Appendix A. Proofs

Proof of Proposition 3

Sufficiency is trivial, as $\mathscr{I} \subseteq \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$. For necessity, assume that $\mathscr{D}$ is exchangeable, so $\mathscr{D}+\mathscr{I} \subseteq \mathscr{D}$. Consider any $\hat{A} \in \mathscr{D}$ and any $\hat{B} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$, then we have to show that $\hat{A}+\hat{B} \in \mathscr{D}$. Since $\hat{B} - \mathrm{pr}_{\mathrm{ex}}(\hat{B}) = \hat{B}$, we find that

$$\hat{B} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}(\hat{B} - \pi^t\hat{B}) \tag{6}$$

and therefore $\hat{A}+\hat{B} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}(\hat{A}+\hat{B}-\pi^t\hat{B})$. Since $\hat{A} \in \mathscr{D}$ and $\hat{B}-\pi^t\hat{B} \in \mathscr{I}$, we infer from $\mathscr{D}+\mathscr{I} \subseteq \mathscr{D}$ that $\hat{A}+\hat{B}-\pi^t\hat{B} \in \mathscr{D}$, and therefore from D3 and D4 that, indeed, also $\hat{A}+\hat{B} \in \mathscr{D}$.

For the second statement, we infer from Equation (6) that $\mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} \subseteq \mathrm{span}(\mathscr{I})$. For the converse inequality, observe that for any $\hat{A} \in \mathscr{H}$ and $\sigma \in \mathbb{P}$,

$$\mathrm{pr}_{\mathrm{ex}}(\hat{A} - \sigma^t\hat{A}) = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\pi^t\hat{A} - \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\pi^t(\sigma^t\hat{A}) = \hat{0},$$

so $\mathscr{I} \subseteq \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$, and therefore also $\mathrm{span}(\mathscr{I}) \subseteq \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$. ■

Proof of Proposition 4

We give a circular proof.

(i)⇒(ii). Assume that there's some exchangeable coherent set of measurements $\mathscr{D}$ such that $\underline{\Lambda} = \underline{\Lambda}_{\mathscr{D}}$, and consider any $\hat{B} \in \mathscr{I}$. Then we infer from Proposition 3 and D2 that $\hat{B}+\alpha\hat{I} \in \mathscr{D}$ for all real $\alpha > 0$, so Equation (1) tells us that $\underline{\Lambda}_{\mathscr{D}}(\hat{B}) \geq 0$. Since also $-\hat{B} \in \mathscr{I}$, we can use the same argument to infer that also $\underline{\Lambda}_{\mathscr{D}}(-\hat{B}) \geq 0$. Using LP4, we then find that $0 \geq -\underline{\Lambda}_{\mathscr{D}}(-\hat{B}) = \overline{\Lambda}_{\mathscr{D}}(\hat{B}) \geq \underline{\Lambda}_{\mathscr{D}}(\hat{B}) \geq 0$.

(ii)⇒(iii). Consider any $\hat{A} \in \mathscr{I}_{\mathrm{pr}_\star}$, then we infer from $\hat{A} = \hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}(\hat{A}-\pi^t\hat{A})$ and from LP1, LP2 and LP4, that

$$\frac{1}{m!}\sum_{\pi\in\mathbb{P}}\underline{\Lambda}(\hat{A}-\pi^t\hat{A}) \leq \underline{\Lambda}(\hat{A}) \leq \overline{\Lambda}(\hat{A}) \leq \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\overline{\Lambda}(\hat{A}-\pi^t\hat{A}),$$

so we infer from (ii) that, indeed, $\underline{\Lambda}(\hat{A}) = \overline{\Lambda}(\hat{A}) = 0$.

(iii)⇒(iv). Consider any $\hat{A} \in \mathscr{H}$ and infer from $\hat{A} = \mathrm{pr}_{\mathrm{ex}}(\hat{A}) + (\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A}))$ and LP5 that

$$\underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(\hat{A})) + \underline{\Lambda}(\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A})) \leq \underline{\Lambda}(\hat{A}) \leq \underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(\hat{A})) + \overline{\Lambda}(\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A})).$$

But, $\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A}) \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$, because $\mathrm{pr}_{\mathrm{ex}}(\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A})) = \mathrm{pr}_{\mathrm{ex}}(\hat{A}) - \mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \hat{0}$, so by (iii), $\underline{\Lambda}(\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A})) = \overline{\Lambda}(\hat{A} - \mathrm{pr}_{\mathrm{ex}}(\hat{A})) = 0$.

(iv)⇒(i). Let $\mathscr{D}^{>}_{\underline{\Lambda}} \coloneqq \{\hat{A} \in \mathscr{H} \colon \underline{\Lambda}(\hat{A}) > 0\}$ and also let $\mathscr{D} \coloneqq \mathscr{D}^{>}_{\underline{\Lambda}} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$. The coherence of $\underline{\Lambda}$ readily implies that $\mathscr{D}$ satisfies D2, D3 and D4. For D1, assume towards contradiction that there are $\hat{A} \in \mathscr{D}^{>}_{\underline{\Lambda}}$ and $\hat{B} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$ such that $\hat{A}+\hat{B} = \hat{0}$, so $\mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \mathrm{pr}_{\mathrm{ex}}(\hat{A}+\hat{B}) = \hat{0}$. Then LP4 and (iv) imply that $0 = \underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(\hat{A})) = \underline{\Lambda}(\hat{A})$, contradicting that $\hat{A} \in \mathscr{D}^{>}_{\underline{\Lambda}}$. Hence, $\mathscr{D}$ is coherent. The exchangeability of $\mathscr{D}$ follows from the fact that $\mathscr{D} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} = \mathscr{D}^{>}_{\underline{\Lambda}} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} = \mathscr{D}^{>}_{\underline{\Lambda}} + \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}} = \mathscr{D}$. We're done if we can prove that $\underline{\Lambda} = \underline{\Lambda}_{\mathscr{D}}$. Fix any $\hat{A} \in \mathscr{H}$, then $\hat{A}-\alpha\hat{I} \in \mathscr{D}$ implies that there are $\hat{B} \in \mathscr{D}^{>}_{\underline{\Lambda}}$ and $\hat{C} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$ such that $\hat{A}-\alpha\hat{I} = \hat{B}+\hat{C}$. Observe that (iv) and LP2 imply that $\underline{\Lambda}(\hat{C}) = \underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(\hat{C})) = \underline{\Lambda}(\hat{0}) = 0$, and similarly, that $\overline{\Lambda}(\hat{C}) = -\underline{\Lambda}(-\hat{C}) = -\underline{\Lambda}(\mathrm{pr}_{\mathrm{ex}}(-\hat{C})) = -\underline{\Lambda}(\hat{0}) = 0$. LP6 and LP5 therefore imply that $\underline{\Lambda}(\hat{A}) = \alpha + \underline{\Lambda}(\hat{B}+\hat{C}) = \alpha + \underline{\Lambda}(\hat{B}) \geq \alpha$, so $\underline{\Lambda}_{\mathscr{D}}(\hat{A}) \leq \underline{\Lambda}(\hat{A})$. Conversely, as $\mathscr{D}^{>}_{\underline{\Lambda}} \subseteq \mathscr{D}$, we also find that $\underline{\Lambda}_{\mathscr{D}}(\hat{A}) \geq \underline{\Lambda}_{\mathscr{D}^{>}_{\underline{\Lambda}}}(\hat{A})$. Now, by Equation (1),

$$\begin{aligned}\underline{\Lambda}_{\mathscr{D}^{>}_{\underline{\Lambda}}}(\hat{A}) &= \sup\{\alpha \in \mathbb{R} \colon \hat{A}-\alpha\hat{I} \in \mathscr{D}^{>}_{\underline{\Lambda}}\} \\ &= \sup\{\alpha \in \mathbb{R} \colon \underline{\Lambda}(\hat{A}-\alpha\hat{I}) > 0\} \\ &= \sup\{\alpha \in \mathbb{R} \colon \underline{\Lambda}(\hat{A}) > \alpha\} = \underline{\Lambda}(\hat{A}),\end{aligned}$$

where the penultimate equality is due to LP6. Then $\underline{\Lambda}_{\mathscr{D}}(\hat{A}) \geq \underline{\Lambda}_{\mathscr{D}^{>}_{\underline{\Lambda}}}(\hat{A}) = \underline{\Lambda}(\hat{A}) \geq \underline{\Lambda}_{\mathscr{D}}(\hat{A})$. ■

Proof of Corollary 5

First, assume that $\underline{\Lambda}$ is exchangeable. If we consider any $\hat{\rho} \in \mathscr{R}_{\underline{\Lambda}}$ and any $\pi \in \mathbb{P}$, then we must show that $\overline{\pi}^t\hat{\rho} = \hat{\rho}$. Consider, to this end, any $\hat{A} \in \mathscr{H}$ and let $\hat{B} \coloneqq \hat{A} - \pi^t\hat{A}$, then Proposition 4(ii) implies that $\underline{\Lambda}(\hat{B}) = \overline{\Lambda}(\hat{B}) = 0$. We then infer from Equation (3) applied to $\hat{B}$ and $-\hat{B}$ that $\underline{\Lambda}(\hat{B}) \leq \mathrm{Tr}(\hat{\rho}\hat{B}) \leq \overline{\Lambda}(\hat{B})$, and therefore $\mathrm{Tr}(\hat{\rho}(\hat{A}-\pi^t\hat{A})) = 0$. By the linearity of the trace and Equation (5), we find that

$$\mathrm{Tr}(\hat{\rho}\hat{A}) = \mathrm{Tr}\big(\hat{\rho}(\pi^t\hat{A})\big) = \mathrm{Tr}\big((\overline{\pi}^t\hat{\rho})\hat{A}\big).$$

Since $\hat{A} \in \mathscr{H}$ is arbitrary, Theorem 2 implies that $\hat{\rho} = \overline{\pi}^t\hat{\rho}$.

Conversely, assume that all density operators $\hat{\rho}$ in $\mathscr{R}_{\underline{\Lambda}}$ are permutation invariant. Consider any $\hat{A} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$, then $\mathrm{pr}_{\mathrm{ex}}(\hat{A}) = \hat{0}$ and therefore, by the linearity of the trace,

$$\begin{aligned}0 = \mathrm{Tr}\big(\hat{\rho}\,\mathrm{pr}_{\mathrm{ex}}(\hat{A})\big) &= \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\mathrm{Tr}\big(\hat{\rho}(\pi^t\hat{A})\big) \\ &= \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\mathrm{Tr}\big((\overline{\pi}^t\hat{\rho})\hat{A}\big) = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\mathrm{Tr}\big(\hat{\rho}\hat{A}\big) = \mathrm{Tr}\big(\hat{\rho}\hat{A}\big),\end{aligned}$$

where the third equality follows from Equation (5), and the last equality from the permutation invariance of $\hat{\rho}$. But then $\underline{\Lambda}(\hat{A}) = \min\{\mathrm{Tr}(\hat{\rho}\hat{A}) \colon \hat{\rho} \in \mathscr{R}_{\underline{\Lambda}}\} = 0$, and similarly, $\overline{\Lambda}(\hat{A}) = 0$, since also $-\hat{A} \in \mathscr{I}_{\mathrm{pr}_{\mathrm{ex}}}$. Proposition 4(iii) now guarantees that $\underline{\Lambda}$ is indeed exchangeable. ■

Proof of Proposition 6

We give the proof for antisymmetric densities; the proof for symmetric densities is similar, but somewhat simpler.

For the first statement, sufficiency is proved in the main text, so we prove necessity. Assume that $\hat{\rho}$ is antisymmetric. Since $\hat{\rho}$ is Hermitian, it has a decomposition $\hat{\rho} = \sum_{k=1}^{n} \lambda_k |a_k\rangle\langle a_k|$, where the $|a_k\rangle$ are its mutually orthonormal eigenstates and the λ_k its real eigenvalues [26, Box 2.2]. It then follows from Proposition 1 that $\lambda_1, \dots, \lambda_n$ constitute a probability mass function over the eigenstates. We may assume without loss of generality that all $\lambda_k > 0$. Fix any $|a_k\rangle$ and any $\pi \in \mathbb{P}$, then on the one hand it follows from the assumption that $\hat{\rho}|a_k\rangle = \operatorname{sgn}(\pi)\hat{\Pi}_\pi\hat{\rho}|a_k\rangle = \lambda_k \operatorname{sgn}(\pi)\hat{\Pi}_\pi|a_k\rangle$, and on the other hand we find that $\hat{\rho}|a_k\rangle = \lambda_k|a_k\rangle$. So $|a_k\rangle = \operatorname{sgn}(\pi)\hat{\Pi}_\pi|a_k\rangle$, and therefore $|a_k\rangle$ is indeed antisymmetric.

The second statement follows from Ref. [26, Theorem 2.6] and the fact that, by Proposition 7, $\mathscr{X}_\mathrm{a}$ is a subspace of $\mathscr{X}$. The basic idea behind this argument is that Ref. [26, Theorem 2.6] guarantees that the state $|\psi_k\rangle$ must be a linear combination of the eigenstates $|a_1\rangle, \dots, |a_n\rangle$ corresponding to positive eigenvalues. The argument above confirms that those eigenstates $|a_1\rangle, \dots, |a_n\rangle$ are antisymmetric, and since $\mathscr{X}_\mathrm{a}$ is a subspace of $\mathscr{X}$, the same holds for any linear combination of them, so indeed $|\psi_k\rangle \in \bar{\mathscr{X}}_\mathrm{a}$. ■

Proof of Proposition 7

We begin by proving the first statement. Observe that, for any $\pi \in \mathbb{P}$, $\hat{\Pi}_\pi\hat{P}_\mathrm{a} = \hat{\Pi}_\pi\frac{1}{m!}\sum_{\sigma\in\mathbb{P}}\operatorname{sgn}(\sigma)\hat{\Pi}_\sigma = \frac{1}{m!}\sum_{\sigma\in\mathbb{P}}\operatorname{sgn}(\sigma)\hat{\Pi}_{\sigma\circ\pi} = \operatorname{sgn}(\pi)\hat{P}_\mathrm{a}$. Therefore $\hat{P}_\mathrm{a}\hat{P}_\mathrm{a} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)\hat{\Pi}_\pi\hat{P}_\mathrm{a} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)^2\hat{P}_\mathrm{a} = \hat{P}_\mathrm{a}$, and $\hat{P}_\mathrm{s}\hat{P}_\mathrm{a} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\hat{\Pi}_\pi\hat{P}_\mathrm{a} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)\hat{P}_\mathrm{a} = \hat{0}$. The proofs for $\hat{P}_\mathrm{s}\hat{P}_\mathrm{s} = \hat{P}_\mathrm{s}$ and $\hat{P}_\mathrm{a}\hat{P}_\mathrm{s} = \hat{0}$ are very similar. It now follows at once that $\hat{P}_\mathrm{o}\hat{P}_\mathrm{o} = (\hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s})(\hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s}) = \hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s} = \hat{P}_\mathrm{o}$, and that $\hat{P}_\mathrm{o}\hat{P}_\mathrm{s} = (\hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s})\hat{P}_\mathrm{s} = \hat{P}_\mathrm{s} - \hat{P}_\mathrm{s} = \hat{0}$. The proofs for the remaining identities $\hat{P}_\mathrm{s}\hat{P}_\mathrm{o} = \hat{P}_\mathrm{o}\hat{P}_\mathrm{a} = \hat{P}_\mathrm{a}\hat{P}_\mathrm{o} = \hat{0}$ are again very similar.

We now turn to the proof of the remaining statements. The identities in the first statement already allow us to conclude that $\hat{P}_\mathrm{s}$, $\hat{P}_\mathrm{a}$ and $\hat{P}_\mathrm{o}$ are projection operators that project onto mutually orthogonal spaces whose direct sum is the state space. We now prove that $\hat{P}_\mathrm{a}$ projects onto $\mathscr{X}_\mathrm{a}$. For any $|\psi\rangle \in \mathscr{X}_\mathrm{a}$, $\hat{P}_\mathrm{a}|\psi\rangle = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)\hat{\Pi}_\pi|\psi\rangle = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)^2|\psi\rangle = |\psi\rangle$, so $\mathscr{X}_\mathrm{a} \subseteq \hat{P}_\mathrm{a}\mathscr{X}$. For the converse inclusion, we have that $\hat{P}_\mathrm{a}|\psi\rangle \in \mathscr{X}_\mathrm{a}$ for any $|\psi\rangle \in \mathscr{X}$, since, as proved above, $\hat{\Pi}_\pi\hat{P}_\mathrm{a} = \operatorname{sgn}(\pi)\hat{P}_\mathrm{a}$ and thus $\operatorname{sgn}(\pi)\hat{\Pi}_\pi\hat{P}_\mathrm{a}|\psi\rangle = \hat{P}_\mathrm{a}|\psi\rangle$ for all $\pi \in \mathbb{P}$. Hence, $\hat{P}_\mathrm{a}\mathscr{X} \subseteq \mathscr{X}_\mathrm{a}$. We conclude that $\hat{P}_\mathrm{a}$ indeed projects onto $\mathscr{X}_\mathrm{a}$. The proof that $\hat{P}_\mathrm{s}$ projects onto $\mathscr{X}_\mathrm{s}$ is very similar, and the rest of the proof is then immediate. ■

Proof of Proposition 8

For all $\pi \in \mathbb{P}$, $\hat{\Pi}_\pi$ is unitary, so $\hat{\rho}\hat{\Pi}_\pi = (\hat{\Pi}_\pi\hat{\rho}\hat{\Pi}_\pi^\dagger)\hat{\Pi}_\pi = \hat{\Pi}_\pi\hat{\rho}\hat{\Pi}_\pi^\dagger\hat{\Pi}_\pi = \hat{\Pi}_\pi\hat{\rho}$. Therefore, $\hat{P}_\mathrm{s}\hat{\rho} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\hat{\Pi}_\pi\hat{\rho} = \hat{\rho}\frac{1}{m!}\sum_{\pi\in\mathbb{P}}\hat{\Pi}_\pi = \hat{\rho}\hat{P}_\mathrm{s}$. Similarly, $\hat{P}_\mathrm{a}\hat{\rho} = \hat{\rho}\hat{P}_\mathrm{a}$ and therefore $\hat{P}_\mathrm{o}\hat{\rho} = (\hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s})\hat{\rho} = \hat{\rho}(\hat{I} - \hat{P}_\mathrm{a} - \hat{P}_\mathrm{s}) = \hat{\rho}\hat{P}_\mathrm{o}$. Now use Proposition 7 to find that, indeed,

$$\begin{aligned}\hat{\rho} &= \hat{\rho}(\hat{P}_\mathrm{o} + \hat{P}_\mathrm{a} + \hat{P}_\mathrm{s}) = \hat{\rho}\hat{P}_\mathrm{o}^2 + \hat{\rho}\hat{P}_\mathrm{a}^2 + \hat{\rho}\hat{P}_\mathrm{s}^2 \\ &= \hat{P}_\mathrm{o}\hat{\rho}\hat{P}_\mathrm{o} + \hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} + \hat{P}_\mathrm{s}\hat{\rho}\hat{P}_\mathrm{s} = \hat{\omega}_\mathrm{o} + \hat{\omega}_\mathrm{a} + \hat{\omega}_\mathrm{s}.\end{aligned}$$

For the second statement, we'll only give a proof for antisymmetric densities. The proof for symmetric densities is similar, if somewhat simpler. For necessity, assume that $\hat{\rho}$ is antisymmetric, so $\hat{\rho} = \operatorname{sgn}(\pi)\hat{\Pi}_\pi\hat{\rho}$ for all $\pi \in \mathbb{P}$. Then

$$\hat{P}_\mathrm{a}\hat{\rho} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)\hat{\Pi}_\pi\hat{\rho} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\hat{\rho} = \hat{\rho},$$

and similarly, $\hat{\rho}\hat{P}_\mathrm{a} = \hat{\rho}$, and therefore, indeed, $\hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} = \hat{\rho}\hat{P}_\mathrm{a} = \hat{\rho}$. For sufficiency, assume that $\hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} = \hat{\rho}$, and fix any $\sigma \in \mathbb{P}$. Observe that

$$\begin{aligned}\operatorname{sgn}(\sigma)\hat{\Pi}_\sigma\hat{P}_\mathrm{a} &= \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\sigma)\operatorname{sgn}(\pi)\hat{\Pi}_\sigma\hat{\Pi}_\pi \\ &= \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}(\pi)\hat{\Pi}_\pi = \hat{P}_\mathrm{a},\end{aligned}$$

and therefore also

$$\operatorname{sgn}(\sigma)\hat{\Pi}_\sigma\hat{\rho} = \operatorname{sgn}(\sigma)\hat{\Pi}_\sigma\hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} = \hat{P}_\mathrm{a}\hat{\rho}\hat{P}_\mathrm{a} = \hat{\rho},$$

so we're done. ■

Proof of Proposition 9

The first statement is an immediate consequence of the second, which itself follows readily from

$$\begin{aligned}\hat{\Pi}_\pi^\dagger\hat{P}_\star &= \frac{1}{m!}\sum_{\sigma\in\mathbb{P}}\operatorname{sgn}^\star(\sigma)\hat{\Pi}_\pi^\dagger\hat{\Pi}_\sigma \\ &= \operatorname{sgn}^\star(\pi)\frac{1}{m!}\sum_{\sigma\in\mathbb{P}}\operatorname{sgn}^\star(\sigma\circ\pi^{-1})\hat{\Pi}_{\sigma\circ\pi^{-1}} \\ &= \operatorname{sgn}^\star(\pi)\hat{P}_\star,\end{aligned} \tag{7}$$

and similarly, $\hat{P}_\star\hat{\Pi}_\pi = \operatorname{sgn}^\star(\pi)\hat{P}_\star$. For the last statement, consider any $\hat{A} \in \mathscr{J}_{\mathrm{pr}_\star}$, then $\hat{A} - \mathrm{pr}_\star(\hat{A}) = \hat{A}$. Now we use the definition of $\mathrm{pr}_\star$ to find that

$$\begin{aligned}&\hat{A} - \mathrm{pr}_\star(\hat{A}) \\ &= \frac{1}{m!^2}\sum_{\pi,\sigma\in\mathbb{P}}\left[\hat{A} - \operatorname{sgn}^\star(\pi\circ\sigma)\hat{\Pi}_\pi^\dagger\hat{A}\hat{\Pi}_\sigma\right] \\ &= \frac{1}{m!^2}\sum_{\pi,\sigma\in\mathbb{P}}\left[\hat{A} - \frac{\operatorname{sgn}^\star(\pi\circ\sigma)}{2}\left(\hat{\Pi}_\pi^\dagger\hat{A}\hat{\Pi}_\sigma + \hat{\Pi}_\sigma^\dagger\hat{A}\hat{\Pi}_\pi\right)\right],\end{aligned}$$

where we reshuffled some terms to get the second equality. Consider that

$$2S^\star_\sigma(S^\star_\pi(\hat{A})) = \frac{\operatorname{sgn}^\star(\sigma\circ\pi)}{2}(\hat{\Pi}^\dagger_{\sigma\circ\pi}\hat{A}+\hat{A}\hat{\Pi}_{\sigma\circ\pi}+\hat{\Pi}^\dagger_\sigma\hat{A}\hat{\Pi}_\pi+\hat{\Pi}^\dagger_\pi\hat{A}\hat{\Pi}_\sigma),$$

and therefore

$$\frac{\operatorname{sgn}^\star(\pi\circ\sigma)}{2}(\hat{\Pi}^\dagger_\pi\hat{A}\hat{\Pi}_\sigma+\hat{\Pi}^\dagger_\sigma\hat{A}\hat{\Pi}_\pi) = 2S^\star_\sigma(S^\star_\pi(\hat{A})) - S^\star_{\sigma\circ\pi}(\hat{A}),$$

so a generic term in the sum above can be rewritten as

$$\hat{A} - \frac{\operatorname{sgn}^\star(\pi\circ\sigma)}{2}(\hat{\Pi}^\dagger_\pi\hat{A}\hat{\Pi}_\sigma+\hat{\Pi}^\dagger_\sigma\hat{A}\hat{\Pi}_\pi) = 2\big[\hat{A}-S^\star_\pi(A)\big]-\big[\hat{A}-S^\star_{\sigma\circ\pi}(\hat{A})\big]+2\big[S^\star_\pi(\hat{A})-S^\star_\sigma S^\star_\pi(\hat{A})\big].$$

Each of the terms in brackets is an element of $\mathscr{I}^\star$, so $\hat{A} = \hat{A} - \operatorname{pr}_\star(\hat{A})$ is a linear combination of elements of $\mathscr{I}^\star$. Hence, $\mathscr{I}_{\operatorname{pr}_\star} \subseteq \operatorname{span}(\mathscr{I}^\star)$.

For the converse inclusion, consider any $\hat{A}\in\mathscr{I}^\star$, so there are $\hat{B}\in\mathscr{H}$ and $\pi\in\mathbb{P}$ such that $\hat{A} = \hat{B} - S^\star_\pi(\hat{B})$. Then

$$\operatorname{pr}_\star(\hat{A}) = \operatorname{pr}_\star(\hat{B} - S^\star_\pi(\hat{B})) = \operatorname{pr}_\star(\hat{B}) - \operatorname{pr}_\star(S^\star_\pi(\hat{B})) = \hat{0},$$

where the last equality holds because

$$\begin{aligned}\operatorname{pr}_\star(S^\star_\pi(\hat{B})) &= \hat{P}_\star\frac{\operatorname{sgn}^\star(\pi)}{2}\Big(\hat{\Pi}^\dagger_\pi\hat{B}+\hat{B}\hat{\Pi}_\pi\Big)\hat{P}_\star\\ &= \frac{\operatorname{sgn}^\star(\pi)}{2}(\hat{P}_\star\hat{\Pi}^\dagger_\pi\hat{B}\hat{P}_\star+\hat{P}_\star\hat{B}\hat{\Pi}_\pi\hat{P}_\star)\\ &= \frac{\operatorname{sgn}^\star(\pi)^2}{2}(\hat{P}_\star\hat{B}\hat{P}_\star+\hat{P}_\star\hat{B}\hat{P}_\star) = \hat{P}_\star\hat{B}\hat{P}_\star,\end{aligned}$$

due to the Equation (7) and its right-sided counterpart, proved above. Hence, $\hat{A}\in\mathscr{I}_{\operatorname{pr}_\star}$, which implies that $\mathscr{I}^\star \subseteq \mathscr{I}_{\operatorname{pr}_\star}$ and therefore indeed also $\operatorname{span}(\mathscr{I}^\star)\subseteq\mathscr{I}_{\operatorname{pr}_\star}$. ■

The argumentation for proving of Propositions 10 and 11 is analogous to that in the proofs of Propositions 3 and 4, respectively, so we won't include them here.

Proof of Corollary 12

Using a similar argument as in the proof of Corollary 5, we find that by Proposition 11, the strong $\star$-symmetry requirement is equivalent to the requirement that $\operatorname{Tr}(\hat{\rho}\hat{A}) = 0$ for all $\hat{\rho}\in\mathscr{R}_{\underline{A}}$ and all $\hat{A}\in\mathscr{I}_{\operatorname{pr}_\star}$, or equivalently, that

$$\operatorname{Tr}\Big(\hat{\rho}\,(\hat{A}-\operatorname{pr}_\star(\hat{A}))\Big) = 0 \text{ for all } \hat{A}\in\mathscr{H}.$$

By the linearity and cyclic property of the trace, we can rewrite this condition, as

$$\operatorname{Tr}(\hat{\rho}\hat{A}) = \operatorname{Tr}\big(\hat{\rho}\hat{P}_\star\hat{A}\hat{P}_\star\big) = \operatorname{Tr}\big(\hat{P}_\star\hat{\rho}\hat{P}_\star\hat{A}\big) \text{ for all } \hat{A}\in\mathscr{H},$$

and therefore, by Theorem 2, the strong $\star$-symmetry requirement is equivalent to

$$\hat{\rho} = \hat{P}_\star\hat{\rho}\hat{P}_\star = \operatorname{pr}_\star(\hat{\rho}) \text{ for all } \hat{\rho}\in\mathscr{R}_{\underline{A}}. \tag{8}$$

To prove necessity for the first statement, observe that this condition, together with Proposition 9(ii), implies that $\hat{\rho} = \operatorname{sgn}^\star(\pi)\hat{\Pi}_\pi\hat{\rho}$ for all $\pi\in\mathbb{P}$, which is the stated condition.

For sufficiency, if the stated condition holds, then

$$\hat{P}_\star\hat{\rho} = \frac{1}{m!}\sum_{\pi\in\mathbb{P}}\operatorname{sgn}^\star(\pi)\hat{\Pi}_\pi\hat{\rho} = \operatorname{sgn}^\star(\pi)^2\hat{\rho} = \hat{\rho},$$

and therefore, $\hat{\rho} = \hat{\rho}^\dagger = (\hat{P}_\star\hat{\rho})^\dagger = \hat{\rho}^\dagger\hat{P}^\dagger_\star = \hat{\rho}\hat{P}_\star$, whence, indeed, $\hat{\rho} = \hat{P}_\star\hat{\rho}\hat{P}_\star$. ■

Proof of Proposition 13

Assume that $\hat{A}\in\mathscr{D}$. Since

$$\operatorname{pr}_\star(\operatorname{pr}_\star(\hat{A})-\hat{A}) = \operatorname{pr}_\star(\operatorname{pr}_\star(\hat{A})) - \operatorname{pr}_\star(\hat{A}) = \hat{0},$$

we see that $\operatorname{pr}_\star(\hat{A})-\hat{A}\in\mathscr{I}_{\operatorname{pr}_\star}$, so the strong $\star$-symmetry of $\mathscr{D}$ implies that $\operatorname{pr}_\star(\hat{A}) = \hat{A} + (\operatorname{pr}_\star(\hat{A})-\hat{A})\in\mathscr{D}$.

Conversely, assume that $\operatorname{pr}_\star(\hat{A})\in\mathscr{D}$. Since, similarly as before, $\hat{A}-\operatorname{pr}_\star(\hat{A})\in\mathscr{I}_{\operatorname{pr}_\star}$, the strong $\star$-symmetry of $\mathscr{D}$ yields $\hat{A} = \operatorname{pr}_\star(\hat{A}) + (\hat{A}-\operatorname{pr}_\star(\hat{A}))\in\mathscr{D}$. ■

Proof of Theorem 14

For sufficiency, assume that there's some coherent $\mathscr{D}_o$ for $\bar{\mathscr{X}}_\star$ such that $\mathscr{D} = \{\hat{A}\in\mathscr{H}\colon \operatorname{Hy}_\star(\hat{A})\in\mathscr{D}_o\}$. First, we prove that $\mathscr{D}$ is then coherent. For D1, assume that $\hat{A}=\hat{0}$, then also $\operatorname{Hy}_\star(\hat{A}) = \hat{0}$ and therefore $\operatorname{Hy}_\star(\hat{A})\notin\mathscr{D}_o$, so $\hat{A}\notin\mathscr{D}$. For D2, assume that $\hat{A}>\hat{0}$, then also $\operatorname{Hy}_\star(\hat{A})>\hat{0}$[15] and therefore $\operatorname{Hy}_\star(\hat{A})\in\mathscr{D}_o$, so $\hat{A}\in\mathscr{D}$. D3 and D4 follow from the linearity of $\operatorname{Hy}_\star$. Next, we show that $\mathscr{D}$ is strongly $\star$-symmetric, using Proposition 10. Consider any $\hat{A}\in\mathscr{D}$ and $\hat{B}\in\mathscr{I}_{\operatorname{pr}_\star}$, then we have to show that $\hat{A}+\hat{B}\in\mathscr{D}$. Since for all $|\psi\rangle\in\bar{\mathscr{X}}_\star$, $|\psi\rangle = \hat{P}_\star|\psi\rangle$ and therefore also $\langle\psi|\hat{B}|\psi\rangle = \langle\psi|\hat{P}_\star\hat{B}\hat{P}_\star|\psi\rangle = \langle\psi|\hat{0}|\psi\rangle = 0$, we find that $\operatorname{Hy}_\star(\hat{B}) = \hat{0}$. Therefore $\operatorname{Hy}_\star(\hat{A}+\hat{B}) = \operatorname{Hy}_\star(\hat{A})\in\mathscr{D}_o$, whence, indeed, $\hat{A}+\hat{B}\in\mathscr{D}$. Finally, consider any $\hat{C}\in\mathscr{H}(\mathscr{X}_\star)$ and observe that for all $|\psi\rangle\in\mathscr{X}_\star$, $\langle\psi|\operatorname{ext}_\star(\hat{C})|\psi\rangle = \langle\psi|\hat{C}\hat{P}_\star|\psi\rangle = \langle\psi|\hat{C}|\psi\rangle$, so $\operatorname{Hy}_\star(\operatorname{ext}_\star(\hat{C})) = \hat{C}$, and therefore indeed,

$$\operatorname{ext}_\star(\hat{C})\in\mathscr{D} \Leftrightarrow \operatorname{Hy}_\star(\operatorname{ext}_\star(\hat{C}))\in\mathscr{D}_o \Leftrightarrow \hat{C}\in\mathscr{D}_o.$$

For necessity, assume that $\mathscr{D}$ is a strongly $\star$-symmetric coherent set of desirable measurements for $\bar{\mathscr{X}}$, then we first prove that $\mathscr{D}_\star$ is a coherent set of desirable measurements for $\bar{\mathscr{X}}_\star$. For D1, since $\operatorname{ext}_\star(\hat{0}) = \hat{0}$ and $\hat{0}\notin\mathscr{D}$, we find that $\hat{0}\notin\mathscr{D}_\star$. For D2, consider any $\hat{C}\in\mathscr{H}(\mathscr{X}_\star)_{>\hat{0}}$ and let

[15] A Hermitian operator $\hat{A}$ on a Hilbert space $\mathscr{X}$ is positive definite if and only if $\langle\psi|\hat{A}|\psi\rangle>0$ for all $|\psi\rangle\in\bar{\mathscr{X}}$.

$\hat{A} \coloneqq \mathrm{ext}_\star(\hat{C}) + \hat{I} - \hat{P}_\star$. As the codomain of $\mathscr{E}(\hat{C}) = \hat{C}\hat{P}_\star$ is $\mathscr{X}_\star$, we find that $\hat{C}\hat{P}_\star = \hat{P}_\star\hat{C}\hat{P}_\star$. Then for all $|\psi\rangle \in \bar{\mathscr{X}}$,

$$\begin{aligned}\langle\psi|\hat{A}|\psi\rangle &= \langle\psi|(\hat{P}_\star\hat{C}\hat{P}_\star + \hat{I} - \hat{P}_\star)|\psi\rangle \\ &= \langle\psi|\hat{P}_\star\hat{C}\hat{P}_\star|\psi\rangle + 1 - \langle\psi|\hat{P}_\star|\psi\rangle.\end{aligned}$$

Now, there are two possibilities, for any $|\psi\rangle \in \bar{\mathscr{X}}$. If $\hat{P}_\star|\psi\rangle \neq 0$, then the positivity of $\hat{C}$ implies that $\langle\psi|\hat{P}_\star\hat{C}\hat{P}_\star|\psi\rangle > 0$ and $1 - \langle\psi|\hat{P}_\star|\psi\rangle \geq 0$, because the eigenvalues of the projector $\hat{P}_\star$ are 0 and 1. If $\hat{P}_\star|\psi\rangle = 0$, then $\langle\psi|\hat{P}_\star\hat{C}\hat{P}_\star|\psi\rangle = 0$ and $1 - \langle\psi|\hat{P}_\star|\psi\rangle = 1 > 0$. This tells us that $\hat{A} > \hat{0}$, and therefore $\hat{A} \in \mathscr{D}$. But, as $\hat{P}_\star$ is a projection operator,

$$\begin{aligned}\mathrm{pr}_\star(\hat{A}) &= \hat{P}_\star\big(\mathrm{ext}_\star(\hat{C}) + \hat{I} - \hat{P}_\star\big)\hat{P}_\star \\ &= \hat{P}_\star\,\mathrm{ext}_\star(\hat{C})\hat{P}_\star + \hat{P}_\star\hat{P}_\star - \hat{P}_\star\hat{P}_\star\hat{P}_\star \\ &= \hat{P}_\star\hat{C}\hat{P}_\star\hat{P}_\star = \hat{P}_\star\hat{C}\hat{P}_\star = \hat{C}\hat{P}_\star = \mathrm{ext}_\star(\hat{C}),\end{aligned}$$

where the penultimate equality follows from the fact that the codomain of $\hat{C}$ is $\mathscr{X}_\star$. But then Proposition 13 guarantees that $\mathrm{pr}_\star(\hat{A}) = \mathrm{ext}_\star(\hat{C})$ also belongs to $\mathscr{D}$, so indeed $\hat{C} \in \mathscr{D}_\star$. D3 and D4, follow from the linearity of $\mathrm{ext}_\star$. To conclude, we prove that $\mathscr{D} = \{\hat{A} \in \mathscr{H} \colon \mathrm{Hy}_\star(\hat{A}) \in \mathscr{D}_\star\}$. Consider any $\hat{A} \in \mathscr{H}$ and observe that $\mathrm{ext}_\star(\mathrm{Hy}_\star(\hat{A})) = \mathrm{pr}_\star(\hat{A})$, so

$$\begin{aligned}\hat{A} \in \mathscr{D} &\Leftrightarrow \mathrm{pr}_\star(\hat{A}) \in \mathscr{D} \Leftrightarrow \mathrm{ext}_\star(\mathrm{Hy}_\star(\hat{A})) \in \mathscr{D} \\ &\Leftrightarrow \mathrm{Hy}_\star(\hat{A}) \in \mathscr{D}_\star,\end{aligned}$$

where the first equivalence follows from Proposition 13. ■