omega}^{\sigma(\omega)}$ [because $\tilde{\omega} \in \Gamma(\omega^{\sigma(\omega)})$] and since $f_{\sigma(\omega)}$ is $\sigma(\omega)$-measurable by assumption, this implies that $\mathscr{M}(\tilde{\omega}^{\sigma(\omega)}) + \epsilon \geq f_{\sigma(\omega)}(\tilde{\omega})$. Then, according to the definition of σ and since $\tilde{\omega} \in \Gamma(s)$ [because $\sigma(\omega) \geq \ell$ and $\omega \in \Gamma(s)$, and therefore $\tilde{\omega} \in \Gamma(\omega^{\sigma(\omega)}) \subseteq \Gamma(s)$], we have that $\sigma(\tilde{\omega}) \leq \sigma(\omega)$. On the other hand, since $\tilde{\omega} \in \Gamma(s)$ and $\omega \in \Gamma(\tilde{\omega}^{\sigma(\tilde{\omega})})$ [because $\omega^{\sigma(\omega)} = \tilde{\omega}^{\sigma(\omega)}$ and $\sigma(\tilde{\omega}) \leq \sigma(\omega)$], we can infer, in exactly the same way as before, that also $\sigma(\omega) \leq \sigma(\tilde{\omega})$. So we conclude that $\sigma(\omega) = \sigma(\tilde{\omega})$ when $\omega \in \Gamma(s)$. If $\omega \notin \Gamma(s)$, then $\tilde{\omega} \notin \Gamma(s)$ because $\sigma(\omega) \geq \ell$ and therefore $\Gamma(\omega^{\sigma(\omega)}) \cap \Gamma(s) = \emptyset$. Then it follows immediately from the definition of σ that $\sigma(\omega) = \sigma(\tilde{\omega})$. So σ is indeed a stopping time and thus a $(\sup\sigma)$-measurable gamble for which it moreover holds that $\sigma \geq \ell$ and $\sigma \geq m$.

For any $\omega \in \Omega$, we now let $u_\sigma(\omega)$ be the unique situation in U_σ such that $\omega \in \Gamma(u_\sigma(\omega))$. This situation $u_\sigma(\omega)$ exists and is unique because U_σ is a complete cut due to Lemma 4.C.4$_{210}$. By definition of U_σ, we also clearly have that $u_\sigma(\omega) = \omega^{\sigma(\omega)}$. Hence, by the definition of $\mathscr{M}_{\dashv\sigma}$, we obtain that

$$\lim_{m\to+\infty} \mathscr{M}_{\dashv\sigma}(\omega^m) = \lim_{m\to+\infty} \mathscr{M}(u_\sigma(\omega)) = \mathscr{M}(u_\sigma(\omega)) = \mathscr{M}(\omega^{\sigma(\omega)}) \text{ for all } \omega \in \Omega.$$

Therefore, by the definition of σ, we have that

$$\lim_{m\to+\infty} \big(\mathscr{M}_{\dashv\sigma}(\omega^m) + \epsilon\big) = \mathscr{M}(\omega^{\sigma(\omega)}) + \epsilon \geq f_{\sigma(\omega)}(\omega) = f_\sigma(\omega) \text{ for all } \omega \in \Gamma(s).$$

Then by Definition 4.7$_{160}$ and taking into account that $\mathscr{M}_{\dashv\sigma} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ by Lemma 4.C.5$_{211}$, and therefore that $\mathscr{M}_{\dashv\sigma}+\epsilon \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ [because the local models $\overline{\mathrm{Q}}_s^{\uparrow}$ satisfy LE5$_{156}$], it follows that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_\sigma|s) \leq \mathscr{M}_{\dashv\sigma}(s) + \epsilon$. Moreover, $\mathscr{M}_{\dashv\sigma}(s) = \mathscr{M}(s)$ because $\sigma \geq \ell$ and therefore $s \not\sqsupset U_\sigma$ or $s \in U_\sigma$, so we also have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_\sigma|s) \leq \mathscr{M}(s) + \epsilon \leq \alpha + \epsilon$. Since this inequality holds for any $\epsilon \in \mathbb{R}_>$, we infer that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_\sigma|s) \leq \alpha$, which together with the fact that σ is a stopping time such that $m \leq \sigma$, establishes the lemma. □

The idea underlying the proof of Theorem 4.7.3$_{182}$ is borrowed from [23, Theorem 3]. However, just as in [8, 94], real supermartingales were adopted there. Moreover, our result here considers sequences of (extended real) finitary variables that are bounded above, instead of sequences of n-measurable gambles.

Proof of Theorem 4.7.3$_{182}$. Note that, because f is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}_0}$ of bounded above variables, f is also bounded above. Because $f_n \geq f_{n+1} \geq f$ for all $n \in \mathbb{N}_0$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is monotone [EC4$_{163}$], the limit $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s)$ exists and we have that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. So we are left to show that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

Consider the sequence $(f_n^{\vee-n})_{n\in\mathbb{N}_0}$ and note that it suffices to show that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n^{\vee-n}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$, where the limit $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n^{\vee-n}|s)$ exists because $(f_n^{\vee-n})_{n\in\mathbb{N}_0}$ is clearly decreasing [since $(f_n)_{n\in\mathbb{N}_0}$ is decreasing] and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is monotone [EC4$_{163}$]. Indeed, it will then follow that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ because $f_n \leq f_n^{\vee-n}$ for all $n \in \mathbb{N}_0$ and therefore, by EC4$_{163}$, that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) \leq \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n^{\vee-n}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

Since $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of finitary variables that converges decreasingly to f, the same holds for the sequence $(f_n^{\vee-n})_{n\in\mathbb{N}_0}$. In fact, $(f_n^{\vee-n})_{n\in\mathbb{N}_0}$ is even a sequence of finitary gambles because each f_n is bounded above. Now let $g_n := f_n^{\vee-n}$ for all $n \in \mathbb{N}_0$ and consider the sequence $(g_n^\xi)_{n\in\mathbb{N}_0}$ defined by the recursive expression in Eq. (4.11)$_{180}$, with g_0^ξ a real constant such that $g_0^\xi \geq \sup g_0$ [this is possible because $g_0 = f_0^{\vee 0}$ is a gamble]. Due to Lemma 4.7.1(i)$_{181}$, $(g_n^\xi)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable variables. Since $(g_n)_{n\in\mathbb{N}_0} = (f_n^{\vee-n})_{n\in\mathbb{N}_0}$ is a sequence of finitary gambles that converges decreasingly to f, it follows from Lemma 4.7.1(iii)$_{181}$, (v)$_{181}$ and (vi)$_{181}$ that $(g_n^\xi)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that converges decreasingly to f. Due to Lemma 4.7.1(vii)$_{181}$, we moreover have that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n^{\vee-n}|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n^\xi|s). \tag{4.29}$$

Consider any real number $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$, which is possible because f is bounded above and therefore, by EC1$_{163}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) < +\infty$. Then since $(g_n^\xi)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that converges decreasingly to f, Lemma 4.C.6$_{212}$ implies that there is a stopping time σ such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_\sigma^\xi|s) \leq \alpha$ [we simply let $m = 0$ in the lemma]. Since $(g_n^\xi)_{n\in\mathbb{N}_0}$ is decreasing and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ is monotone [EC4$_{163}$], and since $\sup\sigma \in \mathbb{N}_0$ due to Lemma 4.C.3$_{210}$, we have that

$$\alpha \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_\sigma^\xi|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_{(\sup\sigma)}^\xi|s) \geq \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n^\xi|s),$$

so we infer that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g_n^\xi|s) \leq \alpha$. Recalling Eq. (4.29), it follows that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n^{\vee-n}|s) \leq \alpha$. Since this holds for any real $\alpha > \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$, we conclude that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n^{\vee-n}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ as desired. ☐

For any net $\{c_{(m,n)}\}_{m,n\in\mathbb{N}_0}$ in $\overline{\mathbb{R}}$, we say that $c := \lim_{(m,n)\to+\infty} c_{(m,n)} \in \overline{\mathbb{R}}$ is the (Moore-Smith) limit [67] of $\{c_{(m,n)}\}_{m,n\in\mathbb{N}_0}$ if, for each neighbourhood A of c, there is a couple $(m^*, n^*) \in \mathbb{N}_0^2$ such that $c_{(m,n)} \in A$ for all $m \geq m^*$ and all $n \geq n^*$. For the definition of a neighbourhood, see [111, Section 2.4]. In our setting, the neighboorhoods of a real number $c \in \mathbb{R}$ are all the sets that include an open ϵ-disk $\{a \in \mathbb{R} : |a - c| < \epsilon\}$ around c [111, Example 4.4(b)], and the neighbourhoods around $+\infty$ are all the sets that include $\{a \in \overline{\mathbb{R}} : a > B\}$ for some $B \in \mathbb{R}$, and similarly for neighbourhoods around $-\infty$; see Section 1.6$_{14}$ for the open sets in $\overline{\mathbb{R}}$. Furthermore, for any net $\{f_{(m,n)}\}_{m,n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}$ such that $\lim_{(m,n)\to+\infty} f_{(m,n)}(\omega)$ exists for all $\omega \in \Omega$, we write $\lim_{(m,n)\to+\infty} f_{(m,n)}$ to denote the variable in $\overline{\mathbb{V}}$ defined by $\lim_{(m,n)\to+\infty} f_{(m,n)}(\omega)$ for all $\omega \in \Omega$.

Lemma 4.C.7. *Consider any sequence $(f_n)_{n\in\mathbb{N}_0}$ in $\mathbb{V}$ that converges pointwise to some variable $f \in \overline{\mathbb{V}}_{\mathrm{b}}$. Then we have that $\lim_{(m,n)\to+\infty} f_n^{\wedge m} = f$.*

Proof. Consider any $\omega \in \Omega$. First consider the case that $f(\omega) \in \mathbb{R}$ and fix any $\epsilon \in \mathbb{R}_>$. Then there is an $n^* \in \mathbb{N}_0$ such that $|f_n(\omega) - f(\omega)| < \epsilon$ for all $n \geq n^*$. Consider any $m^* \geq f(\omega)+\epsilon$. Then for all $n \geq n^*$ and all $m \geq m^*$, we have that $f_n(\omega) < f(\omega)+\epsilon \leq m$, so $f_n^{\wedge m}(\omega) = f_n(\omega)$ and therefore $|f_n^{\wedge m}(\omega) - f(\omega)| = |f_n(\omega) - f(\omega)| < \epsilon$. Since this holds for any $\epsilon > 0$ [and since any neighboorhood of $f(\omega) \in \mathbb{R}$ includes an open ϵ-disk] we have that $\lim_{(m,n)\to+\infty} f_n^{\wedge m}(\omega) = f(\omega)$. If $f(\omega) = +\infty$, fix any $B > 0$. Then there is an $n^* \in \mathbb{N}_0$ such that $f_n(\omega) > B$ for all $n \geq n^*$. If we now take $m^* \geq B$, then clearly also $f_n^{\wedge m}(\omega) \geq B$ for all $n \geq n^*$ and all $m \geq m^*$. Hence, we have that $\lim_{(m,n)\to+\infty} f_n^{\wedge m}(\omega) = f(\omega)$ which, together with our earlier considerations, allows us to conclude that $\lim_{(m,n)\to+\infty} f_n^{\wedge m} = f$. ☐

Proof of Theorem 4.7.4$_{183}$. Fix any $s \in \mathscr{X}^*$ and any $f \in \overline{\mathbb{L}}_{\mathrm{b}}$. According to Proposition 4.7.2$_{182}$, there is a sequence $(f_n)_{n\in\mathbb{N}_0}$ of n-measurable gambles that converges pointwise to f and such that $B \leq f_n \leq \sup f$ for all $n \in \mathbb{N}_0$, where B is any real number if $\inf f = +\infty$ and $B := \inf f$ if $\inf f \in \mathbb{R}$ [$\inf f = -\infty$ is impossible because f is bounded below].

Fix any $\ell \in \mathbb{N}$ and note that the sequence $(f_n^{\wedge\ell})_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that converges pointwise to $f^{\wedge\ell}$ because $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that converges pointwise to f. Moreover, $f^{\wedge\ell}$ is bounded above by ℓ, so Lemma 4.C.6$_{212}$ guarantees that, for any $m \in \mathbb{N}_0$ and any $\alpha \in \mathbb{R}$ such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) < \alpha$, there is some stopping time σ such that $m \leq \sigma$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_\sigma^{\wedge\ell}|s) \leq \alpha$. Since $f^{\wedge\ell}$ is both bounded below and above and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfies EC1$_{163}$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) \in \mathbb{R}$ and therefore, that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) < \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) + 1/\ell$. So in particular, for any $m \in \mathbb{N}_0$, there is a stopping time σ such that $m \leq \sigma$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_\sigma^{\wedge\ell}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) + 1/\ell$. Lemma 4.C.3$_{210}$ moreover implies that $\sup\sigma \in \mathbb{N}_0$ for any such a stopping time.

It follows from the above that there is a sequence $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ of stopping times σ_ℓ such that $\sigma_0 = 0$ and, for all $\ell \in \mathbb{N}$, $\sigma_\ell \geq \sup\sigma_{\ell-1} + 1$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_{\sigma_\ell}^{\wedge\ell}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) + 1/\ell$. We now show that $\{f_{\sigma_\ell}^{\wedge\ell}\}_{\ell\in\mathbb{N}_0}$ is a sequence of finitary gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_{\sigma_\ell}^{\wedge\ell}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$.

Each $f_{\sigma_\ell}^{\wedge\ell}$ is a gamble because it is bounded above by ℓ and because, since each f_n is bounded below by B, $f_{\sigma_\ell}^{\wedge\ell}$ is bounded below by $\min\{B, \ell\}$. It then also follows that $\{f_{\sigma_\ell}^{\wedge\ell}\}_{\ell\in\mathbb{N}_0}$ is uniformly bounded below by $\min\{B, 0\}$. To see that each $f_{\sigma_\ell}^{\wedge\ell}$ is finitary, recall that each σ_ℓ is $(\sup\sigma_\ell)$-measurable, by Lemma 4.C.3$_{210}$. This implies that $\sigma_\ell(\omega) = \sigma_\ell(\tilde{\omega})$ for any $\omega \in \Omega$ and any $\tilde{\omega} \in \Gamma(\omega^{(\sup\sigma_\ell)})$, and therefore that

$$f_{\sigma_\ell}^{\wedge\ell}(\omega) = f_{\sigma_\ell(\omega)}^{\wedge\ell}(\omega) = f_{\sigma_\ell(\tilde{\omega})}^{\wedge\ell}(\omega) = f_{\sigma_\ell(\tilde{\omega})}^{\wedge\ell}(\tilde{\omega}) = f_{\sigma_\ell}^{\wedge\ell}(\tilde{\omega}),$$

where the third equality follows from the fact that $f_{\sigma_\ell(\tilde{\omega})}^{\wedge\ell}$ is $\sigma_\ell(\tilde{\omega})$-measurable [because $(f_n^{\ell})_{n\in\mathbb{N}_0}$ is a sequence of n-measurable gambles] and that $\tilde{\omega} \in \Gamma(\omega^{(\sup\sigma_\ell)}) \subseteq \Gamma(\omega^{\sigma_\ell(\tilde{\omega})})$ [because $\sigma_\ell(\tilde{\omega}) \leq \sup\sigma_\ell$]. As a consequence, each $f_{\sigma_\ell}^{\wedge\ell}$ is indeed $(\sup\sigma_\ell)$-measurable, and therefore finitary.

To see that $\{f_{\sigma_\ell}^{\wedge\ell}\}_{\ell\in\mathbb{N}_0}$ converges pointwise to f, recall that $(f_n)_{n\in\mathbb{N}_0}$ is a sequence of gambles that converges pointwise to $f \in \overline{\mathbb{L}}_{\mathrm{b}}$. So Lemma 4.C.7$_{\leftarrow}$ implies that $\lim_{(\ell,n)\to+\infty} f_n^{\wedge\ell} = f$, meaning that, for any $\omega \in \Omega$ and any neighbourhood A of $f(\omega)$, there is a couple $(\ell^*, n^*) \in \mathbb{N}_0^2$ such that $f_n^{\wedge\ell}(\omega) \in A$ for all $\ell \geq \ell^*$ and all $n \geq n^*$. Then, since $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ is strictly increasing in ℓ [because $\sigma_\ell \geq \sup\sigma_{\ell-1} + 1$ for all $\ell \in \mathbb{N}$], there is an $\ell' \in \mathbb{N}_0$ such that $\ell \geq \ell^*$ and $\sigma_\ell(\omega) \geq n^*$ for all $\ell \geq \ell'$. Together with the above, this implies that $f_{\sigma_\ell}^{\wedge\ell}(\omega) = f_{\sigma_\ell(\omega)}^{\wedge\ell}(\omega) \in A$ for all $\ell \geq \ell'$. Since there is such an $\ell' \in \mathbb{N}_0$ for any $\omega \in \Omega$ and any neighbourhood A of $f(\omega)$, we have that $\lim_{\ell\to+\infty} f_{\sigma_\ell}^{\wedge\ell} = f$.

Finally, to see that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_{\sigma_\ell}^{\wedge\ell}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$, recall that $\{\sigma_\ell\}_{\ell\in\mathbb{N}_0}$ is such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_{\sigma_\ell}^{\wedge\ell}|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) + 1/\ell$ for all $\ell \in \mathbb{N}$. So we have that

$$\limsup_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_{\sigma_\ell}^{\wedge\ell}|s) \leq \limsup_{\ell\to+\infty} \left(\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) + 1/\ell\right) = \limsup_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s),$$

where the last equality follows from Theorem 4.6.1$_{175}$ which we can apply because $\{f^{\wedge\ell}\}_{\ell\in\mathbb{N}_0}$ is an increasing sequence in $\overline{\mathbb{V}}_\mathrm{b}$ [because f is bounded below] that converges pointwise to $f \in \overline{\mathbb{V}}_\mathrm{b}$. On the other hand, we have that $\liminf_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}_{\sigma_\ell}|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ because of Corollary 4.6.2$_{177}$ and the fact that $\{f^{\wedge\ell}_{\sigma_\ell}\}_{\ell\in\mathbb{N}_0}$ is uniformly bounded below by $\min\{B,0\}$ and converges pointwise to f. Hence, we conclude that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}_{\sigma_\ell}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$.

As a final step, we consider the sequence $(f'_\ell)_{\ell\in\mathbb{N}_0} := \big((f^{\wedge\ell}_{\sigma_\ell})^{\xi}\big)_{\ell\in\mathbb{N}_0}$ defined through Eq. (4.11)$_{180}$, with $c = 0$. Then, by Lemma 4.7.1(i)$_{181}$, (iv)$_{181}$–(vi)$_{181}$, we have that $(f'_\ell)_{\ell\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that is uniformly bounded below and converges pointwise to f. Lemma 4.7.1(vii)$_{181}$ and the fact that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge\ell}_{\sigma_\ell}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ moreover imply that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'_\ell|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$. So we conclude that $(f'_\ell)_{\ell\in\mathbb{N}_0}$ is a sequence of n-measurable gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{\ell\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'_\ell|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$. □

— Chapter V —

Measure-theoretic upper expectations

Ever since the release of Kolmogorov's landmark contribution [56][1] in the 1930's, probability theory has become predominately measure-theoretic in nature. This branch of probability theory aims to quantify uncertainty in terms of probability measures: probability charges that, apart from satisfying finite additivity, are assumed to satisfy the axiom of countable additivity [55, Section II.1].[2] So similarly as in Section 2.2_{22} and Section 3.3_{69}, probabilities—in the form of charges or measures—are thus considered to be the primary objects, and linear expectations and upper expectations are only regarded to be the derived secondary objects. The fact, however, that probability measures are assumed to additionally satisfy the axiom of countable additivity changes matters dramatically: it ensures that probability measures and the corresponding measure-theoretic expectations—obtained by Lebesgue integration—possess powerful limit properties, which in turn can be seen as one of the major reasons for the success of the measure-theoretic framework. A second crucial factor for this success, which was largely due to the work of Kolmogorov [56] and, in the context of stochastic processes, due to that of Doob [33], is that the framework can be based entirely on a small number of simple and clean axioms. This axiomatic approach was, at the time, considered to be uniquely elegant, and it provided a much required unified approach to probability theory.

Yet, apart from these undeniable advantages, the measure-theoretic approach also has some serious drawbacks. The most important, to us, being that it is strongly 'precise' in spirit; one posits a single probability measure and all inferences are being deduced from this single probability measure. It therefore hides the inferential nature of probabilistic reasoning. Of course,

[1]Though the work of Kolmogorov is often cited as what gave birth to today's measure-theoretic probability theory, it was itself preceded by a series of impactful advances by Borel, Fréchet, Lévy and many others [87].

[2]The axiom of countable additivity is equivalent to the axiom of continuity [87, Section 5.2.1].

to remain more general, we can instead consider sets of probability measures. However, the properties and especially the continuity properties of the resulting imprecise (upper and lower) expectation operators remain relatively unexplored thus far—yet we know that they will inevitably be weaker than their classical precise counterparts.[3] Another issue is that classical measure-theoretic probability is typically only concerned with events of positive probability, making conditioning on events of probability zero a bit of a nuisance—which is typically circumvented by using a mathematical trick involving equivalence classes. Lastly, the variables for which measure-theoretic expectations are defined are always required to be measurable: an abstract assumption that, though mostly justified in practice, complicates the mathematical analysis considerably.

In this chapter, we try to do away with these issues, and consider a suitably adapted type of measure-theoretic global upper (and lower) expectation whose continuity properties, though less powerful than their precise measure-theoretic counterparts, are still considerably stronger than those of the finitary probability-based upper expectations from Section 3.3_{69}.

We start by considering the precise case, where local dynamics are described by a single precise probability tree, and deal with the latter two issues. Instead of turning the precise probability tree into a single probability measure on Ω—or a σ-algebra on Ω—, as is done classically, we turn the precise probability tree into a global (conditional) probability measure. Such a global probability measure is close in spirit to the notion of a full conditional probability measure [4]; it essentially specifies a probability measure on (an algebra on) Ω for each situation $s \in \mathscr{X}^*$, which describes the global dynamics of the stochastic process if we are sure that the path ω taken by the process will pass through this situation s; see Section 5.1_{220}. This course of reasoning is similar to—and inspired by—the one presented by Lopatatzidis [62, Chapter 3], and it enables us to meaningfully define measure-theoretic expectations conditional on situations that have probability zero. Subsequently, to extend the domain of these conditional measure-theoretic expectations from measurable variables to all global variables $\overline{\mathbb{V}}$, we propose two possible upper expectation operators, which can be seen as variations of the standard upper Lebesgue integral. We argue why they are suitable as extensions, and show that they are equivalent; this will be the topic of Section 5.2_{227}.

After an intermediate section (Section 5.3_{235}) on the properties of this common precise measure-theoretic upper (respectively lower) expectation, we generalise in Section 5.4_{240} towards an imprecise context. We do this in

[3] The continuity properties of measure-theoretic upper and lower **probabilities**, defined as upper and lower envelopes over sets of probability measures, were however already thoroughly studied by Krätschmer [59].

a straightforward and intuitive manner, similar as before in Section 3.3_{69}, by applying the foregoing 'precise' course of reasoning to each compatible precise probability tree, and then taking an upper (lower) envelope of all the measure-theoretic upper (lower) expectations obtained from these compatible precise probability trees. In that way, we obtain our desired 'imprecise' measure-theoretic upper (lower) expectation.

Apart from suggesting a suitable generalisation away from the classical measure-theoretic setting, the current chapter also aims to examine, on the one hand, the characteristic properties of the generalised measure-theoretic upper expectations thus obtained, and, on the other hand, the relation between these operators and the game-theoretic upper expectations discussed in the previous chapter. First, in Section 5.3_{235}, we consider the case where local models are precise; we prove that measure-theoretic upper expectations and game-theoretic upper expectations are then equal on their entire domain, thereby generalising—for finite state spaces—an earlier result [85, Theorem 9.3] by Shafer and Vovk. The most important properties of the 'precise' measure-theoretic upper expectation then follow immediately from this equality and the fact that they are already known to be satisfied by the game-theoretic upper expectation.

Section 5.4_{240} then, considers the imprecise case and, apart from introducing general measure-theoretic upper expectations as discussed above, presents a number of important properties for these measure-theoretic operators; e.g. extended coherence, a relation with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, continuity from below, and two specific types of continuity from above. Finally, in Section 5.5_{249}, we use these properties to establish an equality with the game-theoretic upper expectation on two complementary types of domains: the set of all bounded below measurable variables, and—if the local sets of probability mass functions are closed—the set of all monotone limits of finitary gambles. We argue that, together, these domains cover almost all practically relevant inferences, and therefore that, in practice, the two types of upper expectations—the measure-theoretic and the game-theoretic—can often be regarded as equivalent. We also discuss their relation in case they are not equivalent, and conclude the chapter with an overview on how, and in which aspects, our result generalises the results of Shafer and Vovk in [85, Chapter 9].

Concluding this introductory section, we want to mention that there is also an alternative route one may take in generalising classical measure-theoretic probability in order to deal with the need for imprecision; by using sub- or super-additive probability measures as a starting point instead of global probability measures, as in the classical case, or sets of them, as in our case. This approach was largely initiated by Choquet's work [6] on capacities and non-additive measures, and the study of these objects has

	local model	**global upper expectation**	
		finitary	continuity-based
behavioural	$\mathscr{A}_\bullet$ sets of acceptable gambles	$\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ from sets of acceptable gambles or martingales	$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$, $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ game-theoretic upper expectations
axiomatic	$\overline{\mathrm{Q}}_\bullet$ coherent upper expectations	$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extension under coherence	Chapter 6
probabilistic	$\mathscr{P}_\bullet$ sets of probability mass functions	$\overline{\mathrm{E}}_{\mathscr{P}}$ from finitely additive probabilities	$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ measure-theoretic upper expectations

Figure 5.1 Overview of the global upper expectations treated in this and previous chapters.

grown to be a major topic of interest; see e.g. [28, 31, 42, 43, 59]. However, the setback with using a single non-additive measure instead of a set of probability measures is that it penalizes generality considerably; see [106, Chapter 6] and [31, p.viii] for an elaborate treatment of the topic. Though we do not adopt this approach in the context of the current dissertation, some of our results in Section 5.5$_{249}$ nevertheless rely crucially on results from the theory of non-additive measures.

5.1 From charges to measures

In the current section, we aim to come to grips with the notions of countable additivity and probability measures, and show how they lead us to define global probability charges on domains considerably larger than those considered in Section 3.3$_{69}$.

5.1.1 The requirement of countable additivity

Recall Section 3.3$_{69}$, where we proposed a possible and straightforward method for extending a given imprecise probability tree $\mathscr{P}_\bullet$ to a global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$. The upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ was constructed in three steps; first, for all $p \sim \mathscr{P}_\bullet$, we considered a global—finitely additive—probability charge on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ [Definition 3.2$_{70}$] connected to p by Eq. (3.12)$_{72}$;

then, for each such global probability charge P_p, we defined a corresponding global upper expectation $\overline{\mathrm{E}}_p$ on $\mathbb{V}\times\mathscr{X}^*$ using the (upper) Lebesgue integral (or upper S-integral) [Definition 3.5_{78}]; and finally, we took an upper envelope of these global upper expectations over all $p \sim \mathscr{P}_\bullet$ to obtain our global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ corresponding to $\mathscr{P}_\bullet$. Though elegant as it may seem, this way of defining a global upper expectation—even only on global gambles—was not satisfactory. This became apparent when we considered Example $3.6.1_{99}$; it was shown that $\overline{\mathrm{E}}_p$ may lack elementary continuity properties, and that it therefore sometimes returns overly conservative values.

A possible way to perhaps remedy this issue, is to rely on the notion of **countable additivity** [4, 5, 56, 89] for (global) probability charges; this property strengthens the finite additivity condition [$\mathrm{GP3}_{70}$] to also apply to countable (disjoint) unions of events.

Definition 5.1 (Countable additivity)**.** A(n) (unconditional) probability charge P^{u} on an algebra $\mathcal{A} \subseteq \wp(\Omega)$ is called countably additive—or, σ-additive—if, for any sequence $(A_i)_{i\in\mathbb{N}}$ in $\mathcal{A}$ such that $\cup_{i\in\mathbb{N}}A_i \in \mathcal{A}$,

$$A_i \cap A_j = \emptyset \text{ for any } i \neq j \Rightarrow \mathrm{P}^{\mathrm{u}}(\cup_{i\in\mathbb{N}}A_i) = \textstyle\sum_{i\in\mathbb{N}} \mathrm{P}^{\mathrm{u}}(A_i).$$

Analogously, a global probability charge P on $\mathcal{A}\times\mathscr{X}^*$, with $\mathcal{A}$ an algebra such that $\langle\mathscr{X}^*\rangle \subseteq \mathcal{A} \subseteq \wp(\Omega)$, is called countably additive if, for all $s \in \mathscr{X}^*$, the (unconditional) probability charge $\mathrm{P}(\cdot|s)$ on $\mathcal{A}$ is countably additive. ⊚

Imposing countable additivity typically allows us to—uniquely—extend (global) probability charges to a larger domain, while still preserving fairly strong limit properties. Before we get into the details, let us first gather some intuition and see how the condition of countable additivity can be used to resolve the issue from Example $3.6.1_{99}$.

Example 5.1.1. Reconsider the precise probability tree p from Example $3.6.1_{99}$; then we have that $p(a|s) = 1$ and $p(b|s) = 0$ for all $s \in \mathscr{X}^*$. Let us consider two different global probability charges on $\wp(\Omega)\times\mathscr{X}^*$ that are related to p by Eq. $(3.12)_{72}$; a countably additive one and a finitely additive one. The first global probability charge P_1 is defined, for all $A \subseteq \Omega$ and $s \in \mathscr{X}^*$, by $\mathrm{P}_1(A|s) := 1$ if $saa\cdots \in A$, and $\mathrm{P}_1(A|s) := 0$ otherwise. To see that P_1 is a global probability charge, it suffices to check that $\mathrm{GP1}_{70}$–$\mathrm{GP4}_{70}$ are satisfied.

We only prove $\mathrm{GP4}_{70}$ and leave $\mathrm{GP1}_{70}$–$\mathrm{GP3}_{70}$ to the reader, as they are fairly straightforward. Fix any $A \subseteq \Omega$ and any $s, t \in \mathscr{X}^*$ such that $s \sqsubseteq t$. We need to show that $\mathrm{P}_1(A \cap \Gamma(t)|s) = \mathrm{P}_1(A|t)\mathrm{P}_1(t|s)$. Let $\ell := |t| - |s| \geq 0$ be the difference in length between the situations t and s. Then we either have that $t = sa^\ell$ and then $\Gamma(t) = \Gamma(sa^\ell)$ or otherwise, that $\Gamma(t) \cap \Gamma(sa^\ell) = \emptyset$. Suppose the former is true. Then $saa\cdots \in \Gamma(t)$ and so it follows from the definition of P_1 that $\mathrm{P}_1(t|s) = 1$. For any $A \subseteq \Omega$, we then also have that $taa\cdots = saa\cdots \in A$ if and only if $saa\cdots \in A \cap \Gamma(t)$, and therefore that $\mathrm{P}_1(A|t) = \mathrm{P}_1(A \cap \Gamma(t)|s)$, which together with $\mathrm{P}_1(t|s) = 1$ establishes GP4$_{70}$ for the case that $t = sa^\ell$. Now suppose on the other hand that $\Gamma(t) \cap \Gamma(sa^\ell) = \emptyset$. Then we clearly have that $saa\cdots \notin \Gamma(t)$ and therefore that $\mathrm{P}_1(t|s) = 0$ and that $\mathrm{P}_1(A \cap \Gamma(t)|s) = 0$, which immediately establishes GP4$_{70}$.

So, in summary, P_1 always satisfies GP4$_{70}$ (and GP1$_{70}$–GP3$_{70}$) and is therefore a global probability charge on $\wp(\Omega) \times \mathscr{X}^*$. It can moreover be checked easily that P_1 satisfies Eq. (3.12)$_{72}$. The (unconditional) probability of the event $\Omega \setminus \{aaa\cdots\}$ according to P_1 is equal to 0, and the (unconditional) probability of the singleton $\{aaa\cdots\}$ according to P_1 is furthermore equal to 1; these values are in line with our intuition.

Moreover, observe that P_1 is countably additive, and actually the **unique** countably additive global probability charge on $\wp(\Omega) \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$. That P_1 is countably additive is easy to check. To show that it is unique, consider any global probability charge P on $\wp(\Omega) \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$, and observe that the values of P on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ are uniquely determined by Proposition 3.3.4$_{73}$ [since the restriction of P to $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ is by Definition 3.2$_{70}$ again a global probability charge]. In particular, for any $s \in \mathscr{X}^*$, we have that $\mathrm{P}(sa^n b|s) = 0$ for all $n \in \mathbb{N}_0$. Note that $\Gamma(s) \setminus \{saa\cdots\} = \cup_{n\in\mathbb{N}_0}\Gamma(sa^n b)$, and therefore by countable additivity that $\mathrm{P}(\Gamma(s) \setminus \{saa\cdots\}|s) = \sum_{n\in\mathbb{N}_0} \mathrm{P}(sa^n b|s) = 0$. Hence, it now follows in a straightforward way from GP1$_{70}$–GP3$_{70}$ that $\mathrm{P}(A|s) = 1$ if $saa\cdots \in A$ and $\mathrm{P}(A|s) = 0$ otherwise, and thus indeed that $\mathrm{P} = \mathrm{P}_1$.

We next consider a second global probability charge P_2 on $\wp(\Omega) \times \mathscr{X}^*$ that is **not** countably additive. We start by defining the map P^* on $\mathscr{K} := (\langle\mathscr{X}^*\rangle \times \mathscr{X}^*) \cup \{(\{saa\cdots\}, s) \colon s \in \mathscr{X}^*\}$ by $\mathrm{P}^*(A|s) := \mathrm{P}_1(A|s)$ for any $(A, s) \in \langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ and $\mathrm{P}^*(saa\cdots|s) := 0$ for any $s \in \mathscr{X}^*$. Note that this definition is internally consistent because $\{saa\cdots\} \notin \langle\mathscr{X}^*\rangle$ for all $s \in \mathscr{X}^*$ due to Lemma 3.3.3$_{72}$. We aim to show that P^* can be extended to a global probability charge P_2 on the entire domain $\wp(\Omega) \times \mathscr{X}^*$. To this end, we will use the extension result [62, Theorem 8], which requires us to first show that P^* is a 'coherent conditional probability' according to [62, Definition 5]. This means checking a condition similar to Proposition 3.3.1(iii)$_{71}$: we need to check that, for all $n \in \mathbb{N}$, all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ and all $(A_1, s_1), \ldots, (A_n, s_n) \in \mathscr{K}$,

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i}\left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i)\right) \Big| \cup_{i=1}^{n} \Gamma(s_i)\Big) \geq 0.$$

By definition, the map P^* coincides with P_1 on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$. As a result, because we know that the global probability charge P_1 satisfies the condition above on its entire domain [since it is a global probability charge and due to Proposition 3.3.1], we have that P^* satisfies the inequality above if $(A_1, s_1), \dots, (A_n, s_n) \in \langle \mathscr{X}^* \rangle \times \mathscr{X}^* \subset \mathscr{K}$. So it remains to check that, for any $n \in \mathbb{N}_0$, any $m \in \mathbb{N}$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_m \in \mathbb{R}$, $(A_1, s_1), \dots, (A_n, s_n) \in \langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ and $t_1, \dots, t_m \in \mathscr{X}^*$,

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i) \right) \\ + \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \left(\mathbb{I}_{t_j aa\cdots} - \mathrm{P}^*(t_j aa\cdots|t_j) \right) \Big| \cup_{i=1}^{n} \Gamma(s_i) \bigcup \cup_{j=1}^{m} \Gamma(t_j) \Big) \geq 0.$$

By definition, we have that $\mathrm{P}^*(t_j aa\cdots|t_j) = 0$ for all $j = \{1, \dots, m\}$, and therefore that

$$\sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \left(\mathbb{I}_{t_j aa\cdots} - \mathrm{P}^*(t_j aa\cdots|t_j) \right) = \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \mathbb{I}_{t_j aa\cdots} = \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{I}_{t_j aa\cdots}. \tag{5.1}$$

So, if $n = 0$, we indeed obtain that

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i) \right) \\ + \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \left(\mathbb{I}_{t_j aa\cdots} - \mathrm{P}^*(t_j aa\cdots|t_j) \right) \Big| \cup_{i=1}^{n} \Gamma(s_i) \bigcup \cup_{j=1}^{m} \Gamma(t_j) \Big) \\ = \sup\Big(\sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{I}_{t_j aa\cdots} \Big| \cup_{j=1}^{m} \Gamma(t_j) \Big) \geq 0.$$

If $n \geq 1$, it suffices to show that

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i) \right) \\ + \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \left(\mathbb{I}_{t_j aa\cdots} - \mathrm{P}^*(t_j aa\cdots|t_j) \right) \Big| \big(\cup_{i=1}^{n} \Gamma(s_i) \big) \Big) \geq 0,$$

or, even stronger, to show that

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i) \right) \\ + \sum_{j=1}^{m} \tilde{\lambda}_j \mathbb{1}_{t_j} \left(\mathbb{I}_{t_j aa\cdots} - \mathrm{P}^*(t_j aa\cdots|t_j) \right) \Big| \big(\cup_{i=1}^{n} \Gamma(s_i) \big) \setminus \big(\cup_{j=1}^{m} \{t_j aa\cdots\} \big) \Big) \geq 0.$$

Using Eq. (5.1) and taking into account that the supremum above is taken over a set that does not contain any of the paths $t_j aa\cdots$, the desired inequality follows if we manage to show that

$$\sup\Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i) \right) \Big| \big(\cup_{i=1}^{n} \Gamma(s_i) \big) \setminus \big(\cup_{j=1}^{m} \{t_j aa\cdots\} \big) \Big) \geq 0. \tag{5.2}$$

To this end, observe by Lemma 3.3.3$_{72}$ that the variable $\mathbb{I}_{A_i}$ for all $i = \{1, \dots, n\}$, and thus also the variable $\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i)\right)$ is finitary. So it is ℓ-measurable for some $\ell \in \mathbb{N}_0$. By our earlier considerations, we already know that

$$\sup \Big(\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i)\right) \Big| \cup_{i=1}^{n} \Gamma(s_i) \Big) \geq 0.$$

So since $\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i)\right)$ is ℓ-measurable—and thus only takes a finite number of different values—there is an $x_{1:\ell} \in \mathscr{X}^{\ell}$ such that $\sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i} \left(\mathbb{I}_{A_i} - \mathrm{P}^*(A_i|s_i)\right)$ is larger than or equal to 0 on all paths $\omega \in \Gamma(x_{1:\ell})$. Hence, since there are (infinitely many) paths in $\Gamma(x_{1:\ell})$ different from the paths $t_j aa \cdots$ for $j = \{1, \dots, m\}$, it is clear that Eq. (5.2)$_{\curvearrowleft}$ indeed holds. So P^* is a 'coherent conditional probability' according to [62, Definition 5].

Now, [62, Theorem 8] says that P^* can be extended to a 'coherent conditional probability' P_2 on the entire domain $\wp(\Omega) \times \mathscr{X}^*$. It then follows from [62, Definition 5] that P_2 satisfies Proposition 3.3.1(iii)$_{71}$, and thus by Proposition 3.3.1(i)$_{71}$ that P_2 is a global probability charge on $\wp(\Omega) \times \mathscr{X}^*$. It is also obvious that P_2 satisfies Eq. (3.12)$_{72}$. Since P_1 is the unique countably additive global probability charge on $\wp(\Omega) \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$, and since P_2 cannot be equal to P_1, we infer that P_2 is not countably additive.

So, in summary, there exists a global probability charge P_2 on $\wp(\Omega) \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$ and that is not countably additive, but this global probability charge returns values that contradict our intuition; it assigns (unconditional) probability zero to the path $aaa \cdots$, and [by GP1$_{70}$–GP3$_{70}$] assigns (unconditional) probability one to $\Omega \setminus \{aaa \cdots\}$. On the other hand, the unique countably additive global probability charge P_1 returns values that are in line with our intuition. ◇

In light of our findings above, it seems that a global upper expectation based on countably additive global probability charges will prove to be a more informative and adequate global model compared to the finite-additivity-based upper expectations $\overline{\mathrm{E}}_p$ or $\overline{\mathrm{E}}_{\mathscr{P}}$. In order to define such a global model, we need to delve into the somewhat abstract world that is called measure theory.

5.1.2 Probability measures and Carathéodory's extension theorem

In Example 5.1.1$_{221}$, we considered a countably additive global probability charge P_1 on the domain $\wp(\Omega) \times \mathscr{X}^*$ that is compatible with the (im)precise probability tree p according to Eq. (3.12)$_{72}$. For general precise probability trees, however, such a compatible countably additive global probability charge on $\wp(\Omega) \times \mathscr{X}^*$ does not necessarily exist. This essentially follows from the work of Vitali [108] and others on the nature of non-measurable sets. For instance, consider the precise probability tree p, defined by $p(x|s) := 1/|\mathscr{X}|$ for all $x \in \mathscr{X}$ and all $s \in \mathscr{X}^*$, and the corresponding unique probability charge $\mathrm{P}_p(\cdot|\square)$ on $\langle\mathscr{X}^*\rangle$ that is described by Proposition 3.3.4$_{73}$. This finitely additive probability charge $\mathrm{P}_p(\cdot|\square)$ on $\langle\mathscr{X}^*\rangle$ is automatically countably additive [5, Theorem 2.3], but it cannot be extended to a countably additive probability charge on $\wp(\Omega)$.[4,5] We refer the reader to [70, Chapter 5] and [5, Section 3] for a didactic treatment on this topic.

Though a countably additive global probability charge on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ cannot always be extended to the entire domain $\wp(\Omega) \times \mathscr{X}^*$, we can always extend it to a smaller domain that is still considerably larger than $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$: the domain of all events in the σ-algebra generated by $\langle\mathscr{X}^*\rangle$ (and all situations). In general, a **σ-algebra** $\mathscr{A}$ on a non-empty set $\mathscr{Y}$ is an algebra on $\mathscr{Y}$ that is closed under countable unions:

$$(\forall i \in \mathbb{N})\, A_i \in \mathscr{A} \Rightarrow (\cup_{i\in\mathbb{N}} A_i) \in \mathscr{A}.$$

Since any algebra is closed under taking complements, a σ-algebra is also closed under countable intersections. For any algebra $\mathscr{B}$, we use $\sigma(\mathscr{B})$ to denote the smallest σ-algebra that includes $\mathscr{B}$ and call $\sigma(\mathscr{B})$ the σ-algebra generated by $\mathscr{B}$; the algebra $\sigma(\mathscr{B})$ always exists because any arbitrary intersection of σ-algebras is itself a σ-algebra [5, Section 2].[6] In particular, we use $\sigma(\mathscr{X}^*)$ to denote the σ-algebra generated by $\langle\mathscr{X}^*\rangle$. Any element A of a σ-algebra $\mathscr{A}$ will be called $\mathscr{A}$-measurable. The following definition of a probability measure is standard; see [4, 5, 56, 89].

Definition 5.2 (Probability measures & global probability measures)**.** A countably additive (unconditional) probability charge P^{u} on a σ-algebra $\mathscr{A} \subseteq \wp(\Omega)$ is called a probability measure. Analogously, a countably additive global probability charge P on $\mathscr{A} \times \mathscr{X}^*$, with $\mathscr{A}$ a σ-algebra such that $\langle\mathscr{X}^*\rangle \subseteq \mathscr{A} \subseteq \wp(\Omega)$, is called a global probability measure.[7] ◎

[4] At least, if we adopt the **continuum hypothesis** and the **axiom of choice (AC)**.

[5] For if $\mathrm{P}_p(\cdot|\square)$ would be endowed with a countably additive extension to the entire powerset $\wp(\Omega)$, then it can be derived that each singleton in Ω must have probability zero, and thus by [70, Theorem 5.6] that all sets in $\wp(\Omega)$ must have probability zero. But this is in contradiction with the fact that $\mathrm{P}_p(\cdot|\square)$ is a probability charge, because this demands that $\mathrm{P}_p(\Omega|\square) = 1$; see also [5, p. 45–46].

[6] And, of course, because there is at least one σ-algebra including $\mathscr{B}$, namely the powerset $\wp(\Omega)$.

[7] A general measure, the main object of interest in measure theory, is not necessarily normed and need not take values in $[0, 1]$. Instead, it takes values in $\overline{\mathbb{R}}_{\geq}$ and is only required to satisfy countable additivity. Measures can be seen as generalised notions of 'length' or 'volume' for abstract spaces.

Now, as claimed earlier, given any countably additive global probability charge P on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$, we can extend P to the domain $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ and preserve countable additivity—in fact, such an extension will even always be **unique**. This follows immediately from the famous extension theorem by Constantin Carathéodory [5, 89, 112].

Theorem 5.1.2 (Carathéodory's extension theorem)**.** *For any countably additive probability charge* P' *on an algebra* $\mathcal{A} \subseteq \wp(\Omega)$*, there is a unique probability measure* P *on* $\sigma(\mathcal{A})$ *such that* $\mathrm{P}'(A) = \mathrm{P}(A)$ *for all* $A \in \mathcal{A}$*. In particular, this probability measure* P *is given, for all* $A \in \sigma(\mathcal{A})$*, by*

$$\mathrm{P}(A) = \inf\Big\{\sum_{i\in\mathbb{N}} \mathrm{P}'(A_i) \colon A_i \in \mathcal{A} \text{ and } A \subseteq \cup_{i\in\mathbb{N}} A_i\Big\}.$$

Proof. The existence and uniqueness of P follow from [5, Theorem 3.1]. The expression for P follows from the discussion in [5, p.37–41]. □

Combined with Proposition $3.3.4_{73}$, this extension theorem leads us to the following central conclusion.

Proposition 5.1.3. *For any precise probability tree p, there is a unique global probability measure* P_p *on* $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ *that satisfies Eq.* $(3.12)_{72}$.

Proof. Due to Proposition $3.3.4_{73}$, there is a unique global probability charge P'_p on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ satisfying Eq. $(3.12)_{72}$. It moreover follows from [5, Theorem 2.3] that, for any $s \in \mathscr{X}^*$, the unconditional probability charge $\mathrm{P}'_p(\cdot|s)$ on $\langle\mathscr{X}^*\rangle$ is countably additive. Hence, we can apply Theorem 5.1.2 to each $\mathrm{P}'_p(\cdot|s)$ individually, to obtain a unique global probability measure P_p on $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ that extends the global probability charge P'_p. The global measure P_p then clearly also satisfies Eq. $(3.12)_{72}$. To see that P_p is moreover the only global probability measure P_p on $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ that satisfies Eq. $(3.12)_{72}$, assume that there is a second global probability measure P_2 on $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ satisfying Eq. $(3.12)_{72}$. Then it follows from Proposition $3.3.1(\mathrm{i})_{71}$ [by applying it twice] that the restriction of P_2 to $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ is a global probability charge. This restriction also clearly needs to satisfy Eq. $(3.12)_{72}$. Due to the uniqueness of P'_p, it follows that this restriction is equal to P'_p, and hence by the construction of P_p and Theorem 5.1.2 we have that $\mathrm{P}_2 = \mathrm{P}_p$. □

Observe that the restriction of this unique global probability measure P_p to $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ then still takes the intuitive form as described by Proposition $3.3.4_{73}$. Additionally, one may also note that the probability measure $\mathrm{P}_p(\cdot|\square)$ corresponding to the initial situation $\square$ could have just as well be obtained from applying Ionescu-Tulcea's extension theorem [89, Theorem 2.9.2] to the local probability mass functions $p(\cdot|s)$.

5.2 Measure-theoretic upper expectations for precise probability trees

To obtain global linear expectations and upper expectations from global probability measures, we will use the well-known Lebesgue integral. It is probably the type of integral used most commonly in the modern measure-theoretic probability theory, and largely owes this status to the fact that it has considerably stronger continuity properties compared to other well-known integrals, e.g. the Riemann/Darboux integral. Moreover, our use of the Lebesgue integral here is in line with our approach in Section 3.3.3$_{74}$, where our choice of integral—the S-integral—was completely equivalent to the Lebesgue integral. As was already mentioned there, the Lebesgue integral introduced here for global probability measures and measurable (extended real) variables will—perhaps counter-intuitively—be closer in appearance to the S-integral from Definition 3.3$_{76}$ than to the Lebesgue integral presented in Troffaes & De Cooman [106, Definition 8.27] (see also Proposition 3.3.6$_{76}$).

The measure-theoretic concepts that will be used in the following sections—including the definition of the Lebesgue integral—are immediately adapted to our specific stochastic processes setting; we refer to Appendix 5.A$_{263}$ for a more general, and perhaps more familiar introduction of some of these concepts.

5.2.1 Measurable variables

Central to the definition of the Lebesgue integral is the concept of measurability for extended real variables; for any σ-algebra $\mathcal{A} \subseteq \wp(\Omega)$, a variable $f \in \overline{\mathbb{V}}$ is called **$\mathcal{A}$-measurable** if $\{\omega \in \Omega \colon f(\omega) \leq c\} \in \mathcal{A}$ for all $c \in \mathbb{R}$.[8] If it is clear from the context which algebra we are considering, we will simply call f **measurable**. The notion of measurability for extended real variables with respect to σ-algebras that we have just introduced is in accordance—in the sense that it extends—the earlier notion from Section 3.3.3$_{74}$, where measurability of a **gamble** was introduced as the requirement that it should be the uniform limit of a sequence of simple gambles; this can be deduced from [106, Proposition 1.19] and [106, Definition 1.17] (and the considerations in Appendix 5.A$_{263}$). We also extend the notion of being $\mathcal{A}$-simple to non-negative extended real variables: for any algebra $\mathcal{A} \subseteq \wp(\Omega)$, a non-negative variable $f \in \overline{\mathbb{V}}$ is called **$\mathcal{A}$-simple** if it

[8]As is shown in Appendix 5.A$_{263}$, this notion of measurability for extended real variables is equivalent to the standard notion where one uses the inverse image of sets in the Borel σ-algebra on $\overline{\mathbb{R}}$.

is a finite sum $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ with $a_1, \dots, a_n \in \overline{\mathbb{R}}_{\geq}$ and $A_1, \dots, A_n \in \mathcal{A}$ [102, Definition 1.3.2].

We gather all $\sigma(\mathscr{X}^*)$-measurable global variables in the set $\overline{\mathbb{V}}_\sigma$, and we let $\mathbb{V}_\sigma$ and $\overline{\mathbb{V}}_{\sigma,\mathrm{b}}$ be the respective subsets of all bounded and bounded below ones. The following result shows that the set $\overline{\mathbb{V}}_\sigma$ is closed under pointwise countable infima and suprema, and that it is closed under pointwise convergence. To state the result, for any sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{V}}$, we let $\bigvee_{n\in\mathbb{N}} f_n(\omega) := \sup\{f_n(\omega) : n \in \mathbb{N}\}$ and $\bigwedge_{n\in\mathbb{N}} f_n(\omega) := \inf\{f_n(\omega) : n \in \mathbb{N}\}$ for all $n \in \mathbb{N}$.

Proposition 5.2.1. *For any sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{V}}_\sigma$, we have that*

MV1. $\bigvee_{n\in\mathbb{N}} f_n \in \overline{\mathbb{V}}_\sigma$ *and* $\bigwedge_{n\in\mathbb{N}} f_n \in \overline{\mathbb{V}}_\sigma$.

MV2. $\liminf_{n\to+\infty} f_n \in \overline{\mathbb{V}}_\sigma$, $\limsup_{n\to+\infty} f_n \in \overline{\mathbb{V}}_\sigma$, *and if* $\lim_{n\to+\infty} f_n$ *exists then* $\lim_{n\to+\infty} f_n \in \overline{\mathbb{V}}_\sigma$.

Proof. Since $\{\omega \in \Omega : \bigvee_{n\in\mathbb{N}} f_n(\omega) \leq c\} = \bigcap_{n\in\mathbb{N}}\{\omega \in \Omega : f_n(\omega) \leq c\}$ and $\{\omega \in \Omega : \bigwedge_{n\in\mathbb{N}} f_n(\omega) \leq c\} = \cup_{n\in\mathbb{N}}\{\omega \in \Omega : f_n(\omega) \leq c\}$, MV1 follows immediately from the fact that the class of all $\sigma(\mathscr{X}^*)$-measurable sets is closed under countable unions and intersections; see also [89, p. 209]. MV2 also follows from MV1 since $\liminf_{n\to+\infty} f_n = \bigvee_{n\in\mathbb{N}} \bigwedge_{m\geq n} f_m$ and $\limsup_{n\to+\infty} f_n = \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n} f_m$, and since $\lim_{n\to+\infty} f_n$ is simply a special case of $\liminf_{n\to+\infty} f_n$ or $\limsup_{n\to+\infty} f_n$. □

5.2.2 Measure-theoretic expectations

For the definition of the Lebesgue integral, we follow Billingsley [5, Chapter 3],[9,10] yet immediately limit ourselves to integrals over Ω and with respect to probability measures on $\sigma(\mathscr{X}^*)$:

Definition 5.3 (The Lebesgue integral)**.** Consider any probability measure P on $\sigma(\mathscr{X}^*)$, and any non-negative $f \in \overline{\mathbb{V}}_\sigma$. Then the Lebesgue integral of f with respect to P is defined as

$$\int f \mathrm{dP} := \sup\left\{\sum_{i=1}^{n} \inf(f|A_i)\mathrm{P}(A_i) : A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^{n} \text{ partitions } \Omega\right\}. \tag{5.3}$$

[9] Many slightly different, yet equivalent versions of the definition of the Lebesgue integral can be found in the literature; e.g. compare [5, Chapter 3] with [89, Section 2.6]. That these versions are indeed equivalent is illustrated by Proposition 5.2.2$_{\rightarrow}$, and also clarified at the end of [5, Chapter 3, Section 15] and, for bounded non-negative variables, in [89, Remark 2.6.6].

[10] This version of the Lebesgue integral bases itself (for non-negative variables) solely on the lower (Lebesgue) integral, and does not demand equivalence with the upper (Lebesgue) integral. This is in contrast to the procedures from Definition 3.3$_{76}$ and Proposition 3.3.6$_{76}$. A clarification for this is given in Footnote 12$_{232}$.

For a general $f \in \overline{\mathbb{V}}_\sigma$, we let $f^+ := f^{\vee 0}$ and $f^- := -(f^{\wedge 0})$, and the Lebesgue integral is then defined by

$$\int f \mathrm{dP} := \int f^+ \mathrm{dP} - \int f^- \mathrm{dP},$$

unless $\int f^+ \mathrm{dP} = \int f^- \mathrm{dP} = +\infty$, in which case the Lebesgue integral of f with respect to P is not defined. ◎

We say that the Lebesgue integral $\int f \mathrm{dP}$ of a variable $f \in \overline{\mathbb{V}}_\sigma$ exists, simply if it is defined. If both $\int f^+ \mathrm{dP}$ and $\int f^- \mathrm{dP}$ are real, then f is called P-integrable. Confusingly enough, the integral $\int f \mathrm{dP}$ may still exist if f is not P-integrable; it suffices that either $\int f^+ \mathrm{dP} \neq +\infty$ or $\int f^- \mathrm{dP} \neq +\infty$; note that $\int f^+ \mathrm{dP} = -\infty$ or $\int f^- \mathrm{dP} = -\infty$ is impossible because the Lebesgue integral is clearly non-negative for non-negative variables. Alternatively, we could have also defined the Lebesgue integral (for non-negative variables) as a supremum over all the non-negative $\sigma(\mathscr{X}^*)$-simple variables smaller or equal than f—an expression that is similar to the one in Proposition 3.3.6$_{76}$ and to the one of the Lebesgue integral in [81, 89, 102]. The following result is well-known to hold, yet because we did not find a suitable reference for it, we provide an explicit proof for it here.

Proposition 5.2.2. *For any probability measure* P *on* $\sigma(\mathscr{X}^*)$*, the following statements hold.*

(i) *For any non-negative $\sigma(\mathscr{X}^*)$-simple variable f, and $\sum_{i=1}^n a_i \mathbb{I}_{A_i}$ any representation of f,*

$$\int f \mathrm{dP} = \sum_{i=1}^n a_i \mathrm{P}(A_i)$$

(ii) *For any general non-negative $f \in \overline{\mathbb{V}}_\sigma$,*

$$\int f \mathrm{dP} = \sup\left\{\int g \mathrm{dP} : g \textit{ is } \sigma(\mathscr{X}^*)\textit{-simple and } 0 \leq g \leq f\right\}.$$

Proof. To prove (i), suppose that f is non-negative and $\sigma(\mathscr{X}^*)$-simple and that $\sum_{i=1}^n a_i \mathbb{I}_{A_i}$ is a representation for f. Then by [5, Theorem 15.1.(iv)] we have that $\mathrm{E}(f) = \sum_{i=1}^n a_i \mathrm{E}(\mathbb{I}_{A_i})$, which by [5, Theorem 15.1.(i)] in turn implies that $\mathrm{E}(f) = \sum_{i=1}^n a_i \mathrm{P}(A_i)$ as desired.

To see that (ii) holds, fix any general non-negative $f \in \overline{\mathbb{V}}_\sigma$ and start by observing that, for any partition $(A_i)_{i=1}^n$ of Ω such that $A_i \in \sigma(\mathscr{X}^*)$ for all $i \in \{1, \dots, n\}$, we trivially have that $\sum_{i=1}^n \inf(f|A_i)\mathbb{I}_{A_i}$ is a non-negative $\sigma(\mathscr{X}^*)$-simple variable smaller or equal than f. Due to property (i), we thus have that $\sum_{i=1}^n \inf(f|A_i)\mathrm{P}(A_i) =$

$\int \sum_{i=1}^n \inf(f|A_i)\mathbb{I}_{A_i}\mathrm{dP}$. As a consequence, it follows that

$$\int f\mathrm{dP} = \sup\left\{\sum_{i=1}^n \inf(f|A_i)\mathrm{P}(A_i) \colon A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}$$
$$\leq \sup\left\{\int g\mathrm{dP} \colon g \text{ is } \sigma(\mathscr{X}^*)\text{-simple and } 0 \leq g \leq f\right\}.$$

The converse inequality now trivially follows from the fact that, as can be observed from Definition 5.3_{228}, the Lebesgue integral is monotone with respect to non-negative variables in $\overline{\mathbb{V}}_\sigma$. □

We next use the Lebesgue integral to define the global linear expectation corresponding to a precise probability tree.

Definition 5.4 (Global measure-theoretic expectations)**.** Consider any precise probability tree p, let P_p be the unique global probability measure from Proposition $5.1.3_{226}$, and let $\mathrm{P}_p^{|s} := \mathrm{P}_p(\cdot|s)$ for any $s \in \mathscr{X}^*$. Then the global measure-theoretic expectation $\mathrm{E}_{p,\mathrm{M}}$ is defined by $\mathrm{E}_{p,\mathrm{M}}(f|s) := \int f\mathrm{dP}_p^{|s}$ for all $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that $\int f\mathrm{dP}_p^{|s}$ exists. ⊚

Note that, in defining the expectation $\mathrm{E}_{p,\mathrm{M}}(\cdot|s)$, we integrate with respect to the unconditional probability measure $\mathrm{P}_p(\cdot|s)$, which was obtained from a forward construction only involving the probabilities $p(\cdot|\,t)$ for which t followed s; recall Eq. $(3.12)_{72}$ and Proposition $3.3.4_{73}$. This should be contrasted with the more traditional measure-theoretic approach, where conditional probabilities and expectations are derived from a single unconditional probability measure using Bayes' rule and/or the Radon-Nikodym derivative; also see Appendix $5.\mathrm{A}_{263}$. As mentioned below Proposition $3.3.4_{73}$, our alternative approach allows us to condition in a meaningful way on situations of probability zero. Remark, however, that the unconditional expectation $\mathrm{E}_{p,\mathrm{M}}(\cdot) := \mathrm{E}_{p,\mathrm{M}}(\cdot|\square)$ is completely equivalent to the usual unconditional expectation used in standard measure-theoretic probability theory.

As mentioned earlier, the prominent role of the Lebesgue integral in modern probability theory is mainly due to its mathematically convenient properties—in particular, to its strong continuity properties. Since $\mathrm{E}_{p,\mathrm{M}}$ is defined in terms of this Lebesgue integral, it inherits these properties. We next list some of the properties that will be used in the main text; we refer to Appendix $5.\mathrm{A}_{263}$ for a more complete overview. Note in particular that ME3 below confirms that $\mathrm{E}_{p,\mathrm{M}}$ is linear (in its first argument).

Proposition 5.2.3. *For any precise probability tree p and any $s \in \mathscr{X}^*$, the following statements hold:*

ME2. *if $f \leq g$ then $\mathrm{E}_{p,\mathrm{M}}(f|s) \leq \mathrm{E}_{p,\mathrm{M}}(g|s)$ for all $f, g \in \overline{\mathbb{V}}_\sigma$ such that $\mathrm{E}_{p,\mathrm{M}}(f|s)$ and $\mathrm{E}_{p,\mathrm{M}}(g|s)$ exist.*

ME3. $\mathrm{E}_{p,\mathrm{M}}(af + bg|s) = a\mathrm{E}_{p,\mathrm{M}}(f|s) + b\mathrm{E}_{p,\mathrm{M}}(g|s)$ *for all* $a, b \in \mathbb{R}$ *and all* $f, g \in \overline{\mathbb{V}}_\sigma$ *that are* $\mathrm{P}_p^{|s}$*-integrable.*

ME4. $\mathrm{E}_{p,\mathrm{M}}(f|s)$ *exists for all* $f \in \overline{\mathbb{V}}_\sigma$ *that are bounded below or above.*

ME5. $\mathrm{E}_{p,\mathrm{M}}(f|s)$ *is real and* f *is* $\mathrm{P}_p^{|s}$*-integrable for all (bounded)* $f \in \mathbb{V}_\sigma$.

ME6. $-\mathrm{E}_{p,\mathrm{M}}(f|s) = \mathrm{E}_{p,\mathrm{M}}(-f|s)$ *for all* $f \in \overline{\mathbb{V}}_\sigma$ *such that* $\mathrm{E}_{p,\mathrm{M}}(f|s)$*, or equivalently* $\mathrm{E}_{p,\mathrm{M}}(-f|s)$*, exists.*

ME7. *Consider any sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$ *that converges pointwise to a variable* $f \in \overline{\mathbb{V}}_\sigma$*. If there is a* $\mathrm{P}_p^{|s}$*-integrable variable* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $|f_n| \leq f^*$ *for all* $n \in \mathbb{N}$*, then* f *and all* f_n *are* $\mathrm{P}_p^{|s}$*-integrable and*

$$\lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f_n|s) = \mathrm{E}_{p,\mathrm{M}}(f|s).$$

ME8. *Consider any increasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$*. If there is an* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $\mathrm{E}_{p,\mathrm{M}}(f^*|s) > -\infty$ *and* $f_1 \geq f^*$*, then*

$$\lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f_n|s) = \mathrm{E}_{p,\mathrm{M}}(f|s) \textit{ where } \lim_{n\to+\infty} f_n = f.$$

ME9. *Consider any decreasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$*. If there is an* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $\mathrm{E}_{p,\mathrm{M}}(f^*|s) < +\infty$ *and* $f_1 \leq f^*$*, then*

$$\lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f_n|s) = \mathrm{E}_{p,\mathrm{M}}(f|s) \textit{ where } \lim_{n\to+\infty} f_n = f.$$

ME10. $\mathrm{E}(f + \mu) = \mathrm{E}(f) + \mu$ *for all* $\mu \in \mathbb{R}$ *and all* $f \in \overline{\mathbb{V}}_\sigma$ *that are bounded below.*

Proof. See Lemma 5.A.1$_{264}$ in Appendix 5.A$_{263}$. □

5.2.3 Beyond measurable variables

Next, we want to drop the constraint of $\sigma(\mathscr{X}^*)$-measurability and extend our global measure-theoretic expectation $\mathrm{E}_{p,\mathrm{M}}$ to a global operator that is defined on all global variables (and conditional situations). It is unconventional to do so in standard measure-theoretic probability because this means giving up linearity of the resulting global (upper/lower) expectation operator. However, as already mentioned in the paragraph above Definition 3.5$_{78}$, this does not concern us, since we will work with imprecise local models—and thus also imprecise global models—in the end anyway.

As far as we know, there is however no real consensus on how measure-theoretic expectations should be extended from measurable variables to non-measurable variables. One possible approach would be—if for the moment, we limit ourselves to non-negative variables and upper expectations—to simply use the formula from Eq. (5.3)$_{228}$ and apply it to the entire domain of all non-negative variables. The problem with this approach is that the resulting operator—which is typically called the **lower Lebesgue integral—** would take the form of a lower expectation rather than that of an upper expectation; e.g. it can easily be deduced from the expressions in Proposition 5.2.2$_{229}$ (and the fact that the supremum in Proposition 5.2.2(ii)$_{229}$ remains equal to the supremum in Eq. (5.3)$_{228}$ for non-measurable f) that the lower Lebesgue integral would be super-additive instead of sub-additive. Hence, a possible alternative would then be to use the conjugate **upper Lebesgue integral**, described by

$$\overline{\int} f \mathrm{dP} = \inf \left\{ \sum_{i=1}^{n} \sup(f|A_i)\mathrm{P}(A_i) \colon A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^{n} \text{ partitions } \Omega \right\}$$
$$= \inf \left\{ \int g \mathrm{dP} \colon g \text{ is } \sigma(\mathscr{X}^*)\text{-simple and } g \geq f \right\},$$

for any non-negative $f \in \overline{\mathbb{V}}$ and any general probability measure P on $\sigma(\mathscr{X}^*)$.[11] The problem here is that $\overline{\int} f \mathrm{dP}$ does not necessarily coincide with the standard (lower) Lebesgue integral from Definition 5.3$_{228}$ on non-negative $\sigma(\mathscr{X}^*)$-measurable variables; see [5, Problem 15.1] or [89, p.244].[12] Since the (lower) Lebesgue integral, and thus $\mathrm{E}_{p,\mathrm{M}}$, always exists on this domain, the upper Lebesgue integral too cannot be used as a device for extending the expectation $\mathrm{E}_{p,\mathrm{M}}$.

Because none of the options above are satisfactory, we propose an extension of our own. It is strongly inspired by the upper Lebesgue integral but adapted to not only approximate from above by $\sigma(\mathscr{X}^*)$-simple variables, but also by more general variables from the domain of the Lebesgue integral. Our extension is moreover introduced in two slightly different, yet equivalent ways.

Definition 5.5 (Global measure-theoretic upper expectations for precise probability trees)**.** Consider any precise probability tree p. Let $\overline{\mathrm{E}}^{1}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{2}_{p,\mathrm{M}}$ be defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

(i) $\overline{\mathrm{E}}^{1}_{p,\mathrm{M}}(f|s) := \inf \left\{ \mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f \right\}$;

(ii) $\overline{\mathrm{E}}^{2}_{p,\mathrm{M}}(f|s) := \inf \left\{ \mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma},\ \mathrm{E}_{p,\mathrm{M}}(g|s) \text{ exists and } g \geq f \right\}$.

[11] That these two expressions are equivalent can be deduced in an analogous way as how we proved Proposition 5.2.2$_{229}$.

[12] This is also the reason why, in contrast to Definition 3.3$_{76}$, the Lebesgue integral for unbounded or extended real-valued functions is usually not defined as the common upper/lower Lebesgue integral (if both the upper and lower Lebesgue integral exist and are equal). The latter is sometimes called the Darboux-Young approach [89, p.244].

Then $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^2_{p,\mathrm{M}}$ are equal and the common operator is what we refer to as the global measure-theoretic upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$. ◎

Note that it is valid to write $\mathrm{E}_{p,\mathrm{M}}(g|s)$ for all $g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$ in the definition of $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ because $\mathrm{E}_{p,\mathrm{M}}(g|s)$ always exists for such a bounded below variable g [$\mathrm{ME4}_{231}$].

Our proof of the equality between $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^2_{p,\mathrm{M}}$ uses the following lemma, which guarantees that $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ is an extension of $\mathrm{E}_{p,\mathrm{M}}$.

Lemma 5.2.4. *For any precise probability tree p, we have that $\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) = \mathrm{E}_{p,\mathrm{M}}(f|s)$ for all $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that $\mathrm{E}_{p,\mathrm{M}}(f|s)$ exists.*

Proof. Fix any $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that $\mathrm{E}_{p,\mathrm{M}}(f|s)$ exists. By the monotonicity [$\mathrm{ME2}_{230}$] of $\mathrm{E}_{p,\mathrm{M}}$, we have that

$$\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) = \inf\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\} \geq \mathrm{E}_{p,\mathrm{M}}(f|s). \tag{5.4}$$

To see that the converse inequality holds, observe that, since $\mathrm{E}_{p,\mathrm{M}}(f|s)$—and thus $\int f \mathrm{d}\mathrm{P}_p^{|s}$—exists, we have that $\mathrm{E}_{p,\mathrm{M}}(f^+|s)$ or $\mathrm{E}_{p,\mathrm{M}}(f^-|s)$ is real. Hence, if $\mathrm{E}_{p,\mathrm{M}}(f^+|s) = +\infty$ then $\mathrm{E}_{p,\mathrm{M}}(f^-|s) \in \mathbb{R}$ and therefore $\mathrm{E}_{p,\mathrm{M}}(f|s) = +\infty$, and the converse inequality then follows trivially. We proceed to show that the desired converse inequality also holds if $\mathrm{E}_{p,\mathrm{M}}(f^+|s) < +\infty$.

Consider the decreasing sequence $(f^{\vee -n})_{n\in\mathbb{N}}$ of lower cuts of f. Then, for any $n \in \mathbb{N}$, the global variable $f^{\vee -n}$ is bounded below and $\sigma(\mathscr{X}^*)$-measurable [$\mathrm{MV1}_{228}$]. Hence, indeed,

$$\begin{aligned}\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) &= \inf\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\}\\ &\leq \inf_{n\in\mathbb{N}} \mathrm{E}_{p,\mathrm{M}}(f^{\vee -n}|s) = \lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f^{\vee -n}|s) = \mathrm{E}_{p,\mathrm{M}}(f|s),\end{aligned}$$

where the second equality follows from the decreasing character of the sequence $(f^{\vee -n})_{n\in\mathbb{N}}$ and the monotonicity [$\mathrm{ME2}_{230}$] of $\mathrm{E}_{p,\mathrm{M}}$, and the final equality follows from $\mathrm{ME9}_{231}$, which we can use because $f^{\vee -n} \leq f^{\vee 0} = f^+$ for all $n \in \mathbb{N}$ and $\mathrm{E}_{p,\mathrm{M}}(f^+|s) < +\infty$ by assumption. □

Proof of Definition 5.5$_{\leftarrow}$. Fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. That $\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) \geq \overline{\mathrm{E}}^2_{p,\mathrm{M}}(f|s)$ follows immediately from the fact that $\mathrm{E}_{p,\mathrm{M}}(g|s)$ exists for each $g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$ and therefore, that the infimum in $\overline{\mathrm{E}}^2_{p,\mathrm{M}}(f|s)$ is taken over a set that is at least as large as the set over which the infimum is taken in $\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s)$. On the other hand, we have that

$$\begin{aligned}\overline{\mathrm{E}}^2_{p,\mathrm{M}}(f|s) &= \inf\left\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_\sigma, \mathrm{E}_{p,\mathrm{M}}(g|s) \text{ exists and } g \geq f\right\}\\ &= \inf\left\{\overline{\mathrm{E}}^1_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_\sigma, \mathrm{E}_{p,\mathrm{M}}(g|s) \text{ exists and } g \geq f\right\} \geq \overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s),\end{aligned}$$

where the second equality follows from Lemma 5.2.4 and the inequality follows from the monotonicity of $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$—which is itself a consequence of the definition of $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$. □

It follows that Definition 5.5232 is valid and that, due to Lemma 5.2.4↶, the global measure-theoretic upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is an extension of $\mathrm{E}_{p,\mathrm{M}}$. In fact, it can easily be verified that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is the most conservative extension that satisfies monotonicity [EC4163]; in other words, it is the natural extension of $\mathrm{E}_{p,\mathrm{M}}$ under monotonicity.

Corollary 5.2.5. *For any precise probability tree p, the global upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is the most conservative global upper expectation on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\mathrm{E}_{p,\mathrm{M}}$ and that is monotone [*EC4*163*].*

Proof. Fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ and any global upper expectation $\overline{\mathrm{E}}'$ that is monotone [EC4163] and that coincides with $\mathrm{E}_{p,\mathrm{M}}$ on its domain. Then,

$$\begin{aligned}\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) &= \inf\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\} \\ &= \inf\Big\{\overline{\mathrm{E}}'(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\} \geq \overline{\mathrm{E}}'(f|s),\end{aligned}$$

where the second equality is simply the definition of $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$, the third equality follows from the fact that $\overline{\mathrm{E}}'$ coincides with $\mathrm{E}_{p,\mathrm{M}}$ on the domain of the latter, and the inequality follows from the fact that $\overline{\mathrm{E}}'$ is monotone [EC4163]. The result now follows immediately from the fact that, due to Lemma 5.2.4↶, the global upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is itself an extension of $\mathrm{E}_{p,\mathrm{M}}$ and the fact that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is monotone [EC4163]—due to the definition of $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$. □

We can also extend the measure-theoretic expectation $\mathrm{E}_{p,\mathrm{M}}$ to a 'precise' **measure-theoretic lower expectation** by using expressions analogous to (i)–(ii) in Definition 5.5232, but where the infima are replaced by suprema that range over all variables g that are smaller or equal than f. We immediately see that the resulting common lower expectation is then related to $\overline{\mathrm{E}}_{p,\mathrm{M}}$ by conjugacy, which is why we will continue to only work with the upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$. We use $\overline{\mathbb{V}}_{\sigma,\mathrm{a}}$ in the following definition to denote the set of all **bounded above** variables in $\overline{\mathbb{V}}_\sigma$.

Definition 5.6 (Global measure-theoretic lower expectations for precise probability trees)**.** Consider any precise probability tree p. Let $\underline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}^2_{p,\mathrm{M}}$ be defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

(i) $\underline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) \coloneqq \sup\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{a}} \text{ and } g \leq f\Big\}$;

(ii) $\underline{\mathrm{E}}^2_{p,\mathrm{M}}(f|s) \coloneqq \sup\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma},\ \mathrm{E}_{p,\mathrm{M}}(g|s) \text{ exists and } g \leq f\Big\}$.

Then $\underline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}^2_{p,\mathrm{M}}$ are equal and the common operator is what we refer to as the global measure-theoretic lower expectation $\underline{\mathrm{E}}_{p,\mathrm{M}}$. Moreover, we have that $\underline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = -\overline{\mathrm{E}}_{p,\mathrm{M}}(-f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. ◎

Proof. To see that $\underline{\mathrm{E}}^1_{p,\mathrm{M}}$ is related to $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ by conjugacy, it suffices to observe that for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$,

$$
\begin{aligned}
-\overline{\mathrm{E}}^1_{p,\mathrm{M}}(f|s) &= -\inf\left\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\right\} \\
&= \sup\left\{-\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\right\} \\
&= \sup\left\{\mathrm{E}_{p,\mathrm{M}}(-g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\right\} \\
&= \sup\left\{\mathrm{E}_{p,\mathrm{M}}(-g|s) \colon\ -g \in \overline{\mathbb{V}}_{\sigma,\mathrm{a}} \text{ and } -g \leq -f\right\} \\
&= \sup\left\{\mathrm{E}_{p,\mathrm{M}}(g|s) \colon g \in \overline{\mathbb{V}}_{\sigma,\mathrm{a}} \text{ and } g \leq -f\right\} = \underline{\mathrm{E}}^1_{p,\mathrm{M}}(-f|s),
\end{aligned}
$$

where the third step follows from ME6$_{231}$ and ME4$_{231}$. In a similar way, once again using ME6$_{231}$, we can show that $\underline{\mathrm{E}}^2_{p,\mathrm{M}}$ is related to $\overline{\mathrm{E}}^2_{p,\mathrm{M}}$ by conjugacy. The equality between $\underline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}^2_{p,\mathrm{M}}$ then subsequently follows from the equality between $\overline{\mathrm{E}}^1_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^2_{p,\mathrm{M}}$ [Definition 5.5$_{232}$] and these conjugacy relations. □

A multitude of properties can now be established for the upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$—for instance, extended coherence [EC1$_{163}$–EC6$_{163}$] and a monotone convergence theorem—but, we prefer not to do so just yet. For we will show in the next section that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is equal to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ if $\overline{\mathrm{Q}}_{\bullet}$ is the (upper) expectations tree that agrees with p, and so $\overline{\mathrm{E}}_{p,\mathrm{M}}$ will then inherit all the properties that we have in Chapter 4$_{129}$ established for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$.

5.3 Relation to game-theoretic upper expectations in a precise context

In the current context where the local dynamics are described by a precise probability tree p, we look at how $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is related to the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ with $\overline{\mathrm{Q}}_{\bullet} \coloneqq \mathrm{Q}_{\bullet,p}$ the (upper) expectations tree that agrees with p according to Eq. (3.4)$_{52}$; so

$$\mathrm{Q}_{s,p}(f) = \sum_{x \in \mathscr{X}} f(x)p(x|s) \text{ for all } f \in \mathscr{L}(\mathscr{X}) \text{ and all } s \in \mathscr{X}^*.$$

We immediately state the main result of this section:

Theorem 5.3.1. *Consider any precise probability tree p and the expectations tree $\mathrm{Q}_{\bullet} \coloneqq \mathrm{Q}_{\bullet,p}$ that agrees with p according to Eq.* (3.4)$_{52}$. *Then*

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|s) \textit{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Recall furthermore that, due to Theorem 4.3.6$_{161}$, the theorem above will also hold if we were to replace $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ by the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ deduced from any acceptable gambles tree $\mathscr{A}_\bullet$ that agrees with $\mathrm{Q}_\bullet$ according to Eq. $(3.1)_{50}$.[13]

The proof of Theorem $5.3.1_\curvearrowleft$ is centrally based on a measure-theoretic version of Lévy's zero-one law [Theorem $5.\mathrm{A}.3_{267}$] and an adapted game-theoretic version of Ville's theorem [Lemma $5.\mathrm{B}.3_{272}$]. Ville's (original) theorem [107] characterises a null (or P-null) event in terms of measure-theoretic martingales that converge to infinity. This characterisation is conceptually very close to how null events are defined in the game-theoretic framework [see Section 4.5_{171}], and this is why it becomes a crucial tool when relating both frameworks. Shafer and Vovk were the first to make this relation concrete; we refer to [85, Theorem 9.3] for their most recent version of a result that connects both frameworks. Apart from the fact that we only consider finite state spaces, our Theorem $5.3.1_\curvearrowleft$ generalises [85, Theorem 9.3] in a number of ways: it applies to conditional expectations whereas [85, Theorem 9.3] only considers unconditional expectations; it establishes equality on the entire domain of all extended real variables, whereas [85, Theorem 9.3] only does so for $\sigma(\mathscr{X}^*)$-measurable (bounded) gambles; and, as mentioned above, it also holds if one were to consider game-theoretic upper expectations corresponding to acceptable gambles trees.

Stating Ville's theorem and (a measure-theoretic version of) Lévy's zero-one law, and showing how it leads to Theorem $5.3.1_\curvearrowleft$, would require us to introduce various measure-theoretic notions such as filtrations and Radon-Nikodým derivatives. Hence, in order not to overload the main text with these abstract concepts, we have relegated part of the proof of Theorem $5.3.1_\curvearrowleft$ to Appendix $5.\mathrm{B}_{267}$. More precisely, we start here from the following partial result—whose proof is the topic of Appendix $5.\mathrm{B}_{267}$—which states that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ are equal on $\mathbb{V}_\sigma \times \mathscr{X}^*$. So this result is very similar to [85, Theorem 9.3], but we nevertheless give a self-contained proof for it (in Appendix $5.\mathrm{B}_{267}$) because [85, Theorem 9.3] differs in context and style; and because our result involves conditioning. Moreover, observe that, in contrast with the proof of [85, Theorem 9.3], our proof of Proposition 5.3.2 does not rely on the notion of a measure-theoretic (super)martingale. This, we believe, makes the proof easier to grasp.

Proposition 5.3.2. *Consider any precise probability tree p and the expectations tree $\mathrm{Q}_\bullet := \mathrm{Q}_{\bullet,p}$ that agrees with p according to Eq. $(3.4)_{52}$. Then*

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|s) \textit{ for all } (f,s) \in \mathbb{V}_\sigma \times \mathscr{X}^*.$$

Theorem $5.3.1_\curvearrowleft$ can now be established by combining Proposition 5.3.2

[13] Note that, unlike the relation between precise probability trees p and (upper) expectations trees $\mathrm{Q}_\bullet$, the relation between $\mathrm{Q}_\bullet$ and $\mathscr{A}_\bullet$ in this precise case is not one-to-one.

with the continuity properties of $\mathrm{E}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$, and the representation of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ in terms of limits of finitary variables [Proposition 4.7.6$_{184}$].

Proof of Theorem 5.3.1$_{235}$. We start by showing that the equality is true for all $(f,s) \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$. So fix any $(f,s) \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$. Because f is bounded below and $\sigma(\mathscr{X}^*)$-measurable, the expectation $\mathrm{E}_{p,\mathrm{M}}(f|s)$ exists [ME4$_{231}$] and therefore $\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \mathrm{E}_{p,\mathrm{M}}(f|s)$ due to Corollary 5.2.5$_{234}$. We can moreover assume that f is non-negative without loss of generality because it is bounded below and both $\mathrm{E}_{p,\mathrm{M}}(\cdot|s)$—and thus $\overline{\mathrm{E}}_{p,\mathrm{M}}(\cdot|s)$—and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ are constant additive with respect to real constants; see ME10$_{231}$ and EC5$_{163}$. Consider now the increasing sequence $(f^{\wedge n})_{n\in\mathbb{N}_0}$ of upper cuts and note that each $f^{\wedge n}$ is bounded and $\sigma(\mathscr{X}^*)$-measurable [MV1$_{228}$]. Using ME8$_{231}$—which we are allowed to use because $f^{\wedge 0}$ is $\mathrm{P}^{|s}_p$-integrable [by ME5$_{231}$ and the fact that $f^{\wedge 0}$ is bounded] and because $(f^{\wedge n})_{n\in\mathbb{N}_0}$ is increasing—we have that $\mathrm{E}_{p,\mathrm{M}}(f|s) = \lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f^{\wedge n}|s)$. As a consequence, we infer that

$$\mathrm{E}_{p,\mathrm{M}}(f|s) = \lim_{n\to+\infty} \mathrm{E}_{p,\mathrm{M}}(f^{\wedge n}|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f^{\wedge n}|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^{\wedge n}|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s),$$

where the second equality follows from the fact that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ extends $\mathrm{E}_{p,\mathrm{M}}$ [Corollary 5.2.5$_{234}$], the third equality follows from Proposition 5.3.2$_{\leftarrow}$, and the last equality follows from Theorem 4.6.1$_{175}$. Hence, we conclude that $\mathrm{E}_{p,\mathrm{M}}$—and thus $\overline{\mathrm{E}}_{p,\mathrm{M}}$—and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ coincide on the domain $\overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$.

To see that the equality also holds on the general domain $\overline{\mathbb{V}} \times \mathscr{X}^*$, we fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, and note that

$$\begin{aligned}\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \overset{\text{Def. } 5.5_{232}}{=} \overline{\mathrm{E}}^{1}_{p,\mathrm{M}}(f|s) &= \inf\Big\{\mathrm{E}_{p,\mathrm{M}}(g|s) : g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\}\\ &= \inf\Big\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) : g \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \text{ and } g \geq f\Big\} \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s),\end{aligned}$$

where the third equality follows from the already established equality between $\mathrm{E}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ on $\overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$, and where the inequality follows from Proposition 4.4.3$_{164}$ [EC4]. To show that the converse inequality holds, we will use Proposition 4.7.6$_{184}$.

Consider any $g \in \overline{\mathbb{L}}_{\mathrm{b}}$ that is the pointwise limit of a sequence $(g_n)_{n\in\mathbb{N}_0}$ of finitary gambles $g_n \in \mathbb{F}$. Since any finitary gamble is clearly $\sigma(\mathscr{X}^*)$-measurable, g is the pointwise limit of a sequence of $\sigma(\mathscr{X}^*)$-measurable gambles. Then it follows from MV2$_{228}$ that g itself is also $\sigma(\mathscr{X}^*)$-measurable. Furthermore, by the definition of $\overline{\mathbb{L}}_{\mathrm{b}}$, g is also bounded below. Hence, by the equality of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ on $\overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) = \overline{\mathrm{E}}_{p,\mathrm{M}}(g|s)$. Since this holds for any $g \in \overline{\mathbb{L}}_{\mathrm{b}}$, we infer by Proposition 4.7.6$_{184}$ that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) &= \inf\Big\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) : g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\Big\}\\ &= \inf\Big\{\overline{\mathrm{E}}_{p,\mathrm{M}}(g|s) : g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\Big\} \geq \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s),\end{aligned}$$

where the inequality follows from the monotonicity [EC4$_{163}$] of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ as established by Corollary 5.2.5$_{234}$. □

That the measure-theoretic upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ coincide for precise probability trees, is a powerful result. On the one hand, it allows us to infer that **all** properties of the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ proved in Sections 4.4$_{162}$–4.8$_{186}$ carry over to the measure-theoretic upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$; indeed, since these properties were all proven to hold in a context with general upper expectations trees, they surely hold in the special case where the (upper) expectations trees correspond to precise probability trees. Many of these properties are already known to hold—even in a stronger form—for the standard measure-theoretic (linear) expectation $\mathrm{E}_{p,\mathrm{M}}$ on $\mathbb{V}_\sigma \times \mathscr{X}^*$, but as this operator is usually not extended beyond the domain $\mathbb{V}_\sigma \times \mathscr{X}^*$, little is typically said about the properties of the upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$. We give an overview of the most significant ones.

Corollary 5.3.3. *Consider any precise probability tree p and the expectations tree $\mathrm{Q}_\bullet := \mathrm{Q}_{\bullet,p}$ that agrees with p according to Eq.* (3.4)$_{52}$*. Then the following statements hold:*

(i) *The restriction of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ to $\mathbb{V} \times \mathscr{X}^*$ is coherent.*

(ii) *$\overline{\mathrm{E}}_{p,\mathrm{M}}$ satisfies the extended coherence properties* EC1$_{163}$–EC6$_{163}$.

(iii) *For any $f \in \overline{\mathbb{V}}$ and any $k \in \mathbb{N}_0$,*

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|X_{1:k}) = \overline{\mathrm{E}}_{p,\mathrm{M}}\Big(\overline{\mathrm{E}}_{p,\mathrm{M}}\,(f|X_{1:k+1})\,\Big|X_{1:k}\Big).$$

(iv) *For any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$ and any $(g,t) \in \mathbb{F} \times \mathscr{X}^*$,*

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \leq \overline{\mathrm{E}}_p(f|s) = \overline{\mathrm{E}}_{\mathrm{Q}}(f|s) \;\; \textit{and} \;\; \overline{\mathrm{E}}_{p,\mathrm{M}}(g|t) = \overline{\mathrm{E}}_p(g|t) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g|t).$$

(v) *For any $s \in \mathscr{X}^*$ and any increasing sequence $(f_n)_{n\in\mathbb{N}_0}$ in $\overline{\mathbb{V}}_{\mathrm{b}}$, $\lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{p,\mathrm{M}}(\lim_{n\to+\infty} f_n|s)$. [Continuity from below]*

(vi) *For any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}_0}$ of finitary bounded above variables, $\lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{p,\mathrm{M}}(\lim_{n\to+\infty} f_n|s)$. [Continuity w.r.t. decreasing finitary variables]*

(vii) *For any $s \in \mathscr{X}^*$ and any $f \in \overline{\mathbb{L}}_{\mathrm{b}}$, there is a sequence $(f_n)_{n\in\mathbb{N}_0}$ of n-measurable gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s)$.*

Proof. The properties above follow from combining Theorem 5.3.1$_{235}$ with, respectively,

(i). Corollary 4.4.5$_{167}$;

(ii). Proposition 4.4.3$_{164}$;

(iii). Theorem 4.4.4$_{166}$;

(iv). Corollary 4.4.8$_{170}$, Corollary 4.4.9$_{170}$ and Theorem 3.5.2$_{91}$;

(v). Theorem 4.6.1$_{175}$;

(vi). Theorem 4.7.3$_{182}$;

(vii). Theorem 4.7.4$_{183}$. □

On the other hand, we can also reason in the reverse direction, and use Theorem 5.3.1$_{235}$ and the information about $\mathrm{E}_{p,\mathrm{M}}$ to draw conclusions about $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$. Indeed, $\mathrm{E}_{p,\mathrm{M}}$ is defined using the Lebesgue integral with respect to (countably additive) probability measures, so the extensions $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ inherit all its strong and desirable properties on the subset of $\overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ where $\mathrm{E}_{p,\mathrm{M}}$ exists. We again limit ourselves to formulating the most eminent ones.

Corollary 5.3.4. *Consider any (upper) expectations tree* $\mathrm{Q}_\bullet$ *for which there is a precise probability tree p such that the agreeing tree* $\mathrm{Q}_{\bullet,p}$ *defined by Eq.* (3.4)$_{52}$ *coincides with* $\mathrm{Q}_\bullet$. *Then the following statements hold:*

(i) $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$ *for all bounded below or above* $f \in \overline{\mathbb{V}}_\sigma$ *and all* $s \in \mathscr{X}^*$. *[precision/self-conjugacy]*

(ii) $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(af + bg|s) = a\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) + b\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s)$ *for all* $f \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$, *all* $g \in \mathbb{V}_\sigma$, $s \in \mathscr{X}^*$ *and* $a, b \in \mathbb{R}$. *[linearity]*

(iii) *Consider any* $s \in \mathscr{X}^*$ *and any* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$ *that converges pointwise to a variable* $f \in \overline{\mathbb{V}}_\sigma$. *If there is an* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $|f_n| \leq f^*$ *for all* $n \in \mathbb{N}$ *and* $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) < +\infty$, *then* $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f)$. *[dominated convergence]*

(iv) *Consider any* $s \in \mathscr{X}^*$ *and any decreasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$. *If there is an* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) < +\infty$ *and* $f_1 \leq f^*$, *then* $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lim_{n\to+\infty} f_n|s)$. *[continuity from above]*

Proof. (i) follows from Theorem 5.3.1$_{235}$, Corollary 5.2.5$_{234}$, properties ME6$_{231}$ and ME4$_{231}$, and conjugacy [Corollary 4.3.7$_{162}$].

To see that Property (ii) holds, note that by (i) and Proposition 4.4.3$_{164}$ [EC3] that $a\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(af|s)$ and $b\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(bg|s)$. So it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f' + g'|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'|s) + \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g'|s)$ where $f' := af$ is a bounded below or above variable in $\overline{\mathbb{V}}_\sigma$ and $g' := bg$ is a gamble in $\mathbb{V}_\sigma$. To this end, we already have by Proposition 4.4.3$_{164}$ [EC2] that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f' + g'|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'|s) + \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g'|s).$$

To prove the converse inequality, we can use the self-conjugacy [(i)] of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ on bounded below and above variables. Indeed, $f' + g'$ is bounded below or above because f' is bounded below or above and g' is bounded. Since $-(f' + g') = -f' - g'$ because g' is a gamble and thus real-valued, (i) and Proposition 4.4.3$_{164}$[EC2$_{163}$]

thus imply that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'+g'|s) = -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f'-g'|s) &\geq -\big(\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f'|s) + \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-g'|s)\big)\\ &= -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f'|s) + (-\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-g'|s))\\ &= \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'|s) + \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g'|s),\end{aligned}$$

where in the second equality we used the fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-g'|s) \in \mathbb{R}$, which follows from Proposition 4.4.3$_{164}$[EC1$_{163}$] and the fact that g' is bounded.

To see that Property (iii)$_{\curvearrowleft}$ holds, suppose that there is an $f^* \in \overline{\mathbb{V}}_\sigma$ such that $|f_n| \leq f^*$ for all $n \in \mathbb{N}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) < +\infty$. Since $|f_n| \leq f^*$ for all $n \in \mathbb{N}$, f^* is non-negative, and so by Proposition 4.4.3$_{164}$ [EC1] and the fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) < +\infty$, we find that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) \in \mathbb{R}_{\geq}$. Theorem 5.3.1$_{235}$ therefore guarantees that $\overline{\mathrm{E}}_{p,\mathrm{M}}(f^*|s) \in \mathbb{R}_{\geq}$. Since $f^* \in \overline{\mathbb{V}}_\sigma$ is non-negative, its expectation $\mathrm{E}_{p,\mathrm{M}}(f^*|s)$ exists [ME4$_{231}$] and so it follows from Corollary 5.2.5$_{234}$ that also $\mathrm{E}_{p,\mathrm{M}}(f^*|s) \in \mathbb{R}_{\geq}$. Hence, f^* is $\mathrm{P}_p^{|s}$-integrable. The desired statement now follows from ME7$_{231}$, Corollary 5.2.5$_{234}$ and Theorem 5.3.1$_{235}$.

Finally, to prove Property (iv)$_{\curvearrowleft}$, suppose that $(f_n)_{n\in\mathbb{N}}$ is decreasing and that there is an $f^* \in \overline{\mathbb{V}}_\sigma$ such that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f^*|s) < +\infty$ and $f_1 \leq f^*$. Then by Theorem 5.3.1$_{235}$ we also have that $\overline{\mathrm{E}}_{p,\mathrm{M}}(f^*|s) < +\infty$, which by Definition 5.5(ii)$_{232}$ implies that there is a $g \in \overline{\mathbb{V}}_\sigma$ such that $g \geq f^*$ and $\mathrm{E}_{p,\mathrm{M}}(g|s) < +\infty$. Since $f_1 \leq f^*$, we then also have that $f_1 \leq g$. Hence, combining ME9$_{231}$, Corollary 5.2.5$_{234}$ and Theorem 5.3.1$_{235}$, we indeed find that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(\lim_{n\to+\infty} f_n|s)$ as desired. □

5.4 Measure-theoretic upper expectations for imprecise probability trees

Similarly to what we did in Section 3.3$_{69}$, we will generalise measure-theoretic upper (and lower) expectations from a precise to an imprecise context by taking upper (resp. lower) envelopes of the upper (lower) expectations corresponding to the individual compatible precise probability trees. Concretely, consider the general case where the local dynamics are described by an imprecise probability tree $\mathscr{P}_\bullet$. Recall that a precise probability tree p is called **compatible** with $\mathscr{P}_\bullet$, and that we write $p \sim \mathscr{P}_\bullet$, if $p(\cdot|s) \in \mathscr{P}_s$ for all $s \in \mathscr{X}^*$. For each compatible precise tree $p \sim \mathscr{P}_\bullet$, we can proceed as in Sections 5.1$_{220}$–5.2$_{227}$, constructing a global probability measure P_p on $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ and subsequently using the Lebesgue integral to define the corresponding global expectation $\mathrm{E}_{p,\mathrm{M}}$ and global upper and lower expectations $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}_{p,\mathrm{M}}$. The upper (resp. lower) envelope of the global upper (lower) expectations $\overline{\mathrm{E}}_{p,\mathrm{M}}$ ($\underline{\mathrm{E}}_{p,\mathrm{M}}$) over all the compatible precise trees $p \sim \mathscr{P}_\bullet$ is what defines our global measure-theoretic upper (lower) expectation corresponding to $\mathscr{P}_\bullet$.

Definition 5.7 (Global measure-theoretic upper and lower expectations for imprecise probability trees)**.** For any imprecise probability tree $\mathscr{P}_\bullet$, the global measure-theoretic upper and lower expectation are defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) := \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \text{ and } \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) := \inf_{p\sim\mathscr{P}_\bullet} \underline{\mathrm{E}}_{p,\mathrm{M}}(f|s),$$

with $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}_{p,\mathrm{M}}$ for any $p \sim \mathscr{P}_\bullet$ described by Definition 5.5232 and Definition 5.6234 respectively. ◎

In particular, if we consider measurable variables that are bounded below or above, then these measure-theoretic upper and lower expectations simply reduce to upper and lower envelopes of standard Lebesgue integrals [Definition 5.4230]—as is confirmed by Corollary 5.4.1 below. The extension beyond measurable (bounded below or above) variables set out in Section 5.2.3231—which may appear unconventional to a more traditional measure-theoretic practitioner—thus becomes irrelevant in that case. For any imprecise probability tree $\mathscr{P}_\bullet$, let us denote this simplified measure-theoretic upper and lower global expectation by $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$; so, for any $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that f is bounded below or above, let

$$\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s) := \sup_{p\sim\mathscr{P}_\bullet} \mathrm{E}_{p,\mathrm{M}}(f|s) \text{ and } \underline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s) := \inf_{p\sim\mathscr{P}_\bullet} \mathrm{E}_{p,\mathrm{M}}(f|s),$$

with $\mathrm{E}_{p,\mathrm{M}}$ for any $p \sim \mathscr{P}_\bullet$ described by Definition 5.4230. Recall from ME4231 that $\mathrm{E}_{p,\mathrm{M}}(f|s)$ indeed exists for all $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that f is either bounded below or above.

Corollary 5.4.1. *For any imprecise probability tree $\mathscr{P}_\bullet$ and any $(f,s) \in \overline{\mathbb{V}}_\sigma \times \mathscr{X}^*$ such that f is bounded below or above, we have that*

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s) \text{ and } \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \underline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s).$$

Proof. This follows from Corollary 5.2.5234, and the definitions of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$. □

All properties that will be proved for the more general—but also more complex—upper and lower expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ thus also hold for the simplified upper and lower expectation $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$, as long as these properties are—if possible—restricted to apply only to $\sigma(\mathscr{X}^*)$-measurable variables that are bounded below or above.

Global measure-theoretic upper and lower expectations are again related by conjugacy, and so it suffices to focus mainly on upper expectations.

Corollary 5.4.2 (Conjugacy)**.** *For any imprecise probability tree $\mathscr{P}_\bullet$ and any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, we have that $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = -\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(-f|s)$.*

Proof. Consider any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ and note that

$$-\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = -\inf_{p\sim\mathscr{P}_\bullet} \underline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \sup_{p\sim\mathscr{P}_\bullet} -\underline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(-f|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(-f|s),$$

where the penultimate step follows from the conjugacy between $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\underline{\mathrm{E}}_{p,\mathrm{M}}$ for any precise probability tree p [Definition 5.6_{234}]. □

Contrary to the precise case, where the properties of the (linear) expectations corresponding to probability measures have been thoroughly studied—at least, on the domain of measurable functions—the properties of the (imprecise) measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ as introduced in Definition 5.7_{240} are relatively unknown—even for measurable functions. We now aim to address this imbalance. In particular, we will first focus on establishing basic properties such as coherence, extended coherence axioms, and a relation with the finitary global upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ presented in Chapter 3_{45}. We will then go on to prove that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is continuous from below, continuous with respect to decreasing finitary gambles converging in $\mathbb{V}$, and, under a compactness condition on the local models, continuous with respect to decreasing finitary gambles converging in $\overline{\mathbb{V}}$—these will constitute the measure-theoretic counterparts of Theorems $4.6.1_{175}$ and $4.7.3_{182}$. These properties will then subsequently allow us to establish an equality between $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ on a fairly large domain—that will be the topic of the next section.

5.4.1 Extended coherence and relation to the natural extension $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$

That $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is coherent and satisfies the extended coherence axioms $\mathrm{EC1}_{163}$–$\mathrm{EC6}_{163}$ can be straightforwardly deduced from the fact that this is true for the upper expectation $\overline{\mathrm{E}}_{p,\mathrm{M}}$ corresponding to any precise probability tree $p \sim \mathscr{P}_\bullet$.

Proposition 5.4.3. *For any imprecise probability tree $\mathscr{P}_\bullet$,*

(i) *the restriction of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ to $\mathbb{V} \times \mathscr{X}^*$ is coherent;*

(ii) *$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ satisfies the extended coherence properties $\mathrm{EC1}_{163}$–$\mathrm{EC6}_{163}$.*

Proof. (i). By Corollary $5.3.3(\mathrm{i})_{238}$, we know that, for any $p \sim \mathscr{P}_\bullet$, the restriction of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ to $\mathbb{V} \times \mathscr{X}^*$ is coherent. So for all $p \sim \mathscr{P}_\bullet$, $n \in \mathbb{N}_0$, $\lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{R}_{\geq}$ and $(f_0,s_0), (f_1,s_1), \dots, (f_n,s_n) \in \mathbb{V} \times \mathscr{X}^*$, we have by Definition 3.7_{82} that

$$\sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^{n} \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{p,\mathrm{M}}(f_i|s_i)\right) \,\middle|\, \cup_{i=0}^{n} \Gamma(s_i)\right) \geq 0.$$

Since all λ_i are non-negative, and since $\overline{\mathrm{E}}_{p,\mathrm{M}}(f_i|s_i) \leq \sup_{p' \sim \mathscr{P}_\bullet} \overline{\mathrm{E}}_{p',\mathrm{M}}(f_i|s_i) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)$ for all (f_i, s_i), we then surely also have that

$$\sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \cup_{i=0}^n \Gamma(s_i)\right) \geq 0.$$

This holds for all $p \sim \mathscr{P}_\bullet$, so we find that $\sup(g_p | \cup_{i=0}^n \Gamma(s_i)) \geq 0$ for all $p \sim \mathscr{P}_\bullet$, where each g_p is defined by

$$g_p := \lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right).$$

Since, for any $p \sim \mathscr{P}_\bullet$,

$$0 \leq \sup(g_p | \cup_{i=0}^n \Gamma(s_i)) = \max\left\{\sup(g_p|\Gamma(s_0)), \sup(g_p | \cup_{i=0}^n \Gamma(s_i) \setminus \Gamma(s_0))\right\},$$

we surely either have that $\sup(g_p|\cup_{i=0}^n\Gamma(s_i)\setminus\Gamma(s_0)) \geq 0$ for at least one $p \sim \mathscr{P}_\bullet$, or that $\sup(g_p|\Gamma(s_0)) \geq 0$ for all $p \sim \mathscr{P}_\bullet$. Suppose the former is true. Note from the definition above that $g_p \mathbb{1}_{\cup_{i=0}^n \Gamma(s_i)\setminus\Gamma(s_0)} = g_{p'} \mathbb{1}_{\cup_{i=0}^n \Gamma(s_i)\setminus\Gamma(s_0)}$ for all $p' \sim \mathscr{P}_\bullet$, and therefore also that $g_p \mathbb{1}_{\cup_{i=0}^n \Gamma(s_i)\setminus\Gamma(s_0)} = \inf_{p' \sim \mathscr{P}_\bullet} g_{p'} \mathbb{1}_{\cup_{i=0}^n \Gamma(s_i)\setminus\Gamma(s_0)}$. So since $\sup(g_p | \cup_{i=0}^n \Gamma(s_i) \setminus \Gamma(s_0)) \geq 0$, we have that

$$0 \leq \sup(\inf_{p' \sim \mathscr{P}_\bullet} g_{p'} | \cup_{i=0}^n \Gamma(s_i) \setminus \Gamma(s_0)) \leq \sup(\inf_{p' \sim \mathscr{P}_\bullet} g_{p'} | \cup_{i=0}^n \Gamma(s_i)),$$

which by the definition of all $g_{p'}$ and the fact that $\inf_{p' \sim \mathscr{P}_\bullet} -\overline{\mathrm{E}}_{p',\mathrm{M}}(f_0|s_0) = -\sup_{p' \sim \mathscr{P}_\bullet} \overline{\mathrm{E}}_{p',\mathrm{M}}(f_0|s_0) = -\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0)$, implies that

$$0 \leq \sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \cup_{i=0}^n \Gamma(s_i)\right).$$

So by Definition 3.782 we have that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is coherent on $\mathbb{V} \times \mathscr{X}^*$ if the above also holds for the case that $\sup(g_p|\Gamma(s_0)) \geq 0$ for all $p \sim \mathscr{P}_\bullet$. To show that this is true, note that, for any $p \sim \mathscr{P}_\bullet$, since the supremum is taken over $\Gamma(s_0)$,

$$\begin{aligned}
0 &\leq \sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \Gamma(s_0)\right) \\
&= -\lambda_0 \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0) + \sup\left(\lambda_0 f_0 - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \Gamma(s_0)\right).
\end{aligned}$$

Since this holds for any tree $p \sim \mathscr{P}_\bullet$, and since $\inf_{p \sim \mathscr{P}_\bullet} -\lambda_0 \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0) = -\lambda_0 \sup_{p \sim \mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_0|s_0) = -\lambda_0 \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0)$ [because λ_0 is non-negative] we have that

$$\begin{aligned}
0 &\leq -\lambda_0 \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0) + \sup\left(\lambda_0 f_0 - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \Gamma(s_0)\right) \\
&= \sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \Gamma(s_0)\right) \\
&\leq \sup\left(\lambda_0 \mathbb{1}_{s_0}\left(f_0 - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_0|s_0)\right) - \sum_{i=1}^n \lambda_i \mathbb{1}_{s_i}\left(f_i - \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_i|s_i)\right) \middle| \cup_{i=0}^n \Gamma(s_i)\right),
\end{aligned}$$

where the equality uses once more the fact that the supremum is taken over $\Gamma(s_0)$.

(ii)$_{242}$. Corollary 5.3.3(ii)$_{238}$ states that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ satisfies EC1$_{163}$–EC6$_{163}$ for each $p \sim \mathscr{P}_\bullet$. Using this fact together with the definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, it can then readily be inferred that Properties EC1$_{163}$–EC6$_{163}$ also hold for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$. □

As far as the relation with the finitary upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$ is concerned, it is easy to see that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is always at least as informative as $\overline{\mathrm{E}}_{\mathscr{P}}$ on its domain $\mathbb{V} \times \mathscr{X}^*$.

Proposition 5.4.4. *For any imprecise probability tree $\mathscr{P}_\bullet$, we have that*

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \leq \overline{\mathrm{E}}_{\mathscr{P}}(f|s) \textit{ for all } (f,s) \in \mathbb{V} \times \mathscr{X}^*.$$

Proof. Fix any $(f,s) \in \mathbb{V} \times \mathscr{X}^*$. For any $p \sim \mathscr{P}_\bullet$ and the agreeing (upper) expectations tree $\mathrm{Q}_\bullet := \overline{\mathrm{Q}}_{\bullet,p}$ defined by Eq. (3.4)$_{52}$ [or Eq. (3.3)$_{51}$], we have by Theorem 5.3.1$_{235}$ that[14]

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|s) \leq \overline{\mathrm{E}}_{\mathrm{Q}}(f|s) = \overline{\mathrm{E}}_p(f|s),$$

where the second equality follows from Corollary 4.4.8$_{170}$, and the third from Theorem 3.5.2$_{91}$ and the fact that $\mathrm{Q}_\bullet$ could alternatively be obtained from Eq. (3.3)$_{51}$ if we were to consider an imprecise probability tree that consists for each situation $s \in \mathscr{X}^*$ of the singleton $p(\cdot|s)$. Since the equality above holds for any $p \sim \mathscr{P}_\bullet$, we infer from the definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and the definition [Definition 3.6$_{79}$] of $\overline{\mathrm{E}}_{\mathscr{P}}$ that

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \leq \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_p(f|s) = \overline{\mathrm{E}}_{\mathscr{P}}(f|s).$$ □

The following proposition shows that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ actually coincide on the finitary domain $\mathbb{F} \times \mathscr{X}^*$.

Proposition 5.4.5. *For any imprecise probability tree $\mathscr{P}_\bullet$, we have that*

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}_{\mathscr{P}}(f|s) \textit{ for all } (f,s) \in \mathbb{F} \times \mathscr{X}^*.$$

Proof. Fix any $(f,s) \in \mathbb{F} \times \mathscr{X}^*$. Then, for any $p \sim \mathscr{P}_\bullet$ and the agreeing (upper) expectations tree $\mathrm{Q}_\bullet := \overline{\mathrm{Q}}_{\bullet,p}$ defined by Eq. (3.4)$_{52}$, we have by Theorem 5.3.1$_{235}$ that

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|s) = \overline{\mathrm{E}}_{\mathrm{Q}}(f|s) = \mathrm{E}_p(f|s) = \overline{\mathrm{E}}_p(f|s)$$

where the second equality follows from Corollary 4.4.9$_{170}$ and the fact that $f \in \mathbb{F}$, the third from Corollary 3.5.3$_{92}$, and the last from Proposition 3.3.8$_{79}$. Since the equality above holds for any $p \sim \mathscr{P}_\bullet$, we infer from the definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and the definition of $\overline{\mathrm{E}}_{\mathscr{P}}$ [Definition 3.6$_{79}$] that

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_p(f|s) = \overline{\mathrm{E}}_{\mathscr{P}}(f|s).$$ □

[14]The fact that $\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \leq \overline{\mathrm{E}}_p(f|s)$ could also be deduced from the definitions of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}_p$.

Proposition 5.4.4$_{\leftarrow}$ says that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is always at least as informative as $\overline{\mathrm{E}}_{\mathscr{P}}$, yet since $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ satisfies continuity from below [see Theorem 5.4.7 further on] and $\overline{\mathrm{E}}_{\mathscr{P}}$ sometimes fails to satisfy this type of continuity [Example 3.6.1$_{99}$], and since by Proposition 5.4.5$_{\leftarrow}$ both global upper expectations coincide on $\mathbb{F}\times\mathscr{X}^*$, it can be seen that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ will sometimes be strictly smaller—more informative—than $\overline{\mathrm{E}}_{\mathscr{P}}$.

Example 5.4.6. Reconsider the precise probability tree p from Example 3.6.1$_{99}$. Recall that for the corresponding finitary upper expectation $\overline{\mathrm{E}}_p$—or equivalently, $\overline{\mathrm{E}}_{\mathscr{P}}$ for $\mathscr{P}_{\bullet}$ the imprecise tree consisting out of the single mass function $p(\cdot|s)$ for each $s\in\mathscr{X}^*$—we had that

$$\begin{aligned}\lim_{k\to+\infty}\overline{\mathrm{E}}_p(\mathbb{I}_{H_b^k})=\lim_{k\to+\infty}\overline{\mathrm{P}}_p(H_b^k)=0\neq 1=\overline{\mathrm{P}}_p(\lim_{k\to+\infty}H_b^k)&=\overline{\mathrm{E}}_p(\lim_{k\to+\infty}\mathbb{I}_{H_b^k})\\&=\overline{\mathrm{E}}_p(\mathbb{I}_{H_b}).\end{aligned}$$

In contrast, since $\overline{\mathrm{E}}_{p,\mathrm{M}}$ coincides with $\overline{\mathrm{E}}_p$ by Proposition 5.4.5$_{\leftarrow}$—remember that p is simply a particular type of imprecise probability tree $\mathscr{P}_{\bullet}$—and since $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is continuous from below [see Theorem 5.4.7] we have that

$$0=\lim_{k\to+\infty}\overline{\mathrm{E}}_{p,\mathrm{M}}(\mathbb{I}_{H_b^k})=\overline{\mathrm{E}}_{p,\mathrm{M}}(\mathbb{I}_{H_b}).$$

So, indeed, $\overline{\mathrm{E}}_{p,\mathrm{M}}$ (or more generally $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$) is sometimes strictly smaller than $\overline{\mathrm{E}}_p$ (or $\overline{\mathrm{E}}_{\mathscr{P}}$). ◇

5.4.2 Continuity with respect to two types of monotone sequences

We will now show that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is continuous with respect to increasing sequences in $\overline{\mathbb{V}}_{\mathrm{b}}$, that it is continuous with respect to decreasing sequences in $\mathbb{F}$ that are uniformly bounded below, and that it is continuous with respect to general decreasing sequences in $\mathbb{F}$ if the local sets of mass functions $\mathscr{P}_s$ are closed (or compact). The proof of the first result is relatively straightforward since $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is an upper envelope of operators that are monotone and continuous with respect to increasing sequences in $\overline{\mathbb{V}}_{\mathrm{b}}$.

Theorem 5.4.7 (Continuity from below). *For any imprecise probability tree $\mathscr{P}_{\bullet}$, any $s\in\mathscr{X}^*$ and any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{V}}_{\mathrm{b}}$, we have that*

$$\lim_{n\to+\infty}\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s)=\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s),\ \textit{with}\ f:=\sup_{n\in\mathbb{N}}f_n=\lim_{n\to+\infty}f_n.$$

Proof. For any $p\sim\mathscr{P}_{\bullet}$, Theorem 5.3.1$_{235}$ and Theorem 4.6.1$_{175}$ together imply that

$$\lim_{n\to+\infty}\overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s)=\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s).\tag{5.5}$$

This holds for any $p \sim \mathscr{P}_\bullet$, so we have that

$$\begin{aligned}\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \sup_{n\in\mathbb{N}} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \sup_{n\in\mathbb{N}} \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) &= \sup_{p\sim\mathscr{P}_\bullet} \sup_{n\in\mathbb{N}} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) \\ &= \sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) \\ &= \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) \\ &= \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s),\end{aligned}$$

where the first equality follows from the fact that $(f_n)_{n\in\mathbb{N}}$ is increasing and the monotonicity [EC4$_{163}$] of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ [due to Proposition 5.4.3$_{242}$], where the fourth equality also follows from the fact that $(f_n)_{n\in\mathbb{N}}$ is increasing and the monotonicity [EC4$_{163}$] of $\overline{\mathrm{E}}_{p,\mathrm{M}}$ for each $p \sim \mathscr{P}_\bullet$ [due to Corollary 5.3.3(ii)$_{238}$], and where the penultimate equality follows from Eq. (5.5)$_\curvearrowleft$ above. □

Next, we show that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is also continuous with respect to decreasing sequences in $\mathbb{F}$ that are uniformly bounded below—in other words, sequences in $\mathbb{F}$ that converge decreasingly to a gamble in $\mathbb{V}$. The proof is less straightforward than that of Theorem 5.4.7$_\curvearrowleft$, as it essentially relies on the technical topological results from Appendix 3.E.1$_{120}$. Nonetheless, it suffices to only explicitly use Lemma 3.E.8$_{126}$; a result that by Proposition 5.4.5$_{244}$ continues to hold if we replace the finitary global upper expectations $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ by the measure-theoretic upper expectations $\overline{\mathrm{E}}_{p,\mathrm{M}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$.

Lemma 5.4.8. *For any imprecise probability tree $\mathscr{P}_\bullet$, any $s \in \mathscr{X}^*$, and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that converges to a gamble $f \in \mathbb{V}$,*

$$\sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s).$$

Proof. This follows from Lemma 3.E.8$_{126}$ and the fact that $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ [and thus also $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{p,\mathrm{M}}$ for all $p \sim \mathscr{P}_\bullet$] coincide on $\mathbb{F} \times \mathscr{X}^*$ [Proposition 5.4.5$_{244}$]. □

The desired downward continuity of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ now follows immediately.

Proposition 5.4.9. *For any imprecise probability tree $\mathscr{P}_\bullet$, any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below,*

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \text{ with } f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Proof. Since $f = \inf_{n\in\mathbb{N}} f_n$ is bounded below [because the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded below] and bounded above by $\sup f_1$ [which is real because f_1 is a gamble], we have that f is a gamble. Hence, due to Lemma 5.4.8, we have that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \sup_{p\sim\mathscr{P}_\bullet} \lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s).$$

But by Corollary 5.3.3(vi)$_{238}$ and the definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, the right-hand side is equal to $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s)$, so we indeed find that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s)$. □

The following example shows that if the sequence in Proposition 5.4.9← is not uniformly bounded below, the continuity may no longer hold.

Example 5.4.10. Consider the state space $\mathscr{X} = \{a, b\}$, any imprecise probability tree $\mathscr{P}_\bullet$ such that $\mathscr{P}_\Box = \{\mu \in \mathbb{P}(\mathscr{X})\colon 0 < \mu(a) < 1\}$, and any non-positive 1-measurable variable g. Then, for any $p \sim \mathscr{P}$,

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(g) = \mathrm{E}_{p,\mathrm{M}}(g) = \int g \mathrm{dP}_p^{|\Box} = g(a)\mathrm{P}_p^{|\Box}(a) + g(b)\mathrm{P}_p^{|\Box}(b)$$
$$= g(a)\mathrm{P}_p(a|\Box) + g(b)\mathrm{P}_p(b|\Box)$$
$$= g(a)p(a|\Box) + g(b)p(b|\Box),$$

where the first follows from the fact that $\overline{\mathrm{E}}_{p,\mathrm{M}}$ is an extension of $\mathrm{E}_{p,\mathrm{M}}$ [Corollary 5.2.5$_{234}$] together with the fact that $\mathrm{E}_{p,\mathrm{M}}(g)$ exists because g is 1-measurable and non-positive [ME4$_{231}$], where the third follows from Proposition 5.2.2(i)$_{229}$, the fact that $-g = -g(a)\mathbb{1}_a - g(b)\mathbb{1}_b$ is a non-negative $\sigma(\mathscr{X}^*)$-simple variable and ME6$_{231}$, and where the last equality follows from the fact that P_p satisfies Eq. (3.12)$_{72}$ by assumption [see Proposition 5.1.3$_{226}$]. Hence,

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(g) = \sup_{p \sim \mathscr{P}_\bullet} \big[g(a)p(a|\Box) + g(b)p(b|\Box)\big]$$
$$= \sup_{\mu \in \mathscr{P}_\Box} \big[g(a)\mu(a) + g(b)\mu(b)\big]$$
$$= \sup_{0<\mu(a)<1} \big[g(a)\mu(a) + g(b)(1 - \mu(a))\big], \tag{5.6}$$

where the last equality follows from the construction of $\mathscr{P}_\Box$. Now consider the sequence of non-positive 1-measurable gambles $(f_n)_{n \in \mathbb{N}}$ defined by $f_n(a) = -n$ and $f_n(b) = 0$ for all $n \in \mathbb{N}$. Then by Eq. (5.6) we have that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = \sup_{0<\mu(a)<1}[(-n)\mu(a)] = 0$ for all $n \in \mathbb{N}$. On the other hand, the limit $f := \lim_{n \to +\infty} f_n$ is also a non-positive 1-measurable variable, with $f(a) = -\infty$ and $f(b) = 0$. Hence, Eq. (5.6) also applies here, and so we get that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f) = \sup_{0<\mu(a)<1}[(-\infty)\mu(a)] = \sup_{0<\mu(a)<1}(-\infty) = -\infty$. As a result, we have that

$$\lim_{n \to +\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = 0 \neq -\infty = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\lim_{n \to +\infty} f_n).$$

Note that in contrast the conjugate lower expectation $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is continuous with respect to the decreasing sequence $(f_n)_{n \in \mathbb{N}}$. Indeed, in a similar way, we can infer that $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = \inf_{0<\mu(a)<1}[(-n)\mu(a)] = -n$ for all $n \in \mathbb{N}$, and that $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f) = \inf_{0<\mu(a)<1}[(-\infty)\mu(a)] = -\infty$, which implies that

$$\lim_{n \to +\infty} \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = -\infty = \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f) = \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\lim_{n \to +\infty} f_n).$$

Note that this continuity for $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ could also been deduced from conjugacy [Corollary 5.4.2$_{241}$] and the fact that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is continuous with respect to increasing sequences in $\overline{\mathbb{V}}_{\mathrm{b}}$ [Theorem 5.4.7$_{245}$]. ◇

The downward continuity can still be preserved, though, if we restrict ourselves to imprecise probability trees $\mathscr{P}_{\bullet}$ whose local sets of mass functions $\mathscr{P}_s$ are closed. In accordance with our earlier conventions, we here mean closed with respect to the topology of pointwise convergence [Appendix 3.E.1$_{120}$] on $\mathbb{P}(\mathscr{X})$; also see [24] for more details. Note that, since $\mathbb{P}(\mathscr{X})$ is metrizable and compact [Appendix 3.E.1$_{120}$], the closedness—and thus the compactness [111, Theorem 17.5 (a)]—of a subset $\mathscr{P}_s$ of $\mathbb{P}(\mathscr{X})$ implies its sequential compactness [111, Section 17G.3]. This implies that the limit point of any convergent sequence in $\mathscr{P}_s$ itself also belongs to $\mathscr{P}_s$. This property can be used in conjunction with the following lemma—which is similar to Lemma 3.E.7$_{125}$ but for measure-theoretic upper expectations—to obtain the desired continuity. This lemma uses a notion of convergence for precise probability trees; we consider a sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees to converge if there is some limit tree p such that, for each $s \in \mathscr{X}^*$, the mass functions $(p_i(\cdot|s))_{i\in\mathbb{N}}$ converge (pointwise) to the mass function $p(\cdot|s)$; see Appendix 3.E.1$_{120}$.

Lemma 5.4.11. *For any imprecise probability tree $\mathscr{P}_{\bullet}$, any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ and any $s \in \mathscr{X}^*$,*

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s),$$

where the precise probability tree p is the limit of some convergent sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees, each of which are compatible with the imprecise tree $\mathscr{P}_{\bullet}$.

Proof. This follows from Lemma 3.E.7$_{125}$ and the fact that $\overline{\mathrm{E}}_{\mathscr{P}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ [and thus also $\overline{\mathrm{E}}_p$ and $\overline{\mathrm{E}}_{p,\mathrm{M}}$] coincide on $\mathbb{F} \times \mathscr{X}^*$ [Proposition 5.4.5$_{244}$]. □

Proposition 5.4.12 (Continuity w.r.t. decreasing finitary gambles)**.** *For any imprecise probability tree $\mathscr{P}_{\bullet}$ such that $\mathscr{P}_t$ is closed for all $t \in \mathscr{X}^*$, any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$,*

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \text{ with } f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Proof. Since $(f_n)_{n\in\mathbb{N}}$ is decreasing and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is monotone [EC4$_{163}$] due to Proposition 5.4.3$_{242}$, we immediately have that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) \geq \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s)$. To prove the converse inequality, note that by Lemma 5.4.11,

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{p,\mathrm{M}}(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s),$$

where the precise probability tree p is the limit of some convergent sequence $(p_i)_{i\in\mathbb{N}}$ of precise probability trees, each of which being compatible with the imprecise tree $\mathscr{P}_\bullet$. By Corollary 5.3.3(vi)$_{238}$, this yields

$$\overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s). \tag{5.7}$$

Now note that $p \sim \mathscr{P}_\bullet$. Indeed, for any $t \in \mathscr{X}^*$, since the trees p_i converge to p, the mass functions $p_i(\cdot|t)$ converge to $p(\cdot|t)$. For all $i \in \mathbb{N}$, we have that $p_i(\cdot|t) \in \mathscr{P}_t$ because $p_i \sim \mathscr{P}_\bullet$, and so the sequential compactness of $\mathscr{P}_t$ implies that $\lim_{i\to+\infty} p_i(\cdot|t) = p(\cdot|t) \in \mathscr{P}_t$. Since this holds for any $t \in \mathscr{X}^*$, we infer that $p \sim \mathscr{P}_\bullet$. As a result, we infer from Eq. (5.7) and the definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ that

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \geq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s).$$

as desired. □

5.5 Relation to game-theoretic upper expectations in an imprecise context

We now turn to one of the main subjects of this chapter and, in fact, of the entire dissertation; the relation between the game-theoretic global upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ in a general imprecise context. We already know from Section 5.3$_{235}$ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ are equal on all of $\overline{\mathbb{V}} \times \mathscr{X}^*$ as soon as we limit ourselves to precise probability trees and (linear) expectations trees. It will turn out that, to a large extent, the equality still holds in the imprecise case. Concretely, we will show that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ coincide for all variables (and situations) that are (i) $\sigma(\mathscr{X}^*)$-measurable and bounded below or (ii) if the local sets of mass functions $\mathscr{P}_s$ are closed, decreasing limits of finitary gambles. Observe that variables of type (ii) are not necessarily of type (i) because the former may not be bounded below. As we will discuss later on in Section 5.5.4$_{257}$, an equality on these two types of domains suffices to regard $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ as two interchangeable, equivalent models for almost all practical purposes. Moreover, bear in mind that, in all of this, we can always simply replace $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ by the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ obtained from an agreeing acceptable gambles tree $\mathscr{A}_\bullet$ due to Theorem 4.3.6$_{161}$.

5.5.1 A general inequality

Before we prove any equality between $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, let us first establish an inequality that holds on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$; namely, that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ is always larger than or equal to—at least as conservative as—$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$.

Proposition 5.5.1. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ such that $\mathscr{P}_t$ is closed for all $t \in \mathscr{X}^*$, and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees with $\mathscr{P}_\bullet$ according to Eq.* (3.3)$_{51}$*. Then*

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Proof. Fix any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ and any $p \sim \mathscr{P}_\bullet$. Let $\mathrm{Q}_\bullet$ be the (upper) expectations tree that agrees with the precise probability tree p according to Eq. (3.4)$_{52}$. Then note that $\mathrm{Q}_t(g) \leq \overline{\mathrm{Q}}_t(g)$ for all $g \in \mathscr{L}(\mathscr{X})$ and all $t \in \mathscr{X}^*$ due to how $\overline{\mathrm{Q}}_\bullet$ and $\mathrm{Q}_\bullet$ are related to the trees $\mathscr{P}_\bullet$ and p [resp. Eq. (3.3)$_{51}$ and Eq. (3.4)$_{52}$]. Hence, it then follows from the definition of a supermartingale that $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet) \subseteq \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$, and so by Definition 4.7$_{160}$ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}(f|s)$. By Theorem 5.3.1$_{235}$ and since $\mathrm{Q}_\bullet$ is the expectations tree that agrees with p, this implies that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s)$. Since this holds for any $p \sim \mathscr{P}_\bullet$, we obtain that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \geq \sup_{p\sim\mathscr{P}_\bullet} \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s). \qquad \square$$

It can be observed that the inequality above sometimes becomes strict; see Example 5.5.16$_{259}$ below, where we will also further discuss the relevance of Proposition 5.5.1$_{\curvearrowleft}$.

5.5.2 An equality for bounded below $\sigma(\mathscr{X}^*)$-measurable variables

To prove that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on bounded below $\sigma(\mathscr{X}^*)$-measurable variables, we require the notions of continuity and upper semicontinuity.

Let Ω be endowed with the topology generated by the cylinder events $\Gamma(\mathscr{X}^*) = \{\Gamma(s) \colon s \in \mathscr{X}^*\}$—the smallest topology including $\Gamma(\mathscr{X}^*)$ [111, Problem 5.D]. As we show in Appendix 5.C$_{274}$, a set in this topology is open if and only it is a countable union of cylinder events. This topology is moreover metrizable and compact, and coincides with the product topology on $\Omega = \mathscr{X}^{\mathbb{N}}$ (where $\mathscr{X}$ is endowed with the discrete topology). For any two topological spaces $\mathscr{Y}, \mathscr{Z}$—and hence, in particular, for Ω and $\overline{\mathbb{R}}$ (or $\mathbb{R}$) respectively—a map $f \colon \mathscr{Y} \to \mathscr{Z}$ is **continuous** if the inverse image $f^{-1}(B) = \{y \in \mathscr{Y} \colon f(y) \in B\}$ is an open subset of $\mathscr{Y}$ for each open $B \subseteq \mathscr{Z}$ [37, 53, 111]. A real-valued function $f \colon \mathscr{Y} \to \mathbb{R}$ is called **upper semicontinuous (u.s.c.)** if $\{y \in \mathscr{Y} \colon f(y) < a\}$ is an open subset (or if $\{y \in \mathscr{Y} \colon f(y) \geq a\}$ is a closed subset) of $\mathscr{Y}$ for each $a \in \mathbb{R}$; see [53, p.71 & p.186], [111, Problem 7.K] and [37, p.61]. The function f is called lower semicontinuous (l.s.c.) if $-f$ is u.s.c. A function $f \colon \mathscr{Y} \to \mathbb{R}$ is continuous if and only if it is u.s.c. and l.s.c. [37, p.61]; so if $\{y \in \mathscr{Y} \colon f(y) < a\}$ and $\{y \in \mathscr{Y} \colon f(y) > a\}$ are open for all $a \in \mathbb{R}$. Furthermore, the pointwise limit of any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of continuous real-valued functions f_n on $\mathscr{Y}$ is u.s.c. (if this limit is itself real-valued) [111, Problem 7.K (2)].

We will henceforth use $\mathbb{V}^{\mathrm{u}}$ to denote the set of all real-valued (possibly unbounded) functions on Ω—this in contrast with $\mathbb{V}$ which denotes all bounded real-valued functions (or gambles) on Ω. Note that, since $\sigma(\mathscr{X}^*)$ contains all countable unions of cylinder events, and thus all open subsets of Ω, any u.s.c. global variable in $\mathbb{V}^{\mathrm{u}}$ is $\sigma(\mathscr{X}^*)$-measurable. Moreover, one may check using the topology on Ω that any finitary real-valued variable (or gamble) is a continuous real-valued variable (resp. gamble) on Ω, but not necessarily the other way around. We next show that u.s.c. variables in $\mathbb{V}^{\mathrm{u}}$ can always be obtained as limits of decreasing sequences of finitary variables (or gambles). The proof of this result can be found in Appendix 5.C$_{274}$.

Lemma 5.5.2. *For any $f \in \mathbb{V}^{\mathrm{u}}$, we have that f is u.s.c. if and only if it is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of extended real variables, each of which is finitary and bounded below. Moreover, f is both u.s.c. and bounded above if and only if it is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles.*

Lemma 5.5.2 leads us to two interesting intermediate results; the first is that, since $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on all finitary gambles, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on all u.s.c. gambles. We first state that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on all finitary gambles.

Corollary 5.5.3. *For any imprecise probability tree $\mathscr{P}_\bullet$ and the agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet \coloneqq \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ defined by Eq. (3.3)$_{51}$,*

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \mathbb{F} \times \mathscr{X}^*.$$

Proof. This follows readily from Proposition 5.4.5$_{244}$ and Corollary 4.4.9$_{170}$. □

Corollary 5.5.4. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet \coloneqq \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq. (3.3)$_{51}$. Then $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for any $s \in \mathscr{X}^*$ and any $f \in \mathbb{V}$ that is u.s.c.*

Proof. Corollary 5.5.3 says that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on $\mathbb{F} \times \mathscr{X}^*$. By Proposition 5.4.9$_{246}$ and Theorem 4.7.3$_{182}$, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ are both continuous with respect to decreasing sequences of finitary gambles that converge in $\mathbb{V}$. Hence, since any u.s.c. gamble is a decreasing limit of finitary gambles due to Lemma 5.5.2, we infer that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on all $(f,s) \in \mathbb{V} \times \mathscr{X}^*$ for which f is u.s.c. □

On the other hand, Lemma 5.5.2 also implies that continuity with respect to decreasing (uniformly bounded below) sequences of u.s.c. gambles is actually not stronger than continuity with respect to decreasing (uniformly bounded below) sequences of finitary gambles:

Lemma 5.5.5. *Any operator* $\mathrm{F}\colon \overline{\mathbb{V}} \to \overline{\mathbb{R}}$ *that is monotone and that is continuous with respect to decreasing sequences of finitary gambles that are uniformly bounded below, is also continuous with respect to decreasing sequences* $(f_n)_{n\in\mathbb{N}}$ *of u.s.c. gambles that are uniformly bounded below; i.e.*

$$\lim_{n\to+\infty} \mathrm{F}(f_n) = \mathrm{F}(f), \textit{ with } f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Proof. Consider any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of u.s.c. gambles that is uniformly bounded below and let $f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n$. Then, for all $n \in \mathbb{N}$, it follows from Lemma 5.5.2↶ and the fact that f_n is a gamble [and thus bounded above] that there is a decreasing sequence $(g_{n,m})_{m\in\mathbb{N}}$ of finitary gambles such that $\lim_{m\to+\infty} g_{n,m} = f_n$. Now let $(h_m)_{m\in\mathbb{N}}$ be the sequence of variables defined by

$$h_m(\omega) := \min\{g_{n,m}(\omega)\colon 0 \leq n \leq m\} \text{ for all } \omega \in \Omega.$$

Because each $(g_{n,m})_{m\in\mathbb{N}}$ is decreasing, $(h_m)_{m\in\mathbb{N}}$ is also decreasing. The variables h_m for all $m \in \mathbb{N}$ are clearly bounded—and hence, they are gambles—and they are also finitary because, on the one hand, $g_{n,m}$ is finitary for all $n \in \mathbb{N}$, and on the other hand, the minimum over a finite number of finitary variables is trivially also finitary. So $(h_m)_{m\in\mathbb{N}}$ is a decreasing sequence of finitary gambles. Furthermore, note that $h_m \geq f$ because $g_{n,m} \geq f_n \geq f$ for all $n, m \in \mathbb{N}$, and therefore $\lim_{m\to+\infty} h_m \geq f$. To see that also $\lim_{m\to+\infty} h_m \leq f$, fix any $\omega \in \Omega$ and any $a \in \mathbb{R}$ such that $a > f(\omega)$. Since $\lim_{n\to+\infty} f_n = f$, there is some $n' \in \mathbb{N}$ such that $a > f_{n'}(\omega)$ and since also $\lim_{m\to+\infty} g_{n',m} = f_{n'}$, there is some $m' \geq n'$ such that $a > g_{n',m'}(\omega)$. Then certainly $a > h_{m'}(\omega)$, and since $(h_m)_{m\in\mathbb{N}}$ is decreasing, we have that $a > \lim_{m\to+\infty} h_m(\omega)$. This holds for any $a \in \mathbb{R}$ such that $a > f(\omega)$, so we have that $\lim_{m\to+\infty} h_m(\omega) \leq f(\omega)$, which in turn implies that $\lim_{m\to+\infty} h_m \leq f$ because $\omega \in \Omega$ was chosen arbitrarily. So we have that, indeed, $\inf_{m\in\mathbb{N}} h_m = \lim_{m\to+\infty} h_m = f$. Since f is bounded below because $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded below, $(h_m)_{m\in\mathbb{N}}$ is moreover uniformly bounded below. Then recalling that $(h_m)_{m\in\mathbb{N}}$ is a decreasing sequence of finitary gambles, it follows from the assumptions about F that $\lim_{m\to+\infty} \mathrm{F}(h_m) = \mathrm{F}(f)$. Furthermore, note that, due to the decreasing character of $(g_{n,m})_{m\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$,

$$\begin{aligned} h_m(\omega) &= \min\{g_{n,m}(\omega)\colon 0 \leq n \leq m\} \\ &\geq \min\{f_n(\omega)\colon 0 \leq n \leq m\} = f_m(\omega) \end{aligned}$$

for all $m \in \mathbb{N}$ and all $\omega \in \Omega$. So, $f_m \leq h_m$ for all $m \in \mathbb{N}$, which by the monotonicity of F implies that

$$\lim_{m\to+\infty} \mathrm{F}(f_m) \leq \lim_{m\to+\infty} \mathrm{F}(h_m) = \mathrm{F}(f),$$

where the first limit exists because $(f_n)_{n\in\mathbb{N}}$ is decreasing and F is monotone. The converse inequality—that $\lim_{m\to+\infty} \mathrm{F}(f_m) \geq \mathrm{F}(f)$—follows from the decreasing character of $(f_n)_{n\in\mathbb{N}}$ and the monotonicity of F. □

Since both $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ satisfy the type of continuity described in Lemma 5.5.5↶, we immediately obtain the following result.

Proposition 5.5.6. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of u.s.c. gambles that is uniformly bounded below. Then*

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) \text{ with } f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n,$$

and similarly for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$.

Proof. Since any decreasing sequence of finitary gambles that is uniformly bounded below trivially converges to a gamble, we have by Proposition 5.4.9$_{246}$ that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ is continuous with respect to decreasing sequences of finitary gambles that are uniformly bounded below. This is also true for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ because of Theorem 4.7.3$_{182}$. Hence, since both operators are also monotone [EC4$_{163}$] due to Proposition 4.4.3$_{164}$ and Proposition 5.4.3$_{242}$, the desired statement thus follows from Lemma 5.5.5$_{251}$. □

As a final step towards establishing our desired result, we will use a result called Choquet's capacitability theorem. This theorem can be found in many different textbooks, but we will make use of the specific version of Dellacherie [28]. We do this because Dellacherie's notion of a capacity can directly be applied to an extended real-valued functional—such as $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—whereas most other sources restrict capacities to take the form of set-functions. Let us start by introducing some key concepts and terminology regarding capacitability.

Let $\overline{\mathbb{V}}_{\geq}$ be the set of all variables taking values in $\overline{\mathbb{R}}_{\geq}$ and $\mathbb{V}^{\mathrm{u}}_{\geq}$ the set of all (possibly unbounded) variables taking values in $\mathbb{R}_{\geq}$. The following definition is borrowed from [28, Section II.1.1].[15]

Definition 5.8 (Capacities)**.** A functional $\mathrm{F}\colon \overline{\mathbb{V}}_{\geq} \to \overline{\mathbb{R}}_{\geq}$ is called a **capacity** on Ω if it satisfies the following three properties:

CA1. $f \leq g \Rightarrow \mathrm{F}(f) \leq \mathrm{F}(g)$ for all $f, g \in \overline{\mathbb{V}}_{\geq}$;

CA2. $\lim_{n\to+\infty} \mathrm{F}(f_n) = \mathrm{F}(\lim_{n\to+\infty} f_n)$ for any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{V}}_{\geq}$;

CA3. $\lim_{n\to+\infty} \mathrm{F}(f_n) = \mathrm{F}(\lim_{n\to+\infty} f_n)$ for any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of u.s.c. variables in $\mathbb{V}^{\mathrm{u}}_{\geq}$. ◎

[15] Dellacherie [28] does not explicitly state a definition for a u.s.c. function, yet we suppose that he is using the standard definitions that we also adopt here; for instance, it is mentioned at the bottom of [28, p.4] that the level sets K_n^m are compact (or closed) for f_n^m being u.s.c., and that the converse is true for f_n^m being l.s.c. In any way, from [28, p.3], it is sure that his notion of u.s.c. implies being a decreasing limit of continuous real-valued functions, which, as already mentioned in the beginning of Section 5.5.2$_{250}$, implies being u.s.c. according to our definition. Any capacity according to us is thus surely a capacity according to Dellacherie [28], which is sufficient for all our further results to hold.

Recall from the beginning of this section—Section 5.5.2$_{250}$—that Ω is compact and metrizable, which is in line with Dellacherie's assumption about the set 'E' in [28, Section II.1.1]; see [28, Introduction, Paragraph 2]. Furthermore, observe that CA3$_{\curvearrowleft}$ only applies to sequences in $\mathbb{V}^{\mathrm{u}}_{\geq}$ instead of sequences in $\overline{\mathbb{V}}_{\geq}$; this too corresponds to the definition given in [28, Section II.1.1] because Dellacherie always considers u.s.c. functions to be real-valued [28, Introduction, Paragraph 2]. In fact, one could restate CA3$_{\curvearrowleft}$ so as to only apply to sequences that are uniformly bounded above (and below); this follows immediately from the decreasing character and the following lemma.

Lemma 5.5.7. *Any u.s.c. variable $f \in \mathbb{V}^{\mathrm{u}}_{\geq}$ is bounded and therefore a gamble.*

Proof. f is clearly bounded below, so it suffices to prove that f is bounded above. Recall from Lemma 5.5.2$_{251}$ that f is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of finitary bounded below variables. Assume **ex absurdo** that f is not bounded above. Then, for each $n \in \mathbb{N}$, since $f_n \geq \inf_{m\in\mathbb{N}} f_m = f$, it follows that f_n is also not bounded above. Since each f_n can only take a finite number of different values—because it is finitary and $\mathscr{X}$ is finite—we must have that $f_n(\omega) = +\infty$ for at least one $\omega \in \Omega$. So, for each $n \in \mathbb{N}$, the set $A_n := \{\omega \in \Omega \colon f_n(\omega) = +\infty\}$ is non-empty. Since f_n is finitary, A_n is a finite union of cylinder events, and since $(f_n)_{n\in\mathbb{N}}$ is decreasing, the sequence $(A_n)_{n\in\mathbb{N}}$ is clearly also decreasing. Hence, by Lemma 4.C.2$_{209}$, there exists a path $\omega \in \Omega$ such that $\omega \in A_n$ for all $n \in \mathbb{N}$. This means, by the definition of the sets A_n, that $f_n(\omega) = +\infty$ for all $n \in \mathbb{N}$, and therefore also that $f(\omega) = \lim_{n\to+\infty} f_n(\omega) = +\infty$. But this is in contradiction with the fact that f is real-valued. □

It therefore follows—almost immediately—from the earlier deduced continuity properties for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$, that the restrictions of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|s)$ to $\overline{\mathbb{V}}_{\geq}$ are both capacities on Ω.

Proposition 5.5.8. *For any imprecise probability tree $\mathscr{P}_{\bullet}$, any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, and any $s \in \mathscr{X}^*$, the restrictions of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|s)$ to $\overline{\mathbb{V}}_{\geq}$ are capacities on Ω.*

Proof. Property CA1$_{\curvearrowleft}$ follows for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\cdot|s)$ from Proposition 5.4.3$_{242}$ and for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|s)$ from Proposition 4.4.3$_{164}$. That $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|s)$ satisfy Property CA2$_{\curvearrowleft}$ follows from Theorem 4.6.1$_{175}$ and Theorem 5.4.7$_{245}$, respectively, and the fact that $\overline{\mathbb{V}}_{\geq} \subseteq \overline{\mathbb{V}}_{\mathrm{b}}$. Finally, that they satisfy Property CA3$_{\curvearrowleft}$ follows from Proposition 5.5.6$_{\curvearrowleft}$, together with the fact that, as a consequence of Lemma 5.5.7, u.s.c. variables in $\mathbb{V}^{\mathrm{u}}_{\geq}$ are gambles [and uniformly bounded below by 0]. □

Now, for any capacity F on Ω, we say that a variable $f \in \overline{\mathbb{V}}_{\geq}$ is **F-capacitable** if

$$\mathrm{F}(f) = \sup\big\{\mathrm{F}(g) \colon g \in \mathbb{V}^{\mathrm{u}}_{\geq},\, g \text{ is u.s.c. and } f \geq g\big\}. \tag{5.8}$$

A variable $f \in \overline{\mathbb{V}}_{\geq}$ is called **universally capacitable** if it is F-capacitable for all capacities F on Ω. Now, Choquet's capacitability theorem [28, Theorem II.2.5] states that any **analytic** (non-negative) variable is universally capacitable. The definition of an analytic (non-negative) variable can be found in [28, 53]; we do not explicitly give it here, because it is a rather abstract concept that, in practice, can often be replaced by the simpler and better-known notion of a Borel-measurable (non-negative) variable, which is in our case in turn equivalent to the notion of a $\sigma(\mathscr{X}^*)$-measurable (non-negative) variable; see Corollary 5.C.2$_{278}$ and Proposition 5.C.3$_{278}$. Taking this into account, [28, Theorem II.2.5] allows us to state the following weaker version of Choquet's capacitability theorem:

Theorem 5.5.9 (Choquet's capacitability light)**.** *Any $\sigma(\mathscr{X}^*)$-measurable variable $f \in \overline{\mathbb{V}}_{\geq}$ is universally capacitable.*

It now suffices to combine this theorem with what we already know to obtain the desired type of equality.

Theorem 5.5.10. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet :=$ $\overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq. (3.3)$_{51}$. Then, for any $s \in \mathscr{X}^*$ and any $\sigma(\mathscr{X}^*)$-measurable bounded below variable $f \in \overline{\mathbb{V}}_{\mathrm{b}}$, we have that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.*

Proof. Let $f \in \overline{\mathbb{V}}_{\mathrm{b}}$ be bounded below and $\sigma(\mathscr{X}^*)$-measurable. Since f is bounded below, and both $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ are constant additive [Proposition 4.4.3$_{164}$ EC5], we may assume without loss of generality that f is non-negative—and therefore, that $f \in \overline{\mathbb{V}}_{\geq}$. Then, according to Theorem 5.5.9, the variable f is universally capacitable. Since $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\cdot|s)$ are both capacities on Ω by Proposition 5.5.8$_{\leftarrow}$, this implies that

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \sup\Big\{\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(g|s) \colon g \in \mathbb{V}^{\mathrm{u}}_{\geq},\, g \text{ is u.s.c. and } f \geq g\Big\}$$

and

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \sup\Big\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) \colon g \in \mathbb{V}^{\mathrm{u}}_{\geq},\, g \text{ is u.s.c. and } f \geq g\Big\}.$$

Now recall Corollary 5.5.4$_{251}$, which says that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(h|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(h|s)$ for all u.s.c. gambles $h \in \mathbb{V}$. Since all u.s.c. variables $g \in \mathbb{V}^{\mathrm{u}}_{\geq}$ are gambles due to Lemma 5.5.7$_{\leftarrow}$, we obtain that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. □

Note that, since this equality is valid for bounded below $\sigma(\mathscr{X}^*)$-measurable variables, the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ in the result above can be replaced by its simplified version $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$, which was defined as an upper envelope of standard 'precise' Lebesgue integrals.

Corollary 5.5.11. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq.* (3.3)$_{51}$. *Then, for any $s \in \mathscr{X}^*$ and any $\sigma(\mathscr{X}^*)$-measurable variable $f \in \overline{\mathbb{V}}_{\mathrm{b}}$ that is bounded below, we have that $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.*

Proof. This follows from Corollary 5.4.1$_{241}$ and Theorem 5.5.10$_\curvearrowleft$. □

Also note that the theorem above in particular also implies that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on all limits of increasing sequences of finitary gambles; e.g. hitting times [Example 4.2.2$_{140}$].

Corollary 5.5.12. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq.* (3.3)$_{51}$. *Then $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for any $s \in \mathscr{X}^*$ and any $f \in \overline{\mathbb{V}}$ that is the pointwise limit of an increasing sequence of finitary gambles.*

Proof. Consider any $s \in \mathscr{X}^*$ and any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$, and let $f := \lim_{n\to+\infty} f_n = \sup_{n\in\mathbb{N}} f_n$ be its pointwise limit. Since $(f_n)_{n\in\mathbb{N}}$ is increasing and f_1 is a (bounded) gamble, we have that f is bounded below. To see that f is moreover $\sigma(\mathscr{X}^*)$-measurable, it suffices to observe that any finitary gamble is $\sigma(\mathscr{X}^*)$-measurable [because the level sets will be finite unions of cylinder events] and then use MV2$_{228}$. So we have that $f \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$, and therefore the desired equality follows from Theorem 5.5.10$_\curvearrowleft$. □

5.5.3 An equality for decreasing limits of finitary gambles

The equality in Theorem 5.5.10$_\curvearrowleft$ already covers a great deal of variables, yet if local sets of mass functions are closed, we can extend this equality even further to also involve decreasing—not necessarily bounded below—limits of finitary gambles.

Theorem 5.5.13. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ such that $\mathscr{P}_t$ is closed for all $t \in \mathscr{X}^*$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq.* (3.3)$_{51}$. *Then $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for any $s \in \mathscr{X}^*$ and any $f \in \overline{\mathbb{V}}$ that is the pointwise limit of a decreasing sequence of finitary gambles.*

Proof. Corollary 5.5.3$_{251}$ says that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide on $\mathbb{F} \times \mathscr{X}^*$. By Proposition 5.4.12$_{248}$ and Theorem 4.7.3$_{182}$, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ are both continuous with respect to decreasing sequences of finitary gambles, so $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ also coincide on all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ for which f is the pointwise limit of a decreasing sequence of finitary gambles. □

Again, the equality above can equally well be stated for the simplified upper expectation $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$.

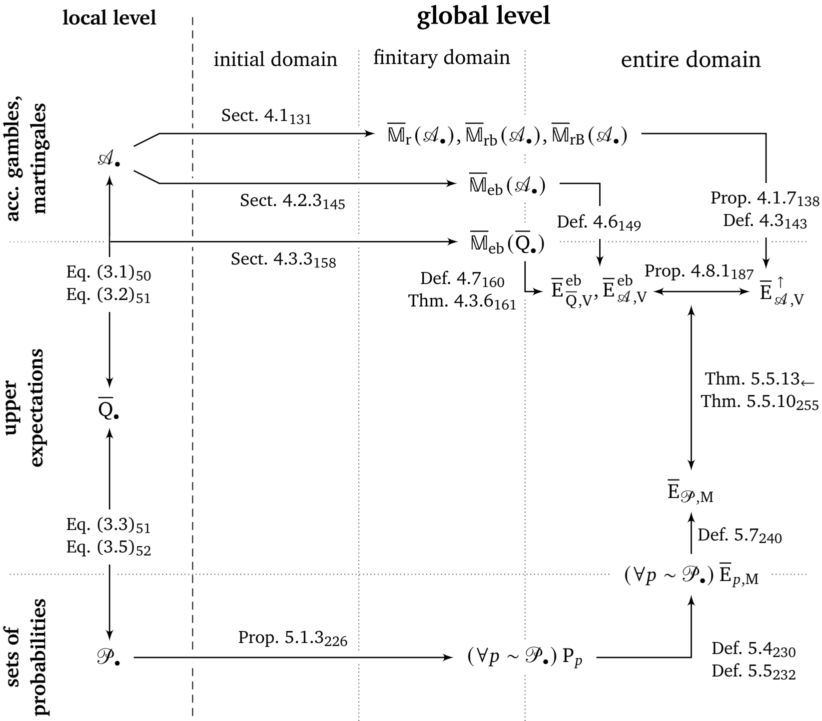

Figure 5.2 Schematic overview of the measure-theoretic approach, and how it relates to the game-theoretic approach.

Corollary 5.5.14. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ such that $\mathscr{P}_t$ is closed for all $t \in \mathscr{X}^*$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees according to Eq.* (3.3)$_{51}$*. Then $\overline{\mathrm{E}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for any $s \in \mathscr{X}^*$ and any $f \in \overline{\mathbb{V}}$ that is the pointwise limit of a decreasing sequence of finitary gambles.*

Proof. Any finitary gamble g is clearly $\sigma(\mathscr{X}^*)$-measurable because the level sets $\{\omega \in \Omega \colon g(\omega) \leq c\}$ are finite unions of cylinder events. So MV2$_{228}$ implies that any pointwise limit of a decreasing sequence of finitary gambles is a $\sigma(\mathscr{X}^*)$-measurable variable. Such a limit is clearly also bounded above, so the desired statement follows from Corollary 5.4.1$_{241}$ and Theorem 5.5.13$_{\leftarrow}$. □

5.5.4 Concluding notes on the relation between $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$

If the local sets of mass functions $\mathscr{P}_s$ are closed, Theorem 5.5.10$_{255}$ can be combined with Theorem 5.5.13$_{\leftarrow}$, and together they establish an equality between $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ for nearly all practically relevant variables. This becomes clear if, apart from upper expectations, we also take a look at the conjugate lower expectations. Indeed, in a practical situation with a fixed variable of interest, we usually want to assess both the corresponding upper and lower expectation at the same time; see e.g. [58, 62]. Theorem 5.5.10$_{255}$ and Theorem 5.5.13$_{256}$ can be extended to this two-sided setting, and what we get can be summarized as follows.

Corollary 5.5.15. *Consider any imprecise probability tree $\mathscr{P}_\bullet$ and let $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{P}}$ be the upper expectations tree that agrees with it according to Eq. (3.3)$_{51}$. Then, for any $s \in \mathscr{X}^*$ and any $\sigma(\mathscr{X}^*)$-measurable gamble $f \in \mathbb{V}$, we have that*

$$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \text{ and } \underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s).$$

If the sets $\mathscr{P}_t$ are moreover closed for all $t \in \mathscr{X}^*$, then the above equalities also hold for any (extended real-valued) variable $f \in \overline{\mathbb{V}}$ that is the pointwise limit of a monotone (decreasing or increasing) sequence of finitary gambles.

Proof. The equality between $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ for $\sigma(\mathscr{X}^*)$-measurable gambles follows from Theorem 5.5.10$_{255}$. The equality for the lower expectations follows from conjugacy and Theorem 5.5.10$_{255}$. Indeed, if f is an $\sigma(\mathscr{X}^*)$-measurable gamble, then so is $-f$, and therefore, by Theorem 5.5.10$_{255}$, we have that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(-f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(-f|s)$. This implies by Corollary 5.4.2$_{241}$ and Corollary 4.3.7$_{162}$ that $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$.

Now suppose that the sets $\mathscr{P}_t$ are moreover closed for all $t \in \mathscr{X}^*$. Then the equality between $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ for decreasing limits of finitary gambles follows from Theorem 5.5.13$_{256}$. On the other hand, the equality (between the upper expectations) for increasing limits of finitary gambles follows from the equality on finitary gambles [Corollary 5.5.3$_{251}$] and the continuity of both operators with respect to increasing bounded below sequences [Theorem 4.6.1$_{175}$ and Theorem 5.4.7$_{245}$].[16] So we have that $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincide for all monotone (decreasing or increasing) limits of finitary gambles. So it remains to prove that this also holds for the lower expectations $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. To that end, note that if f is the pointwise limit of a monotone sequence of finitary gambles, then so is $-f$, and hence, by what we have just proved, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(-f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(-f|s)$. This implies by Corollary 5.4.2$_{241}$ and Corollary 4.3.7$_{162}$ that $-\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = -\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ and therefore that $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. □

Note that the equality for $\sigma(\mathscr{X}^*)$-measurable gambles already covers many practically relevant inferences; limit upper and lower expected time averages [26, 93], hitting probabilities [58] and (bounded) stopping times [see Lemma 4.C.3$_{210}$] to name but a few. Another example is the upper and lower probability of the event that the pathwise time average of a function

[16] It can also be deduced from Theorem 5.5.10$_{255}$ though, since any increasing limit of finitary gambles is clearly bounded below and also $\sigma(\mathscr{X}^*)$-measurable because finitary gambles are $\sigma(\mathscr{X}^*)$-measurable and because the set $\overline{\mathbb{V}}_\sigma$ of $\sigma(\mathscr{X}^*)$-measurable variables is closed under taking pointwise limits [MV2$_{228}$].

$f \in \mathscr{L}(\mathscr{X})$ eventually remains within some given interval; so the event A consisting of all paths $\omega \in \Omega$ such that

$$a \leq \liminf_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(\omega_i) \leq \limsup_{n\to+\infty} \frac{1}{n}\sum_{i=1}^{n} f(\omega_i) \leq b,$$

with a and b two reals such that $a \leq b$. Such events are typically of interest when studying ergodic behaviour; see [26].

The fact that boundedness of the considered variable (or gamble) is required is a serious issue though, since, as we have mentioned before in Section 3.6$_{98}$ and Section 4.2$_{139}$, unbounded and extended real-valued variables often also belong to our field of interest. Most of these unbounded and/or extended real variables can be written as monotone limits of finitary gambles—e.g. hitting times [58]—and as such this clarifies the importance of the second part of Corollary 5.5.15$_{\leftarrow}$. Caution should be taken here, however, because we can only ensure that this equality for monotone limits of finitary gambles holds when the local sets of mass functions are closed.

If local sets of mass functions are not closed, then, as was shown in Example 5.4.10$_{247}$, the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ may fail to be continuous with respect to decreasing sequences of finitary gambles. Since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ on the other hand always satisfies this type of continuity [Theorem 4.7.3$_{182}$], and since the two types of upper expectations coincide on finitary gambles [Corollary 5.5.3$_{251}$], then the inequality in Proposition 5.5.1$_{249}$ must sometimes become strict.

Example 5.5.16. Recall the situation from Example 5.4.10$_{247}$. We had that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = 0 \neq -\infty = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f),$$

with $f := \lim_{n\to+\infty} f_n$ the pointwise limit of the decreasing sequence $(f_n)_{n\in\mathbb{N}}$. Due to Corollary 5.5.3$_{251}$ and Theorem 4.7.3$_{182}$, it then follows that

$$0 = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f_n) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f_n) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f).$$

Hence, we have that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f) > \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f)$. ◇

It follows from Proposition 5.5.1$_{249}$ that, even if game-theoretic upper expectations and measure-theoretic upper expectations do not coincide, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ still always provides a conservative upper bound for the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$. Analogously, by using conjugacy, one can then also easily see that $\underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ provides a conservative lower bound for $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$.

Now, if we take a step back and look at the overall features of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ as global uncertainty models, it seems a done deal that the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ comes out as the better of the two. Our statement is supported by the following four arguments:

(i) The definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is obtained from direct behavioural principles and therefore has an interpretation that is clear, and perhaps even intuitive. The definition of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ in contrast, is rather implicit—since probability charges and measures are then primary objects—and relies crucially on the abstract notion of measurability.

(ii) The previous issue is not only relevant from a philosophical point of view, but also from a mathematical one, because it makes the analysis of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ easier compared to that of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$. We do not have to care about troublesome issues such as checking measurability. Instead, we can simply focus on building supermartingales; a job that is often more intuitive to perform than finding a suitable choice for a (compatible) precise probability tree and determining its corresponding Lebesgue integral (and checking measurability).

(iii) As mentioned above and shown in Proposition 5.5.1$_{249}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ provides a conservative upper bound for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$, and $\underline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ provides a conservative lower bound for $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$. Hence, if there is no good reason to use either game-theoretic upper and lower expectations or measure-theoretic upper and lower expectations, then it is safest—or more robust—to work with game-theoretic upper and lower expectations.

(iv) As shown in Example 5.4.10$_{247}$, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ sometimes lacks continuity with respect to decreasing sequences of finitary gambles, and thus $\underline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ sometimes lacks continuity with respect to increasing sequences of finitary gambles. We consider this to be a deficiency; the limit variables—e.g. hitting times—of such sequences $(f_n)_{n\in\mathbb{N}}$ are simply considered to be abstract idealisations of the individual finitary gambles f_n for arbitrarily large n, and so we typically also want the upper and lower expectations of such limit variables to assess the upper and lower expectations of the finitary gambles f_n for large n. This is only guaranteed if the adopted global upper and lower expectations are continuous.

A possible argument that might be advanced to mitigate the somewhat negative image of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ given above is that, in a precise context, the vast amount of standard measure-theoretic literature provides a broad variety of powerful results for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$—or, better, $\overline{\mathrm{E}}_{p,\mathrm{M}}$—on the domain of measurable variables. Yet, by Theorem 5.3.1$_{235}$, these properties also all hold for the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ obtained from the agreeing (upper) expectations tree $\mathrm{Q}_\bullet$, so there is nothing to be gained in this respect.

Another argument for the defence could be that, compared to the game-theoretic upper expectation, the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ allows us to model uncertainty in a more flexible manner. Indeed, recall from Theorem 4.3.6$_{161}$ that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ could be replaced by $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ with $\overline{\mathrm{Q}}_\bullet := \overline{\mathrm{Q}}_{\bullet,\mathscr{A}}$

the agreeing upper expectations tree—that never takes into account the boundary structure of the sets $\mathscr{A}_s$. A similar thing **cannot** be done for the measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ however. Consider, for instance, the setting from Example 5.4.10$_{247}$ and replace the open set $\mathscr{P}_\square$ by its closure $\mathscr{P}'_\square = \{\mu \in \mathbb{P}(\mathscr{X}) \colon 0 \leq \mu(a) \leq 1\} = \mathbb{P}(\mathscr{X})$; then we obtain the same agreeing upper expectations tree—its initial upper expectation is simply the vacuous upper expectation—but the resulting measure-theoretic upper expectation clearly differs on the variable $f := -\infty\mathbb{1}_a$. The measure-theoretic upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ can thus distinguish between differences in the local models in ways that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is unable to. The reason, however, why we do not find this argument convincing is that, as was raised in point (iv)$_\leftarrow$ above, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ lacks certain basic continuity properties if local sets of mass functions are not closed. We are therefore inclined to only consider $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ as a possible option if local sets of mass functions are closed, and in that case there is no longer a difference in generality compared to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$.

This being said, it should not be forgotten that the comparison we draw here between the game-theoretic and measure-theoretic approaches is focussed on our current setting; that is, a setting where we start from the local models $\overline{\mathrm{Q}}_\bullet$ or $\mathscr{P}_\bullet$ (or $\mathscr{A}_\bullet$), and where the specific global upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ or $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ are the eventual objects of interest. These considerations should by no means be extrapolated to other settings; on the contrary, there are cases where the measure-theoretic approach would definitely be more suited. A great advantage of it, for instance, is its capability to extend general initial—yet precise—assessments, which may come in the form of general global probabilities rather than only local 'one-step' probabilities—as was the case in our treatment. Moreover, the domain of conditioning events can in the measure-theoretic framework easily be extended beyond the set of all situations. The game-theoretic framework lacks such features.

5.5.5 Relation to Shafer and Vovk's work

Before we conclude this chapter, it seems appropriate to spend a few words on how our work here compares to that of Shafer and Vovk in [85, Chapter 9]. As readers that are familiar with their work may have noticed, the idea to use Choquet's capacitability theorem to extend the domain of the equality between $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ from u.s.c. variables to $\sigma(\mathscr{X}^*)$-measurable (or analytic) variables already appears in [85, Chapter 9]. Apart from that, it can also be observed that Lemma 3.E.7$_{125}$—which underlies the proof of Proposition 5.4.9$_{246}$ and Proposition 5.4.12$_{248}$—is also strongly inspired by the proof of [85, Lemma 9.10]. So it is fair to say that [85, Chapter 9] served as an important inspiration for our work in Sections 5.4$_{240}$–5.5$_{249}$. Nonetheless, there are several aspects which make our work here stand apart from that in [85, Chapter 9], and we next aim to highlight the most important of these aspects.

The first and most important difference is that Shafer and Vovk consider supermartingales and game-theoretic upper expectations under the prequential principle. Recall from point (iv)$_{189}$ of the discussion in Section 4.9.1$_{188}$ that, in that case, Forecaster's moves—the specification of the local models $\mathscr{A}_s$ or $\overline{\mathrm{Q}}_s$—are not required to be known beforehand for each situation $s \in \mathscr{X}^*$, but instead are allowed to also depend on previous moves by Skeptic. While this assumption allows them to remain more general—note that, in contrast with (ii)$_{189}$, local state spaces are now assumed finite for both us and them (in [85, Section 9.2])—the benefit that we gain from dropping it is remarkable; it allows us to replace [85, Lemma 9.10] and [85, Theorem 9.7], which require strong topological conditions on how the local models can be chosen (by Forecaster),[17] with respectively Corollary 5.5.4$_{251}$ and Theorem 5.5.10$_{255}$, which are similar, but do not need any topological conditions at all.

A second notable difference is that our results involve larger domains; Theorem 5.5.10$_{255}$ applies to bounded below ($\sigma(\mathscr{X}^*)$-measurable) variables, and Theorem 5.5.13$_{256}$—which holds if local models are closed—applies to any decreasing (extended real-valued) limit of finitary gambles. The equalities established in [85, Lemma 9.10] and [85, Theorem 9.7], however, only apply to bounded variables.[18] The fact that this extension in domain is relevant can be deduced by looking at Corollary 5.5.15$_{258}$; as already mentioned, the second type of equality—for monotone limits of finitary gambles—is of considerable importance, yet it is exactly this class of variables that is missing in Shafer and Vovk's results. Moreover, our results also allow conditioning on situations, whereas the ones in [85, Chapter 9] only apply to unconditional upper expectations.

Finally, though we have stated all of our results for the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$, we know by Theorem 4.3.6$_{161}$ that they all remain to hold if $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is replaced by the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ obtained from an acceptable gambles tree $\mathscr{A}_\bullet$. Recall that the latter are more general than upper expectations trees, so a priori—without Theorem 4.3.6$_{161}$—it is not guaranteed, nor trivial that this can be done. Shafer and Vovk [85, Chapter 9], on the other hand, always limit themselves to the case where local models come in the form of upper expectations.

[17] More specifically, Forecaster is required to choose elements θ from a compact metrizable parameter space Θ, which are then mapped to a corresponding local upper expectation by a upper semicontinuous mapping $\overline{\mathbf{E}}$; see [85, Protocol 9.5].

[18] [85, Theorem 9.7] applies to (bounded) analytic variables, but recall from the discussion above Theorem 5.5.9$_{255}$ that we could have just as well stated Theorem 5.5.10$_{255}$ for analytic variables instead of $\sigma(\mathscr{X}^*)$-measurable variables.

— Appendices —

5.A Basic measure-theoretic concepts

The purpose of this appendix section is, on the one hand, to establish some of the basic properties of measure-theoretic expectations that we have used throughout this chapter, and on the other hand, to set some concepts in place as a preparation for proving Proposition 5.3.2$_{236}$—which will be the topic of the next section. We limit ourselves to the bare essentials; for more contextual information we refer to a multitude of textbooks on the topic of measure-theoretic probability [5, 32, 80, 89, 90, 102].

A **measurable space** $(\mathscr{Y}, \mathcal{A})$ is a couple where $\mathscr{Y}$ is a non-empty set and $\mathcal{A}$ is a σ-algebra on $\mathscr{Y}$. We say that $A \subseteq \mathscr{Y}$ is $\mathcal{A}$**-measurable** if $A \in \mathcal{A}$, and we say that an extended real-valued function $f\colon \mathscr{Y} \to \overline{\mathbb{R}}$ is $\mathcal{A}$**-measurable** if the set $f^{-1}(B) := \{y \in \mathscr{Y} : f(y) \in B\}$ is $\mathcal{A}$-measurable for every $B \in \mathscr{B}(\overline{\mathbb{R}})$. Here, $\mathscr{B}(\overline{\mathbb{R}})$ denotes the Borel σ-algebra on $\overline{\mathbb{R}}$, being the σ-algebra generated by all open—or, by complementation, closed—sets in $\overline{\mathbb{R}}$; recall Section 1.6$_{14}$ for the topology on $\overline{\mathbb{R}}$. A subset B of $\overline{\mathbb{R}}$ is in $\mathscr{B}(\overline{\mathbb{R}})$ if and only if B is the union of a Borel subset of $\mathbb{R}$ and one of the four subsets of $\{+\infty, -\infty\}$ [40, Section 1.4, Problem 17]. This leads to a notion of measurability that is equivalent to the one used by Billingsley [5, p. 184]. The Borel σ-algebra $\mathscr{B}(\overline{\mathbb{R}})$ can also be generated alternatively from the sets $\{x \in \overline{\mathbb{R}} : x \lhd c\}$ or the sets $\{x \in \overline{\mathbb{R}} : x \rhd c\}$ where $c \in \mathbb{R}$ and $\lhd$ takes the form $<$ or $\leq$ and $\rhd$ takes the form of $>$ or $\geq$; see for instance [81, Lemma 8.3]. So, as we have done in the main text, we can alternatively characterise the $\mathcal{A}$-measurable functions as those functions $f\colon \mathscr{Y} \to \overline{\mathbb{R}}$ such that $\{y \in \mathscr{Y} : f(y) \leq c\} \in \mathcal{A}$ for all $c \in \mathbb{R}$ [5, Theorem 13.1. (i)]. Typically, in measure-theoretic probability literature, an $\mathcal{A}$-measurable real-valued function f is called a **random variable**. We gather all (possibly unbounded) random variables in the set $\mathscr{L}^{\mathrm{u}}_{\mathcal{A}}(\mathscr{Y})$, and all $\mathcal{A}$-measurable **extended** real-valued functions f in $\overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$. A non-negative extended real-valued function $f\colon \mathscr{Y} \to \overline{\mathbb{R}}_{\geq}$ is $\mathcal{A}$**-simple** if it is a finite sum $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ with $a_1, \dots, a_n \in \overline{\mathbb{R}}_{\geq}$ and $A_1, \dots, A_n \in \mathcal{A}$. It is therefore trivially an element of $\overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$.

A **probability space** $(\mathscr{Y}, \mathcal{A}, \mathrm{P})$ is a measurable space $(\mathscr{Y}, \mathcal{A})$ equipped with a probability measure P on $\mathcal{A}$ [5, p.23]. We say that an event $A \in \mathcal{A}$ is P**-null** if $\mathrm{P}(A) = 0$, and we say that A is P**-almost sure** if $\mathrm{P}(A) = 1$. We will also say that a property about the elements in $\mathscr{Y}$ holds P**-almost surely (**P**-a.s.)** if the event consisting of all elements for which the property holds is P-almost sure [5, p.60].[19] Note that, since a probability measure

[19] Recall from Section 4.5$_{171}$ that the game-theoretic notion of almost surely (a.s.) is defined

is (countably) additive [Definition 5.1$_{221}$] and since it always takes values between zero and one [5, p.22], the intersection $A \cap B \in \mathcal{A}$ of two P-almost sure events $A, B \in \mathcal{A}$, is itself also P-almost sure.

For any probability space $(\mathscr{Y}, \mathcal{A}, \mathrm{P})$, the corresponding (measure-theoretic) **expectation** E is defined by $\mathrm{E}(f) := \int_{\mathscr{Y}} f \mathrm{dP}$ for all $f \in \overline{\mathscr{L}}_{\mathcal{A}}$ such that $\int_{\mathscr{Y}} f \mathrm{dP}$ is defined (or exists), where the latter is the Lebesgue integral of f over $\mathscr{Y}$ with respect to P. The Lebesgue integral for a general probability space is defined similarly as in Definition 5.3$_{228}$; we nevertheless state it for the sake of completeness.

Definition 5.9 (The Lebesgue integral)**.** Consider any probability space $(\mathscr{Y}, \mathcal{A}, \mathrm{P})$, and any non-negative $f \in \overline{\mathscr{L}}_{\mathcal{A}}$. Then the Lebesgue integral of f with respect to P is defined as

$$\int f \mathrm{dP} := \sup \left\{ \sum_{i=1}^{n} \inf(f|A_i)\mathrm{P}(A_i) \colon A_i \in \mathcal{A} \text{ and } (A_i)_{i=1}^{n} \text{ partitions } \mathscr{Y} \right\}.$$

For a general $f \in \overline{\mathscr{L}}_{\mathcal{A}}$, we let $f^+ := f^{\vee 0}$ and $f^- := -(f^{\wedge 0})$, and the Lebesgue integral is then defined by

$$\int f \mathrm{dP} := \int f^+ \mathrm{dP} - \int f^- \mathrm{dP},$$

unless $\int f^+ \mathrm{dP} = \int f^- \mathrm{dP} = +\infty$, in which case the Lebesgue integral of f with respect to P is not defined (does not exist). ⊚

If both $\int f^+ \mathrm{dP} < +\infty$ and $\int f^- \mathrm{dP} < +\infty$, and thus $\mathrm{E}(f^+) = \int f^+ \mathrm{dP}$ and $\mathrm{E}(f^-) = \int f^- \mathrm{dP}$ are real—neither of them can be equal to $-\infty$ because the Lebesgue integral is clearly non-negative for non-negative variables—then we say that f is P-integrable. The following is a list of convenient properties for measure-theoretic expectations/Lebesgue integrals that we have used in the main text; recall Proposition 5.2.3$_{230}$.

Lemma 5.A.1. *For any probability space* $(\mathscr{Y}, \mathcal{A}, \mathrm{P})$ *and any two* $f, g \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$, $c \in \overline{\mathbb{R}}$ *and* $a, b \in \mathbb{R}$, *the following properties hold:*

ME1. $\mathrm{E}(c) = c$;

ME2. $f \leq g \Rightarrow \mathrm{E}(f) \leq \mathrm{E}(g)$ *if* $\mathrm{E}(f)$ *and* $\mathrm{E}(g)$ *exist;*

ME3. $\mathrm{E}(af + bg) = a\mathrm{E}(f) + b\mathrm{E}(g)$ *if* f, g *are* P*-integrable;*

ME4. $\mathrm{E}(f)$ *exists if* f *is bounded below or above;*

ME5. *If* f *is bounded, then* $\mathrm{E}(f)$ *is real and* f *is* P*-integrable;*

by means of supermartingales that converge to $+\infty$. It is equivalent to demanding that the **lower** probability $\underline{\mathrm{P}}^{\mathrm{eb}}_{\mathrm{Q,V}}(A)$ of the event A of interest is 1, yet does not require A to be measurable.

ME6. $-\mathrm{E}(f) = \mathrm{E}(-f)$ *if* $\mathrm{E}(f)$ *(or equivalently* $\mathrm{E}(-f)$*) exists.*

The following statements hold for any sequence $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$*:*

ME7. *If* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to a variable* $f \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$*, and* $|f_n| \leq f^*$ *for all* $n \in \mathbb{N}$*, with* $f^* \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ *a* P*-integrable variable, then* f *and all* f_n *are* P*-integrable and*

$$\lim_{n\to+\infty} \mathrm{E}(f_n) = \mathrm{E}(f).$$

ME8. *If* $(f_n)_{n\in\mathbb{N}}$ *is increasing and there is an* $f^* \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ *such that* $\mathrm{E}(f^*) > -\infty$ *and* $f_1 \geq f^*$*, then*

$$\lim_{n\to+\infty} \mathrm{E}(f_n) = \mathrm{E}(f) \text{ where } \lim_{n\to+\infty} f_n = f.$$

ME9. *If* $(f_n)_{n\in\mathbb{N}}$ *is decreasing and there is an* $f^* \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ *such that* $\mathrm{E}(f^*) < +\infty$ *and* $f_1 \leq f^*$*, then*

$$\lim_{n\to+\infty} \mathrm{E}(f_n) = \mathrm{E}(f) \text{ where } \lim_{n\to+\infty} f_n = f.$$

The following statements hold for any two bounded below $f, g \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ *and* $\mu \in \mathbb{R}$*:*

ME10. $\mathrm{E}(f + \mu) = \mathrm{E}(f) + \mu$*;*

ME11. $f = g$ P-almost surely $\Rightarrow \mathrm{E}(f) = \mathrm{E}(g)$.

Proof. Property ME1$_{\leftarrow}$ follows straightforwardly from the definition of the Lebesgue integral [Definition 5.9$_{\leftarrow}$].

Property ME2$_{\leftarrow}$ for non-negative f and g also follows trivially from the definition of the Lebesgue integral. That it holds for general f and g, can be inferred from the observation that, if $f \leq g$, then $f^+ \leq g^+$ and $f^- \geq g^-$, and thus [by ME2$_{\leftarrow}$ for non-negative functions] that $\mathrm{E}(f^+) \leq \mathrm{E}(g^+)$ and $\mathrm{E}(f^-) \geq \mathrm{E}(g^-)$. It then indeed follows from the definition of the Lebesgue integral that $\mathrm{E}(f) \leq \mathrm{E}(g)$.

Property ME3$_{\leftarrow}$ is taken directly from [5, Theorem 16.1 (ii)].

Property ME4$_{\leftarrow}$ for bounded below f follows from the fact that $\mathrm{E}(f^-)$ is finite, which is on itself a consequence of f^- being bounded above (and trivially bounded below by 0) and properties ME1$_{\leftarrow}$ and ME2$_{\leftarrow}$. In a similar way, property ME4$_{\leftarrow}$ holds for bounded above f.

To establish ME5$_{\leftarrow}$, suppose that f is bounded (above and below). Then by ME4$_{\leftarrow}$ the expectation $\mathrm{E}(f)$ exists, and due to ME1$_{\leftarrow}$ and ME2$_{\leftarrow}$ we moreover have that $\inf f \leq \mathrm{E}(f) \leq \sup f$. Since f is bounded, it follows that $\mathrm{E}(f)$ is real. In a similar way, one can establish that $\mathrm{E}(f^+)$ and $\mathrm{E}(f^-)$ are real, and therefore that f is P-integrable. This establishes ME5$_{\leftarrow}$.

To prove ME6, assume that $\mathrm{E}(f)$ exists. By the definition of the Lebesgue integral, we then have that

$$-\mathrm{E}(f) = -\mathrm{E}(f^+) + \mathrm{E}(f^-) = -\mathrm{E}((-f)^-) + \mathrm{E}((-f)^+) = \mathrm{E}(-f),$$

and as a consequence, that $\mathrm{E}(-f)$ moreover exists. Furthermore, that the existence of $\mathrm{E}(-f)$ implies the existence of $\mathrm{E}(f) = \mathrm{E}(-(-f))$, follows in the same way.

Property $\mathrm{ME7}_{\curvearrowleft}$ follows from [5, Theorem 16.4].

To prove $\mathrm{ME8}_{\curvearrowleft}$, fix any increasing $(f_n)_{n\in\mathbb{N}}$ for which there is an $f^* \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ such that $\mathrm{E}(f^*) > -\infty$ and $f_1 \geq f^*$. Let $f := \lim_{n\to+\infty} f_n$. Then $(f_n^+)_{n\in\mathbb{N}}$ is a sequence of non-negative extended real-valued functions that converges increasingly to f^+. Hence, by [5, Theorem 15.1 (iii)], we have that $\lim_{n\to+\infty} \mathrm{E}(f_n^+) = \mathrm{E}(f^+)$. On the other hand, since $f^* \leq f_1 \leq f_n \leq f$ for all $n \in \mathbb{N}$ [because $(f_n)_{n\in\mathbb{N}}$ is increasing], we have that $(f^*)^- \geq f_n^- \geq f^- \geq 0$ for all $n \in \mathbb{N}$. Moreover, since $\mathrm{E}(f^*) = \mathrm{E}((f^*)^+) - \mathrm{E}((f^*)^-) > -\infty$ by assumption, we have that $\mathrm{E}((f^*)^-) < +\infty$ [recall that $\mathrm{E}((f^*)^+) = \mathrm{E}((f^*)^-) = +\infty$ is not allowed in the definition of the Lebesgue integral]. Since $(f^*)^-$ is non-negative, $\mathrm{E}((f^*)^-)$ is non-negative too [clearly by Definition 5.9_{264}], and so by the fact that $\mathrm{E}((f^*)^-) < +\infty$ we obtain that $(f^*)^-$ is P-integrable. Hence, by $\mathrm{ME7}_{\curvearrowleft}$ [and taking into account that $|f_n^-| = f_n^- \leq (f^*)^-$ and that $\lim_{n\to+\infty} f_n^- = f^-$], we have that $\lim_{n\to+\infty} \mathrm{E}(f_n^-) = \mathrm{E}(f^-)$. Combining this with the fact that $\lim_{n\to+\infty} \mathrm{E}(f_n^+) = \mathrm{E}(f^+)$, we obtain that

$$\lim_{n\to+\infty} \Big(\mathrm{E}(f_n^+) - \mathrm{E}(f_n^-)\Big) = \lim_{n\to+\infty} \mathrm{E}(f_n^+) - \lim_{n\to+\infty} \mathrm{E}(f_n^-) = \mathrm{E}(f^+) - \mathrm{E}(f^-),$$

where in the first step we were allowed to bring the limits inside because $0 \leq \lim_{n\to+\infty} \mathrm{E}(f_n^-) \leq \mathrm{E}(f_n^-) < +\infty$ due to the monotonicity [$\mathrm{ME2}_{264}$] of E, and the fact that $0 \leq f_n^- \leq (f^*)^-$ and $\mathrm{E}((f^*)^-) < +\infty$. Since thus $\mathrm{E}(f^-) = \lim_{n\to+\infty} \mathrm{E}(f_n^-) \leq \mathrm{E}(f_n^-) < +\infty$, the expectations $\mathrm{E}(f_n) = \mathrm{E}(f_n^+) - \mathrm{E}(f_n^-)$ and $\mathrm{E}(f) = \mathrm{E}(f^+) - \mathrm{E}(f^-)$ surely exist, and thus by the equality above we indeed find that $\lim_{n\to+\infty} \mathrm{E}(f_n) = \mathrm{E}(f)$.

Property $\mathrm{ME9}_{\curvearrowleft}$ follows from $\mathrm{ME8}_{\curvearrowleft}$ by using the fact that $-\mathrm{E}(g) = \mathrm{E}(g^-) - \mathrm{E}(g^+) = \mathrm{E}((-g)^+) - \mathrm{E}((-g)^-) = \mathrm{E}(-g)$ for any $g \in \overline{\mathscr{L}}_{\mathcal{A}}(\mathscr{Y})$ such that $\mathrm{E}(g)$ exists.

To prove $\mathrm{ME10}_{\curvearrowleft}$, we also make use of $\mathrm{ME8}_{\curvearrowleft}$. Fix any $\mu \in \mathbb{R}$. Note that the sequences $(f^{\wedge n})_{n\in\mathbb{N}}$ and $(f^{\wedge n} + \mu)_{n\in\mathbb{N}}$ are both sequences of real-valued functions that converge increasingly to f and $f + \mu$, respectively. Moreover, since f is bounded below, we have that all $f^{\wedge n}$ and $f^{\wedge n} + \mu$ are bounded below by a single real constant $c \in \mathbb{R}$. By $\mathrm{ME1}_{264}$, we have that $\mathrm{E}(c) = c \in \mathbb{R}$. Note moreover that it follows from the $\mathcal{A}$-measurability of f that all $f^{\wedge n}$ and $f^{\wedge n} + \mu$ are $\mathcal{A}$-measurable—e.g. if $\{y \in \mathscr{Y} : f(y) \leq a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$, then clearly also $\{y \in \mathscr{Y} : f^{\wedge n}(y) \leq a\} \in \mathcal{A}$ for all $a \in \mathbb{R}$. Hence, we can apply $\mathrm{ME8}_{\curvearrowleft}$ [with $f^* = c$], to find that

$$\mathrm{E}(f + \mu) = \lim_{n\to+\infty} \mathrm{E}(f^{\wedge n} + \mu) = \lim_{n\to+\infty} \mathrm{E}(f^{\wedge n}) + \mu = \mathrm{E}(f) + \mu,$$

where the second equality follows from $\mathrm{ME1}_{264}$ and the linearity [$\mathrm{ME3}_{264}$] of E, which we can apply because each $f^{\wedge n}$ is bounded below and bounded above, and therefore by $\mathrm{ME5}_{264}$ P-integrable.

To prove $\mathrm{ME11}_{\curvearrowleft}$, we can assume without loss of generality that f and g are both non-negative; indeed, this follows from the fact that they are both bounded below and from $\mathrm{ME10}_{\curvearrowleft}$. Property $\mathrm{ME11}_{\curvearrowleft}$ then follows immediately from [5, Theorem 15.2 (v)]. □

Conditional measure-theoretic expectations are typically defined using the so-called Radon-Nikodým derivative [5, Section 34]. Concretely, the **conditional expectation** $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{B})$ of a P-integrable random variable $f \in \mathcal{L}^{\mathrm{u}}_{\mathcal{A}}(\mathcal{Y})$ with respect to a σ-algebra $\mathcal{B} \subseteq \mathcal{A}$, is any P-integrable random variable that is $\mathcal{B}$-measurable and that satisfies $\int_{\mathcal{Y}} f \mathbb{I}_A \mathrm{dP} = \int_{\mathcal{Y}} \mathrm{E}_{\mathrm{RN}}(f|\mathcal{B}) \mathbb{I}_A \mathrm{dP}$ for all $A \in \mathcal{B}$ or, using a different notation, $\int_A f \mathrm{dP} = \int_A \mathrm{E}_{\mathrm{RN}}(f|\mathcal{B}) \mathrm{dP}$ for all $A \in \mathcal{B}$. It is shown in [5, Section 34] that such a conditional expectation $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{B})$ always exists, and that it is unique up to a P-null set—so any two versions of the conditional expectation $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{B})$ coincide P-almost surely. The value of $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{B})$ on a P-null set can be chosen arbitrarily—provided it remains $\mathcal{B}$-measurable—since it will not change the value of the integral $\int_A \mathrm{E}_{\mathrm{RN}}(f|\mathcal{B}) \mathrm{dP} = \int_{\mathcal{Y}} \mathrm{E}_{\mathrm{RN}}(f|\mathcal{B}) \mathbb{I}_A \mathrm{dP}$; also see Property ME11$_{265}$ above. Recall from ME5$_{264}$ above that, if f is bounded, then f is P-integrable and thus the conditional expectation $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{B})$ always exists. We moreover have the following convenient property.

Lemma 5.A.2 ([5, p.445])**.** *For any probability space $(\mathcal{Y}, \mathcal{A}, \mathrm{P})$ and any bounded random variable $f \in \mathcal{L}^{\mathrm{u}}_{\mathcal{A}}(\mathcal{Y})$, we have that $\mathrm{E}_{\mathrm{RN}}(f|\mathcal{A}) = f$ P-almost surely.*

A (discrete) **filtration** $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$ on a measurable space $(\mathcal{Y}, \mathcal{A})$ is a sequence of (strictly) increasing σ-algebras in $\mathcal{A}$; so $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}$. We will use $\mathcal{A}_\infty$ to denote the smallest σ-algebra that includes the σ-algebras $\mathcal{A}_n$ for all $n \in \mathbb{N}_0$. We say that $(\mathcal{Y}, \mathcal{A}, (\mathcal{A}_n)_{n \in \mathbb{N}_0})$ is a **filtered measurable space** if $(\mathcal{Y}, \mathcal{A})$ is equipped with a filtration $(\mathcal{A}_n)_{n \in \mathbb{N}_0}$, and moreover say that $(\mathcal{Y}, \mathcal{A}, (\mathcal{A}_n)_{n \in \mathbb{N}_0}, \mathrm{P})$ is a **filtered probability space** if it additionally has a σ-additive measure P on $\mathcal{A}$. The following result will be key in establishing a relation between the measure-theoretic and the game-theoretic framework.

Theorem 5.A.3 (Lévy's zero-one law [5, Theorem 35.6])**.** *For any filtered probability space $(\mathcal{Y}, \mathcal{A}, (\mathcal{A}_n)_{n \in \mathbb{N}_0}, \mathrm{P})$ and any P-integrable $f \in \mathcal{L}^{\mathrm{u}}_{\mathcal{A}}(\mathcal{Y})$, we have that $\lim_{n \to +\infty} \mathrm{E}_{\mathrm{RN}}(f|\mathcal{A}_n) = \mathrm{E}_{\mathrm{RN}}(f|\mathcal{A}_\infty)$ P-almost surely.*

5.B Proof of Proposition 5.3.2

We start with establishing a property similar to Bayes' rule [WC4$_{82}$] for the global expectation $\mathrm{E}_{p,\mathrm{M}}$.

Lemma 5.B.1. *For any precise probability tree p, any non-negative $f \in \mathbb{V}_\sigma$, and any $s, t \in \mathcal{X}^*$ such that $t \sqsubseteq s$,*

$$\mathrm{E}_{p,\mathrm{M}}(f \mathbb{1}_s | t) = \mathrm{E}_{p,\mathrm{M}}(f|s) \mathrm{P}_p(s|t).$$

Proof. Recall from Definition 5.4$_{230}$ and Definition 5.3$_{228}$, that for any non-negative

$g \in \mathbb{V}_\sigma$ and any $u \in \mathscr{X}^*$,

$$\mathrm{E}_{p,\mathrm{M}}(g|u) = \sup\left\{\sum_{i=1}^{n} \inf(g|A_i)\mathrm{P}_p(A_i|u) \colon A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\}. \tag{5.9}$$

Since, for any $A_i \in \sigma(\mathscr{X}^*)$, we have that

$$\begin{aligned}\inf(g|A_i)\mathrm{P}_p(A_i|u) &\overset{\text{GP3}_{70}}{=} \inf(g|A_i)\mathrm{P}_p(A_i \cap \Gamma(s)|u) + \inf(g|A_i)\mathrm{P}_p(A_i \setminus \Gamma(s)|u)\\ &\leq \inf(g|A_i \cap \Gamma(s))\mathrm{P}_p(A_i \cap \Gamma(s)|u) + \inf(g|A_i \setminus \Gamma(s))\mathrm{P}_p(A_i \setminus \Gamma(s)|u),\end{aligned}$$

Eq. (5.9) above implies that

$$\begin{aligned}\mathrm{E}_{p,\mathrm{M}}(g|u) \leq \sup\Bigg\{\sum_{i=1}^{n} &\inf(g|A_i\cap\Gamma(s))\mathrm{P}_p(A_i\cap\Gamma(s)|u)+\inf(g|A_i\setminus\Gamma(s))\mathrm{P}_p(A_i\setminus\Gamma(s)|u) \colon\\ &A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Bigg\}.\end{aligned}$$

On the other hand, if $(A_i)_{i=1}^n$ partitions Ω and is such that $A_i \in \sigma(\mathscr{X}^*)$ for all $i = \{1, \dots, n\}$, then since the algebra $\sigma(\mathscr{X}^*)$ is closed under (countable) intersections and taking complements, and since clearly $\Gamma(s) \in \sigma(\mathscr{X}^*)$, we also have that the events $(A_i\cap\Gamma(s))_{i=1}^n$ and $(A_i\setminus\Gamma(s))_{i=1}^n$ together form a partition[20] of Ω where $A_i\cap\Gamma(s) \in \sigma(\mathscr{X}^*)$ and $A_i \setminus \Gamma(s) \in \sigma(\mathscr{X}^*)$ for all $i = \{1, \dots, n\}$. As a result, we have that the supremum on the right-hand side of the inequality above cannot be larger than the one in Eq. (5.9), which thus implies that

$$\begin{aligned}\mathrm{E}_{p,\mathrm{M}}(g|u) = \sup\Bigg\{\sum_{i=1}^{n} &\inf(g|A_i\cap\Gamma(s))\mathrm{P}_p(A_i\cap\Gamma(s)|u)+\inf(g|A_i\setminus\Gamma(s))\mathrm{P}_p(A_i\setminus\Gamma(s)|u) \colon\\ &A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Bigg\}.\end{aligned}$$

Since the equality above holds for any non-negative $g \in \mathbb{V}_\sigma$ and any $u \in \mathscr{X}^*$, and since f is a non-negative $\sigma(\mathscr{X}^*)$-measurable gamble, we have in particular that

$$\begin{aligned}&\mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_s|t)\\ &= \sup\Bigg\{\sum_{i=1}^{n} \inf(f\mathbb{1}_s|A_i \cap \Gamma(s))\mathrm{P}_p(A_i \cap \Gamma(s)|t)\\ &\qquad + \inf(f\mathbb{1}_s|A_i \setminus \Gamma(s))\mathrm{P}_p(A_i \setminus \Gamma(s)|t) \colon A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Bigg\}\\ &= \sup\left\{\sum_{i=1}^{n} \inf(f|A_i \cap \Gamma(s))\mathrm{P}_p(A_i \cap \Gamma(s)|t) \colon A_i \in \sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\right\},\end{aligned} \tag{5.10}$$

where in the last equality we used the fact that $\inf(f\mathbb{1}_s|A_i \setminus \Gamma(s))\mathrm{P}_p(A_i \setminus \Gamma(s)|t) = 0$ for all $i = \{1, \dots, n\}$, which follows from the fact that $f\mathbb{1}_s(\omega) = 0$ for all $\omega \in A_i \setminus \Gamma(s)$ if $A_i \setminus \Gamma(s)$ is non-empty, and which follows from $\mathrm{P}_p(\emptyset|t) = 0$ [GP6$_{71}$] if $A_i \setminus \Gamma(s)$ is

[20]We permit ourselves a slight abuse of terminology by allowing a partition to also contain empty sets.

empty. In a similar way, we obtain that

$$\begin{aligned}
&\mathrm{E}_{p,\mathrm{M}}(f|s)\\
&= \sup\Big\{\sum_{i=1}^{n}\inf(f|A_i\cap\Gamma(s))\mathrm{P}_p(A_i\cap\Gamma(s)|s)\\
&\qquad + \inf(f|A_i\setminus\Gamma(s))\mathrm{P}_p(A_i\setminus\Gamma(s)|s) \colon A_i\in\sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Big\}\\
&= \sup\Big\{\sum_{i=1}^{n}\inf(f|A_i\cap\Gamma(s))\mathrm{P}_p(A_i\cap\Gamma(s)|s) \colon A_i\in\sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Big\}\\
&= \sup\Big\{\sum_{i=1}^{n}\inf(f|A_i\cap\Gamma(s))\mathrm{P}_p(A_i|s) \colon A_i\in\sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Big\}, \quad (5.11)
\end{aligned}$$

where the penultimate equality follows from $\mathrm{GP3}_{70}$ and the fact that, due to $\mathrm{GP3}_{70}$ and $\mathrm{GP2}_{70}$, we have that $\mathrm{P}_p(A_i\setminus\Gamma(s)|s) = \mathrm{P}_p(A_i\cup\Gamma(s)|s) - \mathrm{P}_p(\Gamma(s)|s) = 0$ for any $A_i\in\sigma(\mathscr{X}^*)$; and where the last equality follows from $\mathrm{GP8}_{71}$. But since P_p is a global probability charge, we have by $\mathrm{GP4}_{70}$ and Eq. $(5.10)_{\leftarrow}$ that

$$\begin{aligned}
&\mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_s|t)\\
&= \sup\Big\{\sum_{i=1}^{n}\inf(f|A_i\cap\Gamma(s))\mathrm{P}_p(A_i|s)\mathrm{P}_p(s|t) \colon A_i\in\sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Big\}\\
&= \mathrm{P}_p(s|t)\sup\Big\{\sum_{i=1}^{n}\inf(f|A_i\cap\Gamma(s))\mathrm{P}_p(A_i|s) \colon A_i\in\sigma(\mathscr{X}^*) \text{ and } (A_i)_{i=1}^n \text{ partitions } \Omega\Big\}.
\end{aligned}$$

Hence, by Eq. (5.11), the latter term is equal to $\mathrm{P}_p(s|t)\mathrm{E}_{p,\mathrm{M}}(f|s)$, and thus we have arrived at the desired equality. □

Recall that, for any precise probability tree p and $s\in\mathscr{X}^*$, since P_p denotes the unique global probability measure on $\sigma(\mathscr{X}^*)\times\mathscr{X}^*$ according to Proposition $5.1.3_{226}$, the map $\mathrm{P}_p(\cdot|s) = \mathrm{P}_p^{|s}$ is a(n) (unconditional) probability measure on $\sigma(\mathscr{X}^*)$. This allows us to apply the concepts and results from Appendix $5.\mathrm{A}_{263}$ here, for each individual situation $s\in\mathscr{X}^*$ and the corresponding probability space $(\Omega,\sigma(\mathscr{X}^*),\mathrm{P}_p^{|s})$. Note in particular that, for any $s\in\mathscr{X}^*$, our global measure-theoretic expectation $\mathrm{E}_{p,\mathrm{M}}(\cdot|s)$ from the main text is the same as the standard measure-theoretic expectation $\mathrm{E}^{|s}$ corresponding to $(\Omega,\sigma(\mathscr{X}^*),\mathrm{P}_p^{|s})$. The reason that we use $\mathrm{E}^{|s}$ as an alternative notation for $\mathrm{E}_{p,\mathrm{M}}(\cdot|s)$ is because it reminds one of the fact that we are actually considering an unconditional expectation in the usual measure-theoretic sense. We furthermore use $\mathrm{E}_{\mathrm{RN}}^{|s}$ to denote any version of the conditional expectation corresponding to $(\Omega,\sigma(\mathscr{X}^*),\mathrm{P}_p^{|s})$ according to the Radon-Nikodým derivative [Appendix $5.\mathrm{A}_{263}$].

We moreover equip the measurable space $(\Omega,\sigma(\mathscr{X}^*))$ with the filtration $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ where, for any $n\in\mathbb{N}_0$, $\mathcal{F}_n$ is the σ-algebra generated by the cylinder events $\Gamma(x_{1:n})$ with $x_{1:n}\in\mathscr{X}^n$. Note that, for any $n\in\mathbb{N}_0$, since the cylinder events $\Gamma(x_{1:n})$ form the atoms of $\mathcal{F}_n$, and since there are only finitely many of these atoms (because $\mathscr{X}$ is finite), any element of $\mathcal{F}_n$ can be written as a finite union of such cylinder events $\Gamma(x_{1:n})$. It is then clear that any $\mathcal{F}_n$-measurable function is an n-measurable variable.

The following rather technical lemma is vital in proving Proposition 5.3.2$_{236}$.

Lemma 5.B.2. *Consider any precise probability tree p, any non-negative* $f \in \mathbb{V}_\sigma$ *and any* $s \in \mathscr{X}^*$*, let* $\mathrm{E}^{|s} = \mathrm{E}_{p,\mathrm{M}}(\cdot|s)$ *be the corresponding measure-theoretic expectation, and let* $\mathrm{Q}_\bullet := \mathrm{Q}_{\bullet,p}$ *be the expectations tree that agrees with p according to Eq.* (3.4)$_{52}$*. Then we have that*

(i) *for all* $n \geq |s|$*, the n-measurable variable* $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})$ *is a version of the conditional expectation* $\mathrm{E}_{\mathrm{RN}}^{|s}(f|\mathcal{F}_n)$*.*

(ii) *the (extended) real process* $\mathrm{E}_{p,\mathrm{M}}(f|\cdot)\colon t \in \mathscr{X}^* \mapsto \mathrm{E}_{p,\mathrm{M}}(f|t)$ *is a game-theoretic supermartingale in* $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$*.*

Proof. Fix any $n \geq |s|$. By definition, a version of $\mathrm{E}_{\mathrm{RN}}^{|s}(f|\mathcal{F}_n)$ is any $\mathrm{P}_p^{|s}$-integrable random variable that is $\mathcal{F}_n$-measurable and that satisfies $\int f\mathbb{I}_A \mathrm{dP}_p^{|s} = \int \mathrm{E}_{\mathrm{RN}}^{|s}(f|\mathcal{F}_n)\mathbb{I}_A \mathrm{dP}_p^{|s}$ for all $A \in \mathcal{F}_n$. Let us prove that these conditions are met for the variable $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})$, Due to ME5$_{264}$ and the fact that f is a (bounded) $\sigma(\mathscr{X}^*)$-measurable gamble, $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})$ exists and is real-valued. It is clearly also n-measurable, and therefore, as we have already discussed above, it is $\mathcal{F}_n$-measurable. Obviously, since $\mathcal{F}_n \subset \sigma(\mathscr{X}^*)$, $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})$ is $\sigma(\mathscr{X}^*)$-measurable and thus, by its real-valuedness, a random variable. Moreover, $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})$ can only take a finite number of values [because $\mathscr{X}$ is finite], and it must therefore be bounded, which in turn by ME5$_{264}$ implies that it is $\mathrm{P}_p^{|s}$-integrable. So it remains to prove that $\int f\mathbb{I}_A \mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{I}_A \mathrm{dP}_p^{|s}$ for all $A \in \mathcal{F}_n$.

To this end, we start by proving that $\int f\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s}$ for all $x_{1:n} \in \mathscr{X}^n$. First consider any $x_{1:n} \in \mathscr{X}^n$ such that $x_{1:n} \not\sqsupseteq s$. By the fact that $n \geq |s|$, this implies that $x_{1:n} \parallel s$ and thus that $\Gamma(x_{1:n}) \cap \Gamma(s) = \emptyset$. So then, by ME2$_{264}$ and Proposition 5.2.2(i)$_{229}$,

$$\int f\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} \leq \int (\sup f)\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} = (\sup f)\mathrm{P}_p^{|s}(x_{1:n}) = (\sup f)\mathrm{P}_p(x_{1:n}|s)$$
$$= (\sup f)\mathrm{P}_p(\emptyset|s) = 0,$$

where the penultimate step follows from GP8$_{71}$ and the fact that $\Gamma(x_{1:n}) \cap \Gamma(s) = \emptyset$, and where the last step follows from GP6$_{71}$. On the other hand, we can use the lower bound $\inf f$ instead of the upper bound $\sup f$, and repeat a similar reasoning to infer that $\int f\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} \geq 0$, and so we have that $\int f\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} = 0$. In a completely similar way, we can also deduce that $\int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} = 0$. So we conclude that $\int f\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{x_{1:n}} \mathrm{dP}_p^{|s}$ for any $x_{1:n} \in \mathscr{X}^n$ such that $x_{1:n} \not\sqsupseteq s$.

Next, consider any $x_{1:n} \in \mathscr{X}^n$ such that $x_{1:n} \sqsupseteq s$. Then, by Lemma 5.B.1$_{267}$, we have that

$$\int f\mathbb{1}_{x_{1:n}}\mathrm{dP}_p^{|s} = \mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_{x_{1:n}}|s) = \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})\mathrm{P}_p(x_{1:n}|s)$$
$$= \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})\mathrm{P}_p^{|s}(x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})\int \mathbb{1}_{x_{1:n}}\mathrm{dP}_p^{|s}$$
$$= \int \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})\mathbb{1}_{x_{1:n}}\mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{x_{1:n}}\mathrm{dP}_p^{|s},$$

where the fourth equality follows from Proposition 5.2.2(i)$_{229}$, and the penultimate equality follows from ME3$_{264}$ [which can be applied because $\mathbb{1}_{x_{1:n}}$ is $\mathrm{P}_p^{|s}$-integrable due to ME5$_{264}$].

Now, to prove that $\int f\mathbb{I}_A\mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{I}_A\mathrm{dP}_p^{|s}$ for any general $A \in \mathcal{A}_n$, note that the situations $x_{1:n} \in \mathscr{X}^n$ form the atoms of $\mathcal{A}_n$, which by the fact that there are only finitely many of them, implies that A can be written as a finite union $\cup_{i=1}^m \Gamma(t_i)$ of cylinder sets $\Gamma(t_i)$ of such situations $t_i \in \mathscr{X}^n$. Since we can clearly assume (without loss of generality) that these cylinder sets $\Gamma(t_i)$ are mutually disjoint, we obtain that $\mathbb{I}_A = \sum_{i=1}^m \mathbb{1}_{t_i}$. Hence, by ME3$_{264}$ [which we can apply in the chain below because all the involved terms are bounded and thus, by ME5$_{264}$, $\mathrm{P}_p^{|s}$-integrable] and the fact that $\int f\mathbb{1}_{t_i}\mathrm{dP}_p^{|s} = \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{t_i}\mathrm{dP}_p^{|s}$ for all t_i due to the considerations above,

$$\int f\mathbb{I}_A\mathrm{dP}_p^{|s} = \int f\textstyle\sum_{i=1}^m \mathbb{1}_{t_i}\mathrm{dP}_p^{|s} = \displaystyle\sum_{i=1}^m \int f\mathbb{1}_{t_i}\mathrm{dP}_p^{|s} = \sum_{i=1}^m \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{1}_{t_i}\mathrm{dP}_p^{|s}$$
$$= \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\textstyle\sum_{i=1}^m \mathbb{1}_{t_i}\mathrm{dP}_p^{|s}$$
$$= \int \mathrm{E}_{p,\mathrm{M}}(f|X_{1:n})\mathbb{I}_A\mathrm{dP}_p^{|s}.$$

This establishes (i)$_\leftarrow$.

To prove (ii)$_\leftarrow$, we need to show that $\mathrm{E}_{p,\mathrm{M}}(f|\cdot) : t \in \mathscr{X}^* \mapsto \mathrm{E}_{p,\mathrm{M}}(f|t)$ is an element of $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$. That $\mathrm{E}_{p,\mathrm{M}}(f|\cdot)$ is bounded below follows from the fact that f is bounded and Properties ME2$_{264}$ and ME1$_{264}$. Furthermore, for any $x_{1:n} \in \mathscr{X}^*$, note that $\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_{x_{1:n}}|x_{1:n})$ due to ME11$_{265}$ and the fact that, by GP8$_{71}$ and GP6$_{71}$, $\mathrm{P}_p^{|x_{1:n}}(\Gamma(x_{1:n})^c) = \mathrm{P}_p(\Gamma(x_{1:n})^c|x_{1:n}) = 0$. Hence, by ME3$_{264}$ and ME5$_{264}$,

$$\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_{x_{1:n}}|x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f\textstyle\sum_{x_{n+1}\in\mathscr{X}}\mathbb{1}_{x_{1:n+1}}|x_{1:n})$$
$$= \sum_{x_{n+1}\in\mathscr{X}} \mathrm{E}_{p,\mathrm{M}}(f\mathbb{1}_{x_{1:n+1}}|x_{1:n}).$$

By Lemma 5.B.1$_{267}$, the fact that P_p satisfies Eq. (3.12)$_{72}$ due to Proposition 5.1.3$_{226}$, and the definition of $\mathrm{Q}_{x_{1:n}}$, the latter is equal to

$$\sum_{x_{n+1}\in\mathscr{X}} \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n+1})\mathrm{P}_p(x_{1:n+1}|x_{1:n}) = \sum_{x_{n+1}\in\mathscr{X}} \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n+1})p(x_{n+1}|x_{1:n})$$
$$= \mathrm{Q}_{x_{1:n}}\big(\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}\cdot)\big) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}\big(\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}\cdot)\big),$$

where $\overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}$ is the extension of $\mathrm{Q}_{x_{1:n}}$ defined through CU1$_{143}$ and CU2$_{143}$ as described in Section 4.3$_{152}$. So we obtain that $\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}) = \overline{\mathrm{Q}}^{\uparrow}_{x_{1:n}}\big(\mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}\cdot)\big)$ for all $x_{1:n} \in \mathscr{X}^*$ and thus, together with its bounded belowness, this implies that the process $\mathrm{E}_{p,\mathrm{M}}(f|\cdot)$ is in $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$. □

The last intermediate result that we need in order to prove Proposition 5.3.2$_{236}$ is a result similar to Ville's theorem [86, Proposition 8.14(2)]. The main differences are that our statement is specifically adapted to the context of discrete-time stochastic processes, and that it involves game-theoretic supermartingales rather than measure-theoretic martingales. Nonetheless, many of the ideas that we use to prove the result below are borrowed from [86, Proposition 8.14].

Lemma 5.B.3. *Consider any precise probability tree p and the agreeing expectations tree $\mathrm{Q}_\bullet$ defined according to Eq. (3.4)$_{52}$. Then, for any $A \in \sigma(\mathscr{X}^*)$ and $s \in \mathscr{X}^*$ such that $\mathrm{P}_p(A|s) = 0$, there is a non-negative game-theoretic supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$ such that $\mathscr{M}(s) \in \mathbb{R}$ and such that $\mathscr{M}$ converges to $+\infty$ on A.*

Proof. Recall from Proposition 5.1.3$_{226}$ that P_p is the unique global probability measure on $\sigma(\mathscr{X}^*) \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$. Let P_p^* be the restriction of P_p to $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$—this is clearly a global probability charge on $\langle\mathscr{X}^*\rangle \times \mathscr{X}^*$ that also satisfies Eq. (3.12)$_{72}$. Then it follows from the proof of Proposition 5.1.3$_{226}$ that, for any $s \in \mathscr{X}^*$, the (unconditional) probability measure $\mathrm{P}_p(\cdot|s)$ on $\sigma(\mathscr{X}^*)$ is arrived at by applying Theorem 5.1.2$_{226}$ to the (unconditional) probability charge $\mathrm{P}_p^*(\cdot|s)$ on $\langle\mathscr{X}^*\rangle$. Hence, by the expression in Theorem 5.1.2$_{226}$, we have that, for all $A \in \sigma(\mathscr{X}^*)$,

$$\begin{aligned}\mathrm{P}_p(A|s) &= \inf\Big\{\sum_{i\in\mathbb{N}} \mathrm{P}_p^*(A_i|s) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } A \subseteq \cup_{i\in\mathbb{N}} A_i\Big\} \\ &= \inf\Big\{\sum_{i\in\mathbb{N}} \mathrm{P}_p(A_i|s) \colon A_i \in \langle\mathscr{X}^*\rangle \text{ and } A \subseteq \cup_{i\in\mathbb{N}} A_i\Big\}.\end{aligned}$$

Now consider any $A \in \sigma(\mathscr{X}^*)$ such that $\mathrm{P}_p(A|s) = 0$. Then, for any $\epsilon > 0$ and any $\ell \in \mathbb{N}$, due to the expression above, there is a collection $(A_{\ell,i})_{i\in\mathbb{N}}$ of events in $\langle\mathscr{X}^*\rangle$ such that $\sum_{i\in\mathbb{N}} \mathrm{P}_p(A_{\ell,i}|s) \leq 2^{-\ell}\epsilon$ and $A \subseteq \cup_{i\in\mathbb{N}} A_{\ell,i}$.

For any $i \in \mathbb{N}$, let $\mathscr{M}_{\ell,i}$ be the (extended) real process defined by $\mathscr{M}_{\ell,i}(t) := \mathrm{P}_p(A_{\ell,i}|t)$ for all $t \in \mathscr{X}^*$. By Proposition 5.2.2(i)$_{229}$ [and since $A_{\ell,i} \in \langle\mathscr{X}^*\rangle \subseteq \sigma(\mathscr{X}^*)$], we have that $\mathrm{P}_p(A_{\ell,i}|t) = \mathrm{E}_{p,\mathrm{M}}(\mathbb{I}_{A_{\ell,i}}|t)$ for all $t \in \mathscr{X}^*$, and therefore by Lemma 5.B.2(ii)$_{270}$ [and since $A_{\ell,i} \in \sigma(\mathscr{X}^*)$ and thus $\mathbb{I}_{A_{\ell,i}} \in \mathbb{V}_\sigma$] that $\mathscr{M}_{\ell,i} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$. Since $A_{\ell,i} \in \langle\mathscr{X}^*\rangle$, we have by Lemma 3.3.3$_{72}$ that $A_{\ell,i}$ is k-measurable for some $k \in \mathbb{N}_0$, in the sense that $A_{\ell,i} = \cup_{z_{1:k}\in C}\Gamma(z_{1:k})$ for some $C \subseteq \mathscr{X}^k$. Then, for all $m \geq k$ and $y_{1:m} \in \mathscr{X}^m$, due to GP2$_{70}$ we have that $\mathrm{P}_p(A_{\ell,i}|y_{1:m}) = \mathrm{P}_p(\cup_{z_{1:k}\in C}\Gamma(z_{1:k})|y_{1:m}) = 1$ if $y_{1:k} \in C$. Or, since $A_{\ell,i} = \cup_{z_{1:k}\in C}\Gamma(z_{1:k})$ [and thus since the indicator $\mathbb{I}_{A_{\ell,i}}$ is k-measurable], for all $\omega \in \Omega$ and $m \geq k$, we have that $\mathrm{P}_p(A_{\ell,i}|\omega_{1:m}) = 1$ if $\mathbb{I}_{A_{\ell,i}}(\omega_{1:k}) = \mathbb{I}_{A_{\ell,i}}(\omega) = 1$. Hence, by the definition of $\mathscr{M}_{\ell,i}$, we have that $\liminf \mathscr{M}_{\ell,i} \geq \mathbb{I}_{A_{\ell,i}}$.

For all $i \in \mathbb{N}$, by the definition of $\mathscr{M}_{\ell,i}$ and GP1$_{70}$, we know that $\mathscr{M}_{\ell,i}$ [and thus also $\liminf \mathscr{M}_{\ell,i}$] is non-negative. Hence, the process $\sum_{i\in\mathbb{N}} \mathscr{M}_{\ell,i}$ and the variable

$\sum_{i\in\mathbb{N}} \liminf \mathscr{M}_{\ell,i}$ exist, and thus by the considerations above, for all $\omega \in \Omega$,

$$\begin{aligned}\sum_{i\in\mathbb{N}} \mathbb{I}_{A_{\ell,i}}(\omega) \leq \sum_{i\in\mathbb{N}} \liminf \mathscr{M}_{\ell,i}(\omega) = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \liminf \mathscr{M}_{\ell,i}(\omega) \\ = \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \lim_{m\to+\infty} \inf_{j\geq m} \mathscr{M}_{\ell,i}(\omega^j) = \sup_{n\in\mathbb{N}} \lim_{m\to+\infty} \sum_{i=1}^{n} \inf_{j\geq m} \mathscr{M}_{\ell,i}(\omega^j) \leq \sup_{n\in\mathbb{N}} \liminf_{j\to+\infty} \sum_{i=1}^{n} \mathscr{M}_{\ell,i}(\omega^j) \\ \leq \liminf_{j\to+\infty} \sup_{n\in\mathbb{N}} \sum_{i=1}^{n} \mathscr{M}_{\ell,i}(\omega^j) = \liminf \sum_{i\in\mathbb{N}} \mathscr{M}_{\ell,i}(\omega), \quad (5.12)\end{aligned}$$

where we used the non-negativity of $\mathscr{M}_{\ell,i}$ in the third equality. Since $A \subseteq \cup_{i\in\mathbb{N}} A_{\ell,i}$, we have that $\mathbb{I}_A \leq \sum_{i\in\mathbb{N}} \mathbb{I}_{A_{\ell,i}}$, and therefore the inequality above implies that $\mathbb{I}_A \leq \liminf \sum_{i\in\mathbb{N}} \mathscr{M}_{\ell,i}$. So if we let $\mathscr{M}_\ell := \sum_{i\in\mathbb{N}} \mathscr{M}_{\ell,i}$, then $\mathbb{I}_A \leq \liminf \mathscr{M}_\ell$. Moreover, since $\mathscr{M}_{\ell,i} \in \overline{\mathbb{M}}_{\text{eb}}(\text{Q}_\bullet)$ for all $i \in \mathbb{N}$ and since $\mathscr{M}_{\ell,i}$ is non-negative for all $i \in \mathbb{N}$, we have by Lemma 4.4.2$_{163}$ that $\mathscr{M}_\ell$ is a non-negative supermartingale in $\overline{\mathbb{M}}_{\text{eb}}(\text{Q}_\bullet)$. On top of this, since $\sum_{i\in\mathbb{N}} \text{P}_p(A_{\ell,i}|s) \leq 2^{-\ell}\epsilon$ by assumption, and since $\mathscr{M}_{\ell,i}(s) = \text{P}_p(A_{\ell,i}|s)$ by definition, we also have that $\mathscr{M}_\ell(s) = \sum_{i\in\mathbb{N}} \mathscr{M}_{\ell,i}(s) \leq 2^{-\ell}\epsilon$.

The above holds for any $\ell \in \mathbb{N}$, so if we let $\mathscr{M} := \sum_{\ell\in\mathbb{N}} \mathscr{M}_\ell$ [which is possible because each $\mathscr{M}_\ell$ is non-negative], then again by Lemma 4.4.2$_{163}$ we infer that $\mathscr{M}$ is a non-negative supermartingale in $\overline{\mathbb{M}}_{\text{eb}}(\text{Q}_\bullet)$. Moreover, since $\mathscr{M}_\ell(s) \leq 2^{-\ell}\epsilon$ for all $\ell \in \mathbb{N}$, we have that $\mathscr{M}(s) \leq \epsilon$ and therefore, together with its non-negativity, we obtain that $\mathscr{M}(s) \in \mathbb{R}$. Finally, since $\mathbb{I}_A \leq \liminf \mathscr{M}_\ell$ for all $\ell \in \mathbb{N}$, we can infer in a similar way as we have done in Eq. (5.12) that

$$\sum_{\ell\in\mathbb{N}} \mathbb{I}_A \leq \sum_{\ell\in\mathbb{N}} \liminf \mathscr{M}_\ell \leq \liminf \sum_{\ell\in\mathbb{N}} \mathscr{M}_\ell = \liminf \mathscr{M}.$$

Since the variable $\sum_{\ell\in\mathbb{N}} \mathbb{I}_A$ is equal to $+\infty$ for all $\omega \in A$, we conclude that $\mathscr{M}$ indeed converges to $+\infty$ on the event A. □

Proof of Proposition 5.3.2$_{236}$. Fix any $f' \in \mathbb{V}_\sigma$ and any $x_{1:n} \in \mathscr{X}^*$. First observe that, because $f' \in \mathbb{V}_\sigma$ is bounded, $\text{E}_{p,\text{M}}(f'|x_{1:n}) = \text{E}^{|x_{1:n}}(f')$ exists [ME4$_{264}$] and so $\text{E}_{p,\text{M}}(f'|x_{1:n}) = \overline{\text{E}}_{p,\text{M}}(f'|x_{1:n})$ because $\overline{\text{E}}_{p,\text{M}}$ extends $\text{E}_{p,\text{M}}$ according to Corollary 5.2.5$_{234}$. Hence, it suffices to show that $\text{E}_{p,\text{M}}(f'|x_{1:n}) = \overline{\text{E}}^{\text{eb}}_{\text{Q,V}}(f'|x_{1:n})$. We will prove that $\text{E}_{p,\text{M}}(f|x_{1:n}) = \overline{\text{E}}^{\text{eb}}_{\text{Q,V}}(f|x_{1:n})$ for the non-negative $\sigma(\mathscr{X}^*)$-measurable gamble $f := f' - \inf f'$ (the variable f is indeed a gamble because f' is a gamble and therefore $\inf f' \in \mathbb{R}$), which then implies that $\text{E}_{p,\text{M}}(f'|x_{1:n}) = \overline{\text{E}}^{\text{eb}}_{\text{Q,V}}(f'|x_{1:n})$ because $\text{E}_{p,\text{M}}(\cdot|x_{1:n})$ and $\overline{\text{E}}^{\text{eb}}_{\text{Q,V}}$ both satisfy the constant additivity property; see ME10$_{265}$ and Proposition 4.4.3$_{164}$ [EC5].

We first show that $\overline{\text{E}}^{\text{eb}}_{\text{Q,V}}(f|x_{1:n}) \leq \text{E}_{p,\text{M}}(f|x_{1:n})$. To do so, we will prove that there is some $c \in \mathbb{R}$ such that, for all $\epsilon > 0$, there is a game-theoretic supermartingale $\mathscr{M}_\epsilon \in \overline{\mathbb{M}}_{\text{eb}}(\overline{\text{Q}}_\bullet)$ such that $\mathscr{M}_\epsilon(x_{1:n}) = \text{E}_{p,\text{M}}(f|x_{1:n}) + \epsilon c$ and $\liminf \mathscr{M}_\epsilon \geq f$. Indeed, the desired inequality then follows immediately from the definition of $\overline{\text{E}}^{\text{eb}}_{\text{Q,V}}$.

Consider the filtered probability space $(\Omega, \sigma(\mathscr{X}^*), (\mathcal{F}_m)_{m\in\mathbb{N}_0}, \text{P}_p^{|x_{1:n}})$ and the corresponding measure-theoretic expectation $\text{E}^{|x_{1:n}} = \text{E}_{p,\text{M}}(\cdot|x_{1:n})$. Since f is bounded and $\sigma(\mathscr{X}^*)$-measurable, it is surely $\text{P}_p^{|x_{1:n}}$-integrable [ME5$_{264}$], and therefore, by Theorem 5.A.3$_{267}$, we have that

$$\lim_{m\to+\infty} \text{E}_{\text{RN}}^{|x_{1:n}}(f|\mathcal{F}_m) = \text{E}_{\text{RN}}^{|x_{1:n}}(f|\mathcal{F}_\infty) \quad \text{P}_p^{|x_{1:n}}\text{-almost surely.}$$

Note that $\mathcal{A}_\infty = \sigma(\cup_{m\in\mathbb{N}_0}\mathcal{A}_m)$ is the smallest σ-algebra $\sigma(\mathscr{X}^*)$ generated by all cylinder events, which, by Lemma 5.A.2$_{267}$ [and the boundedness of f], implies that $\mathrm{E}_{\mathrm{RN}}^{|x_{1:n}}(f|\mathcal{A}_\infty) = f$, $\mathrm{P}_p^{|x_{1:n}}$-almost surely. Hence, since the intersection of two $\mathrm{P}_p^{|x_{1:n}}$-almost sure events is itself also $\mathrm{P}_p^{|x_{1:n}}$-almost sure [p.264], we have that $\lim_{m\to+\infty}\mathrm{E}_{\mathrm{RN}}^{|x_{1:n}}(f|\mathcal{A}_m) = f$ $\mathrm{P}_p^{|x_{1:n}}$-almost surely.

Due to Lemma 5.B.2(i)$_{270}$ and since f is non-negative, we know that $\mathrm{E}_{p,\mathrm{M}}(f|X_{1:m})$ is a version of $\mathrm{E}_{\mathrm{RN}}^{|x_{1:n}}(f|\mathcal{A}_m)$ for all $m \geq n$, so we obtain that $\lim_{m\to+\infty}\mathrm{E}_{p,\mathrm{M}}(f|X_{1:m}) = f$ $\mathrm{P}_p^{|x_{1:n}}$-almost surely. So, if we let $\mathscr{M}$ be the extended real process defined by $\mathscr{M}(t) := \mathrm{E}_{p,\mathrm{M}}(f|t)$ for all $t \in \mathscr{X}^*$, then $\liminf \mathscr{M} = \lim_{m\to+\infty}\mathscr{M}(X_{1:m}) = f$ $\mathrm{P}_p^{|x_{1:n}}$-almost surely. Moreover, since f is non-negative, it follows from Lemma 5.B.2(ii)$_{270}$ and the definition of $\mathscr{M}$, that $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$. Furthermore, consider Lemma 5.B.3$_{272}$ and note that it ensures that there is a non-negative supermartingale $\mathscr{M}' \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$ such that $c := \mathscr{M}'(x_{1:n}) \in \mathbb{R}$ and that converges to $+\infty$ on all paths $\omega \in \Omega$ such that $\liminf \mathscr{M}(\omega) \neq f(\omega)$. Indeed, the set of all such paths ω has probability zero because $\liminf \mathscr{M} = f$ $\mathrm{P}_p^{|x_{1:n}}$-almost surely.

Consider now any $\epsilon > 0$ and let $\mathscr{M}_\epsilon$ be the process defined by $\mathscr{M}_\epsilon(s) := \mathscr{M}(s) + \epsilon\mathscr{M}'(s)$ for all $s \in \mathscr{X}^*$. Then $\mathscr{M}_\epsilon \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$ because of Lemma 4.4.2$_{163}$. Furthermore, note that $\liminf \mathscr{M}_\epsilon(\omega) \geq f(\omega)$ for all $\omega \in \Omega$. Indeed, if $\liminf \mathscr{M}(\omega) = f(\omega)$ for some $\omega \in \Omega$, then also $\liminf \mathscr{M}_\epsilon(\omega) \geq f(\omega)$ because ϵ and $\mathscr{M}'$ are non-negative. If $\liminf \mathscr{M}(\omega) \neq f(\omega)$ for some $\omega \in \Omega$, then $\mathscr{M}'$, and therefore also $\epsilon\mathscr{M}'$, converges to $+\infty$, which, together with the fact that $\mathscr{M}$ is bounded below [due to $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\mathrm{Q}_\bullet)$], implies that $\mathscr{M}_\epsilon$ converges to $+\infty$ on ω. Hence, also in this case, we have that $\liminf \mathscr{M}_\epsilon(\omega) \geq f(\omega)$, so we can conclude that $\liminf \mathscr{M}_\epsilon \geq f$. Moreover, recall that $c = \mathscr{M}'(x_{1:n}) \in \mathbb{R}$ and that $\mathscr{M}(x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})$, so we have that $\mathscr{M}_\epsilon(x_{1:n}) = \mathscr{M}(x_{1:n}) + \epsilon\mathscr{M}'(x_{1:n}) = \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n}) + \epsilon c$. Hence, $\mathscr{M}_\epsilon$ satisfies all the desired conditions and we conclude that indeed $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|x_{1:n}) \leq \mathrm{E}_{p,\mathrm{M}}(f|x_{1:n})$, and thus $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f'|x_{1:n}) \leq \mathrm{E}_{p,\mathrm{M}}(f'|x_{1:n})$.

Then we are left to show that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \geq \mathrm{E}_{p,\mathrm{M}}(f|s)$ for any $f \in \mathbb{V}_\sigma$ and any $s \in \mathscr{X}^*$. However, this can be easily deduced from the already obtained inequality and the self-conjugacy of $\mathrm{E}_{p,\mathrm{M}}$. Indeed, $-f$ is $\sigma(\mathscr{X}^*)$-measurable and bounded, and therefore $\mathrm{P}_p^{|x_{1:n}}$-integrable [ME5$_{264}$], so we can apply ME3$_{264}$ to find that $\mathrm{E}_{p,\mathrm{M}}(f|s) = \mathrm{E}^{|s}(f) = -\mathrm{E}^{|s}(-f) = -\mathrm{E}_{p,\mathrm{M}}(-f|s)$. Since we have already shown that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(g|s) \leq \mathrm{E}_{p,\mathrm{M}}(g|s)$ for all $g \in \mathbb{V}_\sigma$, we have in particular that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f|s) \leq \mathrm{E}_{p,\mathrm{M}}(-f|s)$, which implies that $\mathrm{E}_{p,\mathrm{M}}(f|s) = -\mathrm{E}_{p,\mathrm{M}}(-f|s) \leq -\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(-f|s) = \underline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s)$, where the last inequality follows from Proposition 4.4.3$_{164}$ [EC1]. □

5.C Topological results for the sample space Ω

Consider the distance function δ on Ω defined by

$$\delta(\omega,\omega') := 2^{-n} \text{ with } n := \inf\{k \in \mathbb{N} \colon \omega_k \neq \omega'_k\}, \tag{5.13}$$

for all $\omega, \omega' \in \Omega$. Then it can easily be checked that δ is a metric on Ω. Furthermore, as is shown by the lemma below, the topology on Ω corresponding to this metric δ is the same as the topology that we have adopted throughout the main text; that is, the smallest topology containing (generated by) the set of all cylinder events $\Gamma(\mathscr{X}^*) = \{\Gamma(s) : s \in \mathscr{X}^*\}$ or, equivalently [111, Problem 5.D], the (unique) topology for which $\Gamma(\mathscr{X}^*)$ is a subbase.[21] Recall that a base for a topology is a collection $\mathcal{B}$ of sets (in this topology) such that any open set can be written as an arbitrary union of sets in $\mathcal{B}$ [111, Definition 5.1], and that a subbase is a collection $\mathcal{S}$ of sets (in this topology) such that all finite intersections of elements in $\mathcal{S}$ form a base for this topology [111, Definition 5.5]. The above confirms our claim that Ω is metrizable.

Furthermore, the lemma below also shows that this metric topology coincides with the product topology on $\Omega = \mathscr{X}^{\mathbb{N}}$, with $\mathscr{X}$ (being finite) given the topology consisting of all its subsets; that is, the discrete topology [111, Example 3.2 C]. Since this discrete topology on $\mathscr{X}$ is finite, it is clear by the definition of compactness [111, Definition 17.1] that $\mathscr{X}$ is compact. Hence, since the product of compact spaces is itself compact in the product topology—Tychonoff's theorem [111, Theorem 17.8]—we have that $\Omega = \mathscr{X}^{\mathbb{N}}$ is compact.

Lemma 5.C.1. *The topology on Ω generated by $\Gamma(\mathscr{X}^*)$, the metric topology on Ω corresponding to δ, and the product topology on Ω are all equal. Moreover, a set in this common topology is open if and only if it is a (possibly empty) countable union of cylinder events.*

Proof. First recall that the set of all open ϵ-disks form a subbase for the metric topology. Indeed, [111, Definition 2.5] says that a set $A \subseteq \Omega$ is open if and only if, for each $\omega \in A$, there is an open ϵ-disk about ω contained in A, which by the second part of [111, Definition 5.1] implies that the open ϵ-disks form a base for the metric topology on Ω. Any topology—and thus in particular the metric topology—is closed under taking finite intersections, so the metric topology contains all finite intersections of open ϵ-disks [and also all possible unions of these intersections, since any topology is closed under taking unions]. Furthermore, since the ϵ-disks form a base, any open set can be written as an arbitrary union of ϵ-disks, and thus surely as an arbitrary union of finite intersections of ϵ-disks. Hence, the set of all finite intersections of ϵ-disks forms a base for the metric topology, and thus the set of all ϵ-disks forms a subbase for this topology.

Next, consider any open ϵ-disk; that is, for any $\epsilon > 0$ and any $\omega \in \Omega$, consider the set $\{\omega' \in \Omega : \delta(\omega, \omega') < \epsilon\}$. If $\epsilon > 1$, let $\ell := 0$; otherwise, let $\ell \in \mathbb{N}_0$ be the unique natural number such that $2^{-\ell-1} < \epsilon \leq 2^{-\ell}$. Then, for all $\omega' \in \Gamma(\omega^\ell)$, since $\inf\{k \in \mathbb{N} : \omega'_k \neq \omega_k\} \geq \ell + 1$, we have by Equation (5.13) that $\delta(\omega, \omega') \leq 2^{-\ell-1} < \epsilon$. On the other hand, for any $\omega' \notin \Gamma(\omega^\ell)$, we infer in a similar way that $\delta(\omega, \omega') \geq 2^{-\ell} \geq \epsilon$. Hence, both facts taken together, we obtain that $\Gamma(\omega^\ell) = \{\omega' \in \Omega : \delta(\omega, \omega') < \epsilon\}$

[21] The experienced reader may also understand that this topological space Ω is homeomorphic to the Cantor space; see [53, Theorem 7.4] and Lemma 5.C.4$_{278}$.

is the open ϵ-disk around ω. Conversely, one can see that any cylinder event $\Gamma(x_{1:\ell})$ with $x_{1:\ell} \in \mathscr{X}^*$, is an open ϵ-disk around any $\omega \in \Gamma(x_{1:\ell})$ if $\epsilon > 0$ is such that $2^{-\ell-1} < \epsilon \leq 2^{-\ell}$. As a consequence, the family of open ϵ-disks in Ω is the same as the set $\Gamma(\mathscr{X}^*)$ of all cylinder events and therefore, since the former is a subbase of the metric topology, the set $\Gamma(\mathscr{X}^*)$ is a subbase of the metric topology. As a result, due to [111, Problem 5.D], this establishes the equivalence between the metric topology and the smallest topology containing $\Gamma(\mathscr{X}^*)$ (or the topology generated by $\Gamma(\mathscr{X}^*)$).

Let us show that the same holds for the product topology on $\Omega = \mathscr{X}^{\mathbb{N}}$. Since the set $\{\{x\} : x \in \mathscr{X}\}$ forms a (sub)base for the discrete topology on $\mathscr{X}$ [111, Example 5.2 (b)], it follows from the discussion below [111, Definition 8.3] that the sets $U_{n,y} := \{\omega \in \Omega : \omega_n = y\}$ with $n \in \mathbb{N}$ and $y \in \mathscr{X}$ form a subbase for the product topology on Ω. Clearly, any such set $U_{n,y}$ is the union of the cylinder events $\Gamma(x_{1:n-1}y)$ with $x_{1:n-1} \in \mathscr{X}^{n-1}$, so the topology generated by the cylinder events $\{\Gamma(s) : s \in \mathscr{X}^*\}$ is finer than (includes) the product topology. On the other hand, any cylinder event $\Gamma(x_{1:n})$ with $x_{1:n} \in \mathscr{X}^*$ is the finite intersection of the sets U_{i,x_i} with $i \in \{1, \ldots, n\}$, so we also have that the product topology is finer than the one generated by $\{\Gamma(s) : s \in \mathscr{X}^*\}$. All together, we conclude that the topology generated by the cylinder events $\{\Gamma(s) : s \in \mathscr{X}^*\}$ coincides with the product topology.

It remains to prove the second statement, which says that a set in this common topology is open if and only if it is empty or a countable union of cylinder events. In other words, we have to prove that $\tau := \{\cup_{i\in\mathbb{N}}\Gamma(s_i) : (\forall i \in \mathbb{N})\, s_i \in \mathscr{X}^*\} \cup \emptyset$ is the topology generated by the subbase $\Gamma(\mathscr{X}^*)$. That τ is closed under arbitrary unions follows from the fact that the set $\mathscr{X}^*$ of all situations is countable (since $\mathscr{X}$ is finite). Indeed, any union of elements of τ is a (possibly empty) union of cylinder events, and since $\mathscr{X}^*$—and therefore also $\{\Gamma(s) : s \in \mathscr{X}^*\}$—is countable, this union can always be written as a (possibly empty) countable union, therefore implying that it is an element of τ. Now, consider any finite intersection $\bigcap_{j\in\{1,\ldots,n\}} A_j$ of elements of τ and let us check that this too is an element of τ. If at least one A_j is equal to the empty set $\emptyset$, then the intersection $\bigcap_{j\in\{1,\ldots,n\}} A_j$ is also equal to $\emptyset$ and thus in τ. If not, then by the definition of τ each A_j is equal to some union $\bigcup_{i\in\mathbb{N}} \Gamma(s_{i,j})$ of cylinder events. So we have that $\bigcap_{j\in\{1,\ldots,n\}} A_j = \bigcap_{j\in\{1,\ldots,n\}} \bigcup_{i\in\mathbb{N}} \Gamma(s_{i,j})$. Using distributivity, this finite intersection can be rewritten as $\bigcup_{i_1\in\mathbb{N}} \bigcup_{i_2\in\mathbb{N}} \cdots \bigcup_{i_n\in\mathbb{N}} \bigcap_{j\in\{1,\ldots,n\}} \Gamma(s_{i_j,j})$; a countable union of finite intersections of cylinder events. So we have that this countable union is an element of τ if we manage to show that any finite intersection of cylinder events is itself a cylinder event or empty. In order to do so, consider the intersection of any two cylinder events $\Gamma(x_{1:n})$ and $\Gamma(y_{1:m})$ with $x_{1:n} \in \mathscr{X}^*$ and $y_{1:m} \in \mathscr{X}^*$. Note that this intersection is non-empty if and only if, either, $n \leq m$ and $x_{1:n} = y_{1:n}$, or, if $n > m$ and $x_{1:m} = y_{1:m}$. In the first case, we have that $\Gamma(x_{1:n}) \cap \Gamma(y_{1:m}) = \Gamma(y_{1:m})$ and, in the second case, we have that $\Gamma(x_{1:n}) \cap \Gamma(y_{1:m}) = \Gamma(x_{1:n})$. Hence, the intersection of any two cylinder events is either empty or itself a cylinder event and therefore, any finite intersection of cylinder events is empty or a cylinder event. By our previous considerations, this implies that τ is indeed closed under finite intersections. Together with the fact that τ is closed under arbitrary unions—and trivially includes Ω and the empty subset $\emptyset \subset \Omega$—we may conclude that τ is a topology on Ω. It is moreover clear from the definition of τ, that τ is contained in the topology generated by

$\Gamma(\mathscr{X}^*)$ and, conversely, since τ is a topology containing all cylinder events, that the (smallest) topology generated by $\Gamma(\mathscr{X}^*)$ is contained in τ. Hence, both topologies are equal. □

Proof of Lemma 5.5.2$_{251}$. We start by proving the two direct implications. Let $f \in \mathbb{V}^{\mathrm{u}}$ be u.s.c. and let $(f_n)_{n\in\mathbb{N}}$ be defined by

$$f_n(\omega) := \sup(\{-n\} \cup \{f(\omega') : \omega' \in \Gamma(\omega^n)\}),$$

for all $\omega \in \Omega$ and all $n \in \mathbb{N}$. Then $(f_n)_{n\in\mathbb{N}}$ is clearly a decreasing sequence of n-measurable—and thus finitary—bounded below variables (since $f_n \geq -n$). If f is bounded above, then each f_n is clearly also bounded above, so in that case $(f_n)_{n\in\mathbb{N}}$ is a sequence of gambles. So it only remains to show that $\lim_{n\to+\infty} f_n(\omega) = f(\omega)$ for any $\omega \in \Omega$. That $\lim_{n\to+\infty} f_n(\omega) \geq f(\omega)$ holds, follows from the fact that, due to the definition of the variables f_n, $f_n(\omega) \geq f(\omega)$ for all $n \in \mathbb{N}$. To prove the converse inequality, fix any real $a > f(\omega)$ [remember that f is real-valued]. Since f is u.s.c., the set $\{\omega' \in \Omega : f(\omega') < a\}$ is an open set, which moreover contains ω. According to Lemma 5.C.1$_{275}$, any open set in Ω is a countable union of cylinder events. Since ω belongs to $\{\omega' \in \Omega : f(\omega') < a\}$, one of these cylinder events contains ω. This implies that there is some $n \in \mathbb{N}_0$ such that $f(\omega') < a$ for all $\omega' \in \Gamma(\omega^n)$. Then, for any $k \geq n$, since $\Gamma(\omega^k) \subseteq \Gamma(\omega^n)$, we obviously also have that $f(\omega') < a$ for all $\omega' \in \Gamma(\omega^k)$. Hence, $f_k(\omega) \leq a$ for all $k \geq \max\{|a|, n\}$, which implies that $\lim_{k\to+\infty} f_k(\omega) \leq a$. This holds for any real $a > f(\omega)$, so we find that $\lim_{k\to+\infty} f_k(\omega) \leq f(\omega)$, as desired.

To prove the two converse implications, consider any $f \in \mathbb{V}^{\mathrm{u}}$ that is the pointwise limit of a decreasing sequence $(f_n)_{n\in\mathbb{N}}$ of finitary bounded below variables. We show that, for any $a \in \mathbb{R}$, the set $A := \{\omega \in \Omega : f(\omega) < a\}$ is open, and therefore that f is a u.s.c. variable. It will then be clear that f is moreover bounded above if $(f_n)_{n\in\mathbb{N}}$ is a sequence of gambles, because in that case $f \leq f_1 \leq \sup f_1 \in \mathbb{R}$. So fix any $a \in \mathbb{R}$ and note that the sequence $(A_n)_{n\in\mathbb{N}}$ of events defined by $A_n := \{\omega \in \Omega : f_n(\omega) < a\}$ for all $n \in \mathbb{N}$, is increasing and converges to A because $(f_n)_{n\in\mathbb{N}}$ converges decreasingly to f. So we have that $A = \bigcup_{n\in\mathbb{N}} A_n$. Moreover, for any $n \in \mathbb{N}$, because f_n is finitary, there is a $k \in \mathbb{N}$ such that f_n only depends on the first k states, and so the set A_n is a (possibly empty) finite union of cylinder events of the form $\Gamma(x_{1:k})$ with $x_{1:k} \in \mathscr{X}^k$. So, by Lemma 5.C.1$_{275}$, each set A_n is open. Since any union of open sets is open again, we obtain that $A = \bigcup_{n\in\mathbb{N}} A_n$ is indeed open. □

The last part of this section is devoted to proving that the $\sigma(\mathscr{X}^*)$-measurable (non-negative) variables are the same as the Borel-measurable (non-negative) variables, and that these are in turn a subset of the analytic (non-negative) variables; a result that we have used in the main text to restate Choquet's capacitability theorem [28, Theorem II.2.5] in the form of Theorem 5.5.9$_{255}$.

Let $\mathscr{B}(\Omega)$ be the (smallest) σ-algebra generated by all open sets in Ω; that is, the Borel σ-algebra on Ω [53, Section 11.A]. A global variable $f \in \overline{\mathbb{V}}$

is called **Borel-measurable** if it is measurable with respect to the σ-algebra $\mathscr{B}(\Omega)$ (or simply $\mathscr{B}(\Omega)$-measurable); that is, if the inverse image $f^{-1}(B) := \{\omega \in \Omega \colon f(\omega) \in B\}$ is in $\mathscr{B}(\Omega)$ for every $B \in \mathscr{B}(\overline{\mathbb{R}})$; recall Appendix 5.A$_{263}$. It is clear that this notion of Borel-measurability is in accordance with the general one in [53, Section 11.C]. Moreover, also infer from Appendix 5.A$_{263}$ that the Borel-measurable variables can alternatively be characterised as those variables $f \in \overline{\mathbb{V}}$ for which $\{\omega \in \Omega \colon f(\omega) \leq c\} \in \mathscr{B}(\Omega)$ for all $c \in \mathbb{R}$.

The following result follows almost immediately from Lemma 5.C.1$_{275}$.

Corollary 5.C.2. *The Borel σ-algebra $\mathscr{B}(\Omega)$ coincides with the σ-algebra $\sigma(\mathscr{X}^*)$ generated by the cylinder events. A variable $f \in \overline{\mathbb{V}}$ is thus Borel-measurable if and only if it is $\sigma(\mathscr{X}^*)$-measurable.*

Proof. By Lemma 5.C.1$_{275}$, any open set in Ω is a (possibly empty) countable union of cylinder events. So, by definition of $\sigma(\mathscr{X}^*)$, all open sets are included in the σ-algebra $\sigma(\mathscr{X}^*)$ and therefore, $\sigma(\mathscr{X}^*)$ includes the Borel σ-algebra $\mathscr{B}(\Omega)$. Conversely, it is clear that $\sigma(\mathscr{X}^*)$ is not larger than the Borel σ-algebra $\mathscr{B}(\Omega)$ because each cylinder event is itself open. The last statement then simply follows from the definition of measurability (Borel-measurability or $\sigma(\mathscr{X}^*)$-measurability). □

Without explicitly stating the definition of an analytic function [28, Definition I.1.4], we next show that any $\sigma(\mathscr{X}^*)$-measurable non-negative variable $f \in \overline{\mathbb{V}}_{\geq}$ is analytic. The fact that any Borel-measurable non-negative function (on Ω) is analytic, and thus by Corollary 5.C.2 that any $\sigma(\mathscr{X}^*)$-measurable non-negative global variable is analytic, is already stated in [28, Corollary I.6], but we nevertheless give an independent proof because Dellacherie [28] characterises Borel-measurable functions in a somewhat different—presumably equivalent—way.

Proposition 5.C.3. *Any $\sigma(\mathscr{X}^*)$-measurable non-negative variable $f \in \overline{\mathbb{V}}_{\geq}$ is analytic according to [28, Definition I.1.4].*

The proof relies on the following topological fact about Ω—the definitions of a separable space and a zero-dimensional space can be found in [53, p.3] and [53, p.35], respectively.

Lemma 5.C.4. *The space Ω is metrizable, separable and zero-dimensional.*

Proof. The metrizability follows from Lemma 5.C.1$_{275}$. To prove that Ω is separable, we need to show that there is a countable subset of Ω that is dense in Ω. Let $O \subseteq \Omega$ be any set of paths obtained by including, for each situation $s \in \mathscr{X}^*$, a single path ω from the cylinder event $\Gamma(s)$; such a set O exists and is countable because $\mathscr{X}^*$ is countable.[22] To see that O is dense in Ω, recall from Lemma 5.C.1$_{275}$ that any open

[22]And by evoking the Axiom of Dependent Choice.

set in Ω is a countable union of cylinder events. Since, for any cylinder event $\Gamma(s)$, the set O contains by definition at least one path from $\Gamma(s)$, it is clear that O also contains at least one path from each open set in Ω, and thus that O is dense in Ω.

Furthermore, Ω is zero-dimensional if it is Hausdorff and has a base of sets that are both open and closed (that is, clopen sets). That this is true is mentioned on [53, p.35], but we can also easily derive it ourselves. Indeed, since Ω is metrizable, and since any metrizable space is Hausdorff [53, p.18], Ω is Hausdorff. The cylinder events $\Gamma(\mathscr{X}^*)$ moreover form a base of Ω [by Lemma 5.C.1$_{275}$], and any cylinder event $\Gamma(x_{1:k})$ is both open [Lemma 5.C.1$_{275}$] and, because $\Gamma(x_{1:k})$ is the complement of the finite union $\cup_{z_{1:k} \in \mathscr{X}^k \setminus \{x_{1:k}\}} \Gamma(z_{1:k})$, closed. □

Proof of Proposition 5.C.3$_{\leftarrow}$. We prove that any Borel-measurable non-negative variable $f \in \overline{\mathbb{V}}_{\geq}$ is analytic; the desired statement then follows from Corollary 5.C.2$_{\leftarrow}$. To this end, we start by using [53, Theorem 24.3] which guarantees that, if Ω and $\overline{\mathbb{R}}$ are metrizable and $\overline{\mathbb{R}}$ is separable, then the set $\overline{\mathbb{V}}_{\mathscr{B}}$ of Borel-measurable variables is the union $\cup_{\xi} \mathcal{B}_{\xi}$ of all sets $\mathcal{B}_{\xi}$ of variables of Baire class ξ. Without going into detail, ξ here is any ordinal number such that $1 \leq \xi < \omega_1$ where ω_1 is the first uncountable ordinal, and Baire classes are recursively defined by starting with variables of Baire class 1 and then iteratively defining new larger Baire classes by including pointwise limits of sequences of variables in the preceding Baire classes; see [53, Definition 24.1]. Let us first check that Ω and $\overline{\mathbb{R}}$ are metrizable and that $\overline{\mathbb{R}}$ is separable. The metrizability of Ω is guaranteed by Lemma 5.C.4$_{\leftarrow}$, and the metrizability of $\overline{\mathbb{R}}$ follows from [40, Problem C.11]—recall from Section 1.6$_{14}$ that our topology on $\overline{\mathbb{R}}$ corresponds to the one in [40, Example C.2.1]. Taking into account this topology, it is also clear that the rational numbers $\mathbb{Q}$ are dense in $\overline{\mathbb{R}}$ and thus that $\overline{\mathbb{R}}$ is separable [53, p.3]. So by [53, Theorem 24.3] the set $\overline{\mathbb{V}}_{\mathscr{B}}$ of Borel-measurable variables is equal to the union $\cup_{\xi} \mathcal{B}_{\xi}$.

Next, let us show that the union $\cup_{\xi} \mathcal{B}_{\xi}$ is the smallest subset of $\overline{\mathbb{V}}$ that contains all continuous variables and that is closed under taking pointwise limits. Let V be any subset of $\overline{\mathbb{V}}$ that contains all continuous variables and that is closed under taking pointwise limits. To see that $\cup_{\xi} \mathcal{B}_{\xi} \subseteq V$, we will apply the principle of transfinite induction [44, p.66] on the ordinal numbers ξ (see [44, p.56] for the definition of an 'initial segment'). We are allowed to use this principle because the set of all ordinal numbers ξ such that $1 < \xi < \omega_1$ is well-ordered by [44, p.79]. To start the induction, we prove that $\mathcal{B}_1$ is in V. This follows from [53, Theorem 24.10], which implies that any variable in $\mathcal{B}_1$ is the pointwise limit of continuous variables, and thus an element of V. Note that we can indeed use [53, Theorem 24.10] because, as shown earlier, $\overline{\mathbb{R}}$ is metrizable and separable, and Ω is metrizable, separable and zero-dimensional by Lemma 5.C.4$_{\leftarrow}$. To prove the induction step, consider any ordinal number ξ such that $1 < \xi < \omega_1$, and assume that $\cup_{\xi' < \xi} \mathcal{B}_{\xi'} \subseteq V$. Any variable f in $\mathcal{B}_{\xi}$ is by [53, Definition 24.1] the pointwise limit of a sequence $\{f_n\}_{n \in \mathbb{N}}$ of variables in $\cup_{\xi' < \xi} \mathcal{B}_{\xi'}$, and thus also the pointwise limit of a sequence $\{f_n\}_{n \in \mathbb{N}}$ of variables in V. Since V is closed under taking pointwise limits, we have that $f \in V$, and since f is any variable in $\mathcal{B}_{\xi}$, we obtain that $\mathcal{B}_{\xi} \subseteq V$ and therefore that $\cup_{\xi' \leq \xi} \mathcal{B}_{\xi'} \subseteq V$. This proves the induction step, and so by the principle of transfinite induction, we infer that $\cup_{\xi} \mathcal{B}_{\xi} \subseteq V$ [where

ξ ranges over all ordinal numbers such that $1 < \xi < \omega_1$]. It now suffices to observe that $\cup_\xi \mathcal{B}_\xi$ itself contains all continuous variables [53, p.190] and that it is closed under taking pointwise because, as discussed above, [53, Theorem 24.3] implies that it is equal to the set $\overline{\mathbb{V}}_{\mathscr{B}}$ of Borel-measurable variables, which is itself closed under taking pointwise limits; see [53, Example 11.2 (i)] and take into account that $\overline{\mathbb{R}}$ is metrizable. So $\cup_\xi \mathcal{B}_\xi$ is the smallest set of variables in $\overline{\mathbb{V}}$ that contains all continuous variables and that is closed under taking pointwise limits. Hence, by [53, Theorem 24.3], the same is thus true for the set $\overline{\mathbb{V}}_{\mathscr{B}}$ of all Borel-measurable variables.

We continue to show that the set $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ of all Borel-measurable non-negative variables is the smallest subset of $\overline{\mathbb{V}}_{\geq}$ that contains all continuous non-negative variables and that is closed under taking pointwise limits. It is clear that, since $\overline{\mathbb{V}}_{\mathscr{B}}$ contains all continuous variables and is closed under taking pointwise limits, that the subset of $\overline{\mathbb{V}}_{\mathscr{B}}$ of all non-negative variables—the set $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ of all Borel-measurable non-negative variables—contains all non-negative continuous variables and is also closed under taking pointwise limits. To see that $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ is also the smallest such set, consider any second set $\mathscr{K} \subseteq \overline{\mathbb{V}}_{\geq}$ that contains all non-negative continuous variables and that is closed under taking pointwise limits. Then it is clear that the set $\{f \in \overline{\mathbb{V}} \colon f^+, f^- \in \mathscr{K}\}$ contains all continuous variables in $\overline{\mathbb{V}}$ and is also closed under taking pointwise limits; the former follows from the fact that, for any continuous variable f in $\overline{\mathbb{V}}$, the variables f^+ and f^- are also continuous (and non-negative), and the latter follows from the fact that $\lim_{n\to+\infty} f_n = \lim_{n\to+\infty} f_n^+ - \lim_{n\to+\infty} f_n^-$ for any converging sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{V}}$ (and the fact that $\mathscr{K}$ is closed under taking pointwise limits). Since $\overline{\mathbb{V}}_{\mathscr{B}}$ is the smallest set that contains all continuous variables and is closed under taking pointwise limits, it follows that $\overline{\mathbb{V}}_{\mathscr{B}} \subseteq \{f \in \overline{\mathbb{V}} \colon f^+, f^- \in \mathscr{K}\}$. This implies that for any $f \in \overline{\mathbb{V}}_{\mathscr{B},\geq}$, since $f = f^+$ and $f \in \overline{\mathbb{V}}_{\mathscr{B}}$, we have that $f \in \mathscr{K}$, and thus that $\overline{\mathbb{V}}_{\mathscr{B},\geq} \subseteq \mathscr{K}$. So $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ is indeed the smallest subset of $\overline{\mathbb{V}}_{\geq}$ that contains all continuous non-negative variables and is closed under taking pointwise limits.

It now only suffices to establish that the set $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ of all analytic (non-negative) variables in $\overline{\mathbb{V}}_{\geq}$ [28, Definition I.1.4] contains all continuous non-negative variables and is closed under taking pointwise limits, because we can then combine this with our observation that $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ is the smallest subset of $\overline{\mathbb{V}}_{\geq}$ that contains all continuous non-negative variables and is closed under taking pointwise limits, to infer the desired statement that the set $\overline{\mathbb{V}}_{\mathscr{B},\geq}$ of Borel-measurable non-negative variables is a subset of the analytic non-negative variables. That $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ is closed under taking pointwise limits follows from [28, Theorem I.2.5]. To see that $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ contains all continuous non-negative variables, observe from the text below [28, Definition I.1.4] that all 'elementary Borel functions' are analytic. The notion of an elementary Borel function is defined in [28, Definition I.1.3]; in particular, it follows from this definition that any continuous non-negative real-valued variable in $\overline{\mathbb{V}}_{\geq}$ is an elementary Borel function,[23] and is therefore analytic—for recall from [28, Introduction, Paragraph 2] that continuous functions according to Dellacherie [28] take values in the non-negative reals $\mathbb{R}_{\geq}$. So $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ contains all continuous non-negative real-valued variables. Finally,

[23]This is also explicitly mentioned by Dellacherie [28] himself, in the proof of [28, Corollary I.2.6].

let us check that any continuous non-negative (not necessarily real-valued) variable f is the pointwise limit of a sequence of continuous non-negative real-valued variables, and therefore, since we have already shown that $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ is closed under taking pointwise limits, that $\overline{\mathbb{V}}_{\mathrm{A},\geq}$ contains all continuous non-negative (not necessarily real-valued) variables. Since f is continuous, we also have that $f^{\wedge n}$ for any $n \in \mathbb{N}$ is continuous; indeed, $f^{\wedge n}$ is u.s.c. because f is u.s.c. (since it is continous) and the sublevel sets $\{\omega \in \Omega \colon f^{\wedge n}(\omega) < a\}$ for all $a \in \mathbb{R}$ are either equal to $\{\omega \in \Omega \colon f(\omega) < a\}$ (if $a \leq n$) or equal to Ω (if $n < a$), and similarly we can infer that $f^{\wedge n}$ is l.s.c. So $f^{\wedge n}$ is continuous for all $n \in \mathbb{N}$, and it is clearly also real-valued and non-negative (because f is non-negative). Hence, since $(f^{\wedge n})_{n\in\mathbb{N}}$ converges pointwise to f, we obtain that f is the pointwise limit of a sequence of continuous non-negative real-valued variables. □

— Chapter VI —

Axiomatic upper expectations

The previous two chapters each introduced and studied a particular type of global upper expectation. Each of these operators turned out to have fairly nice continuity properties, yet their construction also relied on a very specific language and interpretation. Game-theoretic upper expectations take local sets of acceptable gambles as a starting point, and then use notions such as supermartingales and superhedging to construct a global model. Measure-theoretic upper expectations on the other hand take local sets of mass functions as a starting point, and then use probability measures and upper integrals to extend beyond the local level.

In the current chapter, we take a more direct route. Starting from an upper expectations tree, we will construct a global model solely by imposing some basic requirements—axioms—and by using conservativity arguments. In that respect, the approach is similar to the one that leads to the natural extensions $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ under coherence; recall Section 3.4_{80}. However, a crucial difference is that we will now only impose coherence on the domain $\mathbb{F} \times \mathscr{X}^*$, and use some type of continuity axiom to extend beyond this finitary domain. Doing so seems to be necessary in order to obtain desirable continuity behaviour of the resulting global upper expectation—recall from Section 3.6_{98} that the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ under coherence alone lacked a rather basic type of continuity from below. Choosing the type of continuity axiom is a delicate matter however; preferably, we want it to be as weak and intuitive as possible, yet it should also be sufficiently strong in order to result in a global upper expectation with satisfactory properties. The specific continuity axiom that we will suggest is fairly weak, and will follow intuitively from an approximation argument that regards global non-finitary variables as representing idealised finitary gambles (see $\mathrm{Co2}_{286}$ further below). However, the global upper expectation that—through conservativity arguments—will result from imposing this weaker type of continuity, will exhibit strong continuity behaviour nonetheless. In fact, the axiomatic model thus obtained will turn out to be equal to the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and hence, for a large part, also to the measure-theoretic

	local model	**global upper expectation**	
		finitary	continuity-based
behavioural	$\mathscr{A}_\bullet$ sets of acceptable gambles	$\overline{\mathrm{E}}_{\mathscr{A}}, \overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ from sets of acceptable gambles or martingales	$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}, \overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$ game-theoretic upper expectations
axiomatic	$\overline{\mathrm{Q}}_\bullet$ coherent upper expectations	$\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ extension under coherence	$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}, \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ extension under monotonicity + continuity
probabilistic	$\mathscr{P}_\bullet$ sets of probability mass functions	$\overline{\mathrm{E}}_{\mathscr{P}}$ from finitely additive probabilities	$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ measure-theoretic upper expectations

Figure 6.1 Overview of all the global upper expectations treated in this dissertation.

upper expectation $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$.

Though comparable to game-theoretic or measure-theoretic upper expectations in terms of continuity behaviour, it is the simple and universal character of its definition that really sets our axiomatic model apart from these other two global models. In that respect, it is very similar to our definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$, which solely relied on upper expectations trees, coherence and conservativity arguments; none of these notions require any particular interpretation; they can instead be motivated or given meaning starting from multiple points of view. The only notable difference with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ is that our new axiomatic model hinges on an additional continuity axiom, but, here too, no particular interpretation for an upper expectation is required. Moreover, we will also provide a wide variety of alternative but equivalent characterisations for our axiomatic model, including a full axiomatisation that does not rely on any additional conservativity arguments. Apart from their obvious mathematical benefits, these alternative characterisations allow readers to interpret and motivate our axiomatic model in an even more flexible way.

The structure of this chapter is rather straightforward: in Section 6.1$_{\rightarrow}$, we introduce and argue for the use of some specific axioms, and then subsequently use these axioms to define a global upper expectation. We then continue to study the properties—and the existence—of this axiomatic global upper expectation in Sections 6.2$_{290}$ and 6.3$_{294}$. More precisely, in Section 6.2$_{290}$, we prove the existence of our axiomatic upper expectation, and show that it is equal to the game-theoretic upper expectation, and hence, for a large part, also to the measure-theoretic upper expectation. In Section 6.3_{294} then, we establish several alternative characterisations for our axiomatic model, including a full axiomatisation—not based on additional conservativity arguments—and a constructive characterisation. The latter is of considerable importance on a practical level, since it explicitly tells us how to calculate the value of our axiomatic upper expectation for any kind of variable.

Finally, in Section 6.4_{302}, we study whether the continuity axiom that characterises our global model can be relaxed even further. We consider one specific weakening of the continuity axiom and show that the newly obtained global upper expectation is essentially a trivial adaptation of the upper integral proposed by Daniell [19]. In general contexts however, the properties of this Daniell-like global upper expectation are not quite satisfactory, and so we are—in general—inclined to stick with the original stronger continuity axiom. Nevertheless, there are three particular instances where it does perform well; if we restrict ourselves to monotone limits of finitary gambles, if we restrict ourselves to the domain of all indicators, and—for all possible variables and situations—if the local models are precise.

6.1 Natural extension under a continuity axiom and a monotonicity axiom

In order to propose a global model that is, in its interpretation, as universal as possible, we start from an upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ to describe the local dynamics of our stochastic process; recall that upper expectations trees can always be seen as to directly result from acceptable gambles trees or imprecise probability trees through Eqs. 3.1_{50} and 3.3_{51}, respectively. One of the simplest and most intuitive ways to then extend $\overline{\mathrm{Q}}_{\bullet}$—or better $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$—is by using the natural extension under coherence, or equivalently under $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$; recall Definition 3.8_{86}. The downside of this approach, however, is that the resulting model $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ lacks some basic but desirable continuity properties, which is why we dismissed it at the end of Chapter 3_{45} and went on to study more involved types of global upper expectations in the subsequent chapters.

Yet, as we also remarked at the end of Section 3.6_{98}, no such continuity issues arise if we restrict our attention to the finitary domain $\mathbb{F} \times \mathscr{X}^*$. Moreover, the restriction of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ to this finitary domain—or simply the natural extension $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ under coherence to $\mathbb{F} \times \mathscr{X}^*$ [Corollary $3.4.7_{89}$]—coincides not only with all the other finitary global upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$, $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P}}$ on $\mathbb{F} \times \mathscr{X}^*$ [Theorems $3.5.1_{90}$ and $3.5.2_{91}$], but also with the continuity-based upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ on $\mathbb{F} \times \mathscr{X}^*$ [Corollary 4.4.9$_{170}$ and Proposition 5.4.5$_{244}$]. Hence, it seems that the extension from the preliminary upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$ to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ is a done deal, and so we will henceforth take $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ on $\mathbb{F} \times \mathscr{X}^*$ as a starting point for defining our axiomatic global upper expectation. Nevertheless, our axiomatic model will later on also be given alternative characterisations that are based on directly extending—for instance, using coherence and continuity—the initial upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\mathrm{Q}}$; see Section 6.3$_{294}$.

The continuity axioms and monotonicity

Given a global upper expectation $\overline{\mathrm{E}} \colon \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$, one of the weakest types of continuity axioms that we can and will want to impose on $\overline{\mathrm{E}}$ is the following:

Co1. For any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$, and any $s \in \mathscr{X}^*$,

$$\limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) \geq \overline{\mathrm{E}}(f|s) \text{ with } f := \sup_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Note that Co1 is weak because it only applies to sequences that are **increasing** and that consist of **finitary gambles**—a motivation for this axiom will be given shortly. Apart from this continuity axiom, we will moreover always impose that $\overline{\mathrm{E}}$ should be monotone on $\overline{\mathbb{V}} \times \mathscr{X}^*$:

EC4$^{\Omega}$. $f \leq g \Rightarrow \overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}(g|s)$ for any $f, g \in \overline{\mathbb{V}}$ and $s \in \mathscr{X}^*$.

This monotonicity property is weaker than EC4$_{163}$ because it requires an inequality between two variables f and g on their entire domain Ω, rather than only on the cylinder set $\Gamma(s)$. Note that, if we assume a global upper expectation to satisfy EC4$^{\Omega}$, imposing Axiom Co1 becomes equivalent to imposing continuity with respect to increasing sequences in $\mathbb{F}$—which is, for readers who are familiar with Choquet integration, perhaps more comfortable or natural to impose as an axiom:

Co1$^{=}$. For any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$, and any $s \in \mathscr{X}^*$,

$$\lim_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s) \text{ with } f := \sup_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Though Axiom Co1 is elegant and weak, we also consider an alternative but stronger version of the axiom, which applies to all converging sequences in $\mathbb{F}$ that are uniformly bounded below:

Co2. For any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that converges pointwise and that is uniformly bounded below, and any $s \in \mathscr{X}^*$,

$$\limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) \geq \overline{\mathrm{E}}(f|s) \text{ with } f := \lim_{n\to+\infty} f_n.$$

Clearly, Co2$_{\leftharpoonup}$ implies Co1$_{\leftharpoonup}$, and therefore also implies Co1$^{=}{}_{\leftharpoonup}$ if combined with monotonicity [EC4$^{\Omega}{}_{\leftharpoonup}$]. The converse—that Co1$_{\leftharpoonup}$ or Co1$^{=}{}_{\leftharpoonup}$ implies Co2$_{\leftharpoonup}$—is not true though; see Section 6.4.3$_{310}$ later on.

A motivation for the axioms

First and foremost, our motivation for the axioms above, and thus also for the global axiomatic upper expectations introduced below, is based—as we will clarify in the next paragraph—on the interpretational convention that we regard higher upper expectations to be more conservative (or less informative). We feel this convention is appropriate because it is true for all global upper expectations that we have encountered so far, whether they are of the probability-based type, or the behavioural/game-theoretic type. Indeed, if an upper expectation denotes an upper bound on the linear expectations corresponding to a set of probability charges/measures, then a higher upper bound is obviously more conservative (or less informative). On the other hand, if an upper expectation denotes the infimum selling (or hedging) prices corresponding to a set of acceptable gambles (resp. supermartingales) then, again, higher selling (hedging) prices are more conservative (or less informative). Note also that this convention is in line with what we have already said in Section 2.6.3$_{38}$ and Section 3.4.2$_{85}$, where we argued for the use of the natural extension—the most conservative or largest extension—under coherence.

Now, apart from the convention above, our motivation for Co1$_{\leftharpoonup}$ and Co2$_{\leftharpoonup}$ additionally stems from our difference in interpretation between finitary gambles and non-finitary (extended real) variables. For we consider finitary gambles to be the only global variables that have direct practical significance; they depend on the states of the stochastic process up until some finite time horizon, and they can only take real values. Global variables that depend on the entire infinite path taken by the process, or take the values $+\infty$ or $-\infty$, are only given an implicit interpretation in the sense that they are considered to be abstract idealisations of finitary gambles that lie arbitrarily close.[1] In particular, if f is the pointwise limit of a sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles—and if f is itself not a finitary gamble—, then f is considered to be an idealisation of f_n for **large** n; think of e.g. hitting times [Example 4.2.2$_{140}$], hitting probabilities [Example 3.6.1$_{99}$], stopping times, infinite time averages [26], ... As a result, we typically desire the upper expectation $\overline{\mathrm{E}}(f|s)$ of such a global limit variable to give information about the upper expectation $\overline{\mathrm{E}}(f_n|s)$ for a generically large value of n. Given this point of view, Co2$_{\leftharpoonup}$ is then only a minimal requirement; it simply demands

[1] Recall that a similar point was raised in the paragraph below Definition 4.3$_{143}$, where we said that extended real variables are being regarded as idealised—bounded—gambles.

that, for any converging (uniformly bounded below) sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles, and the corresponding sequence $\big(\overline{\mathrm{E}}(f_n|s)\big)_{n\in\mathbb{N}}$ of upper expectations, the upper expectation $\overline{\mathrm{E}}(f|s)$ of the limit variable f should **not** exceed the 'maximum' value that $\big(\overline{\mathrm{E}}(f_n|s)\big)_{n\in\mathbb{N}}$ attains **at large** n. If this condition were not satisfied[2] then, since we regard higher upper expectations to be less informative, $\overline{\mathrm{E}}(f|s)$ would not take into account information given by the limit values of $\big(\overline{\mathrm{E}}(f_n|s)\big)_{n\in\mathbb{N}}$, which we consider to be unwarranted. Furthermore, since Co1$_{286}$ is simply a weakened version of Co2$_{286}$, it can be argued for on similar grounds.

Though the reasoning above essentially applies to **any** converging sequence in $\mathbb{F}$, Co2$_{286}$ (and in particular Co1$_{286}$) nonetheless only applies to sequences that are uniformly bounded below. Mathematically speaking, this is no problem since we ideally want to impose axioms that are as weak as possible anyway—as long as we obtain a global model with desirable features. From an interpretational point of view, however, the condition of being uniformly bounded below seems rather arbitrary and, based on mere intuition, we would be inclined to drop it. This is not a good idea though, because it would make global upper expectations sometimes attain extremely low values; we will come back to this issue in Chapter 7$_{323}$.

Lastly, it remains to motivate monotonicity [EC4$^{\Omega}$$_{286}$]. In general, this property is satisfied by almost all upper expectations; for one, all the global upper expectations that we have studied so far satisfy it. It therefore does not appear to be very controversial. On the domain $\mathbb{F}\times\mathscr{X}^*$, monotonicity will automatically be satisfied because $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ satisfies WC5$_{84}$ and we require our desired global upper expectation to extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$. That monotonicity [EC4$^{\Omega}$$_{286}$] should also hold on the more general domain $\overline{\mathbb{V}}\times\mathscr{X}^*$ can then be argued in a somewhat similar way as before, using approximation arguments; since we have that larger finitary gambles lead to higher upper expectations, we also assume that larger—abstract and idealised—(not necessarily finitary) extended real-valued variables return higher upper expectations.

Axiomatic global upper expectations

Given that we choose to accept Co1$_{286}$ and EC4$^{\Omega}$$_{286}$, we can take the natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ under these two axioms, which if it exists, delivers us with a first candidate for a possible axiomatic global upper expectation. The **natural extension** here is defined similarly as in Section 3.4.2$_{85}$, but extended to apply to general operators on $\overline{\mathbb{V}}\times\mathscr{X}^*$; so it is once more the

[2]This is for instance the case in Example 3.6.1$_{99}$ for the upper expectations $\big(\overline{\mathrm{E}}_p(\mathbb{I}_{H^k_y})\big)_{n\in\mathbb{N}}$ and $\overline{\mathrm{E}}_p(\mathbb{I}_{H_y})$. In fact, it can be seen that the sequence $(\mathbb{I}_{H^k_y})_{k\in\mathbb{N}_0}$ there is increasing, and thus that the corresponding upper expectation $\overline{\mathrm{E}}_p$ does not even satisfy Co1$_{286}$.

pointwise **largest** or—since we consider higher upper expectations to be more conservative—the **most conservative** extension satisfying a specific set of axioms.

Definition 6.1. For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, we let $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ be, if it exists, the natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$ under $\mathrm{Co1}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$. ⊚

We will see later on—in Section 6.4_{302}—that this natural extension $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ always exists, and that it moreover can be identified as an imprecise version of Daniell's upper integral. Furthermore, we will also show there that it can be elegantly characterised as an extension of the preliminary upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$.

If we moreover choose to accept the stronger axiom $\mathrm{Co2}_{286}$ instead of $\mathrm{Co1}_{286}$, then we obtain the following axiomatic global upper expectation.

Definition 6.2. For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, we let $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ be, if it exists, the natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$ under $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$. ⊚

Once more, as we will show in Sections $6.2_{\curvearrowright}$ and 6.3_{294}, this global upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ exists and can be given various alternative characterisations.

The reason that we let $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ be defined as the natural extensions—the most conservative extensions—under their respective axioms, is the same as in Section $3.4.2_{85}$. Taking any smaller global upper expectation would mean adding information on top of the already accepted axioms (and what $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ says), which we do not consider to be necessary. Moreover, if one nevertheless desires to impose further axioms or add more information, then the natural extension still provides conservative (upper) bounds.

The **axiomatic lower expectations** $\underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\underline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ are obtained from the upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ by conjugacy; so, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$,

$$\underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) := -\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(-f|s) \text{ and } \underline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) := -\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(-f|s). \tag{6.1}$$

Using conjugacy to define axiomatic lower expectations is justified—and perhaps even intuitive—because all other types of upper and lower expectations satisfy it. Nonetheless, if one desires so, one could equivalently define these axiomatic lower expectations independently, starting from a (conjugate) lower expectations tree, and then following a similar reasoning to how we defined $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, but where all steps are 'conjugate'; e.g. we do not take the pointwise largest extension, but the pointwise smallest extension (of some initial lower expectation).

When it comes to choosing between $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, the former appears to be more appealing—at least, at first sight—simply because its definition is based on the weaker and simpler axiom $\mathrm{Co1}_{286}$. Of course, much depends on whether this operator $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ exhibits satisfactory continuity behaviour, and in particular we would want it to satisfy $\mathrm{Co2}_{286}$—since we have just argued it to be desirable. Unfortunately however, as we will show in Section 6.4_{302}, this is not necessarily the case: $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ may fail to satisfy $\mathrm{Co2}_{286}$. We are therefore inclined to work with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ instead of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$.

Besides it satisfying a stronger continuity axiom—and actually more than one—another major reason why we prefer to use $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ instead of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ will turn out to coincide with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ for an agreeing tree $\mathscr{A}_\bullet$—on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$. Hence, by Theorems $5.5.13_{256}$ and $5.5.10_{255}$, it will also coincide with $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$—for an agreeing tree $\mathscr{P}_\bullet$—on a rather large domain. It is clear that this considerably increases the relevance of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ compared to that of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$. We will therefore mainly study the properties of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ in the coming sections, and only come back to those of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ at the end.

Furthermore, all this being said, we want to stress that the definitions of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, or rather, the justification of these definitions, do not hinge on any particular interpretation for a global upper expectation. Indeed, we started from an upper expectations tree to parametrise the local dynamics of a stochastic process; this parametrisation can on itself already be regarded as resulting from either a behavioural approach involving sets of acceptable gambles, or from a probability-based approach involving sets of probability mass functions. We then continued to consider $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ as a first extension of the local models; this extension was simply obtained from—only—adopting $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$ or, equivalently, coherence; properties that can once more be motivated from a behavioural point of view and from a probability-based point of view. The two axioms that we then additionally impose, $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$ for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, or $\mathrm{Co1}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$ for $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, follow only from an approximation argument; no particular interpretation is required here either.

6.2 Relation to game-theoretic and measure-theoretic upper expectations

Let us start by proving that, for any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ satisfies all the defining properties of the axiomatic upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, which will therefore imply that the latter exists and coincides with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ on all of $\overline{\mathbb{V}} \times \mathscr{X}^*$. Afterwards, in Section 6.3_{294}, we will establish various alternative characterisations for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, one of which will be a full axiomatisation, without a conservativity argument.

The game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is surely smaller than or equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$—if it exists. This follows immediately from the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and the fact that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, and satisfies $\mathrm{EC4}^{\Omega}{}_{286}$ and $\mathrm{Co2}_{286}$, due to Corollary $4.4.9_{170}$, Proposition $4.4.3_{164}$ and Corollary $4.6.2_{177}$. To prove that they are in fact equal, we will crucially rely on Proposition $4.7.6_{184}$ and Theorem $4.7.4_{183}$; the former expresses $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ in terms of its values on the domain $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$ of all bounded below limits of finitary variables (and situations), whereas the latter provides a crucial continuity property for $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ on this domain. We start with the following lemma.

Lemma 6.2.1. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies* $\mathrm{Co2}_{286}$ *and* $\mathrm{EC4}^{\Omega}{}_{286}$*. Moreover, for any other upper expectation $\overline{\mathrm{E}}'$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that satisfies these conditions, we have that*

$$\overline{\mathrm{E}}'(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \text{ for all } (f,s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*.$$

Proof. Observe that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ by Corollary $4.4.9_{170}$, and that it satisfies $\mathrm{Co2}_{286}$ by Corollary $4.6.2_{177}$ and $\mathrm{EC4}^{\Omega}{}_{286}$ by Proposition $4.4.3_{164}$. So it remains to prove that, for any second global upper expectation $\overline{\mathrm{E}}'$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$, that $\overline{\mathrm{E}}'(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for all $(f,s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. Fix any $(f,s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. By Theorem $4.7.4_{183}$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$. Since both $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and $\overline{\mathrm{E}}'$ extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ coincides with $\overline{\mathrm{E}}'$ on $\mathbb{F} \times \mathscr{X}^*$, and so we have that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}'(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s).$$

Applying $\mathrm{Co2}_{286}$ to the left-hand side, we obtain that $\overline{\mathrm{E}}'(f|s) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$, as desired. □

The remaining step, which is showing that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is also the most conservative global upper expectation on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$ among all those that extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfy $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$, now follows trivially from Proposition $4.7.6_{184}$.

Theorem 6.2.2. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ exists and is equal to the upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$.*

Proof. We simply show that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is the pointwise largest global upper expectation on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$. Since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ exists by its very definition, and since the natural extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is defined as the largest extension under these conditions, this immediately implies the desired statement. We know by Lemma 6.2.1 that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies $\mathrm{Co2}_{286}$ and $\mathrm{EC4}^{\Omega}{}_{286}$, so it suffices to prove that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is the (pointwise) largest global upper expectation satisfying these properties.

To this end, consider any global upper expectation $\overline{\mathrm{E}}'$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies Co2$_{286}$ and EC4$^{\Omega}{}_{286}$, and fix any $(f, s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. Then, by Proposition 4.7.6$_{184}$, we have that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) = \inf\left\{\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(g|s) : g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\}$$
$$\geq \inf\left\{\overline{\mathrm{E}}'(g|s) : g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \geq \overline{\mathrm{E}}'(f|s),$$

where the first inequality follows from Lemma 6.2.1$_{\curvearrowleft}$ and our assumptions about $\overline{\mathrm{E}}'$, and where the second inequality follows from the fact that $\overline{\mathrm{E}}'$ satisfies EC4$^{\Omega}{}_{286}$ by assumption. □

An immediate consequence of this main theorem is that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ also coincides with $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ on a fairly large domain, given that $\overline{\mathrm{Q}}_\bullet$ and $\mathscr{P}_\bullet$ agree.

Theorem 6.2.3. *For any imprecise probability tree $\mathscr{P}_\bullet$ and upper expectations tree $\overline{\mathrm{Q}}_\bullet$ that agree according to Eq. (3.3)$_{51}$, we have that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s)$ for all $(f, s) \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}} \times \mathscr{X}^*$. If $\mathscr{P}_t$ is moreover closed for all $t \in \mathscr{X}^*$, then also $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}(f|s)$ for all $(f, s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ such that f is the pointwise limit of a decreasing sequence of finitary gambles.*

Proof. The first statement follows immediately from Theorem 6.2.2$_{\curvearrowleft}$ and Theorem 5.5.10$_{255}$. The second follows from Theorem 6.2.2$_{\curvearrowleft}$ and Theorem 5.5.13$_{256}$. □

Theorems 6.2.2$_{\curvearrowleft}$ and 6.2.3 can be seen as two of the most important results of this dissertation—if not the most important. They have considerable merit both at a philosophical level and at a mathematical level. Philosophically speaking, it is most interesting that game-theoretic upper expectations, which are based on behavioural notions such as supermartingales and superhedging, coincide with the upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ resulting from a direct and interpretation-free axiomatic approach. Due to Theorem 6.2.3, the same can be said, to a large extent, for measure-theoretic upper expectations. The axiomatic model $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ serves as an alternative characterisation that is neutral in interpretation and conceptually much more direct than either of the two other types of global upper expectations. From a mathematical point of view, Theorems 6.2.2$_{\curvearrowleft}$ and 6.2.3 provide—and hopefully will continue to provide—a large number of additional insights about the involved global upper expectations. In particular, these theorems ensure that all the considered global upper expectations share the same properties and features. So properties that were previously only known to hold for only one or two types of global models, suddenly are seen to hold for all three of them, and similarly for any additional properties that might be discovered in future work. We have already extensively used a similar mechanism in

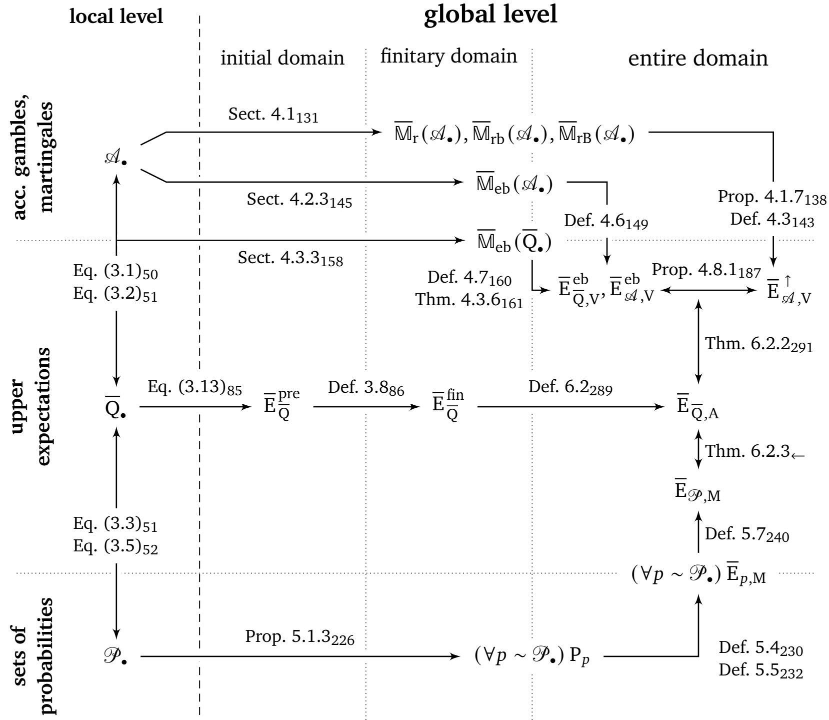

Figure 6.2 Schematic overview of the possible continuity-based approaches and their connections.

Chapter 5[217], where we deduced a large number of properties for measure-theoretic upper expectations (especially in the precise case) from the fact that they are known to hold for game-theoretic upper expectations. We can now do the same for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, establishing properties for it by exploiting its equality with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$.

Corollary 6.2.4. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the following statements hold:*

(i) *The restriction of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ to $\mathbb{V} \times \mathscr{X}^*$ is coherent.*

(ii) *$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies the extended coherence properties* EC1[163]–EC6[163].

(iii) *For any $f \in \overline{\mathbb{V}}$ and any $k \in \mathbb{N}_0$,*

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|X_{1:k}) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}\Big(\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}\,(f|X_{1:k+1})\,\Big|\,X_{1:k}\Big).$$

(iv) *For any* $(f, s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \inf \left\{ \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f \right\};$$

(v) *For any* $s \in \mathscr{X}^*$ *and any increasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_{\mathrm{b}}$, $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\lim_{n\to+\infty} f_n|s)$. *[Continuity from below]*

(vi) *For any* $s \in \mathscr{X}^*$ *and any decreasing sequence* $(f_n)_{n\in\mathbb{N}}$ *of finitary bounded above variables,* $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\lim_{n\to+\infty} f_n|s)$.
[Continuity w.r.t. decreasing finitary variables]

(vii) *For any* $s \in \mathscr{X}^*$ *and any* $f \in \overline{\mathbb{L}}_{\mathrm{b}}$, *there is a sequence* $(f_n)_{n\in\mathbb{N}}$ *of n-measurable gambles that is uniformly bounded below and that converges pointwise to* f *such that* $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$.

(viii) *For any* $s \in \mathscr{X}^*$ *and any sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_{\mathrm{b}}$ *that is uniformly bounded below,* $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s) \leq \liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f_n|s)$ *with* $f := \liminf_{n\to+\infty} f_n$. *[Fatou's lemma]*

Proof. The properties above follow from combining Theorem $6.2.2_{291}$ with, in order,

(i). Corollary $4.4.5_{167}$;

(ii). Proposition $4.4.3_{164}$;

(iii). Theorem $4.4.4_{166}$;

(iv). Proposition $4.7.6_{184}$;

(v). Theorem $4.6.1_{175}$;

(vi). Theorem $4.7.3_{182}$;

(vii). Theorem $4.7.4_{183}$.

(viii). Corollary $4.6.2_{177}$. □

6.3 Alternative characterisations for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$

Our starting point for the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ was the finitary upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, a global upper expectation that itself results from accepting $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$ (or coherence)—and these properties alone—on the finitary domain $\mathbb{F} \times \mathscr{X}^*$. We considered this to be the simplest and most convincing way of introducing our axiomatic model, since there seems to be no disagreement on how a global upper expectation should be defined on the domain $\mathbb{F}\times\mathscr{X}^*$—all the global upper expectations we have considered so far coincide with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F}\times\mathscr{X}^*$. Yet, on the other hand, this approach is somewhat indirect in the sense that we consider two extensions: one from the local models $\overline{\mathrm{Q}}_{\bullet}$—or, equivalently, from $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$—to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, and then subsequently from $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$. One may therefore be inclined to desire a definition of

$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ as a one-step extension of the local models $\overline{\mathrm{Q}}_{\bullet}$; this is what we set out to explore next.

We will moreover give a full axiomatisation of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$—and thus also $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and, for a large part, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$—without any conservativity arguments, similar to what we did for $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ in Section 3.4.3$_{87}$. Lastly, we will mold these characterising axioms together into a single formula; this provides us with a more constructive characterisation of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ that does not rely on any existence arguments.

Let us already give an overview of the possible properties for global upper expectations that, on top of Co2$_{286}$ and EC4$^{\Omega}{}_{286}$, will be used in the results about to come. They are stated for a general global upper expectation $\overline{\mathrm{E}} : \overline{\mathbb{V}} \times \mathscr{X}^* \to \overline{\mathbb{R}}$ and a general upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$:

NE1. $\overline{\mathrm{E}}(f(X_{k+1})|x_{1:k}) = \overline{\mathrm{Q}}_{x_{1:k}}(f)$ for all $f \in \mathscr{L}(\mathscr{X})$ and $x_{1:k} \in \mathscr{X}^*$;

NE2. $\overline{\mathrm{E}}(f|s) = \overline{\mathrm{E}}(f\mathbb{1}_s|s)$ for all $(f,s) \in \mathbb{F} \times \mathscr{X}^*$;

NE3. $\overline{\mathrm{E}}(f|X_{1:k}) = \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k})$ for all $f \in \mathbb{F}$ and $k \in \mathbb{N}_0$ such that $\overline{\mathrm{E}}(f|X_{1:k+1})$ is real-valued.

NE3$^{\leq}$. $\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k})$ for all $f \in \mathbb{F}$ and $k \in \mathbb{N}_0$;

NE4$^{\mathrm{L}}$. $\overline{\mathrm{E}}(f|s) = \inf\left\{\overline{\mathrm{E}}(g|s) : g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\}$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$;

Co3. For any $s \in \mathscr{X}^*$ and any sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that converges pointwise and that is uniformly bounded below,

$$\lim_{m\to+\infty} \overline{\mathrm{E}}(\inf_{n\geq m} f_n|s) = \overline{\mathrm{E}}(f|s) \text{ with } f := \lim_{n\to+\infty} f_n;$$

Co4. For any $s \in \mathscr{X}^*$ and any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathbb{L}}_{\mathrm{b}}$,

$$\lim_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s), \text{ with } f := \sup_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n.$$

Co5. For any $s \in \mathscr{X}^*$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below,

$$\lim_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) = \overline{\mathrm{E}}(f|s), \text{ with } f := \inf_{n\in\mathbb{N}} f_n = \lim_{n\to+\infty} f_n;$$

Properties NE1$_{88}$–NE3$_{88}$ were already introduced in Section 3.4.3$_{87}$, and they fully characterise $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ [Theorem 3.4.6$_{88}$]. Property NE3$^{\leq}$ is a weakened version of the law of iterated upper expectations [NE3$_{88}$],[3] and NE4$^{\mathrm{L}}$

[3]Strictly speaking, NE3$^{\leq}$ is not weaker than NE3$_{88}$ because the latter only applies if $\overline{\mathrm{E}}(f|X_{1:k+1})$ is real-valued. Nonetheless, since all the global upper expectations on which we will impose NE3$^{\leq}$ and/or NE3$_{88}$ will always be real-valued anyway, we will always pretend, for the sake of simplicity, as if NE3$^{\leq}$ is weaker than NE3$_{88}$.

is similar to NE4$_{88}$ but where the approximation is with respect to variables that are in $\overline{\mathbb{L}}_{\mathrm{b}}$—instead of $\mathbb{F}$—and that moreover lie above the considered variable f on their entire domain—rather than only on $\Gamma(s)$ as is the case for NE4$_{88}$. Finally, Co3$_{\curvearrowleft}$–Co5$_{\curvearrowleft}$ are three specific continuity properties; note in particular that Co3$_{\curvearrowleft}$ is a type of continuity with respect to increasing sequences. Properties Co2$_{286}$–Co4$_{\curvearrowleft}$ can be ordered depending on how strong they are; Co4$_{\curvearrowleft}$ is the strongest, Co2$_{286}$ the weakest (under monotonicity [EC4$^{\Omega}$$_{286}$]).

Lemma 6.3.1. *For any global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$, we have that* Co4$_{\curvearrowleft}$ *implies* Co3$_{\curvearrowleft}$. *If $\overline{\mathrm{E}}$ is moreover monotone [*EC4$^{\Omega}$$_{286}$*], then* Co3$_{\curvearrowleft}$ *implies* Co2$_{286}$.

Proof. Fix any global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$. First suppose that $\overline{\mathrm{E}}$ satisfies Co4$_{\curvearrowleft}$. To show that $\overline{\mathrm{E}}$ satisfies Co3$_{\curvearrowleft}$, consider any $s \in \mathscr{X}^*$ and any converging sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below. Let $f := \lim_{n\to+\infty} f_n$ and, for all $k \in \mathbb{N}$, let g_k be the global variable defined by $g_k := \inf_{k\leq n} f_n$. Since $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded below, each g_k is bounded below, and since each g_k is clearly also the pointwise limit of the finitary gambles $\inf_{k\leq n\leq m} f_n$ for $m \to +\infty$, we have that $g_k \in \overline{\mathbb{L}}_{\mathrm{b}}$ for all $k \in \mathbb{N}$. The sequence $(g_k)_{k\in\mathbb{N}}$ is moreover increasing [due to its definition], so we have by Co4$_{\curvearrowleft}$ that

$$
\begin{aligned}
\lim_{k\to+\infty} \overline{\mathrm{E}}\big(\inf_{k\leq n} f_n \big| s\big) = \lim_{k\to+\infty} \overline{\mathrm{E}}(g_k|s) = \overline{\mathrm{E}}\big(\lim_{k\to+\infty} g_k \big| s\big) &= \overline{\mathrm{E}}\big(\lim_{k\to+\infty} \inf_{k\leq n} f_n \big| s\big)\\
&= \overline{\mathrm{E}}\big(\liminf_{n\to+\infty} f_n \big| s\big)\\
&= \overline{\mathrm{E}}\big(\lim_{n\to+\infty} f_n \big| s\big) = \overline{\mathrm{E}}(f|s),
\end{aligned}
$$

establishing that Co3$_{\curvearrowleft}$ also holds.

To establish the second claim, suppose that $\overline{\mathrm{E}}$ satisfies EC4$^{\Omega}$$_{286}$ and Co3$_{\curvearrowleft}$. Again, fix any $s \in \mathscr{X}^*$ and any converging sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below, and let $f := \lim_{n\to+\infty} f_n$. By Co3$_{\curvearrowleft}$, we have that

$$
\lim_{k\to+\infty} \overline{\mathrm{E}}\big(\inf_{k\leq n} f_n \big| s\big) = \overline{\mathrm{E}}(f|s).
$$

Due to the monotonicity [EC4$^{\Omega}$$_{286}$] of $\overline{\mathrm{E}}$, we have that $\limsup_{k\to+\infty} \overline{\mathrm{E}}(f_k|s) \geq \lim_{k\to+\infty} \overline{\mathrm{E}}(\inf_{k\leq n} f_n|s)$, and therefore that

$$
\limsup_{k\to+\infty} \overline{\mathrm{E}}(f_k|s) \geq \overline{\mathrm{E}}(f|s).
$$

This establishes Co2$_{286}$ for $\overline{\mathrm{E}}$. □

6.3.1 Alternative natural extensions of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ or $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$

We start by expressing $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ as a natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ [Eq. (3.13)$_{85}$] under a series of axioms that are as weak as possible. The alternative characterisations of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ that we will then give afterwards, will all be natural extensions of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ (or $\overline{\mathrm{E}}^{\mathrm{fin}}$) under some stronger—but also fairly intuitive—axioms.

Proposition 6.3.2. *For any upper expectations tree* $\overline{\mathrm{Q}}_{\bullet}$*, the upper expectation* $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ *is the natural extension of* $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ *under* NE2*88*, NE3$^{\leq}$*295*, Co2*286* *and* EC4$^{\Omega}$*286*.

Proof. It follows from Definition 6.2$_{289}$ that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies EC4$^{\Omega}{}_{286}$ and Co2$_{286}$. Definition 6.2$_{289}$ also says that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, which is itself an extension of the preliminary upper expectation $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ [Definition 3.8$_{86}$], so $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$. That $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ moreover satisfies the properties NE2$_{88}$ and NE3$^{\leq}{}_{295}$ follows from the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, from Theorem 3.4.6$_{88}$[NE2$_{88}$] and from Corollary 3.5.7$_{94}$. So, according to the definition of the natural extension, it suffices to prove that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is larger or equal than any other global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and that satisfies NE2$_{88}$, NE3$^{\leq}{}_{295}$, EC4$^{\Omega}{}_{286}$ and Co2$_{286}$. To this end, let us first show that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$—and thus $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$—is always larger or equal than $\overline{\mathrm{E}}$ on $\mathbb{F} \times \mathscr{X}^*$.

Start by noting that, for any $x_{1:i} \in \mathscr{X}^*$ and any $(i+1)$-measurable gamble $g(X_{1:i+1})$,

$$\begin{aligned}\overline{\mathrm{E}}(g(X_{1:i+1})|x_{1:i}) \overset{\mathrm{NE2}}{=} \overline{\mathrm{E}}(g(x_{1:i}X_{i+1})|x_{1:i}) &= \overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}(g(x_{1:i}X_{i+1})|x_{1:i}) \\ &= \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g(x_{1:i}X_{i+1})|x_{1:i}) \\ &\overset{\mathrm{NE2}}{=} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g(X_{1:i+1})|x_{1:i}),\end{aligned} \tag{6.2}$$

where the second equality follows from the fact that $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, and where the third equality follows from the fact that, by definition, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$. Now fix any $(f,s) \in \mathbb{F} \times \mathscr{X}^*$ and let $k := |s|$ be the length of s. Since f is finitary, there surely is some $\ell \geq k$ such that f is $(\ell+1)$-measurable. Since $\overline{\mathrm{E}}$ satisfies NE3$^{\leq}{}_{295}$ and EC4$^{\Omega}{}_{286}$, we have that

$$\begin{aligned}\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+1})|X_{1:k}) &\leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\overline{\mathrm{E}}(f|X_{1:k+2})|X_{1:k+1})|X_{1:k}) \\ &\leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\cdots\overline{\mathrm{E}}(f|X_{1:\ell})\cdots|X_{1:k+1})|X_{1:k}).\end{aligned}$$

Applying Eq. (6.2) to the inner upper expectation of the rightmost term, we obtain that

$$\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\cdots\overline{\mathrm{E}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|X_{1:\ell})|X_{1:\ell-1})\cdots|X_{1:k+1})|X_{1:k}).$$

Since $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|X_{1:\ell})$ is real-valued [by Corollary 3.4.2$_{83}$ and Corollary 3.4.5$_{87}$] and clearly ℓ-measurable, it is automatically bounded and thus an ℓ-measurable gamble. Hence, by once more applying Eq. (6.2), we obtain that

$$\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}(\overline{\mathrm{E}}(\cdots\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|X_{1:\ell})|X_{1:\ell-1})\cdots|X_{1:k+1})|X_{1:k}).$$

We can do the same with the other upper expectations, working our way outwards to find that

$$\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\cdots\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|X_{1:\ell})\cdots|X_{1:k+1})|X_{1:k}).$$

It now suffices to recall that Corollary 3.5.7$_{94}$ holds for $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, and therefore that the above inequality implies that $\overline{\mathrm{E}}(f|X_{1:k}) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|X_{1:k})$. Since s is a situation of length k, we have in particular that $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|s)$, and since this holds for any $(f,s) \in \mathbb{F} \times \mathscr{X}^*$, we have that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is always larger or equal than $\overline{\mathrm{E}}$ on $\mathbb{F} \times \mathscr{X}^*$. The upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, so we consequently have that $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for all $(f,s) \in \mathbb{F} \times \mathscr{X}^*$.

We next show that this inequality also holds on the domain $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. Fix any couple $(f,s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. By Corollary 6.2.4(vii)$_{294}$, there is a sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles that is uniformly bounded below and that converges pointwise to f such that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$. Since $\overline{\mathrm{E}}(g|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s)$ for all $g \in \mathbb{F}$ as we have just proved above, and since $\overline{\mathrm{E}}$ satisfies Co2$_{286}$, we have that

$$\overline{\mathrm{E}}(f|s) \leq \limsup_{n\to+\infty} \overline{\mathrm{E}}(f_n|s) \leq \limsup_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s).$$

Hence, we indeed have that $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for all $(f,s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. Finally, to see that this inequality also holds on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$, consider any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ and note that by Corollary 6.2.4(iv)$_{293}$,

$$\begin{aligned}\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) &= \inf\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \\ &\geq \inf\left\{\overline{\mathrm{E}}(g|s) \colon g \in \overline{\mathbb{L}}_{\mathrm{b}} \text{ and } g \geq f\right\} \geq \overline{\mathrm{E}}(f|s),\end{aligned}$$

where the second step follows from the fact that, as we have just proved above, $\overline{\mathrm{E}}(g|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s)$ for all $g \in \overline{\mathbb{L}}_{\mathrm{b}}$, and where the final step follows the fact that $\overline{\mathrm{E}}$ satisfies EC4$^{\Omega}_{286}$ by assumption. □

The following result establishes a characterisation of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ as being a natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ under coherence on $\mathbb{F} \times \mathscr{X}^*$ and Axioms Co2$_{286}$ and EC4$^{\Omega}_{286}$. One may check that coherence on $\mathbb{F} \times \mathscr{X}^*$ is a—strictly—stronger condition than NE2$_{88}$ and NE3$^{\leq}_{295}$,[4] and so the result can easily be seen to follow from Proposition 6.3.2$_{\curvearrowleft}$ above. The characterisation may nonetheless be convenient for those who consider coherence to be a basic requirement for upper expectations.

Corollary 6.3.3. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is the natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ under coherence on $\mathbb{F} \times \mathscr{X}^*$, Co2$_{286}$ and EC4$^{\Omega}_{286}$.*

Proof. Proposition 6.3.2$_{\curvearrowleft}$ says that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfies EC4$^{\Omega}_{286}$ and Co2$_{286}$. $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is also coherent on $\mathbb{F} \times \mathscr{X}^*$ because it extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ [by definition], and because $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is coherent by Corollary 3.4.5$_{87}$. Any other global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$, that is coherent on $\mathbb{F} \times \mathscr{X}^*$, and that satisfies EC4$^{\Omega}_{286}$ and Co2$_{286}$, also satisfies NE2$_{88}$ and NE3$^{\leq}_{295}$ due to Theorem 3.4.3$_{84}$

[4]Even when a global upper expectation extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ and satisfies NE2$_{88}$ and NE3$^{\leq}_{295}$, there is nothing that bounds its values from below on couples $(f, x_{1:n}) \in \mathbb{F} \times \mathscr{X}^*$ with f not $(n+1)$-measurable. We leave it as an exercise for the reader to check that this permits such a global upper expectation to violate coherence. On the other hand, due to Theorem 3.4.3$_{84}$ and Proposition 3.4.4$_{84}$, coherence implies NE2$_{88}$ and NE3$^{\leq}_{295}$.

and Proposition 3.4.4$_{84}$. Hence, by Proposition 6.3.2$_{297}$, $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. □

The next characterisation of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is based on continuity with respect to increasing sequences [Co4$_{295}$]; a property that is satisfied by upper integrals/expectations of all sorts—also upper integrals/expectations that we have not discussed in this dissertation; e.g. Choquet integrals [6, 28, 29]. We express it as a natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, because we want to emphasize the extension from $\mathbb{F} \times \mathscr{X}^*$ to $\overline{\mathbb{V}} \times \mathscr{X}^*$ through Co4$_{295}$ (and EC4$^{\Omega}{}_{286}$), but one may also express it as a natural extension of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ as we did in the previous two results.

Corollary 6.3.4. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is the natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ under* Co4$_{295}$ *and* EC4$^{\Omega}{}_{286}$.

Proof. By definition, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies EC4$^{\Omega}{}_{286}$. Due to Corollary 6.2.4(v)$_{294}$, $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ also satisfies Co4$_{295}$. Any other global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and that satisfies EC4$^{\Omega}{}_{286}$ and Co4$_{295}$, also satisfies Co2$_{286}$ due to Lemma 6.3.1$_{296}$. Hence, since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is by definition the largest extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ satisfying EC4$^{\Omega}{}_{286}$ and Co2$_{286}$, we indeed obtain that $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$. □

6.3.2 Full axiomatisation

What properties or axioms suffice in order for a global upper expectation to be equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$? Or, equivalently, what properties suffice in order for a global upper expectation to be equal to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and, for a large part, $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$? So far, our characterisations for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ always established that it is the most conservative upper expectation under some given set of properties. A full axiomatisation of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ without any conservativity arguments is still lacking at this point; we now provide such an axiomatisation, with the goal of proposing a series of axioms that is as weak and simple as possible—though the latter is admittedly sometimes somewhat of a subjective matter.

We start by axiomatising $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on the domain $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. Recall that Co3$_{295}$, though it may look abstract at first sight, simply imposes continuity with respect to some very specific increasing sequences [see e.g. Lemma 6.3.1$_{296}$].

Lemma 6.3.5. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, a global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$ if and only if it satisfies* NE1$_{88}$–NE3$_{88}$, Co3$_{295}$ *and* Co5$_{295}$.

Proof. First note that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ itself satisfies the axioms above. Indeed, Theorem 3.4.6$_{88}$ and the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ imply that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies NE1$_{88}$–NE3$_{88}$. Axiom Co5$_{295}$ follows from Corollary 6.2.4(vi)$_{294}$, and Axiom Co3$_{295}$ follows from the fact that

$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies Co4$_{295}$ by Corollary 6.3.4, and since Co4$_{295}$ implies Co3$_{295}$ by Lemma 6.3.1$_{296}$. Since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies the axioms above, a global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$ will also satisfy these axioms, because these axioms solely involve variables in $\overline{\mathbb{L}}_{\mathrm{b}}$. It is clear that this is the case for NE1$_{88}$–NE3$_{88}$ and Co5$_{295}$. It is also the case for Co3$_{295}$ because, for any converging sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below, $\inf_{n\geq m} f_n$ for any $m \in \mathbb{N}$ and $f := \lim_{n\to+\infty} f_n$ are both bounded below and limits of finitary gambles. This establishes necessity of NE1$_{88}$–NE3$_{88}$, Co3$_{295}$ and Co5$_{295}$.

To prove sufficiency, suppose that $\overline{\mathrm{E}}$ is any global upper expectation on $\overline{\mathbb{V}} \times \mathscr{X}^*$ satisfying NE1$_{88}$–NE3$_{88}$, Co5$_{295}$ and Co3$_{295}$. Then $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$ by Theorem 3.4.6$_{88}$, which by the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ [by definition] implies that $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $\mathbb{F} \times \mathscr{X}^*$. Now fix any $(f, s) \in \overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. According to Proposition 4.7.2$_{182}$, the variable f is the pointwise limit of a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ that is uniformly bounded below.

Let us first show that $\overline{\mathrm{E}}(\inf_{n\geq m} f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\inf_{n\geq m} f_n|s)$ for any $m \in \mathbb{N}$. For any $m \in \mathbb{N}$, let $(g^m_n)_{n\in\mathbb{N}}$ be the sequence in $\overline{\mathbb{V}}$ defined by $g^m_n := \inf_{n\geq \ell\geq m} f_\ell$ for all $n \geq m$, and $g^m_n := f_m$ for all $n < m$. Then it can easily be checked that, since $(f_n)_{n\in\mathbb{N}}$ is a sequence of finitary gambles, $(g^m_n)_{n\in\mathbb{N}}$ is a sequence of finitary gambles. Moreover, $(g^m_n)_{n\in\mathbb{N}}$ clearly converges decreasingly to $\inf_{n\geq m} f_n$, and $(g^m_n)_{n\in\mathbb{N}}$ is uniformly bounded below because $(f_n)_{n\in\mathbb{N}}$ is uniformly bounded below. Hence, since $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ both satisfy Co5$_{295}$, and since we already know $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ to coincide on $\mathbb{F} \times \mathscr{X}^*$, we obtain that

$$\overline{\mathrm{E}}\big(\inf_{n\geq m} f_n\big|s\big) = \lim_{n\to+\infty} \overline{\mathrm{E}}(g^m_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g^m_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}\big(\inf_{n\geq m} f_n\big|s\big).$$

Since the equality above holds for all $m \in \mathbb{N}$, and since $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfy Co3$_{295}$, it follows that

$$\overline{\mathrm{E}}(f|s) = \lim_{m\to+\infty} \overline{\mathrm{E}}\big(\inf_{n\geq m} f_n\big|s\big) = \lim_{m\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}\big(\inf_{n\geq m} f_n\big|s\big) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s).$$ □

To axiomatise $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$, we only need to add Axiom NE4$^{\mathrm{L}}_{295}$ to the list in Lemma 6.3.5; it simply says that the values of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ are obtained by approximating from above using the variables in the domain $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$.

Proposition 6.3.6. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is the unique global upper expectation satisfying* NE1$_{88}$–NE3$_{88}$*,* Co5$_{295}$*,* Co3$_{295}$ *and* NE4$^{\mathrm{L}}_{295}$*.*

Proof. By Lemma 6.3.5, we know that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies NE1$_{88}$–NE3$_{88}$, Co5$_{295}$ and Co3$_{295}$. That it satisfies NE4$^{\mathrm{L}}_{295}$ follows from Proposition 4.7.6$_{184}$ and Theorem 6.2.2$_{291}$. To prove the uniqueness of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, consider a second global upper expectation $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ satisfying the axioms above. Then Lemma 6.3.5 says that $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide on $\overline{\mathbb{L}}_{\mathrm{b}} \times \mathscr{X}^*$. The fact that they coincide on all of $\overline{\mathbb{V}} \times \mathscr{X}^*$ then follows immediately from the fact that they both satisfy NE4$^{\mathrm{L}}_{295}$. □

6.3.3 A direct formula

Though the axiomatic characterisations presented previously are attractive from a theoretical point of view, pragmatically oriented readers will still be on the lookout for some formula that allows them to derive the values of the upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ more directly. The expression for the game-theoretic upper expectation $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ in Definition 4.6$_{149}$ is in that sense convenient to work with, because it directly expresses $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(f|s)$ for any—general—couple $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ in terms of the allowable supermartingales $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Similarly, Proposition 3.5.10$_{97}$—or, even better, Corollary 3.5.12$_{98}$—shows that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}(f|s)$ for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ can simply be obtained by looking at the values of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on all finitary gambles that are larger or equal than f on $\Gamma(s)$.

The result below provides a similar practical formula for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$. It deliberately takes the values of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ as a starting point, because these can easily be obtained from the local models; one method to do so is to use the formula in Lemma 3.D.5$_{116}$. For particular types of finitary gambles, and particular types of trees, more efficient methods can be found in [58, 63, 100]. Furthermore, as we have already mentioned a few times [e.g. Section 4.7$_{180}$ and Section 5.5.4$_{257}$], we are often interested in limits of finitary gambles, or even more specifically, monotone limits of finitary gambles. In those cases, instead of using the formula below, it is more convenient to use the continuity properties Corollary 6.2.4(v)$_{294}$ and (vi)$_{294}$ in conjunction with the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$.

Proposition 6.3.7. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, we have that*

$$\begin{aligned}
&\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)\\
&\quad= \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) : g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \textit{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}\\
&\quad= \inf_{B\in\mathbb{R}} \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) : g_n \in \mathbb{F}, g_n \geq B \textit{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}.
\end{aligned}$$

Proof. It follows from Theorem 6.2.2$_{291}$ and Proposition 4.7.7$_{185}$, that, for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$,

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \inf\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g_n|s) : g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}.$$

Recalling that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ [by definition], we immediately obtain the first desired equality. The second equality follows trivially from the first, because

$$\begin{aligned}
&\left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) : g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}\\
&\qquad= \bigcup_{B\in\mathbb{R}} \left\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) : g_n \in \mathbb{F}, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\right\}. \qquad \square
\end{aligned}$$

6.4 The natural extension under a weaker continuity axiom

In this final section, we examine whether the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ as the most conservative extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ under $\mathrm{EC4}^{\Omega}{}_{286}$ and $\mathrm{Co2}_{286}$ can be further relaxed by replacing $\mathrm{Co2}_{286}$ with $\mathrm{Co1}_{286}$. In other words, we study whether $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$—if it exists—is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and, more generally, what the main characteristics of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ are. The upper expectation $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is particularly interesting because, as we will discuss in Section $6.4.2_{304}$, it can be seen as a imprecise-probabilistic generalisation of Daniell's upper integral.

6.4.1 Existence of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and a direct formula

We start by establishing the existence of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, and by giving a formula that is similar to—but more elegant than—the one given for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ in Proposition $6.3.7_{\curvearrowleft}$.

Proposition 6.4.1. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the natural extension $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ exists and, for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$,*

$$\begin{aligned}\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) &= \inf\Big\{\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq_s f\Big\}\\ &= \inf\Big\{\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\Big\}\\ &= \inf\Big\{\sup_{n\in\mathbb{N}} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F},\, g_n \leq g_{n+1},\, \sup_{n\in\mathbb{N}} g_n \geq f\Big\}.\end{aligned}$$

Proof. We start by proving the first equality. Let $\overline{\mathrm{E}}$ on $\overline{\mathbb{V}} \times \mathscr{X}^*$ be defined, for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$, by

$$\overline{\mathrm{E}}(f|s) := \inf\Big\{\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq_s f\Big\},$$

where the limit $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s)$ indeed exists for any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ because $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is monotone by Proposition $3.4.4_{84}$ [$\mathrm{WC5}_{84}$]. We show that $\overline{\mathrm{E}}$ is the most conservative—pointwise largest—global upper expectation on $\overline{\mathbb{V}} \times \mathscr{X}^*$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and satisfies $\mathrm{EC4}^{\Omega}{}_{286}$ and $\mathrm{Co1}_{286}$. By the definition of the natural extension, this will then automatically imply the existence of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and the equality of $\overline{\mathrm{E}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$.

Let us first check that $\overline{\mathrm{E}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$, or in other words that $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. That $\overline{\mathrm{E}}(f|s) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f|s)$ for any $(f,s) \in \mathbb{F} \times \mathscr{X}^*$ follows immediately from the definition of $\overline{\mathrm{E}}$; we can simply consider the (increasing) sequence in $\mathbb{F}$ that is equal to f for all indices. To prove the converse inequality, observe that, for any $(f,s) \in \mathbb{F} \times \mathscr{X}^*$,

$$\begin{aligned}\overline{\mathrm{E}}(f|s) &= \inf\Big\{\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq_s f\Big\}\\ &\geq \inf\Big\{\liminf_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, (\exists B \in \mathbb{R})\, g_n \geq B \text{ and } \lim_{n\to+\infty} g_n \geq_s f\Big\},\end{aligned}$$

because any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ is bounded below (by $\inf g_1$), and the corresponding limit $\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s) = \liminf_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s)$ always exists due to the monotonicity [WC5$_{84}$ in Proposition 3.4.4$_{84}$] of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$. Proposition 6.3.7$_{301}$ says that the right-hand side in the inequality above is equal to $\overline{\mathrm{E}}_{\mathrm{Q,A}}(f|s)$, so we obtain that $\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}_{\mathrm{Q,A}}(f|s)$. Since $\overline{\mathrm{E}}_{\mathrm{Q,A}}$ moreover extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$, we infer that $\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f|s)$ for all $(f,s)\in\mathbb{F}\times\mathscr{X}^*$.

Furthermore, that $\overline{\mathrm{E}}$ satisfies EC4$^{\Omega}$$_{286}$ follows straightforwardly from its definition. To see that it also satisfies Co1$_{286}$, consider any $s\in\mathscr{X}^*$ and any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$. Let $g := \sup_{n\in\mathbb{N}} g_n = \lim_{n\to+\infty} g_n$. Then we have that

$$\overline{\mathrm{E}}(g|s) \leq \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s) = \lim_{n\to+\infty}\overline{\mathrm{E}}(g_n|s) = \limsup_{n\to+\infty}\overline{\mathrm{E}}(g_n|s),$$

where the inequality follows from the definition of $\overline{\mathrm{E}}$, and the first equality follows from the fact that $\overline{\mathrm{E}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$. Hence, $\overline{\mathrm{E}}$ is a global upper expectation that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ and satisfies EC4$^{\Omega}$$_{286}$ and Co1$_{286}$.

To prove that $\overline{\mathrm{E}}$ is the largest such global upper expectation, consider any global upper expectation $\overline{\mathrm{E}}'$ that extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ and satisfies EC4$^{\Omega}$$_{286}$ and Co1$_{286}$. Fix any $(f,s)\in\overline{\mathbb{V}}\times\mathscr{X}^*$, and consider any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} g_n \geq_s f$. Let $(\tilde{g}_n)_{n\in\mathbb{N}}$ be the sequence defined by $\tilde{g}_n := g_n\mathbb{I}_{\Gamma(s)} + n\mathbb{I}_{\Gamma(s)^c}$ for all $n\in\mathbb{N}$. Then we clearly have that $(\tilde{g}_n)_{n\in\mathbb{N}}$ is an increasing sequence in $\mathbb{F}$ such that $\lim_{n\to+\infty}\tilde{g}_n = \lim_{n\to+\infty} g_n\mathbb{I}_{\Gamma(s)} + (+\infty)\mathbb{I}_{\Gamma(s)^c} \geq f$. Moreover,

$$\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s) = \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\tilde{g}_n|s) = \lim_{n\to+\infty}\overline{\mathrm{E}}'(\tilde{g}_n|s) \geq \overline{\mathrm{E}}'(\lim_{n\to+\infty}\tilde{g}_n|s) \geq \overline{\mathrm{E}}'(f|s),$$

where the first equality follows from NE2$_{88}$ in Theorem 3.4.6$_{88}$ and the fact that $\tilde{g}_n\mathbb{I}_{\Gamma(s)} = g_n\mathbb{I}_{\Gamma(s)}$ for all $n\in\mathbb{N}$, the first inequality from the fact that $\overline{\mathrm{E}}'$ satisfies Co1$_{286}$, and the last inequality from the monotonicity [EC4$^{\Omega}$$_{286}$] of $\overline{\mathrm{E}}'$. Since the inequality above holds for any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} g_n \geq_s f$, we obtain from the definition of $\overline{\mathrm{E}}$ that $\overline{\mathrm{E}}(f|s) \geq \overline{\mathrm{E}}'(f|s)$. Hence, $\overline{\mathrm{E}}$ is indeed the largest global upper expectation on $\overline{\mathbb{V}}\times\mathscr{X}^*$ that coincides with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ and that satisfies EC4$^{\Omega}$$_{286}$ and Co1$_{286}$. This establishes the first equality in the statement above.

To prove the second equality, note that obviously, for any $(f,s)\in\overline{\mathbb{V}}\times\mathscr{X}^*$,

$$\begin{aligned}\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f|s) &= \inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n \geq_s f\right\}\\ &\leq \inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\right\}.\end{aligned}$$

To establish the converse inequality, consider any increasing sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} h_n \geq_s f$. Let $(\tilde{h}_n)_{n\in\mathbb{N}}$ be defined by $\tilde{h}_n := h_n\mathbb{I}_{\Gamma(s)} + n\mathbb{I}_{\Gamma(s)^c}$ for all $n\in\mathbb{N}$. Then, since $\lim_{n\to+\infty} h_n \geq_s f$, we have that $\lim_{n\to+\infty}\tilde{h}_n = \lim_{n\to+\infty} h_n\mathbb{I}_{\Gamma(s)} + (+\infty)\mathbb{I}_{\Gamma(s)^c} \geq f$. Moreover, the sequence $(\tilde{h}_n)_{n\in\mathbb{N}}$ still is an increasing sequence of finitary gambles, so we get that

$$\begin{aligned}\inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\right\} &\leq \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\tilde{h}_n|s)\\ &= \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(h_n|s),\end{aligned}$$

where the last step follows from the fact that $\tilde{h}_n\mathbb{I}_{\Gamma(s)} = h_n\mathbb{I}_{\Gamma(s)}$ for all $n \in \mathbb{N}$, and the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ satisfies NE2$_{88}$. Since the above holds for any increasing sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} h_n \geq_s f$, we infer that

$$\inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n\geq f\right\}$$
$$\leq \inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n\geq_s f\right\} = \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s).$$

The final equality in the statement above now follows trivially from the increasing character of the sequences $(g_n)_{n\in\mathbb{N}}$, and the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is monotone by Proposition 3.4.4$_{84}$ [WC5$_{84}$]. □

The following result shows that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is additive with respect to real constants; a property that we will need later on.

Proposition 6.4.2. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, we have that* $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f+\mu|s) = \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)+\mu$ *for all* $(f,s)\in\overline{\mathbb{V}}\times\mathscr{X}^*$ *and* $\mu\in\mathbb{R}$.

Proof. Consider any $(f,s)\in\overline{\mathbb{V}}\times\mathscr{X}^*$ and $\mu\in\mathbb{R}$. According to Proposition 6.4.1$_{302}$, we have that

$$\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f+\mu|s) = \inf\left\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s)\colon g_n\in\mathbb{F}, g_n\leq g_{n+1}, \lim_{n\to+\infty} g_n\geq f+\mu\right\}.$$

Note that a sequence of global variables $(g_n)_{n\in\mathbb{N}}$ is an increasing sequence of finitary gambles such that $\lim_{n\to+\infty} g_n \geq f+\mu$, if and only if $(g_n-\mu)_{n\in\mathbb{N}}$ is an increasing sequence of finitary gambles such that $\lim_{n\to+\infty}(g_n-\mu)\geq f$. Moreover, by Proposition 3.4.4 [WC7$_{84}$], we have that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n-\mu|s)+\mu$ for any such a sequence $(g_n)_{n\in\mathbb{N}}$ and all $n\in\mathbb{N}$. Hence,

$$\begin{aligned}&\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f+\mu|s)\\&=\inf\Big\{\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n-\mu|s)+\mu\colon (g_n-\mu)\in\mathbb{F},\\&\qquad (g_n-\mu)\leq(g_{n+1}-\mu), \lim_{n\to+\infty}(g_n-\mu)\geq f\Big\} = \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)+\mu.\end{aligned}$$ □

6.4.2 Daniell-like upper expectations

One of the reasons why we find the extension $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ interesting is that it can be seen as an imprecise adaptation of Daniell's [19] method for extending a linear expectation. This method uses similar ideas as those that are used in standard measure theory to extend the domain of a measure, with the difference that linear expectations are now immediately considered to be the initial objects and that continuity arguments are directly applied to linear expectations rather than probability measures.

The classical Daniell extension

Concretely, Daniell's [19] method for extending a linear expectation consists of the following three steps, of which we will only sketch the essentials; we refer to [103, Chapter 6] and [38, Section 5.1] for more details.

(i) The initial object that we aim to extend ought to be an elementary integral I on a vector lattice $\mathscr{K}$. A **vector lattice** is a non-empty set of real-valued functions $f\colon \mathscr{Y} \to \mathbb{R}$ closed under pointwise addition, (finite) minima, (finite) maxima, and scaling with a real number; e.g. $\mathbb{F}$ is a vector lattice. An **elementary integral** $I\colon \mathscr{K} \to \mathbb{R}$ on a vector lattice $\mathscr{K}$ is a functional that is linear [103, (a), (b) on p.283], that takes non-negative values on (pointwise) non-negative functions [103, (c) on p.283], and that additionally satisfies [103, (d) on p.283]:

$$\lim_{n\to+\infty} I(f_n) = 0 \text{ if } (f_n)_{n\in\mathbb{N}} \text{ is decreasing in } \mathscr{K} \text{ and } \lim_{n\to+\infty} f_n = 0. \quad (6.3)$$

(ii) The integral I is extended to the domain of all over- and under-functions by imposing continuity. An **over-function** $f \in \mathscr{K}^{\mathrm{o}}$ is the (extended real-valued) pointwise limit of an increasing sequence of functions in $\mathscr{K}$; an **under-function** $f \in \mathscr{K}_{\mathrm{u}}$ is the (extended real-valued) pointwise limit of a decreasing sequence of functions in $\mathscr{K}$. The integral I is then defined, for any $f \in \mathscr{K}^{\mathrm{o}}$, by

$$I(f) \coloneqq \lim_{n\to+\infty} I(f_n) \text{ if } (f_n)_{n\in\mathbb{N}} \text{ is increasing in } \mathscr{K} \text{ and } \lim_{n\to+\infty} f_n = f,$$

and similarly for any under-function $f \in \mathscr{K}_{\mathrm{u}}$. It can be shown that, due to the assumptions about I and $\mathscr{K}$, this definition does not suffer from ambiguity.

(iii) The upper integral $\overline{I}(f)$ of any extended real-valued function $f\colon \mathscr{Y} \to \overline{\mathbb{R}}$ for which there is a $g \in \mathscr{K}^{\mathrm{o}}$ such that $g \geq f$, is now defined by the following upper approximation:

$$\overline{I}(f) \coloneqq \inf\left\{I(g)\colon g \in \mathscr{K}^{\mathrm{o}} \text{ and } g \geq f\right\}.$$

Analogously, the lower integral $\underline{I}(f)$ of any extended real-valued function f for which there is a $g \in \mathscr{K}_{\mathrm{u}}$ such that $g \leq f$, is defined by

$$\underline{I}(f) \coloneqq \sup\left\{I(g)\colon g \in \mathscr{K}_{\mathrm{u}} \text{ and } g \leq f\right\}.$$

If $\overline{I}(f)$ and $\underline{I}(f)$ are both defined, real-valued and coincide, then the common value $I(f) \coloneqq \overline{I}(f) = \underline{I}(f)$ is called the integral of f. It can again be proved that this definition of I is consistent with its earlier definition on $\mathscr{K}^{\mathrm{o}} \cup \mathscr{K}_{\mathrm{u}}$, and therefore that this new integral I extends the earlier one.

The measure-theoretic enthousiast may indeed spot great similarities between the reasoning above and how probability measures—sometimes called pre-measures—are usually extended; see e.g. [102, Section 1.7]. Daniell's [19] method can therefore be seen as the functional analysis analogon of the classical measure-theoretic approach. In our context, where (upper) expectations are the main objects of interest, this method is therefore much more natural and direct than any of the measure-theoretic procedures described in Chapter 5_{217}. Of course, since it only involves linear functionals, the applicability of this method is restricted to a precise context. The adaptation[5] to our imprecise setting, however, seems rather straightforward. As we will see, this adaptation furthermore results in the same operator as the axiomatic upper expectation $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ defined earlier. Let us attempt to make this clear.

An imprecise Daniell-like extension

In our context, we use $\mathbb{F}$ as our initial vector lattice, and, for any fixed $s \in \mathscr{X}^*$, we let $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\cdot|s) \colon \mathbb{F} \to \mathbb{R}$ be our 'elementary' upper expectation. We put 'elementary' between quotation marks because $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\cdot|s)$ satisfies all the characteristic properties of an elementary integral, apart from the fact that it is sublinear instead of linear [103, (a), (b) on p.283]; sublinearity follows from the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ satisfies $\mathrm{WC2}_{82}$ and $\mathrm{WC3}_{82}$ by definition; the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is non-negative on non-negative gambles [103, (c) on p.283] follows from Proposition 3.4.4[$\mathrm{WC6}_{84}$]; Eq. $(6.3)_{\curvearrowleft}$, finally, follows from Corollary 6.2.4$(\mathrm{vi})_{294}$, the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, and the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(0|s) = 0$ due to Proposition 3.4.4[$\mathrm{WC6}_{84}$].

Next, similar to step $(\mathrm{ii})_{\curvearrowleft}$, we extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\cdot|s)$ to all over-functions $\mathbb{F}^{\mathrm{o}}$ and under-functions $\mathbb{F}_{\mathrm{u}}$ by imposing continuity; that is, we let the extension $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(\cdot|s)$ on $\mathbb{F}^{\mathrm{o}} \cup \mathbb{F}_{\mathrm{u}}$ be defined by $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) \coloneqq \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f_n|s)$ for any $f \in \mathbb{F}^{\mathrm{o}}$ and any increasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} f_n = f$, and similarly for any $f \in \mathbb{F}_{\mathrm{u}}$ and any decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} f_n = f$. It can easily be checked, using Corollary 6.2.4$(\mathrm{v})_{294}$ and $(\mathrm{vi})_{294}$ and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, that this definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ does not suffer from ambiguity and that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ indeed extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. In fact, it follows from the same arguments that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $(\mathbb{F}^{\mathrm{o}} \cup \mathbb{F}_{\mathrm{u}}) \times \mathscr{X}^*$, and therefore by Corollary 6.2.4$(\mathrm{ii})_{293}$ that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ is monotone—we will use this property shortly. Furthermore, observe that the set of all real-valued over-functions is exactly the class of bounded below lower semicontinuous (l.s.c.) variables in $\mathbb{V}^{\mathrm{u}}$, and that the set of all real-valued under-functions is

[5]We do not call it a generalisation because we restrict our attention to the setting of discrete-time finite-state processes; in that respect, Daniell's [19] approach is much more general since it considers abstract possibility spaces, vector lattices and elementary integrals.

the class of bounded above upper semicontinuous (u.s.c.) variables in $\mathbb{V}^{\mathrm{u}}$; this follows from Lemma 5.5.2$_{251}$ (and the definition of an l.s.c. variable).

Finally, similar as in step (iii)$_{305}$, we extend $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(\cdot|s)$ one step further to the entire domain $\overline{\mathbb{V}}$, by using an outer approximation:

$$\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) := \inf\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(g|s) \colon g \in \mathbb{F}^{\mathrm{o}} \text{ and } g \geq f\right\} \text{ for all } f \in \overline{\mathbb{V}}.^{6}$$

The obtained upper expectation $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ is then an extension of the previous Daniell-like expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$.

Corollary 6.4.3. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, the upper expectation $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ extends $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$.*

Proof. Consider any $(f,s) \in (\mathbb{F}^{\mathrm{o}} \cup \mathbb{F}_{\mathrm{u}}) \times \mathscr{X}^{*}$. If $f \in \mathbb{F}^{\mathrm{o}}$, then clearly $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s)$ because $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(\cdot|s)$ is monotone as mentioned above. If $f \in \mathbb{F}_{\mathrm{u}}$, then the inequality that $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) \geq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s)$ follows once more from the monotonicity of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(\cdot|s)$. To prove the converse inequality, note that $f = \lim_{n\to+\infty} f_n = \inf_{n\in\mathbb{N}} f_n$ for some decreasing sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathbb{F} \subseteq \mathbb{F}^{\mathrm{o}}$ [because f is an under-function]. Hence, by the definition of $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and since $f_n \geq f$ for all $n \in \mathbb{N}$,

$$\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) \leq \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s),$$

where the equality (and also the existence of the limit) follows from the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$. □

The Daniell-like upper expectation $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ that we have defined above can now easily be seen to coincide with our axiomatic upper expectation $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^{*}$; indeed, it follows from the continuity of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ with respect to increasing sequences in $\mathbb{F}$—which itself follows immediately from how $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ was defined on $\mathbb{F}^{\mathrm{o}}$—and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, that, for any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^{*}$,

$$\begin{aligned}
\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) &= \inf\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(g|s) \colon g \in \mathbb{F}^{\mathrm{o}} \text{ and } g \geq f\right\} \\
&= \inf\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(\lim_{n\to+\infty} g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\right\} \\
&= \inf\left\{\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\right\} \\
&= \inf\left\{\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n|s) \colon g_n \in \mathbb{F}, g_n \leq g_{n+1}, \lim_{n\to+\infty} g_n \geq f\right\} \\
&= \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s),
\end{aligned}$$

where the last step follows from Proposition 6.4.1$_{302}$.

[6] In accordance with step (iii)$_{305}$, the infimum in this definition is always taken over a non-empty set; e.g. it is easy to see that the constant $g := +\infty$ is in $\mathbb{F}^{\mathrm{o}}$ and always satisfies $g \geq f$.

That $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincide ought not to surprise us as they are essentially built on the same principles: first adopting continuity with respect to increasing sequences in $\mathbb{F}$ (this corresponds to $\mathrm{Co1}_{286}$ or $\mathrm{Co1}^{=}{}_{286}$ in the case of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$) and then approximating from above (this corresponds to taking the most conservative extension or natural extension under $\mathrm{EC4}^{\Omega}{}_{286}$). This makes this way of extending $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ very elegant. Nevertheless, it remains to be seen how well the resulting common operator $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ performs in terms of continuity properties. The Daniell (outer/inner) integral is well-understood and known to satisfy many strong continuity properties in a precise context—similar to those of the Lebesgue integral with respect to a probability measure—but we are unaware of any existent results about an imprecise variant such as $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$. In Sections 6.4.3₃₁₀ and 6.4.4₃₁₄, we will investigate the properties of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and how it relates to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$.

A note about the Daniell lower extension

Before we continue to study the properties of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, we want to point out an interesting fact about the final step in our construction of $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$. We took the upper integral $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ as our object of interest, but said nothing about the lower integral $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ which approximates from the inside:

$$\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) \coloneqq \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}(g|s) \colon g \in \mathbb{F}_{\mathrm{u}} \text{ and } g \leq f\right\} \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^{*}.$$

The reason is that, as we will clarify shortly, $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ do not coincide in many cases, and then $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ clearly is the more intuitive choice when it comes to defining a global upper expectation operator. Yet, though its definition may be somewhat counter-intuitive, it actually turns out that on a large part of its domain, $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, and therefore also with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ [Theorem 6.2.2₂₉₁] and, actually, if $\mathscr{P}_t$ is closed for all $t \in \mathscr{X}^{*}$, then when $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide, $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ coincide as well [Theorem 6.2.3₂₉₂].

Proposition 6.4.4. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$ and any $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^{*}$ such that f is bounded below and $\sigma(\mathscr{X}^{*})$-measurable, or the limit of a decreasing sequence in $\mathbb{F}$, we have that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$.*

Proof. First recall from the discussion above that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on $(\mathbb{F}^{\mathrm{o}} \cup \mathbb{F}_{\mathrm{u}}) \times \mathscr{X}^{*}$; this followed from, on the one hand, the definition of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$ which starts from $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and assumes continuity with respect to monotone sequences of finitary gambles, and on the other hand, Corollary 6.2.4(v)₂₉₄ and (vi)₂₉₄ and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. Hence, we have that

$$\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) \colon g \in \mathbb{F}_{\mathrm{u}} \text{ and } g \leq f\right\} \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^{*}. \tag{6.4}$$

Next, note from the above equality and the monotonicity [$\mathrm{EC4}^{\Omega}{}_{286}$] of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on the under functions (and situations) $\mathbb{F}_{\mathrm{u}} \times \mathscr{X}^{*}$. Hence, since any limit $f \in \overline{\mathbb{V}}$ of a decreasing sequence in $\mathbb{F}$ is—by definition—in $\mathbb{F}_{\mathrm{u}}$, we have that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for any such f and any $s \in \mathscr{X}^*$.

So it remains to prove that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for any $f \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$ and any $s \in \mathscr{X}^*$. To this end, we will once more use Choquet's capacitability theorem [Theorem 5.5.9$_{255}$]. First observe that, since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincides with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ [Theorem 6.2.2$_{291}$], we have by Proposition 5.5.8$_{254}$ that the restriction of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\cdot|s)$ to $\overline{\mathbb{V}}_{\geq}$ is a capacity on Ω. Since f is bounded below and $\sigma(\mathscr{X}^*)$-measurable, the variable $f' := f - \inf f$ is non-negative and clearly still $\sigma(\mathscr{X}^*)$-measurable. Hence, Theorem 5.5.9$_{255}$ says that

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f'|s) = \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) : g \in \mathbb{V}^{\mathrm{u}}_{\geq}, g \text{ is u.s.c. and } g \leq f'\right\}.$$

We next show that, for any fixed $g \in \overline{\mathbb{V}}$, we have that $g \in \mathbb{V}^{\mathrm{u}}_{\geq}$ and g is u.s.c. if and only if $g \geq 0$ and $g \in \mathbb{F}_{\mathrm{u}}$. Suppose that $g \in \mathbb{V}^{\mathrm{u}}_{\geq}$ and g is u.s.c. Then by Lemma 5.5.7$_{254}$ g is bounded above (and clearly below) and thus by Lemma 5.5.2$_{251}$ the pointwise limit of a decreasing sequence $(g_n)_{n\in\mathbb{N}}$ of finitary gambles. So then we indeed have that $g \geq 0$ and $g \in \mathbb{F}_{\mathrm{u}}$. Conversely, suppose that $g \geq 0$ and $g \in \mathbb{F}_{\mathrm{u}}$. Then by Lemma 5.5.2$_{251}$ g is u.s.c. and bounded above, and thus also real-valued because g is non-negative. Hence, $g \in \mathbb{V}^{\mathrm{u}}_{\geq}$ and g is u.s.c. as desired.

So, by the equality above, we have that

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f'|s) = \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) : g \in \mathbb{F}_{\mathrm{u}}, g \geq 0 \text{ and } g \leq f'\right\} \leq \overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f'|s),$$

where the inequality follows from Eq. (6.4)$_{\leftarrow}$. To prove the converse inequality, consider any $g \in \mathbb{F}_{\mathrm{u}}$ such that $g \leq f'$. Since f' is non-negative, we also have that $g^+ \leq f'$ with $g^+ = g^{\vee 0}$. Moreover, since $g \in \mathbb{F}_{\mathrm{u}}$, we also have that $g^+ \in \mathbb{F}_{\mathrm{u}}$; indeed, if $(g_n)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathbb{F}$ such that $\lim_{n\to+\infty} g_n = g$, then we also have that $((g_n)^+)_{n\in\mathbb{N}}$ is a decreasing sequence in $\mathbb{F}$ such that $\lim_{n\to+\infty}(g_n)^+ = g^+$. Moreover, it follows from $g \leq g^+$ and the monotonicity of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g^+|s)$. Since this holds for any $g \in \mathbb{F}_{\mathrm{u}}$ such that $g \leq f'$, we indeed find that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f'|s) &= \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) : g \in \mathbb{F}_{\mathrm{u}} \text{ and } g \leq f'\right\} \\ &\leq \sup\left\{\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(g|s) : g \in \mathbb{F}_{\mathrm{u}}, g \geq 0 \text{ and } g \leq f'\right\} = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f'|s).\end{aligned}$$

So we conclude that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f'|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f'|s)$. It remains to show that this implies that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}(f|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$. Since $f' = f - \inf f$ with $\inf f \in \mathbb{R}$ [because f is assumed to be bounded below], it suffices to check that $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ are additive with respect to real constants. For $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, this follows from Corollary 6.2.4(ii)$_{293}$[EC5$_{163}$]; for $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$, this then follows from Eq. (6.4)$_{\leftarrow}$, the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is constant additive and the fact that, for any $g \in \overline{\mathbb{V}}$ and any $\mu \in \mathbb{R}$, we clearly have that $g \in \mathbb{F}_{\mathrm{u}}$ if and only if $g + \mu \in \mathbb{F}_{\mathrm{u}}$. □

Since $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on the domain of couples $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$ such that $f \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$ or f is the limit of a decreasing sequence in $\mathbb{F}$, it has the same desirable properties as $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ on this domain. Most practically relevant inferences are included in this domain (recall Section 5.5.4$_{257}$), and so, technically speaking, $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ is a suitable global upper expectation. However, since its definition is rather unconventional and not very intuitive, we are inclined to regard $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ mainly as a technical construct, which may operate as an alternative characterisation for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—or perhaps even $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$—in many cases, but not necessarily convincing as a definition of a global model for discrete-time stochastic processes.

Furthermore, our claim that $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ do not coincide in many cases is illustrated by two examples further on [Example 6.4.5 and Example 6.4.6$_{314}$]. They show that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$—which is equal to $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$—may already differ from $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ for bounded pointwise limits of finitary gambles, which since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincide on such limits (because, as clarified near the end of Example 6.4.6$_{314}$, such a limit is always in $\overline{\mathbb{V}}_{\sigma,\mathrm{b}}$), implies that $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ differ on such limits. Hence, the domain where the Daniell-like upper integral $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and Daniell-like lower integral $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincide, and thus where a common Daniell-like integral as in (iii)$_{305}$ can be defined, is in some cases too small to be practically relevant. Moreover, such a common—upper and lower—integral is in the classical 'precise' setting important because it is a linear operator, yet, in our case, even on the domain where $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ and $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$ coincide, neither of these operators need to be linear (since the upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ itself already isn't linear). Hence, there is no good reason to restrict our attention to such a common operator, instead of working with $\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}$ (or $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$).

6.4.3 $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ fails to satisfy a crucial continuity axiom

A first obvious question that one may pose about the nature of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is whether it satisfies the continuity property Co2$_{286}$; for if it did, then it would follow from the definitions of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ that both upper expectations are equal, and therefore that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ has the same—and desirable—characteristics as $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$. We would then preferably adopt $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ as the main characterisation of this common global upper expectation, simply because it relies on a weaker continuity argument; one that is moreover the same as Daniell's continuity argument in the precise case.

Unfortunately, and somewhat remarkably, this is not the case. This is shown by the following example.

Example 6.4.5. Consider the state space $\mathscr{X} := \{a, b, c\}$, and let $\mathscr{P}_\bullet$ be the imprecise probability tree where, for each $s \in \mathscr{X}^*$, $\mathscr{P}_s$ is the set of all probability mass functions p such that $p(b) = \alpha$ and $p(a) = p(c) = (1-\alpha)/2$ for some $0 \leq \alpha \leq 1$. The agreeing upper expectations tree $\overline{\mathrm{Q}}_\bullet$ is then described by Eq. (3.3)$_{51}$; for all $s \in \mathscr{X}^*$ and all $f \in \mathscr{L}(\mathscr{X})$, it is given by

$$\overline{\mathrm{Q}}_s(f) := \sup_{0 \leq \alpha \leq 1} \big[\alpha f(b) + (1-\alpha)(f(a) + f(c))/2\big].$$

The trees $\mathscr{P}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$ model the case where a subject has vacuous beliefs about whether the next state value will be b or an element of $\{a, c\}$, but where if she knows the latter is true she deems it equally likely that either a or c will be the next state value.

Now let us look at the values of $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}$ for the sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles defined, for all $n \in \mathbb{N}$, by

$$f_n(\omega) := \begin{cases} 1 & \text{if } \omega^k = b^{k-1}a \text{ for some } 1 \leq k \leq n; \\ 0 & \text{if } \omega^n = b^n; \\ -1 & \text{if } \omega^k = b^{k-1}c \text{ for some } 1 \leq k \leq n, \end{cases} \qquad \text{for all } \omega \in \Omega.$$

So f_n for any $n \in \mathbb{N}$ depends on the first n states of the process; it is equal to 1 if the first state different from b in this (finite) sequence is a; it is equal to -1 if the first state different from b in this (finite) sequence is c; and it is equal to 0 if all states in this sequence are b. So it is clear that $(f_n)_{n\in\mathbb{N}}$ is a sequence of finitary gambles—in fact, each f_n is n-measurable. Moreover, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to the global gamble f defined by

$$f(\omega) := \begin{cases} 1 & \text{if } \omega^k = b^{k-1}a \text{ for some } k \in \mathbb{N}; \\ 0 & \text{if } \omega = bbb\cdots; \\ -1 & \text{if } \omega^k = b^{k-1}c \text{ for some } k \in \mathbb{N}, \end{cases} \qquad \text{for all } \omega \in \Omega.$$

So $f(\omega)$ for any $\omega \in \Omega$ is equal to 1 if the first state different from b in ω is a; it is equal to -1 if the first state different from b in ω is c; and it is equal to 0 if all states in ω are b. We will show that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n) = 0$ and $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f) \geq 1/2$, thus establishing that Co2$_{286}$ does not hold for $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}$.

We first prove that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n) = 0$ for all $n \in \mathbb{N}$—and therefore that $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n) = 0$. Fix any $n \in \mathbb{N}$ and note that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n)$ due to the definition of $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}$ and the fact that f_n is a finitary gamble. So it suffices to show that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n) = 0$.

Start by observing that, due to Proposition 3.5.9$_{96}$ and because f_n is n-measurable,

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-1}) = \overline{\mathrm{Q}}_{b^{n-1}}(f_n(b^{n-1}\cdot)) &= \overline{\mathrm{Q}}_{b^{n-1}}(\mathbb{I}_a - \mathbb{I}_c) \\ &= \sup_{0\leq\alpha\leq1} \left[\alpha\, 0 + (1-\alpha)(1-1)/2\right] = 0.\end{aligned}$$

Furthermore, since f_n is constant and equal to 1 on the cylinder set $\Gamma(b^{n-2}a)$, we have by Theorem 3.4.6 [NE2$_{88}$] and Proposition 3.4.4$_{84}$ [WC6$_{84}$] that

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-2}a) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n \mathbb{1}_{b^{n-2}a}|b^{n-2}a) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(1\mathbb{1}_{b^{n-2}a}|b^{n-2}a) = 1.$$

In an analogous way, we can deduce that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-2}c) = -1$. Hence, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-2}\cdot)$ is equal to the local gamble $\mathbb{I}_a - \mathbb{I}_c \in \mathscr{L}(\mathscr{X})$, and so it follows in a similar way as before—note that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|X_{1:n-1})$ is $(n-1)$-measurable and a gamble because of Proposition 3.4.4$_{84}$ [WC6$_{84}$]—that

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|X_{1:n-1})|b^{n-2}) = \overline{\mathrm{Q}}_{b^{n-2}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-2}\cdot)) = \overline{\mathrm{Q}}_{b^{n-2}}(\mathbb{I}_a - \mathbb{I}_c) = 0.$$

Using the law of iterated upper expectations for $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ [Corollary 3.5.7$_{94}$], we obtain that

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-2}) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|X_{1:n-1})|b^{n-2}) = 0.$$

We can now simply repeat this entire reasoning; we have that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-3}a) = 1$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-3}c) = -1$ because of Theorem 3.4.6 [NE2$_{88}$] and Proposition 3.4.4$_{84}$ [WC6$_{84}$], and therefore that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-3}\cdot) = \mathbb{I}_a - \mathbb{I}_c$. This again implies that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-3}) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|X_{1:n-2})|b^{n-3}) &= \overline{\mathrm{Q}}_{b^{n-3}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^{n-3}\cdot)) \\ &= \overline{\mathrm{Q}}_{b^{n-3}}(\mathbb{I}_a - \mathbb{I}_c) = 0.\end{aligned}$$

Applying this reasoning over and over again, eventually yields that indeed

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|b^0) = 0.$$

To show that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f) \geq 1/2$, we use the formula in Proposition 6.4.1$_{302}$. Fix any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} g_n \geq f$, and any $\epsilon > 0$. Since $\lim_{n\to+\infty} g_n(bbb\cdots) \geq f(bbb\cdots) = 0$, there is an $\tilde{n} \in \mathbb{N}$ such that $g_{\tilde{n}}(bbb\cdots) \geq -\epsilon$. The gamble $g_{\tilde{n}}$ is finitary and therefore m-measurable for some $m \in \mathbb{N}$, so this implies that $g_{\tilde{n}}(\omega) \geq -\epsilon$ for all $\omega \in \Gamma(b^m)$. Since $(g_n)_{n\in\mathbb{N}}$ is increasing, we also have that $g_n(\omega) \geq -\epsilon$ for all $n \geq \tilde{n}$ and all $\omega \in \Gamma(b^m)$.

Let us now focus on the values that the gambles $(g_n)_{n>\tilde{n}}$ take on the cylinder event $\Gamma(b^m a)$. Since f is equal to the constant 1 on this entire cylinder event, we know that $(g_n(\omega))_{n>\tilde{n}}$ for any $\omega \in \Gamma(b^m a)$ converges to a value larger than or equal to 1. Let $A_n := \{\omega \in \Gamma(b^m a) \colon g_n(\omega) < 1-\epsilon\}$ for all $n \in \mathbb{N}$ such that $n > \tilde{n}$. Then, by what we have previously said, $\lim_{n\to+\infty} A_n = \emptyset$. We now show that in fact $A_{n^*} = \emptyset$ for some finite $n^* > \tilde{n}$.

Observe that $(A_n)_{n>\tilde{n}}$ is a decreasing sequence of events because $(g_n)_{n>\tilde{n}}$ is increasing. Moreover, each A_n is a finite union of cylinder events; indeed, this follows from the fact that each g_n is finitary and the finiteness of $\mathscr{X}$. As a result, since $\lim_{n\to+\infty} A_n$ is empty, we infer by Lemma 4.C.2$_{209}$ that A_{n^*} must be empty for at least one $n^* > \tilde{n}$ [and consequently also for all n larger than n^*].

Since A_{n^*} is empty, we have that $g_{n^*}(\omega) \geq 1-\epsilon$ for all $\omega \in \Gamma(b^m a)$, and therefore that $g_{n^*}\mathbb{1}_{b^m a} \geq (1-\epsilon)\mathbb{1}_{b^m a}$. On the other hand, since $n^* > \tilde{n}$, we also know from before that $g_{n^*}(\omega) \geq -\epsilon$ for all $\omega \in \Gamma(b^m)$. In particular, we have that $g_{n^*}(\mathbb{1}_{b^m b} + \mathbb{1}_{b^m c}) \geq -\epsilon(\mathbb{1}_{b^m b} + \mathbb{1}_{b^m c})$. So we find by adding these two inequalities that

$$g_{n^*}\mathbb{1}_{b^m} \geq (1-\epsilon)\mathbb{1}_{b^m a} - \epsilon(\mathbb{1}_{b^m b} + \mathbb{1}_{b^m c}). \tag{6.5}$$

The variable on the right-hand side is an $(m+1)$-measurable gamble, and therefore Proposition 3.5.9$_{96}$ guarantees that

$$\begin{aligned}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}\big((1-\epsilon)\mathbb{1}_{b^m a} - \epsilon(\mathbb{1}_{b^m b} + \mathbb{1}_{b^m c})\big|\, b^m\big) &= \overline{\mathrm{Q}}_{b^m}\big((1-\epsilon)\mathbb{I}_a - \epsilon(\mathbb{I}_b + \mathbb{I}_c)\big)\\ &= \sup_{0\leq\alpha\leq 1}\big[\alpha\,(-\epsilon) + (1-\alpha)(1-\epsilon-\epsilon)/2\big]\\ &= \sup_{0\leq\alpha\leq 1}\big[-\alpha/2 + 1/2 - \epsilon\big] = 1/2 - \epsilon.\end{aligned}$$

Hence, by Eq. (6.5), Proposition 3.4.4$_{84}$ [WC5$_{84}$] and Theorem 3.4.6 [NE2$_{88}$], we have that

$$1/2 - \epsilon \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}\mathbb{1}_{b^m}|b^m) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^m). \tag{6.6}$$

Next, note that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^m) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*})$. Indeed, using the law of iterated upper expectations [Corollary 3.5.7$_{94}$] and Lemma 3.D.4$_{116}$—which we can apply due to Proposition 3.4.4$_{84}$ [WC11$_{85}$] and the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ extends $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ by definition—we have for any $0 \leq i \leq m-1$ that

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^i) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|X_{1:i+1})|b^i) = \overline{\mathrm{Q}}_{b^i}\big(\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^i\cdot)\big) \geq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^{i+1}),$$

where the last inequality follows from the definition of $\overline{\mathrm{Q}}_{b^i}$ [simply consider the case where $\alpha = 1$]. Since this holds for all $0 \leq i \leq m-1$, we obtain that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^m) \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*}|b^0) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*})$. Combining this with Eq. (6.6), we infer that $1/2 - \epsilon \leq \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_{n^*})$.

The rest of the proof is now straightforward: Since $(g_n)_{n\in\mathbb{N}}$ is increasing, and since $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ is monotone by Proposition 3.4.4$_{84}$ [WC5$_{84}$], the previous inequality implies that $1/2 - \epsilon \leq \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n)$. This holds for any $\epsilon > 0$, so we get that $1/2 \leq \lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(g_n)$. This holds for any increasing sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathbb{F}$ such that $\lim_{n\to+\infty} g_n \geq f$, so by the formula in Proposition 6.4.1$_{302}$ we obtain that $1/2 \leq \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f)$.

So we conclude that $\lim_{n\to+\infty}\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n) = 0$ but that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f) \geq 1/2$, and therefore that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ does not satisfy Co2$_{286}$. ◇

As already argued in Section 6.1$_{285}$, we believe Axiom Co2$_{286}$ to be a desirable property for global upper expectations to have. Since $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ does not satisfy this axiom, we regard this global upper expectation as somewhat inadequate. Our belief grows even stronger if we compare the behaviour of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ to that of the three other main upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ (or $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$

for any agreeing tree $\mathscr{A}_\bullet$) and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ in the example above; these three global models all return the value 0 for the gamble f.

Example 6.4.6. Reconsider the upper expectations tree $\overline{\mathrm{Q}}_\bullet$ and the variables $(f_n)_{n\in\mathbb{N}}$ and f from Example 6.4.5$_{310}$. Recall that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(f_n) = 0$ for all $n \in \mathbb{N}$. Since $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, this implies that also $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n) = 0$ for all $n \in \mathbb{N}$, and therefore that $\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n) = 0$. Since $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{F}$ that is clearly uniformly bounded below (by -1), Co2$_{286}$ implies that

$$0 = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n) \geq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f). \tag{6.7}$$

By conjugacy [Eq. (6.1)$_{289}$], this yields $0 \leq \underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(-f)$, which by EC1$_{163}$ in Corollary 6.2.4(ii)$_{293}$ in turn implies that $0 \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(-f)$. It follows from symmetry considerations of the tree $\overline{\mathrm{Q}}_\bullet$ and the variable f, that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(-f) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f)$. As a result, we obtain that $0 \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f)$, which together with Eq. (6.7) allows us to infer that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f) = 0$. So we conclude that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n) = 0 = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f),$$

and therefore that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is continuous with respect to the sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles.

To see that the same is true for the upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$—for any agreeing acceptable gambles tree $\mathscr{A}_\bullet$—it suffices to use Theorem 6.2.2$_{291}$ and Theorem 4.3.6$_{161}$. Finally, that it holds for $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ for any agreeing imprecise probability tree $\mathscr{P}_\bullet$ can be deduced from Theorem 6.2.3$_{292}$, and the fact that f is in $\overline{\mathbb{V}}_{\sigma,\mathrm{b}}$. Indeed, f is clearly bounded below, and it is moreover $\sigma(\mathscr{X}^*)$-measurable because each finitary gamble f_n is $\sigma(\mathscr{X}^*)$-measurable [since the level sets $\{\omega \in \Omega\colon f(\omega) \leq c\}$ for all $c \in \mathbb{R}$ are finite unions of cylinder events] and because of MV2$_{228}$. ◇

Another reason why one may not want to use $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ as a global model, is the fact that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ does not satisfy Co4$_{295}$ or any other stronger form of continuity from below (with respect to increasing sequences). Indeed, for if it did, then, since $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is monotone [EC4$^{\Omega}_{286}$] by definition, $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ would satisfy Co2$_{286}$ due to Lemma 6.3.1$_{296}$. But this is impossible as we just shown above. Though we consider Co4$_{295}$ or any other stronger form of continuity from below to be less intuitive and compelling than Co2$_{286}$, one might still want to impose it simply because it is a property common to all sorts of (upper) expectations.

6.4.4 $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ as a suitable alternative for $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ in three special cases

Though $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ does not really qualify as a suitable global model in general contexts, there are still three particular instances where $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ has all the nice features that we would like it to have; (i) if the global variable of interest is the limit of a monotone sequence of finitary gambles, (ii) in a precise context where the local dynamics are described by linear expectations trees and/or precise probability trees, (iii) and in a context where we are solely interested in global upper (and lower) probabilities rather than global upper (and lower) expectations. It turns out that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equivalent to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ in these three special cases, and therefore that it is then also equivalent to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and, for a large part, to $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$. Moreover, it follows trivially from the definitions of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ that, even when $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ does not coincide with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, it still provides a conservative bound for the latter.

Corollary 6.4.7. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, we have that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) \geq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$ for all $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.*

Proof. This follows immediately from the fact that Co2$_{286}$ is stronger than Co1$_{286}$, and Definition 6.1$_{289}$ and Definition 6.2$_{289}$. □

An equality for monotone limits of finitary gambles

Let us start by establishing that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide on all pointwise limits of monotone (increasing or decreasing) sequences of finitary gambles. As already mentioned in Section 4.7$_{180}$ and 5.5.4$_{257}$, these limits already make up a fair deal of all the variables that are relevant for practical purposes. Any hitting time, for instance, can be written as the limit of an increasing sequence of 'stopped' hitting times; see Example 4.2.2$_{140}$ for a case where we consider the hitting time τ_a of a single state $a \in \mathscr{X}$.

Corollary 6.4.8. *For any upper expectations tree $\overline{\mathrm{Q}}_{\bullet}$, any $s \in \mathscr{X}^*$ and any monotone sequence $(f_n)_{n\in\mathbb{N}}$ of finitary gambles that converges to some $f \in \overline{\mathbb{V}}$,*

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) \text{ and } \underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) = \underline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s)$$

Proof. $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide on $\mathbb{F} \times \mathscr{X}^*$ because they are both equal to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ by definition. Then, if the sequence of finitary gambles $(f_n)_{n\in\mathbb{N}}$ is increasing, the desired equality follows from Corollary 6.2.4(v)$_{294}$ and the continuity of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ with respect to $(f_n)_{n\in\mathbb{N}}$ due to Co1$^{=}$$_{286}$ [because it satisfies Co1$_{286}$ and EC4$^{\Omega}$$_{286}$ by definition]. On the other hand, if $(f_n)_{n\in\mathbb{N}}$ is decreasing, then $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is continuous with respect to $(f_n)_{n\in\mathbb{N}}$ due to Corollary 6.2.4(vi)$_{294}$ [since gambles are always bounded above]. Then, because $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is always equal to or larger than $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ due to Corollary 6.4.7,

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) \leq \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s) \leq \lim_{n\to+\infty} \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f_n|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(f|s),$$

where the second inequality follows from EC4$^{\Omega}$$_{286}$ and the decreasing character of $(f_n)_{n\in\mathbb{N}}$, and where the first equality follows from the fact that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ are both by definition equal to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$. The equality between the lower expectations then follows from conjugacy [Eq. (6.1)$_{289}$]. □

Note in particular that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ satisfies, apart from continuity with respect to increasing sequences of finitary gambles [Co1$^{=}_{286}$], also continuity with respect to **decreasing** sequences of finitary gambles; this is due to the proof above, or by Corollary 6.4.8 and Corollary 6.2.4(vi)$_{294}$.

An equality for the precise setting

To show that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincides with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ for precise local models, it suffices to simply check that $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equal to the traditional 'precise' Daniell integral as described in Section 6.4.2(i)$_{305}$–(iii)$_{305}$, with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\cdot|s)$ now fulfilling the role of an actual—linear—elementary integral (and not merely a sublinear one as in the discussion below Section 6.4.2(i)$_{305}$–(iii)$_{305}$). The desired equality then follows immediately from the fact that the standard Daniell integral satisfies continuity properties similar to those of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$. We concretize this reasoning in the proof of the following lemma.

Lemma 6.4.9. *For any precise probability tree p and the agreeing (upper) expectations tree* $\mathrm{Q}_{\bullet}$ *according to Eq.* (3.4)$_{52}$, $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ *satisfies* Co4$_{295}$.

Proof. Let us check that our upper expectation $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ for any $s \in \mathscr{X}^*$ is equal to the 'outer Daniell extension' $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}}$ from [38, Section 5.1.3], with P a specific countably additive probability charge [Definition 5.1$_{221}$]. As we will see, the desired result for $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ will then follow from the 'monotone convergence theorem' [38, Theorem 5.10]. Though the monotone convergence theorem for the Daniell integral is well-established, and can also be found in earlier textbooks—e.g. [103, Chapter 6]—we choose to base ourselves on the work in [38, Section 5.1] because it is adapted to also allow extended real-valued expectations rather than only real-valued ones.

The construction of $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}}$ in [38, Section 5.1] relies on a countably additive (unconditional) probability charge P on an algebra (or field) of events $\mathcal{F}$. It follows from [38, Lemma C.3] and Definition 5.1$_{221}$ that the—somewhat unconventional—definition of countable additivity in [38, Section 5.1] is equivalent to countable additivity in our—traditional—sense. The elementary integral in [38, Section 5.1] is denoted by E_{P}, and is defined by [38, Eq. (2.19)]; that is, for all $\mathcal{F}$-simple gambles,

$$\mathrm{E}_{\mathrm{P}}(f) := \sum_{i=1}^{n} a_i \mathrm{P}(A_i), \tag{6.8}$$

with $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$ any representation of f [recall Section 3.3.3$_{74}$]. A reasoning entirely the same as in Section 6.4.2(i)$_{305}$–(iii)$_{305}$ is then further followed to define the upper Daniell integral—or outer Daniell extension—$\mathrm{E}^{\mathrm{o}}_{\mathrm{P}}$ in [38, Section 5.1].

For any arbitrary but fixed $s \in \mathscr{X}^*$, consider the global probability charge P_p on $\langle \mathscr{X}^* \rangle \times \mathscr{X}^*$ that satisfies Eq. (3.12)$_{72}$—according to Proposition 3.3.4$_{73}$ this global probability charge exists and is unique. The unconditional probability charge $\mathrm{P}_p(\cdot|s)$ on $\langle \mathscr{X}^* \rangle$ is then countably additive according to [5, Theorem 2.3]. So we can apply the procedure above to define the corresponding elementary integral $\mathrm{E}_{\mathrm{P}_p}(\cdot|s)$ [according to Eq. (6.8)] on all $\langle \mathscr{X}^* \rangle$-simple gambles, and subsequently the outer Daniell extension $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ [according to Section 6.4.2(i)$_{305}$–(iii)$_{305}$] on all global variables $f \in \overline{\mathbb{V}}$. Then $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ satisfies the continuity described in [38, Theorem 5.10], because this continuity solely concerns 'D-integrable' variables and the 'Daniell expectation $\mathrm{E}^{\mathrm{D}}_{\mathrm{P}}$' in [38, Theorem 5.10] is simply a restriction of the outer Daniell extension to 'D-integrable' variables.

We next show that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ is equal to $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$, and therefore that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ also satisfies the continuity in [38, Theorem 5.10]. Let $\mathrm{E}_p(\cdot|s)$ be defined from P_p according to Definition 3.4$_{77}$. Then its restriction to $\mathbb{F}$ coincides with the elementary integral $\mathrm{E}_{\mathrm{P}_p}(\cdot|s)$ (on $\mathbb{F}$): indeed, the $\langle\mathscr{X}^*\rangle$-simple gambles are equal to the finitary gambles $\mathbb{F}$ [Lemma 3.3.5$_{75}$], and it is clear from Definition 3.4$_{77}$ and Proposition 3.3.6(i)$_{76}$ that $\overline{\mathrm{E}}_p(\cdot|s)$ on $\mathbb{F}$ is deduced from $\mathrm{P}_p(\cdot|s)$ in agreement with Eq. (6.8)$_{\leftarrow}$, and thus in the same way as how $\mathrm{E}_{\mathrm{P}_p}(\cdot|s)$ was deduced from $\mathrm{P}_p(\cdot|s)$. By Corollary 3.5.3$_{92}$, and since $\mathrm{Q}_\bullet$ agrees with p according to Eq. (3.4)$_{52}$ (or Eq. (3.3)$_{51}$), $\mathrm{E}_p(\cdot|s)$ is also equal to $\overline{\mathrm{E}}_{\mathrm{Q}}(\cdot|s)$, and thus by Corollary 3.4.7$_{89}$ equal to $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\cdot|s)$ on $\mathbb{F}$, so we infer that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(\cdot|s)$ is equal to $\mathrm{E}_{\mathrm{P}_p}(\cdot|s)$. Since $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ is the outer Daniell extension of $\mathrm{E}_{\mathrm{P}_p}(\cdot|s)$ [according to Section 6.4.2(i)$_{305}$–(iii)$_{305}$] and since, as we have already shown, $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ coincides with the outer Daniell extension—or upper Daniell integral—$\overline{\mathrm{E}}^{\mathrm{o}}_{\mathrm{Q},\mathrm{D}}(\cdot|s)$ deduced from $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ according to Section 6.4.2(i)$_{305}$–(iii)$_{305}$, we obtain that $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ and $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ are equal.

Hence, for any $s \in \mathscr{X}^*$, since $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ satisfies the continuity in [38, Theorem 5.10], we have that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$ satisfies the continuity in [38, Theorem 5.10]. Axiom Co4$_{295}$ then follows as a special case, because all variables in $\overline{\mathbb{L}}_{\mathrm{b}}$ are 'D-integrable' according to Proposition 4.7.2$_{182}$ and [38, Theorem 5.12], and because $\mathrm{E}^{\mathrm{o}}_{\mathrm{P}_p}(\cdot|s)$ [and hence, $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(\cdot|s)$] can never take the value $-\infty$ for a D-integrable variable that is bounded below due to [38, Theorem 5.9 (DE3)]. □

Theorem 6.4.10. *Consider any (upper) expectations tree $\mathrm{Q}_\bullet$ for which there is a precise probability tree p such that $\mathrm{Q}_\bullet$ is equal to the agreeing tree $\mathrm{Q}_{\bullet,p}$ defined by Eq.* (3.4)$_{52}$*. Then we have that*

$$\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}(f|s) = \overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}(f|s) \text{ for all } (f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*.$$

Proof. Since $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ satisfies Co4$_{295}$ by Lemma 6.4.9$_{\leftarrow}$, and moreover extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ and satisfies EC4$^{\Omega}_{286}$ by definition, it follows from Corollary 6.3.4 that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ is always smaller than or equal to $\overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}$. Hence, combined with Corollary 6.4.7$_{315}$, we obtain that $\overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ are equal. □

The fact that $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ coincides with $\overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}$ (in the precise case) guarantees that it possesses all the same features as $\overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}$, and thus also that it coincides with $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q},\mathrm{V}}$ and $\overline{\mathrm{E}}_{p,\mathrm{M}}$ (for an agreeing precise tree p); see (i)$_{\curvearrowright}$ below. So in particular $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q},\mathrm{A}}$ satisfies the properties from Corollary 6.2.4$_{293}$—which we will not present separately for the sake of brevity—and, as a result of its equality with $\overline{\mathrm{E}}_{p,\mathrm{M}}$, it satisfies some additional strong continuity properties which we list next.

Corollary 6.4.11. *For any precise probability tree p and the agreeing (upper) expectations tree* $\mathrm{Q}_\bullet$ *according to Eq.* $(3.4)_{52}$*, the following statements hold:*

(i) $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f|s) = \overline{\mathrm{E}}^{\mathrm{eb}}_{\mathrm{Q,V}}(f|s) = \overline{\mathrm{E}}_{p,\mathrm{M}}(f|s)$ *for all* $(f,s) \in \overline{\mathbb{V}} \times \mathscr{X}^*$.

(ii) $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f|s) = \underline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f|s)$ *for all bounded below or above* $f \in \overline{\mathbb{V}}_\sigma$ *and all* $s \in \mathscr{X}^*$. *[being precise]*

(iii) $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(af + bg|s) = a\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f|s) + b\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(g|s)$ *for all* $f \in \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$*, all* $g \in \mathbb{V}_\sigma$*,* $s \in \mathscr{X}^*$ *and* $a, b \in \mathbb{R}$. *[linearity]*

(iv) *Consider any* $s \in \mathscr{X}^*$ *and any* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$ *that converges pointwise to a variable* $f \in \overline{\mathbb{V}}_\sigma$*. If there is a* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $|f_n| \leq f^*$ *for all* $n \in \mathbb{N}$ *and* $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f^*|s) \leq +\infty$*, then* $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n) = \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f)$.
[dominated convergence]

(v) *Consider any* $s \in \mathscr{X}^*$ *and any decreasing sequence* $(f_n)_{n\in\mathbb{N}}$ *in* $\overline{\mathbb{V}}_\sigma$*. If there is a* $f^* \in \overline{\mathbb{V}}_\sigma$ *such that* $\overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f^*|s) < +\infty$ *and* $f_1 \leq f^*$*, then* $\lim_{n\to+\infty} \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(f_n|s) = \overline{\mathrm{E}}^{\nearrow}_{\mathrm{Q,A}}(\lim_{n\to+\infty} f_n|s)$. *[monotone convergence]*

Proof. Property (i) follows from Theorem 6.4.10$_{\curvearrowleft}$, Theorem 6.2.2$_{291}$ and Theorem 5.3.1$_{235}$. Properties (ii)–(v) follow from (i) and Corollary 5.3.4$_{239}$. □

An equality for upper and lower probabilities

Another, perhaps surprising instance where $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equal to $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ occurs when we consider general imprecise local models but restrict our attention to global indicators. Or in other words, if we only look at the upper and lower probabilities associated with $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$. Recall from Section 3.1.3$_{52}$ that, for any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, the (global) upper probability $\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and (global) lower probability $\underline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}$ associated with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ are defined by

$$\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) := \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) \text{ and } \underline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) := \underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s),$$

for all $A \subseteq \Omega$ and $s \in \mathscr{X}^*$. The upper and lower probabilities $\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ are defined similarly.

Theorem 6.4.12. *For any upper expectations tree* $\overline{\mathrm{Q}}_\bullet$*, any* $A \subseteq \Omega$ *and any* $s \in \mathscr{X}^*$*, we have that*

$$\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) \textit{ and } \underline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s)$$

The theorem above can be deduced straightforwardly from the following lemma, which is expressed in terms of game-theoretic upper expectations.

Lemma 6.4.13. *For any upper expectations tree $\overline{\mathrm{Q}}_\bullet$, any $A \subseteq \Omega$, any $t \in \mathscr{X}^*$, and any $\epsilon > 0$, there is a countable collection S of pairwise incomparable situations—a tree cut—such that*

$$A \cap \Gamma(t) \subseteq \bigcup_{s \in S} \Gamma(s) \ \textit{and} \ \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\textstyle\sum_{s\in S} \mathbb{1}_s|t) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{1}_A|t) + \epsilon.$$

Proof. Since $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{1}_A|t)$ is real by Proposition 4.4.3 (EC1$_{163}$), it follows the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ that there is a bounded below supermartingale $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$ such that $\mathscr{M}(t) \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{1}_A|t) + \epsilon/2$ and $\liminf \mathscr{M} \geq_t \mathbb{1}_A$. So, for all $\omega \in A \cap \Gamma(t)$, there is a $k_\omega \geq |t|$ such that $\mathscr{M}(\omega^k) \geq 1 - \epsilon/2$ for all $k \geq k_\omega$. Let S' be the set of situations $s' \sqsupseteq t$ such that $s' = \omega^{k_\omega}$ for some $\omega \in A \cap \Gamma(t)$, and let S be defined by

$$S := \{s \in S' \colon (\nexists s' \in S')\, s' \sqsubset s\}.$$

Then note that $\bigcup_{s'\in S'} \Gamma(s) = \bigcup_{s\in S} \Gamma(s)$. Indeed, for any $s' \in S' \setminus S$, there is at least one $\tilde{s} \in S'$ such that $\tilde{s} \sqsubset s'$. Let $\tilde{s}_1 \in S'$ be such a situation with minimal length; then there are no situations $\tilde{s} \in S'$ such that $\tilde{s} \sqsubset \tilde{s}_1$. Hence, we must have that $\tilde{s}_1 \in S$. Moreover, $\Gamma(\tilde{s}_1) \supset \Gamma(s')$ because $\tilde{s}_1 \sqsubset s'$. Since this holds for any $s' \in S' \setminus S$ [and since clearly $S' \supseteq S$] we thus have that

$$\bigcup_{s'\in S'} \Gamma(s) = \bigcup_{s\in S} \Gamma(s). \tag{6.9}$$

Furthermore, note that the situations in S are pairwise incomparable: for any two situations $s_1, s_2 \in S$, $s_1 \sqsubset s_2$ is impossible, because if this would hold then $s_1, s_2 \in S'$ and $s_1 \sqsubset s_2$, implying that s_2 cannot be in S. Moreover, recall that the situations $\mathscr{X}^* = \cup_{i\in\mathbb{N}} \mathscr{X}^i$ are countable because the state space is finite.[7] Hence, $S \subset \mathscr{X}^*$ is a countable collection of pairwise incomparable situations.

Since S is made up out of pairwise incomparable situations, it is a (possibly partial) cut, and so we can let $\mathscr{M}_{\dashv S}$ be the supermartingale $\mathscr{M}$ stopped at the cut S. Since $\mathscr{M} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$, we have by Lemma 4.C.5$_{211}$ that also $\mathscr{M}_{\dashv S} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_\bullet)$. Furthermore, for any $s \in S$, since $s \in S'$ and by the definition of S', we know that $\mathscr{M}(s) \geq 1 - \epsilon/2$. Hence, since $\mathscr{M}_{\dashv S}$ remains constant for all situations that follow s, we have that $\liminf \mathscr{M}_{\dashv S}(\omega) \geq 1 - \epsilon/2$ for all $\omega \in \Gamma(s)$. Since this holds for all $s \in S$,

$$\liminf \mathscr{M}_{\dashv S}(\omega) + \epsilon/2 \geq 1 \text{ for all } \omega \in \bigcup_{s\in S} \Gamma(s).$$

Recall that $\liminf \mathscr{M} \geq_t \mathbb{1}_A$, which by Lemma 4.4.1$_{163}$ implies that $\mathscr{M}$, and thus—because every $s \in S \subseteq S'$ follows t—also $\mathscr{M}_{\dashv S}$, is non-negative for all situations that follow t. Hence, by the inequality above,

$$\liminf \mathscr{M}_{\dashv S} + \epsilon/2 \geq_t \mathbb{1}_{\bigcup_{s\in S}\Gamma(s)}.$$

Since the situations in S are pairwise incomparable, we have that $\mathbb{1}_{\bigcup_{s\in S}\Gamma(s)} = \sum_{s\in S} \mathbb{1}_s$. Plugging this back into the inequality above, gives us

$$\liminf(\mathscr{M}_{\dashv S} + \epsilon/2) = \liminf \mathscr{M}_{\dashv S} + \epsilon/2 \geq_t \textstyle\sum_{s\in S} \mathbb{1}_s.$$

[7]This continues to hold for countable state spaces.

Taking into account that $\mathscr{M}_{\dashv S} \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}})$, and thus by LE5$_{156}$ in Proposition 4.3.4$_{156}$, that $(\mathscr{M}_{\dashv S} + \epsilon/2) \in \overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}})$, we obtain by the definition of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\textstyle\sum_{s\in S} \mathbb{1}_s|t) \leq (\mathscr{M}_{\dashv S} + \epsilon/2)(t) = \mathscr{M}_{\dashv S}(t) + \epsilon/2.$$

Furthermore observe that $\mathscr{M}_{\dashv S}(t) = \mathscr{M}(t)$ because all situations in S follow t [by the definition of S' and S]. The inequality above and our assumptions about $\mathscr{M}$ then imply that

$$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\textstyle\sum_{s\in S} \mathbb{1}_s|t) \leq \mathscr{M}(t) + \epsilon/2 \leq \overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(\mathbb{I}_A|t) + \epsilon.$$

It remains to check that $A \cap \Gamma(t) \subseteq \bigcup_{s\in S} \Gamma(s)$. Since for any $\omega \in A \cap \Gamma(t)$, there is a $k_\omega \geq |t|$ such that $\omega^{k_\omega} \in S'$, it is clear that $A \cap \Gamma(t) \subseteq \bigcup_{s'\in S'} \Gamma(s')$. Hence, the desired inclusion follows from Eq. (6.9)$_\curvearrowleft$. □

Proof of Theorem 6.4.12$_{318}$. We first prove the equality for the upper probabilities. We trivially have that

$$\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) \leq \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) = \overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s),$$

where the inequality follows from Corollary 6.4.7$_{315}$. To prove the converse inequality, recall from Theorem 6.2.2$_{291}$ that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equal to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$. Hence, for any ϵ, there is by Lemma 6.4.13$_{318}$ a countable collection S of pairwise incomparable situations such that

$$A \cap \Gamma(s) \subseteq \cup_{t\in S}\Gamma(t) \ \text{ and } \ \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\textstyle\sum_{t\in S} \mathbb{1}_t|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) + \epsilon.$$

The countable sum $\sum_{t\in S} \mathbb{1}_t$ can be written as the limit $\lim_{n\to+\infty} \sum_{t\in S,|t|\leq n} \mathbb{1}_t$ of the increasing sequence $(\sum_{t\in S,|t|\leq n} \mathbb{1}_t)_{n\in\mathbb{N}}$. Note, moreover, that $(\sum_{t\in S,|t|\leq n} \mathbb{1}_t)_{n\in\mathbb{N}}$ is a sequence of finitary gambles. Since $A \cap \Gamma(s) \subseteq \cup_{t\in S}\Gamma(t)$, we also have that $\lim_{n\to+\infty} \sum_{t\in S,|t|\leq n} \mathbb{1}_t \geq_s \mathbb{I}_A$. Moreover, due to Corollary 6.2.4(v)$_{294}$ and the fact that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ extends $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$, we have that

$$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\textstyle\sum_{t\in S} \mathbb{1}_t|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\sum_{t\in S,|t|\leq n} \mathbb{1}_t|s) = \lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\sum_{t\in S,|t|\leq n} \mathbb{1}_t|s).$$

Combined with $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\sum_{t\in S} \mathbb{1}_t|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) + \epsilon$, we get that

$$\lim_{n\to+\infty} \overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}(\textstyle\sum_{t\in S,|t|\leq n} \mathbb{1}_t|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) + \epsilon.$$

Hence, taking into account the fact that $(\sum_{t\in S,|t|\leq n} \mathbb{1}_t)_{n\in\mathbb{N}}$ is an increasing sequence of finitary gambles such that $\lim_{n\to+\infty} \sum_{t\in S,|t|\leq n} \mathbb{1}_t \geq_s \mathbb{I}_A$, we have by Proposition 6.4.1$_{302}$ that

$$\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) + \epsilon.$$

This holds for any $\epsilon > 0$, so $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) \leq \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s)$. Together with the earlier deduced inequality, we obtain that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s) = \overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_A|s)$ and therefore that $\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s)$.

The equality for the lower probabilities can then be deduced from conjugacy [Eq. $(6.1)_{289}$] and the equality between the upper probabilities; indeed, for any $B \subseteq \Omega$,

$$1 - \underline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(B|s) = 1 - \underline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_B|s) = 1 + \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(-\mathbb{I}_B|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(1 - \mathbb{I}_B|s) = \overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\mathbb{I}_{B^c}|s) = \overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}(B^c|s),$$

where the third step follows from $\mathrm{EC5}_{163}$ in Corollary $6.2.4(\mathrm{ii})_{293}$. A similar equality can then be deduced for the lower probability $\underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(B|s)$ using conjugacy [Eq. $(6.1)_{289}$] and Proposition $6.4.2_{304}$. Hence, since the upper probabilities $\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ are equal, the lower probabilities $\underline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ are also equal. □

Since $\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equal to $\overline{\mathrm{P}}_{\overline{\mathrm{Q}},\mathrm{A}}$, it follows from Theorem $6.2.2_{291}$ and $6.2.3_{292}$ that $\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is also equal to the game-theoretic upper probability $\overline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ and, for all $\sigma(\mathscr{X}^*)$-measurable events, equal to the measure-theoretic upper probability $\overline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$ or its simplified variant $\overline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ (for an agreeing tree $\mathscr{P}_\bullet$), and similarly, due to conjugacy, for the lower probabilities.

Corollary 6.4.14. *Let $\overline{\mathrm{Q}}_\bullet$ and $\mathscr{P}_\bullet$ be any upper expectations tree and imprecise probability tree that agree according to Eq. $(3.3)_{51}$. Then, for any $A \subseteq \Omega$ and any $s \in \mathscr{X}^*$, we have that*

$$\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \overline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A|s) \text{ and } \underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \underline{\mathrm{P}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}(A|s).$$

If A is moreover $\sigma(\mathscr{X}^*)$-measurable, then

$$\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \overline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}(A|s) = \overline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(A|s);$$
$$\underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(A|s) = \underline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}(A|s) = \underline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}(A|s).$$

Proof. The first statement follows from Theorems $6.4.12_{318}$ and $6.2.2_{291}$, the conjugacy of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ [Corollary $4.3.7_{162}$], and the fact that upper and lower probabilities are specific instances of upper and lower expectations. The equality between $\overline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$, and between $\underline{\mathrm{P}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\underline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$, follows from Theorems $6.4.12_{318}$ and $6.2.3_{292}$, the conjugacy of $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ [Corollary $5.4.2_{241}$], and the fact that $\mathbb{I}_A$ is a $\sigma(\mathscr{X}^*)$-measurable gamble if the event A is $\sigma(\mathscr{X}^*)$-measurable. The remaining two equalities, between $\overline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$ and $\overline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$, and between $\underline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$, follow from the fact that $\overline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{P}}^{\downarrow}_{\mathscr{P},\mathrm{M}}$ are restrictions of respectively $\overline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$ and $\underline{\mathrm{P}}_{\mathscr{P},\mathrm{M}}$ to $\sigma(\mathscr{X}^*)$-measurable events; this follows from Corollary $5.4.1_{241}$. □

We furthermore obviously have that all the properties of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ in Corollary $6.2.4_{293}$ also hold for $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$, if we restrict ourselves to the domain of indicators. In order not to overload this text with excessively many similar results, we will not state this as a separate result.

— Chapter VII —

Conclusions

Our work has focused on global upper expectations for discrete-time stochastic processes with a finite state space. We have considered a wide variety of them; some were entirely new, some already existed or were based on methods frequently used in different contexts. We examined the characteristic features of all these global upper expectations in considerable depth, and laid great emphasis on the problem of how these global models are related to each other. This was done not only to present a technical overview of the matter; above all else, we hope this manuscript provides a unifying guideline on how and why we should use certain types of global upper expectations. Let us give a brief summary of our findings, and highlight some of them a bit more.

We distinguished amongst six different classes of global upper expectation, and within each of these classes we often further distinguished between several possible definitions for a global upper expectation. The first three classes that we studied were the finitary ones [Chapter 3_{45}]; they consisted of the finitary behavioural or betting-based upper expectations $\overline{\mathrm{E}}_{\mathscr{A}}$ and $\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$, the finitary probability-based upper expectation $\overline{\mathrm{E}}_{\mathscr{P}}$, and the finitary axiomatic or coherence-based upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$ and $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$. Each of these three classes are based on different types of local models—$\mathscr{A}_\bullet$, $\mathscr{P}_\bullet$ and $\overline{\mathrm{Q}}_\bullet$, respectively—and subsequently use extension procedures that are unique to the framework that is associated with the corresponding type of local models—sets of acceptable gambles/martingales, sets of probabilities and upper expectations, respectively. Since none of these extension procedures rely on a continuity assumption, the finitary upper expectations can be interpreted in a direct and intuitive way, and their mathematical analysis is, compared to the continuity-based global upper expectations, relatively straightforward. Moreover, as we have shown in Section 3.5_{90}, all these different types of finitary global upper expectations coincide if the local models are chosen in accordance with each other, which can perhaps be seen as the most profound advantage of (any of) these finitary upper expectations. Unfortunately however, these finitary upper expectations are only defined on the domain $\mathbb{V} \times \mathscr{X}^*$ and, as was discussed in Section 3.6_{98}, are—in our opinion—only suited for use on the even smaller domain $\mathbb{F} \times \mathscr{X}^*$ because they lack basic but important continuity properties.

The subsequent chapters were then concerned with three types of global upper expectations that can be seen as the continuity-based counterparts of the three finitary global upper expectations. These upper expectations are defined on the entire space $\overline{\mathbb{V}} \times \mathscr{X}^*$ and are marked by relatively strong (continuity) properties, therefore allowing us to deal with extended real-valued—not necessarily finitary—global variables in a meaningful way. However, their more complex and involved design implies that introducing, interpreting and studying these global models becomes harder.

Chapter 4_{129} studied game-theoretic upper expectations; global operators that express a gambler's infimum starting capital such that he is able to hedge the global variable of interest—possibly by playing for an infinitely long time. These operators were first introduced by Shafer and Vovk [85, 86] but have since then appeared in many different forms, and within many different contexts [8, 26, 60, 88, 101, 109]. We discussed a multitude of possible definitions, and argued why the versions $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$—which coincide—are to be preferred over the other ones. Moreover, we also showed in Section 4.3_{152} that $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ can be alternatively characterised in terms of upper expectations trees $\overline{\mathrm{Q}}_{\bullet}$, and that the resulting operator $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ is then often equivalent to the one used by Shafer and Vovk in their latest book [85]; see Section 4.9_{187}. The remaining part of Chapter 4_{129} was devoted to establishing a host of properties for these global operators, with a heavy focus on proving or disproving continuity properties.

Chapter 5_{217} then treated global upper expectations deduced from the framework of measure-theoretic probability. We started with the precise case; yet the construction of our global measure-theoretic (upper) expectation in this traditional context already differed from the classical approach in two notable ways: we use (conditional) global probability measures instead of a single (unconditional) probability measure in order to meaningfully condition on events of probability zero; and we extended the global expectation beyond the domain of measurable variables, due to which it became an upper expectation instead of an expectation. We showed that in this precise case, the measure-theoretic upper expectation coincides with the game-theoretic upper expectation on the entire domain $\overline{\mathbb{V}} \times \mathscr{X}^*$ [Theorem $5.3.1_{235}$], and therefore that properties of either one can be borrowed and applied to the other one. Subsequently, in the general imprecise setting, we defined the measure-theoretic upper expectation as the upper envelope of the 'precise' measure-theoretic upper expectations corresponding to the compatible precise probability trees. Several strong continuity properties were established, and this in turn lead us to conclude that these global measure-theoretic upper expectations are for many practically relevant variables equal to global game-theoretic upper expectations, especially if the local sets of probability mass functions are closed; see Corollary $5.5.15_{258}$.

Finally, it is the material presented in Chapter 6_{283} that I believe to be the most compelling of all, and certainly the material that I am most proud of. The axiomatic approach described there is simple and straightforward, yet, has to our knowledge never been attempted before—or at least not in this imprecise discrete-time stochastic processes setting. We took the finitary upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on $\mathbb{F} \times \mathscr{X}^*$ as our starting point, because its definition is based on the simple and weak Axioms $\mathrm{WC1}_{82}$–$\mathrm{WC4}_{82}$ (which are equivalent to conditional coherence) and because all the global upper expectations that we have discussed in this dissertation turn out to coincide with $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ on the restricted domain $\mathbb{F} \times \mathscr{X}^*$. Our subsequent extension then simply relied on imposing monotonicity [$\mathrm{EC4}^{\Omega}{}_{286}$] in addition to continuity with respect to specific sequences of finitary gambles. Two versions of the latter were considered; $\mathrm{Co1}_{286}$, which solely concerns increasing sequences, and $\mathrm{Co2}_{286}$, which concerns not necessarily increasing but still bounded below sequences. We then argued to take as global upper expectation $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ or $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, which are the most conservative ones among all those that extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ and that satisfy $\mathrm{EC4}^{\Omega}{}_{286}$ and $\mathrm{Co1}_{286}$, or $\mathrm{EC4}^{\Omega}{}_{286}$ and $\mathrm{Co2}_{286}$, respectively. It quickly turned out that $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ is equal to $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$ [Theorem $6.2.2_{291}$] and therefore that it also inherits all the powerful properties of $\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}$—one of the most important being that it is for a large part equal to $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ [Theorem $6.2.3_{292}$]. On the other hand, though the definition of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is conceptually even more attractive than that of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$, and though it can moreover be seen as an imprecise generalisation of Daniell's integration approach [19], it may for some limits of finitary gambles return overly conservative values. This is why, in a general context and considering general global variables, we prefer the use of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ over $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$.

Nonetheless, we do want to stress that the three instances considered in Section $6.4.4_{314}$ where $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide, are encountered frequently. Indeed, the first is where the considered variable of interest is a finitary gamble or the limit of a monotone sequence of finitary gambles. Most of the practically relevant variables that we know of, such as hitting times, stopping times or averages over a finite time interval [58, 100], are of this type. A second situation where $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide is where local models are assumed to be precise; an assumption that is still often made—rightly or not. Finally, both these operators are also equivalent on the domain of all indicators, which means that they give rise to the same upper and lower probabilities. It requires little explanation that we are sometimes only interested in such upper and lower probabilities rather than general upper and lower expectations; an important and commonly encountered inference is for instance the hitting (upper and lower) probability of a certain subset $A \subseteq \mathscr{X}$; see Example 3.6.1$_{99}$ and [58]. Interestingly enough, the equality on the domain of all indicators can also be seen to imply that, if the restrictions of the (unconditional) upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\cdot|s)$ to $\mathbb{V}$ are 2-monotone [6, 13, 106] for all $s \in \mathscr{X}^*$, then the global upper expectations $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ coincide on the domain $\mathbb{V} \times \mathscr{X}^*$ of all gambles and situations. This can be deduced from the representation result [106, Theorem 6.22], which implies that 2-monotonicity of an unconditional coherent upper expectation, together with the domain of this upper expectation being a set of gambles with a structure that is rich enough, is sufficient for this upper expectation to be fully determined by its restriction to the indicators. Such an upper expectation can then be written as a Choquet integral with respect to its corresponding upper probabilities. It remains to be seen, however, whether the 2-monotonicity of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}(\cdot|s)$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}(\cdot|s)$ (or their restrictions to $\mathbb{V}$) can be characterised in an elegant and useful way, for instance using the form of the local models $\overline{\mathrm{Q}}_{\bullet}$.

All things considered, the axiomatic upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ can in practice often be replaced by $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$. It is then preferable to do so because, as already mentioned, the definition of $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ is more direct, it requires a user to accept weaker axioms, and it agrees with Daniell's traditional approach to extending integrals.

Now, if we take a step back and think about the simple but central question posed in the beginning of Chapter 3$_{45}$ about how to extend imprecise local models to a single global uncertainty model, we have now made up our minds; the axiomatic upper expectation $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$—often to be replaced by $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$—is what we will go with. This because of its simple and direct definition, and because of its strong continuity properties; but the most important reason, we feel, is its universal character. Indeed, as pointed out at the end of Section 6.1$_{285}$, the definitions of $\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$ and $\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$ do not hinge on any particular interpretation, nor do they require a user to quantify uncertainty in one specific way, or within one specific framework. The finitary upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$—the starting point for these global operators—is defined in terms of Axioms WC1$_{82}$–WC4$_{82}$, or equivalently coherence, which can be motivated from both a behavioural point of view, and a probability-based point of view; the subsequent extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ then relies on a monotonicity axiom and a continuity axiom, which can both be argued for on the basis of a neutral approximation argument. This is in sheer contrast with the (continuity-based) global upper expectations $\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$ and $\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$ which are each constructed from a single and distinct point of view; game-theoretic upper expectations start from local sets of acceptable gambles, and use the language of gambling to extend beyond this point; measure-theoretic upper expectations start from local sets of probability mass functions, and use probability charges and measures to extend towards a global level. This distinction between probability-free (behavioural) approaches and probability-based approaches has a long-standing history and is also present on a more general scale, not only in our specific context of discrete-time stochastic processes. Our axiomatic approach reconciles both worlds, at least in our context of discrete-time stochastic processes, and therefore frees the pragmatic user from the controversial debate about how uncertainty ought to be interpreted and quantified.

Lastly, to finish this plea in favour of the axiomatic approach, we want the reader to think of the success of Kolmogorov's work [56] in the field of measure theory; his approach was purely axiomatic and provided the mathematical foundations for a theory of probability in a clear and elegant fashion—which was apparently much needed at the time it was published [87]. By no means, we compare the value, relevance or scale of our work to that of Kolmogorov's, but we do hope that, for the field of imprecise discrete-time stochastic processes, our work may serve a similar purpose.

Future outlook

As tradition will have it, we conclude this chapter with a discussion of some topics that may be worthwhile investigating further. A first one concerns the definition of $\overline{\mathrm{E}}_{\mathrm{Q},\mathrm{A}}$, and more specifically the form of Axiom Co2$_{286}$. Recall that it applies only to converging sequences in $\mathbb{F}$ that are uniformly bounded below. The condition of being uniformly bounded below makes the axiom weaker, which is desirable from an abstract mathematical perspective. But our motivation for Axiom Co2$_{286}$ just as well applies to general converging sequences in $\mathbb{F}$ as it applies to uniformly bounded below ones, so why can't we modify Co2$_{286}$ to apply to general sequences? Well, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ itself does not necessarily satisfy such a generalised type of continuity, and so there need not exist an extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ that satisfies such a continuity property.[1]

Example 7.0.1. Let $\mathscr{X} := \{a, b\}$ and consider the expectations tree $\mathrm{Q}_\bullet$ defined by $\mathrm{Q}_s(f) := (f(a) + f(b))/2$ for all $f \in \mathscr{L}(\mathscr{X})$ and $s \in \mathscr{X}^*$. Then it can easily be derived from the law of iterated (upper) expectations [Proposition 3.5.9$_{96}$] that, for any $f \in \mathbb{F}$ and any $k \in \mathbb{N}$ for which f is k-measurable, the upper expectation $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f)$ is the average of f's values over all situations of length k:

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f) = \frac{1}{|\mathscr{X}|^k} \sum_{x_{1:k} \in \mathscr{X}^k} f(x_{1:k}) = \frac{1}{2^k} \sum_{x_{1:k} \in \mathscr{X}^k} f(x_{1:k}). \tag{7.1}$$

[1] Of course, one could overthrow this argument by simply suggesting that a global upper expectation should not always extend $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$, but we honestly do not see any good reasons to do so.

Now let $(f_n)_{n\in\mathbb{N}}$ be the sequence defined by $f_n := -2^{2n}\mathbb{1}_{a^n b}$ for all $n \in \mathbb{N}$. It is clear that $f_n \in \mathbb{F}$ for all $n \in \mathbb{N}$, and it can also be checked that $\lim_{n\to+\infty}\Gamma(a^n b) = \emptyset$ and therefore that $\lim_{n\to+\infty} f_n = 0$. But by Eq. (7.1)$_\curvearrowleft$ and the fact that each f_n is clearly $(n+1)$-measurable, we have that

$$\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n) = \frac{1}{2^{n+1}} \sum_{x_{1:n+1}\in\mathscr{X}^{n+1}} f_n(x_{1:n+1}) = \frac{1}{2^{n+1}} \sum_{x_{1:n+1}\in\mathscr{X}^{n+1}} -2^{2n}\mathbb{1}_{a^n b}(x_{1:n+1})$$
$$= -\frac{1}{2^{n+1}}\, 2^{2n} = -2^{n-1}.$$

As a result, $\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|s) = -\infty$. But $(f_n)_{n\in\mathbb{N}}$ converges to the finitary gamble 0, so we have that

$$\lim_{n\to+\infty}\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(f_n|s) = -\infty \ngeq 0 = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}(0) = \overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}\big(\lim_{n\to+\infty} f_n\big),$$

where the second equality follows from the fact that $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ satisfies WC3$_{82}$ by definition. Hence, $\overline{\mathrm{E}}^{\mathrm{fin}}_{\mathrm{Q}}$ cannot satisfy a generalised type of Co2$_{286}$ that applies to general sequences in $\mathbb{F}$. ◇

It is clear from the example above that modifying Co2$_{286}$ to apply to not necessarily bounded below sequences is not a good idea—probably not even if an extension were to exist. Still, the bounded below requirement in Co2$_{286}$ seems somewhat arbitrary from a philosophical point of view, and it begs the question whether Co2$_{286}$ cannot be modified in one way or another in order to arrive at an axiom that is more natural altogether. Perhaps it should apply to sequences that converge in a stronger way than simply pointwise, and perhaps the bounded below requirement can then be dropped? Food for thought.

Two other possible routes for future research concern generalising the present theory in two ways. The first is to generalise to a setting where the state space $\mathscr{X}$ is (countably or uncountably) infinite. The ideas and principles that give rise to the definitions of the game-theoretic and measure-theoretic global upper expectations would in such a context remain essentially the same. For instance, Shafer and Vovk [85] allow for general state spaces, and the definition of their global game-theoretic upper expectation relies on entirely the same concepts as ours. In the measure-theoretic case, Ionescu-Tulcea's extension theorem [89, Theorem 2.9.2] would allow us to extend local probability measures similar to how Proposition 5.1.3$_{226}$ allows us to extend local probability mass functions.

As far as the properties of these hypothetical game-theoretic and measure-theoretic global upper expectations are concerned, much shall depend on the additional conditions that we impose on the local models. For instance, in order to obtain continuity of the game-theoretic upper expectation with respect to increasing sequences, we shall at least need to impose this upward continuity on the local upper expectations. This becomes clear if we look at the proof of Theorem $4.6.1_{175}$, which explicitly relies on the continuity property LE6_{156} of the local upper expectations. We did not need to impose it in our treatment here—or at least not on the set $\mathscr{L}(\mathscr{X})$ of gambles—because our state space $\mathscr{X}$ was assumed finite. The measure-theoretic upper expectation, on the other hand, shall probably remain continuous with respect to increasing sequences, simply because it will be an upper envelope of Lebesgue integrals with respect to σ-additive probability measures—here too, there is a continuity assumption at the local level, because we would start from sets of local probability measures (which are each σ-additive).

The question under which conditions the game-theoretic and measure-theoretic upper expectations will remain to be continuous with respect to decreasing finitary gambles [the counterparts of Theorem $4.7.3_{182}$ and Proposition $5.4.9_{246}$] is more tricky, though. A first issue would of course be what we understand under 'finitary' gambles or variables if the state space is infinite—the continuity would surely not hold if we allow a finitary gamble to simply be any bounded variable that depends on the process state at a finite number of time instances. An alternative could be to define them as bounded variables that are both continuous and only depend on the process state at a finite number of time instances—note that, due to Lemma $5.5.2_{251}$ and the paragraph above it, this is in line with our treatment here. Yet, even then, we are convinced that some additional conditions will need to be satisfied before one can guarantee the downward continuity of these global upper expectations. For instance, Lemma $4.C.2_{209}$, which is crucial for proving Theorem $4.7.3_{182}$, can be extended to infinite state spaces, but only if the considered state space remains to be compact. Apart from that, we suspect that, for game-theoretic upper expectations, either supermartingales should be restricted in how they are allowed to behave or some more continuity conditions should be imposed on the local upper expectations, and that for measure-theoretic upper expectations, the sets of local probability measures will need to be compact or satisfy some other topological condition. All this is no more than a calculated guess, though, and we certainly did not look into the details. Yet, if game-theoretic and measure-theoretic global upper expectations would satisfy comparable continuity properties in the case of infinite state spaces as they do here in the case of finite state spaces, then Choquet's capacitability theorem [Theorem $5.5.9_{255}$] could again be used to establish an equality between the two operators.

Lastly, we did not talk about how our axiomatic global upper expectations should be adapted in order to appropriately deal with infinite state spaces. This is difficult to predict, though, since many of its characterising concepts and properties are specifically adapted to the finitary setting; e.g. what are the sequences of finitary gambles/variables in this new context? Can we guarantee existence and uniqueness in this new context? Here too, additional assumptions for the local upper expectations shall need to be made in order to arrive at satisfactory results.

Another interesting way in which this work can perhaps be generalised, is to consider or develop global upper expectations that allow us to condition on more general events than only (cylinder events of) situations. We have little knowledge of existing work done on this part, especially in the game-theoretic case—all the more compelling to look into it.

Of course, though our work was mainly aimed at examining the theoretical aspects of imprecise stochastic processes, we hope that it can also play its part in more practically oriented research. Continuity properties, for instance, can be combined with backwards recursive algorithms [62, 100] to obtain methods for computing the upper expectations of (finitary and) non-finitary variables; in [58], such a reasoning is used to obtain upper and lower expected hitting times and probabilities. On the other hand, one may also evoke our results on the connections between the different types of global models to borrow algorithms and techniques specifically developed for one type of global model and apply them to any other type.

Finally, it would also be worthwhile to further investigate how our work compares to the material in some neighbouring research fields. In particular, the work of Denk et al. [30] on Daniell-Stone type of (global) upper expectations seems interesting; comparing the form of our lower Daniell extension $\overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$—which due to Proposition 6.4.4$_{308}$ is for a large part equal to our axiomatic (and thus also the game-theoretic and measure-theoretic) upper expectation—to the extension described in [30, Theorem 3.10], it seems that a close connection must exist, at least for bounded measurable variables. Equally compelling seems to be the relation with the sublinear expectations proposed by Cohen et al. [7]; as [7, Theorem 2.1] and [7, Definition 2.4] show, the expectation operators treated there must in some sense be similar to our measure-theoretic global upper expectations.

The present is in every age merely the shifting point at which past and future meet, and we can have no quarrel with either. There can be no world without traditions; neither can there be any life without movement. We cannot bathe twice in the same stream, though, as we know to-day, the stream still flows in an unending circle. There is never a moment when the new dawn is not breaking over the earth, and never a moment when the sunset ceases to die. It is well to greet serenely even the first glimmer of the dawn when we see it, not hastening towards it with undue speed, nor leaving the sunset without gratitude for the dying light that once was dawn.

Havelock Ellis.

List of symbols

Number sets

$\emptyset$	the empty set • Chapter 2_{17}
$\mathbb{N}$	the set of natural numbers without zero • Section 1.6_{14}
$\mathbb{N}_0$	the set of natural numbers with zero • Section 1.6_{14}
$\mathbb{R}$	the set of real numbers • Section 1.6_{14}
$\mathbb{R}_{\geq}, \mathbb{R}_{>}, \mathbb{R}_{<}$	the set of non-negative/positive/negative real numbers • Section 1.6_{14}
$\overline{\mathbb{R}}$	the set of extended real numbers • Section 1.6_{14}
$\overline{\mathbb{R}}_{\geq}, \overline{\mathbb{R}}_{>}$	the set of non-negative/positive extended real numbers • Section 1.6_{14}

Modelling uncertainty

Y	uncertain outcome of an experiment • Introduction of Chapter 2_{17}
$\mathscr{Y}$	possibility space • Introduction of Chapter 2_{17}
x, y, z	outcomes in $\mathscr{Y}$ • Introduction of Chapter 2_{17}
$\wp(\mathscr{Y})$	the powerset of $\mathscr{Y}$; that is, the set of all subsets of $\mathscr{Y}$ • Section 2.1_{18}
A, B	events in $\wp(\mathscr{Y})$ • Section 2.1_{18}
A^c	complement of the event A in $\wp(\mathscr{Y})$ • Section $3.3.1_{69}$
$\bigvee_{n\in\mathbb{N}} c_n$	supremum of the sequence $(c_n)_{n\in\mathbb{N}}$ of extended real numbers • Section $5.2.1_{227}$
$\bigwedge_{n\in\mathbb{N}} c_n$	infimum of the sequence $(c_n)_{n\in\mathbb{N}}$ of extended real numbers • Section $5.2.1_{227}$

Stochastic processes

X_k	uncertain state of a stochastic process at time $k \geq 1$ • Introduction of Chapter 3_{45}

$\mathscr{X}$	finite state space • Section 3.1.1$_{47}$
x, y, z	states in $\mathscr{X}$ • Section 3.1.1$_{47}$
$X_{1:k}$	first k (uncertain) state values of a stochastic process • Section 3.1.2$_{48}$
$x_{1:0}, \square$	the initial situation; an empty string • Section 3.1.1$_{47}$
$\mathscr{X}^*$	the set of all situations • Section 3.1.1$_{47}$
s, t, u, v	situations in $\mathscr{X}^*$ • Section 3.1.1$_{47}$
$x_{1:k}$	a situation of length k • Section 3.1.1$_{47}$
$\lvert s\rvert$	length of a situation s • Section 3.1.1$_{47}$
$u \sqsubseteq s$	u precedes s; s follows u • Section 3.1.1$_{47}$
$u \sqsubset s$	$u \sqsubseteq s$ and $u \neq s$ • Section 3.1.1$_{47}$
$u \parallel s$	u and s are incomparable • Section 3.1.1$_{47}$
Ω	set of all infinite state sequences • Section 3.1.1$_{47}$
ω	a path in Ω • Section 3.1.1$_{47}$
ω_k	k-th component of ω • Section 3.1.1$_{47}$
$\omega^k, \omega_{1:k}$	situation consisting of ω's first k components • Section 3.1.1$_{47}$
$\omega_{k:\ell}$	situation consisting of ω's k-th to ℓ-th components • Section 3.1.1$_{47}$
$\Gamma(s)$	cylinder event of a situation s • Section 3.1.1$_{47}$
$\Gamma(\mathscr{X}^*)$	the set of all cylinder events • Section 3.3.2$_{72}$
U, V	tree cuts • p. 199
$s \sqsubset U, s \sqsubseteq U$	$s \sqsubset u$ for all $u \in U$; $s \sqsubseteq u$ for all $u \in U$ • p. 199
$U \sqsubset s$	$u \sqsubset s$ for some $u \in U$ • p. 199

Gambles and variables

Variables on a general space

f, g, h	extended real-valued variables on $\mathscr{Y}$ • Section 1.6$_{14}$
$\mathscr{L}(\mathscr{Y})$	the set of gambles on $\mathscr{Y}$ • Section 1.6$_{14}$
$\overline{\mathscr{L}}(\mathscr{Y})$	the set of extended real(-valued) variables on $\mathscr{Y}$ • Section 1.6$_{14}$ and Section 3.1.3$_{52}$
$\overline{\mathscr{L}}_{\mathrm{b}}(\mathscr{Y})$	the set of bounded below extended real(-valued) variables on $\mathscr{Y}$ • Section 1.6$_{14}$
$\mathscr{L}_{\geq}(\mathscr{Y})$	the set of all non-negative gambles on $\mathscr{Y}$ • Section 1.6$_{16}$
$\mathscr{L}_{\gneq}(\mathscr{Y})$	$\mathscr{L}_{\geq}(\mathscr{Y})$ without the zero gamble 0 • Section 1.6$_{16}$
$\mathscr{L}_{>}(\mathscr{Y})$	the set of all positive gambles on $\mathscr{Y}$ • Section 1.6$_{16}$
$\mathscr{L}_{\leq}(\mathscr{Y})$	the set of all non-positive gambles on $\mathscr{Y}$ • Section 1.6$_{16}$

$\mathscr{L}_{\lneqq}(\mathscr{Y})$	$\mathscr{L}_{\leq}(\mathscr{Y})$ without the zero gamble 0 • Section 1.6_{16}
$\mathscr{L}_{<}(\mathscr{Y})$	the set of all negative gambles on $\mathscr{Y}$ • Section 1.6_{16}
$\mathbb{I}_A$	indicator of an event A in $\wp(\mathscr{Y})$ • Section 1.6_{14}
$\mathscr{A}$	a subset of $\mathscr{L}(\mathscr{Y})$ • Section $2.6.2_{37}$
$\mathscr{K}$	a subset of $\mathscr{L}(\mathscr{Y})$; typically the domain of an upper expectation • Section $2.6.3_{38}$

Global variables

$\mathbb{V}, \overline{\mathbb{V}}$	the set of global gambles/variables • Section $3.1.3_{52}$
$\overline{\mathbb{V}}_{\mathrm{b}}$	the set of bounded below global variables • Section $4.2.2_{142}$
$\overline{\mathbb{V}}_{\geq}$	the set of non-negative global variables • p. 253
$\mathbb{V}^{\mathrm{u}}_{\geq}$	the set of non-negative real-valued global variables • p. 253
$\mathbb{F}, \overline{\mathbb{F}}$	the set of finitary gambles/variables • Section $3.1.3_{52}$
$\overline{\mathbb{L}}_{\mathrm{b}}$	the set of bounded below pointwise limits of sequences of finitary gambles • Section $4.7.1_{182}$
$\mathbb{V}_{\sigma}, \overline{\mathbb{V}}_{\sigma}, \overline{\mathbb{V}}_{\sigma,\mathrm{b}}$	the set of all (bounded below) $\sigma(\mathscr{X}^*)$-measurable gambles/variables • Section $5.2.1_{227}$
$\mathscr{K}$	a subset of $\mathbb{V} \times \mathscr{X}^*$; typically the domain of a global upper expectation • Section $3.1.3_{52}$
$\mathscr{I}$	a subset of $\mathbb{V}$ • Section $3.4.1_{81}$
$\mathbb{1}_s$	the indicator of the cylinder event $\Gamma(s)$ of a situation s • Section $3.1.3_{52}$
τ_A	the hitting time of $A \subseteq \mathscr{X}$ • Section $3.1.3_{52}$ and Section 3.6_{98}
σ	a stopping time • Appendix $4.\mathrm{C}.1_{209}$

Relations and operations for extended real-valued variables

$f = g$	$f(y) = g(y)$ for all y in $\mathscr{Y}$ • Section 1.6_{14}
$f \leq g$	$f(y) \leq g(y)$ for all y in $\mathscr{Y}$ • Section 1.6_{14}
$f < g$	$f(y) < g(y)$ for all y in $\mathscr{Y}$ • Section 1.6_{14}
$f \leq_s g$	$f(\omega) \leq g(\omega)$ for all $\omega \in \Gamma(s)$ • Section $3.1.3_{52}$
$\sup f, \inf f$	pointwise supremum/infimum of f • Section 1.6_{14}
$\sup(f \vert s)$	pointwise supremum of f over $\Gamma(s)$ • Section $3.1.3_{52}$
$\lim_{n\to+\infty} f_n$	pointwise limit of a sequence $(f_n)_{n\in\mathbb{N}}$ • Section 1.6_{14}
$\limsup_{n\to+\infty} f_n$	pointwise limit superior of a sequence $(f_n)_{n\in\mathbb{N}}$ • Section 1.6_{14}
$\liminf_{n\to+\infty} f_n$	pointwise limit inferior of a sequence $(f_n)_{n\in\mathbb{N}}$ • Section 1.6_{14}
$\Vert\cdot\Vert_{\infty}$	supremum norm on $\mathscr{L}(\mathscr{Y})$ • p. 121

cl ($\mathscr{A}$)	uniform closure of a set $\mathscr{A}$ of gambles • Section 1.6$_{14}$
posi ($\mathscr{A}$)	positive linear span of a set $\mathscr{A}$ of gambles • Section 1.6$_{14}$
span ($\mathscr{A}$)	linear span of a set $\mathscr{A}$ of gambles • Section 3.3.3$_{74}$
$f^{\wedge c}, f^{\vee c}$	pointwise minimum/maximum of f and c • Section 4.2.2$_{142}$
f^{+}, f^{-}	positive/negative part of f • Definition 5.3$_{228}$

General and axiomatic upper and lower expectations

$\overline{\mathrm{E}}, \underline{\mathrm{E}}$	(un)conditional upper/lower expectation • Sections 2.4$_{31}$ and 3.1.3$_{52}$
$\overline{\mathrm{E}}(f \mid x_{1:k}X_{k+1:k+\ell})$	finitary variable assuming the value $\overline{\mathrm{E}}(f \mid x_{1:k}\omega_{k+1:k+\ell})$ in ω • Proposition 3.4.4$_{84}$
$\overline{\mathrm{E}}(\overline{\mathrm{E}}(f \mid X_{1:k+1}) \mid X_{1:k})$	finitary variable assuming the value $\overline{\mathrm{E}}(\overline{\mathrm{E}}(f \mid X_{1:k+1}) \mid \omega_{1:k})$ in ω • Proposition 3.4.4$_{84}$
$\overline{\mathrm{Q}}_{\bullet}$	upper expectations tree • Section 3.1.2$_{48}$
$\overline{\mathrm{Q}}_{s}$	local upper expectation corresponding to the tree $\overline{\mathrm{Q}}_{\bullet}$ and the situation s • Section 3.1.2$_{48}$
$\overline{\mathrm{Q}}^{\uparrow}_{\bullet}, \overline{\mathrm{Q}}^{\uparrow}_{s}, \overline{\mathrm{Q}}^{\uparrow}$	extended upper expectations tree/extended local upper expectation • Section 4.3.1$_{153}$
$\overline{\mathrm{Q}}_{\bullet,\mathscr{A}}, \overline{\mathrm{Q}}_{s,\mathscr{A}}$	upper expectations tree/local upper expectation corresponding to the acceptable gambles tree $\mathscr{A}$ • Eq. (3.1)$_{50}$
$\overline{\mathrm{Q}}_{\bullet,\mathscr{P}}, \overline{\mathrm{Q}}_{s,\mathscr{P}}$	upper expectations tree/local upper expectation corresponding to the imprecise probability tree $\mathscr{P}$ • Eq. (3.3)$_{51}$
$\mathrm{Q}_{\bullet,p}, \mathrm{Q}_{s,p}$	linear expectations tree/local linear expectation corresponding to the precise probability tree p • Eq. (3.4)$_{52}$
$\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$	preliminary global upper expectation corresponding to $\overline{\mathrm{Q}}_{\bullet}$ • Eq. (3.13)$_{85}$
$\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}, \overline{\mathrm{E}}_{\overline{\mathrm{Q}}}$	natural extension under coherence of $\overline{\mathrm{E}}^{\mathrm{pre}}_{\overline{\mathrm{Q}}}$ to $\mathbb{F}\times\mathscr{X}^{*}$ or $\mathbb{V}\times\mathscr{X}^{*}$ • Definition 3.8$_{86}$
$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{A}}$	natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ to $\overline{\mathbb{V}}\times\mathscr{X}^{*}$ under a continuity and a monotonicity axiom • Definition 6.2$_{289}$
$\overline{\mathrm{E}}^{\nearrow}_{\overline{\mathrm{Q}},\mathrm{A}}$	natural extension of $\overline{\mathrm{E}}^{\mathrm{fin}}_{\overline{\mathrm{Q}}}$ to $\overline{\mathbb{V}}\times\mathscr{X}^{*}$ under a monotone continuity and a monotonicity axiom • Definition 6.1$_{289}$
$\overline{\mathrm{E}}_{\overline{\mathrm{Q}},\mathrm{D}}$	Daniell-like upper expectation on $(\mathbb{F}^{\mathrm{o}}\cup\mathbb{F}_{\mathrm{u}})\times\mathscr{X}^{*}$ corresponding to the tree $\overline{\mathrm{Q}}_{\bullet}$ • p. 306
$\overline{\mathrm{E}}^{\mathrm{o}}_{\overline{\mathrm{Q}},\mathrm{D}}, \overline{\mathrm{E}}^{\mathrm{u}}_{\overline{\mathrm{Q}},\mathrm{D}}$	outer/inner Daniell-like upper expectation on $\overline{\mathbb{V}}\times\mathscr{X}^{*}$ corresponding to the tree $\overline{\mathrm{Q}}_{\bullet}$ • p. 307 and p. 308

Acceptability, martingales and game-theoretic upper expectations

Sets of acceptable gambles

$\mathscr{A}, \mathscr{D}$	set of acceptable gambles • Section 2.3.1$_{26}$ and Section 2.6$_{36}$
$\mathscr{E}(\mathscr{A}), \mathscr{E}(\mathscr{D})$	natural extension of $\mathscr{A}$ or $\mathscr{D}$ • Section 2.6.2$_{37}$
$\mathscr{D}(\overline{\mathrm{E}})$	smallest coherent set of acceptable gambles associated with $\overline{\mathrm{E}}$ • Eq. (2.4)$_{35}$
$\mathscr{A}(\overline{\mathrm{E}})$	set of acceptable gambles associated with $\overline{\mathrm{E}}$ • Eq. (2.6)$_{41}$
$\mathscr{A}_\bullet$	acceptable gambles tree • Section 3.1.2$_{48}$
$\mathscr{A}_s$	local set of acceptable gambles corresponding to the tree $\mathscr{A}_\bullet$ and the situation s • Section 3.1.2$_{48}$
$\mathscr{A}_\bullet^{\uparrow}, \mathscr{A}_s^{\uparrow}, \mathscr{A}^{\uparrow}$	extended acceptable gambles tree/extended local set of acceptable gambles • Section 4.2.3$_{145}$
$\mathscr{D}_{\mathscr{A}}$	global set of acceptable gambles associated with $\mathscr{A}_\bullet$ • Eq. (3.6)$_{56}$

Extended real processes, super- and submartingales

$\mathscr{C}, \mathscr{G}$	(extended) real process/betting process • Section 3.2.3$_{61}$ and Section 4.2.3$_{145}$
$\mathscr{C}^{\mathscr{G}}$	cumulative process corresponding to $\mathscr{G}$ • Section 3.2.3$_{61}$ and Section 4.2.3$_{145}$
$\Delta\mathscr{C}$	process difference of $\mathscr{C}$ • Section 3.2.3$_{61}$ and Section 4.2.3$_{145}$
$\limsup \mathscr{C}$	pathwise limit superior of $\mathscr{C}$ • Section 4.1$_{131}$
$\liminf \mathscr{C}$	pathwise limit inferior of $\mathscr{C}$ • Section 4.1$_{131}$
$\mathscr{C}^{\wedge B}$	pointwise minimum of $\mathscr{C}$ and B • Section 4.1.3$_{136}$
$\mathscr{C}_{\dashv U}$	process $\mathscr{C}$ stopped at the cut U • Section 4.C.2$_{211}$
$\mathscr{M}$	(extended real) sub- or supermartingale • Section 3.2.3$_{61}$, Section 4.2.3$_{145}$ and Section 4.3.3$_{158}$
$\overline{\mathbb{M}}(\mathscr{A}_\bullet), \underline{\mathbb{M}}(\mathscr{A}_\bullet)$	the set of real super-/submartingales corresponding to $\mathscr{A}_\bullet$ • Section 3.2.3$_{61}$
$\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_\bullet)$	alternative notation for $\overline{\mathbb{M}}(\mathscr{A}_\bullet)$ • Section 4.1$_{131}$
$\overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_\bullet)$	the set of real bounded below supermartingales corresponding to $\mathscr{A}_\bullet$ • Section 4.1.3$_{136}$
$\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_\bullet)$	the set of (real) bounded supermartingales corresponding to $\mathscr{A}_\bullet$ • Section 4.1.3$_{136}$
$\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_\bullet)$	the set of bounded below extended real supermartingales corresponding to $\mathscr{A}_\bullet$ • Section 4.2.3$_{145}$

$\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$	the set of bounded below extended real supermartingales corresponding to $\overline{\mathrm{Q}}_{\bullet}$ • Section 4.3.3$_{158}$
$\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$	alternative set of bounded below extended real supermartingales corresponding to $\overline{\mathrm{Q}}_{\bullet}$ • Section 4.3.3$_{158}$

Acceptability-based and game-theoretic upper expectations

$\overline{\mathrm{E}}_{\mathscr{A}}$	upper expectation as infimum selling prices corresponding to the set $\mathscr{E}(\mathscr{D}_{\mathscr{A}})$ • Eq. (3.10)$_{60}$
$\overline{\mathrm{E}}^{\mathrm{f}}_{\mathscr{A},\mathrm{V}}$	finitary game-theoretic upper expectation corresponding to the tree $\mathscr{A}_{\bullet}$ • Eq. (3.11)$_{63}$
$\overline{\mathrm{E}}^{\mathrm{r}}_{\mathscr{A},\mathrm{V}}$	game-theoretic upper expectation corresponding to $\overline{\mathbb{M}}_{\mathrm{r}}(\mathscr{A}_{\bullet})$ • Definition 4.1$_{132}$
$\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}, \overline{\mathrm{E}}^{\mathrm{rB}}_{\mathscr{A},\mathrm{V}}$	game-theoretic upper expectation corresponding to $\overline{\mathbb{M}}_{\mathrm{rb}}(\mathscr{A}_{\bullet})$ or $\overline{\mathbb{M}}_{\mathrm{rB}}(\mathscr{A}_{\bullet})$ • Definition 4.2$_{137}$
$\overline{\mathrm{E}}^{\uparrow}_{\mathscr{A},\mathrm{V}}$	game-theoretic upper expectation obtained from extending $\overline{\mathrm{E}}^{\mathrm{rb}}_{\mathscr{A},\mathrm{V}}$ through continuity w.r.t. upper and lower cuts • Definition 4.3$_{143}$
$\overline{\mathrm{E}}^{\mathrm{eb}}_{\mathscr{A},\mathrm{V}}$	game-theoretic upper expectation corresponding to $\overline{\mathbb{M}}_{\mathrm{eb}}(\mathscr{A}_{\bullet})$ • Definition 4.6$_{149}$
$\overline{\mathrm{E}}^{\mathrm{eb}}_{\overline{\mathrm{Q}},\mathrm{V}}, \overline{\mathrm{E}}^{\mathrm{eb},\mathscr{G}}_{\overline{\mathrm{Q}},\mathrm{V}}$	game-theoretic upper expectation corresponding to $\overline{\mathbb{M}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ or $\overline{\mathbb{M}}^{\mathscr{G}}_{\mathrm{eb}}(\overline{\mathrm{Q}}_{\bullet})$ • Definition 4.7$_{160}$

Algebras, probabilities and linear expectations

Algebras and measurability

$\mathfrak{A}, \mathfrak{B}$	algebras/fields of events • Section 3.3.1$_{69}$
$\mathfrak{B}^{\circ}$	algebra $\mathfrak{B}$ without the empty set $\emptyset$ • Section 3.3.1$_{69}$
$\langle \mathscr{X}^{*} \rangle$	the (smallest) algebra generated by the cylinder events $\Gamma(\mathscr{X}^{*})$ • Section 3.3.2$_{72}$
$\sigma(\mathscr{X}^{*})$	the (smallest) σ-algebra generated by the cylinder events $\Gamma(\mathscr{X}^{*})$ • Section 5.1.2$_{224}$
$\sigma(\mathfrak{A})$	the (smallest) σ-algebra generated by the subset $\mathfrak{A}$ of $\wp(\Omega)$. • Section 5.1.2$_{224}$
$\mathscr{B}(\overline{\mathbb{R}})$	Borel σ-algebra on $\overline{\mathbb{R}}$ • Appendix 5.A$_{263}$
$\mathrm{span}\,(\mathfrak{A})$	the set of $\mathfrak{A}$-simple gambles • Section 3.3.3$_{74}$

Probability charges and measures

p	probability mass function or precise probability tree • Definition 2.2$_{19}$ and Section 3.1.2$_{48}$

$p(\cdot\|s)$	local probability mass function corresponding to the tree p and the situation s • Section 3.1.2$_{48}$
$\mathscr{P}_\bullet$	imprecise probability tree • Section 3.1.2$_{48}$
$\mathscr{P}_s$	local set of mass functions corresponding to the tree $\mathscr{P}_\bullet$ and the situation s • Section 3.1.2$_{48}$
$p \sim \mathscr{P}_\bullet$	p is compatible with $\mathscr{P}_\bullet$ • Section 3.3.4$_{79}$
$\mathbb{P}(\mathscr{Y})$	set of all probability mass functions on a finite space $\mathscr{Y}$ • Section 2.1$_{18}$
$\mathscr{P}(\overline{\mathrm{E}})$	largest set of probability mass functions corresponding to the upper expectation $\overline{\mathrm{E}}$ • Eq. (2.5)$_{40}$
P	(conditional/global) probability charge • Definition 2.1$_{19}$, Definition 3.1$_{70}$ and Definition 3.2$_{70}$
P_p	global probability charge/measure corresponding to the tree p • Section 3.3.2$_{72}$ and Proposition 5.1.3$_{226}$
$\mathrm{P}_p^{\|s}$	unconditional probability measure corresponding to P_p and the situation s • Definition 5.4$_{230}$

Integrals and linear/upper expectations

$\int f \mathrm{dP}$	S-integral or Lebesgue integral of f with respect to P • Definition 3.3$_{76}$ and Definition 5.3$_{228}$
$\overline{\int} f \mathrm{dP}, \underline{\int} f \mathrm{dP}$	upper/lower S-integral or Lebesgue integral of f with respect to P • Definition 3.3$_{76}$ and Section 5.2.3$_{231}$
E_p	linear expectation corresponding to a probability mass function p • Section 2.1$_{18}$
$\mathrm{E}_p, \overline{\mathrm{E}}_p$	finitary (upper) expectation corresponding to a precise probability tree p • Definition 3.4$_{77}$ and Definition 3.5$_{78}$
$\overline{\mathrm{E}}_{\mathscr{P}}$	finitary upper expectation corresponding to an imprecise probability tree $\mathscr{P}_\bullet$ • Definition 3.6$_{79}$
$\mathrm{E}_{p,\mathrm{M}}, \overline{\mathrm{E}}_{p,\mathrm{M}}$	measure-theoretic (upper) expectation corresponding to the precise probability tree p • Definition 5.4$_{230}$ and Definition 5.5$_{232}$
$\overline{\mathrm{E}}_{\mathscr{P},\mathrm{M}}$	measure-theoretic upper expectation corresponding to the imprecise probability tree $\mathscr{P}_\bullet$ • Definition 5.7$_{240}$
$\mathrm{E}_{\mathrm{RN}}(f\|\mathscr{B})$	Radon-Nikodým derivative of f conditional on the σ-algebra $\mathscr{B}$ • Appendix 5.A$_{263}$

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