Modelling uncertainty in quantum mechanics using imprecise probabilities

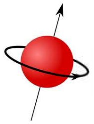
Keano De Vos, Gert de Cooman, Jasper De Bock, Alexander Erreygers, Natan T'Joens

Foundations Lab for imprecise probabilities, Ghent University, Belgium

GHENT UNIVERSITY

We describe uncertainty in quantum mechanics using coherent partial orderings based on utility functions.

Quantum mechanics



$|\Psi\rangle$

$\hat{A}$

λ

System: A *state* $|\psi\rangle$ in a finite dimensional Hilbert space $\mathcal{H}$.

Measurement: A Hermitian operator $\hat{A}$. The *possible outcomes* are its real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$.

Probability

① **Density operator**: $\hat{\rho} := \sum_{i=1}^{k} p(|\psi_i\rangle)|\psi_i\rangle\langle\psi_i|$ with set of possible states $\mathcal{S} = \{|\psi_1\rangle, \ldots, |\psi_p\rangle\}$. and probability mass function $p : \mathcal{S} \to [0,1]$.

② **Born's rule**: $p(\lambda) = \langle\psi|\hat{P}_{\mathcal{E}}|\psi\rangle$ with $\hat{P}_{\mathcal{E}}$ the projector on the eigenspace $\mathcal{E}$ of $\hat{A}$ corresponding to the eigenvalue λ.

Problems

Different probability distributions result in the same density operator.

Born's rule is assumed in a postulate. Where do these probabilities come from and how can they be interpreted?

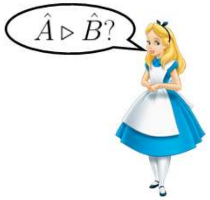
① ~ ①

② ~ ②

Decision-theoretic framework

What

Preference ordering $\triangleright$



How

$\hat{A} \triangleright \hat{B}$

$u_{\hat{A}}(|\Psi\rangle) \triangleright u_{\hat{B}}(|\Psi\rangle)$

Axioms

Rationality criteria on $\triangleright$

(O1) $\hat{A} \not\triangleright \hat{A}$ [Irreflexivity]

(O2) $\hat{A} \triangleright \hat{B}$ and $\hat{B} \triangleright \hat{C} \implies \hat{A} \triangleright \hat{C}$ [Transitivity]

(O3) $\hat{A} \gtrsim \hat{B} \implies \hat{A} \triangleright \hat{B}$ [Monotonicity]

(O4) $\hat{A} \triangleright \hat{B} \implies \hat{B} + \hat{C} \triangleright \hat{B} + \hat{C}$ [Additivity]

(O5) $\hat{A} \triangleright \hat{B}$ and $c > 0 \implies c\hat{A} \triangleright c\hat{B}$ [Positive scaling]

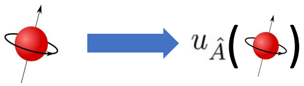
Result

Coherent partial preference ordering $\triangleright$

What

Hermitian operator $\hat{A}$

Utility function $u_{\hat{A}}$



Axioms

Decision-theoretic postulates

(DT1) If $|a\rangle$ is an eigenket with eigenvalue λ, then $u_{\hat{A}}(|a\rangle) = \lambda$.

(DT2) If $\hat{B}$ and $|\psi_{\hat{B}}\rangle$ are the basis permuted equivalents of $\hat{A}$ and $|\psi_{\hat{A}}\rangle$, then $u_{\hat{A}}(|\psi_{\hat{A}}\rangle) = u_{\hat{B}}(|\psi_{\hat{B}}\rangle)$.

(DT3) If $\hat{B}$ corresponds to $\hat{A}$ extended with the identical measurement $\hat{I}$, then $u_{\hat{A}}(|\psi\rangle) = u_{\hat{B}}(|\psi\rangle \otimes |\phi\rangle)$.

(DT4) If $\hat{A}, \hat{B}$ are simultaneously measurable (or thus commutating) operators, then $u_{\hat{A}+\hat{B}}(|\psi\rangle) = u_{\hat{A}}(|\psi\rangle) + u_{\hat{B}}(|\psi\rangle)$.

(DT5) The utility function is continuous: $\lim_{i\to+\infty} |\psi_i\rangle = |\psi\rangle \implies \lim_{i\to+\infty} u_{\hat{A}}(|\psi_i\rangle) = u_{\hat{A}}(|\psi\rangle)$.

Result

$u_{\hat{A}}(|\psi\rangle) = \langle\psi|\hat{A}|\psi\rangle$

Duality

Polar $\mathscr{M}$

Set of density operators

$\mathscr{M} = \left\{\hat{A} : (\forall\hat{B})(\underline{P}(\hat{B}) \leq \mathrm{Tr}(\hat{A}\hat{B}))\right\}$

$\underline{P}(\hat{A}) = \inf_{\hat{\rho}\in\mathscr{M}} \mathrm{Tr}(\hat{\rho}\hat{A})$

$\mathscr{M} = \{\hat{\rho} : (\forall \hat{A} \triangleright 0)$ $\mathrm{Tr}(\hat{\rho}(\hat{A})) \geq 0\}$

Lower expectation $\underline{P}(\hat{A})$

Normalised bounded super-linear real functional

$\underline{P}(\hat{A}) := \sup\{\alpha \in \mathbb{R} : \hat{A} \triangleright \alpha\hat{I}\}$

Precise case

Polar is singleton

$\mathscr{M} = \{\hat{\rho}\}$ ①

Lower expectation is linear

$\underline{P}(\hat{A}) = P(\hat{A})$
$= \mathrm{Tr}(\hat{\rho}\hat{A})$ ②

Reference

(Benavoli et al., 2019) also implemented desirable gambles in quantum mechanics.

Differences: Assumptions, Interpretation.

Similarities: Polar, Lower expectation.