



Jasper De Bock  
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A convenient characterization of  
convergent upper transition  
operators





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## upper transition operator

$$\bar{T}: \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}} : f \rightarrow \bar{T}f$$

$$\text{T1. } \bar{T}(f + g) \leq \bar{T}f + \bar{T}g$$

$$\text{T2. } \bar{T}(\lambda f) = \lambda \bar{T}f \text{ if } \lambda \geq 0$$

$$\text{T3. } \bar{T}f \leq \max f$$



# ergodicity

for all  $f \in \mathbb{R}^{\mathcal{X}}$

$(\overline{T}^n f)_{n \in \mathbb{N}}$  converges  
to a constant



# convergence

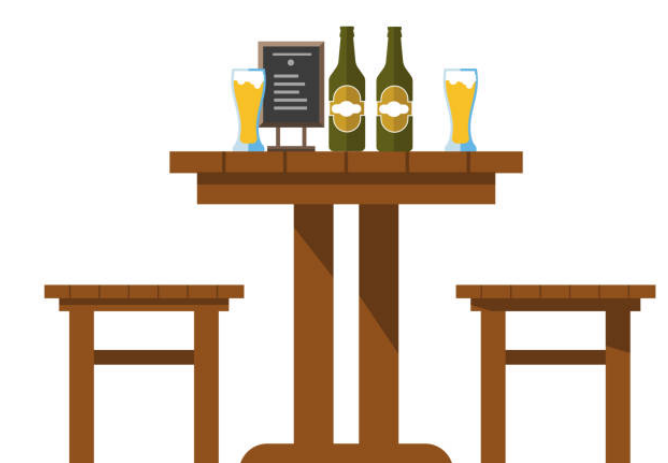
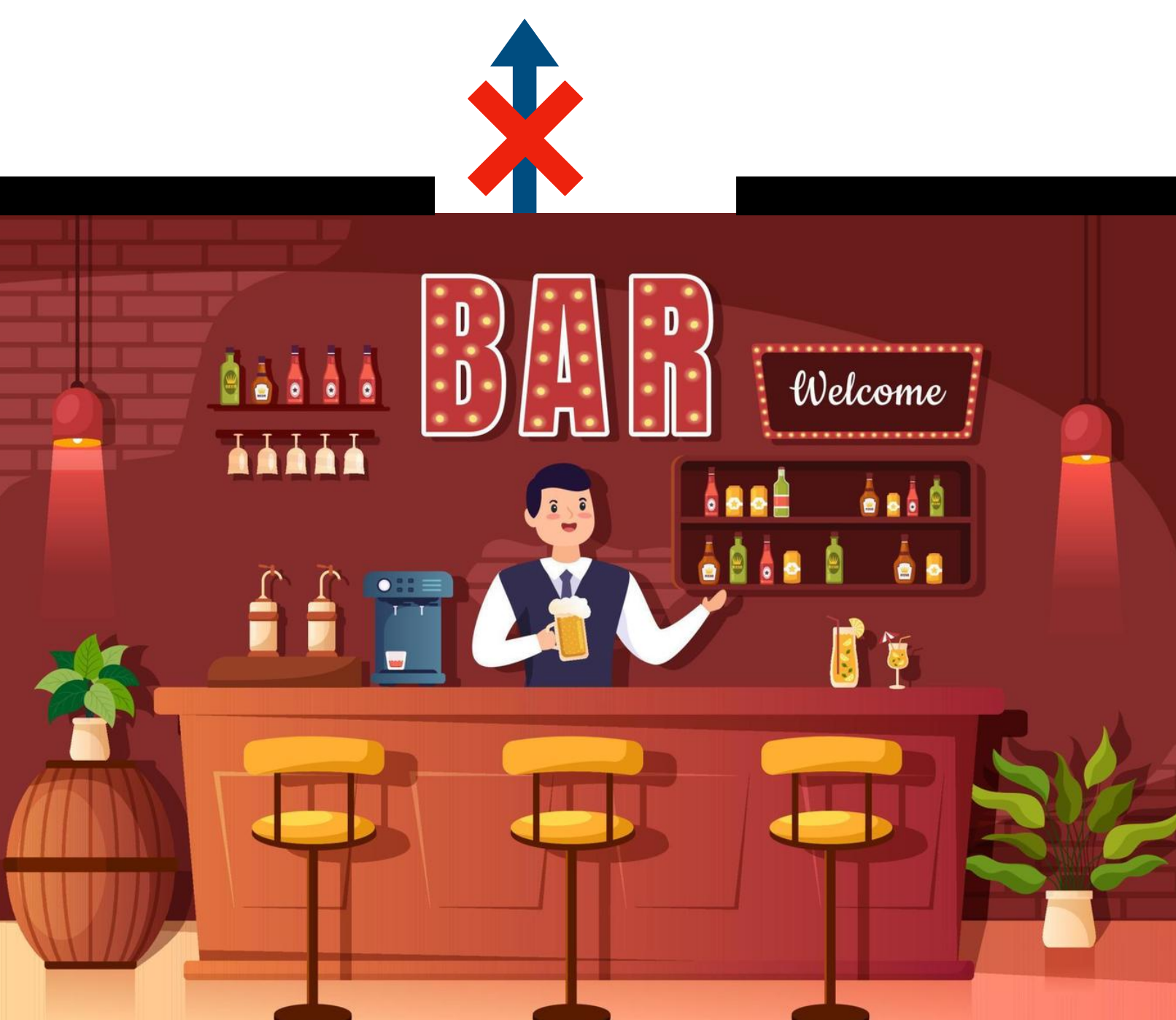
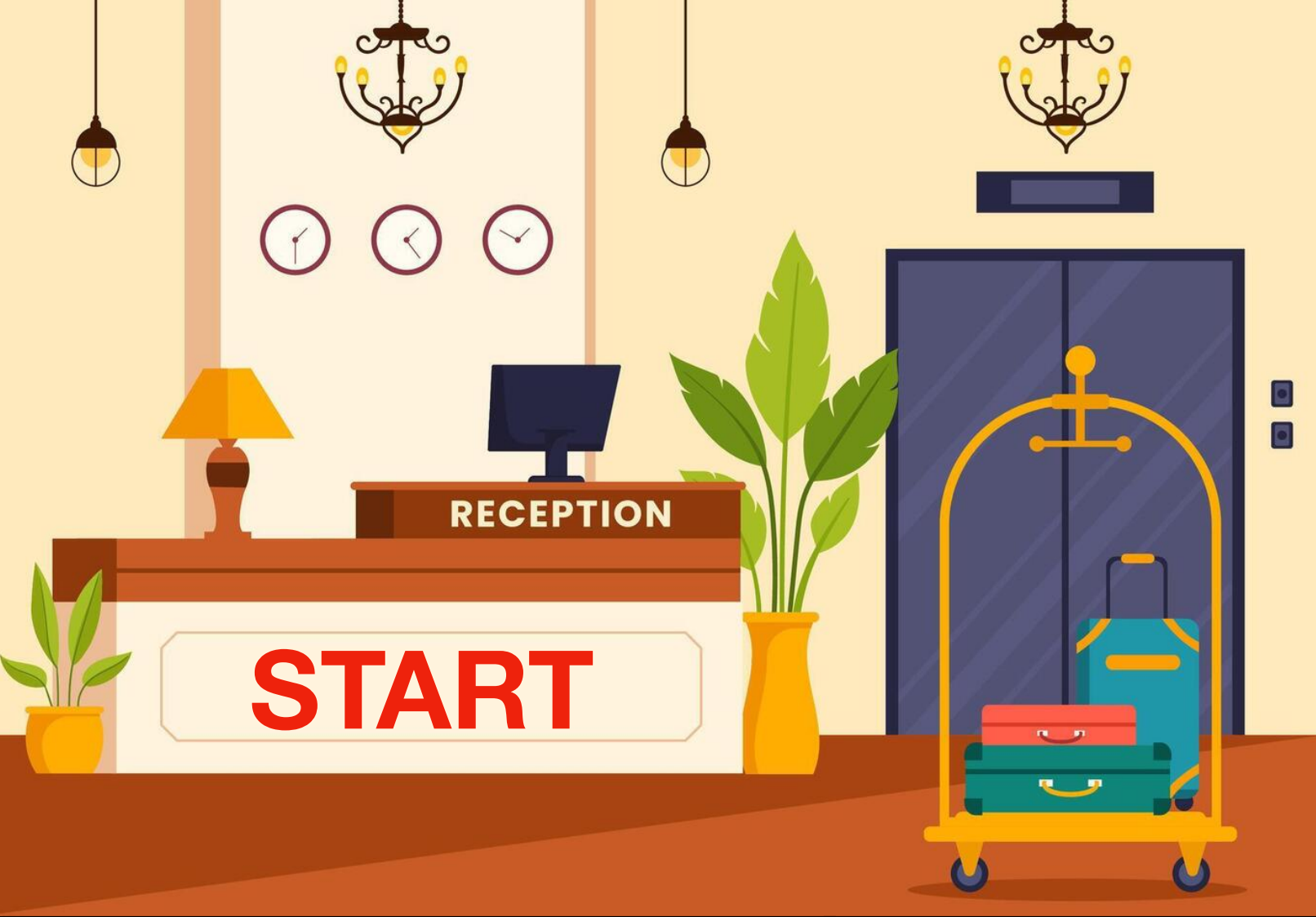
for all  $f \in \mathbb{R}^{\mathcal{X}}$

$(\bar{T}^n f)_{n \in \mathbb{N}}$  converges

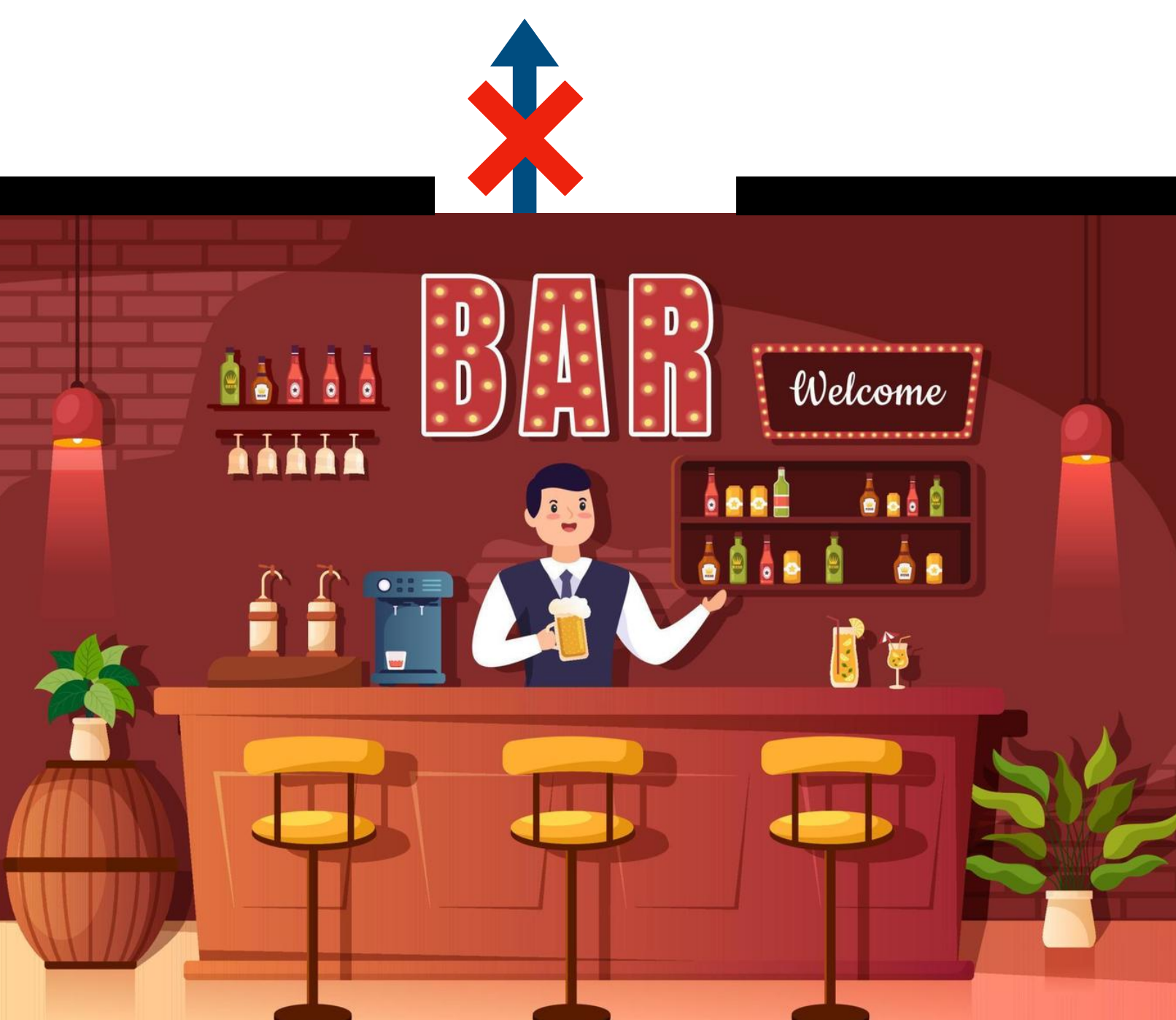
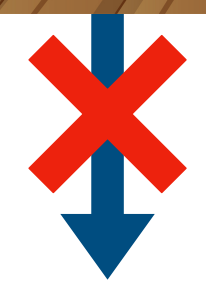
~~to a constant~~



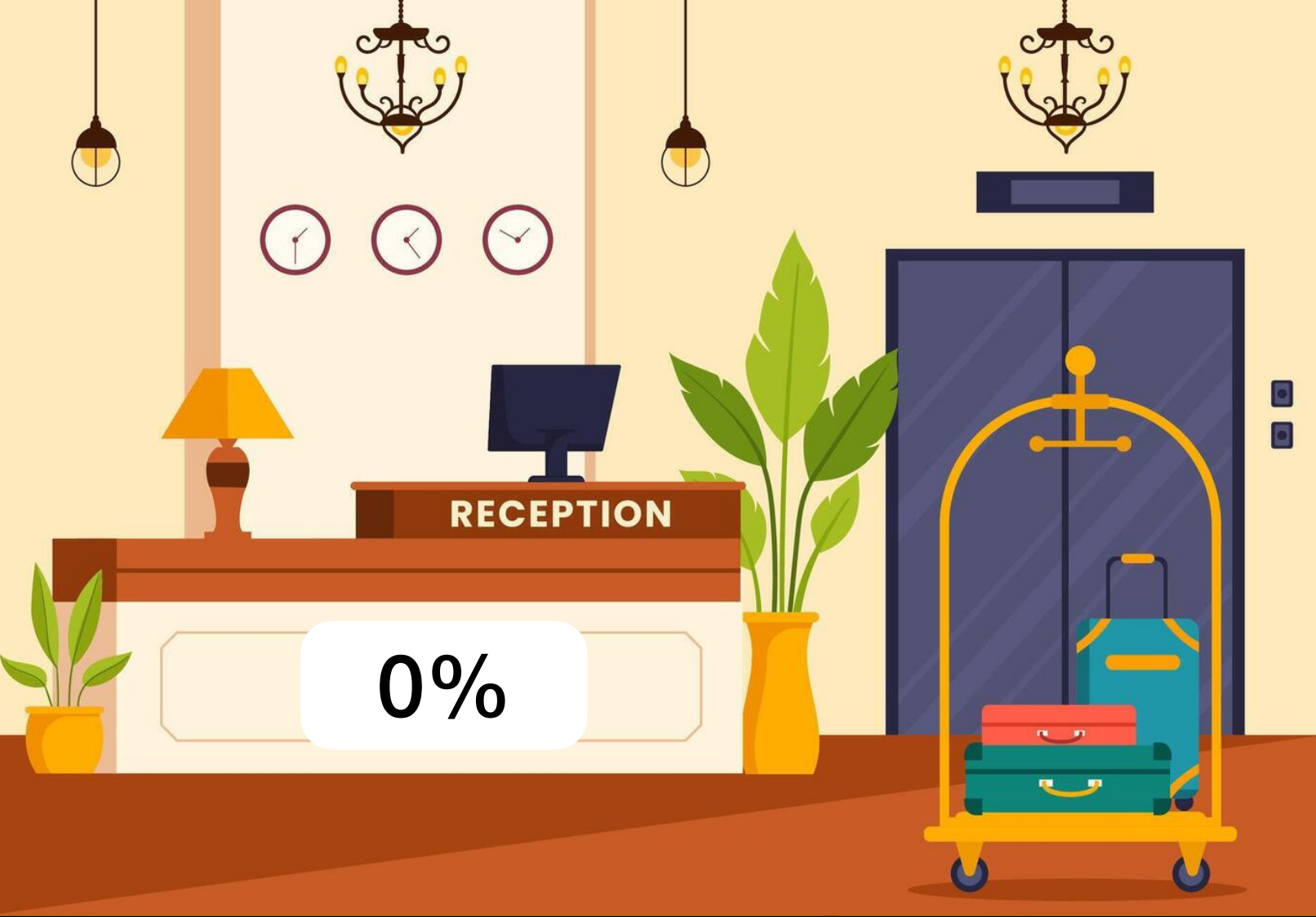




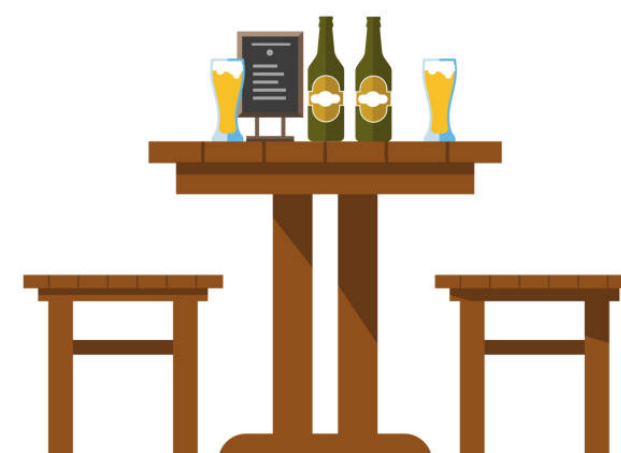




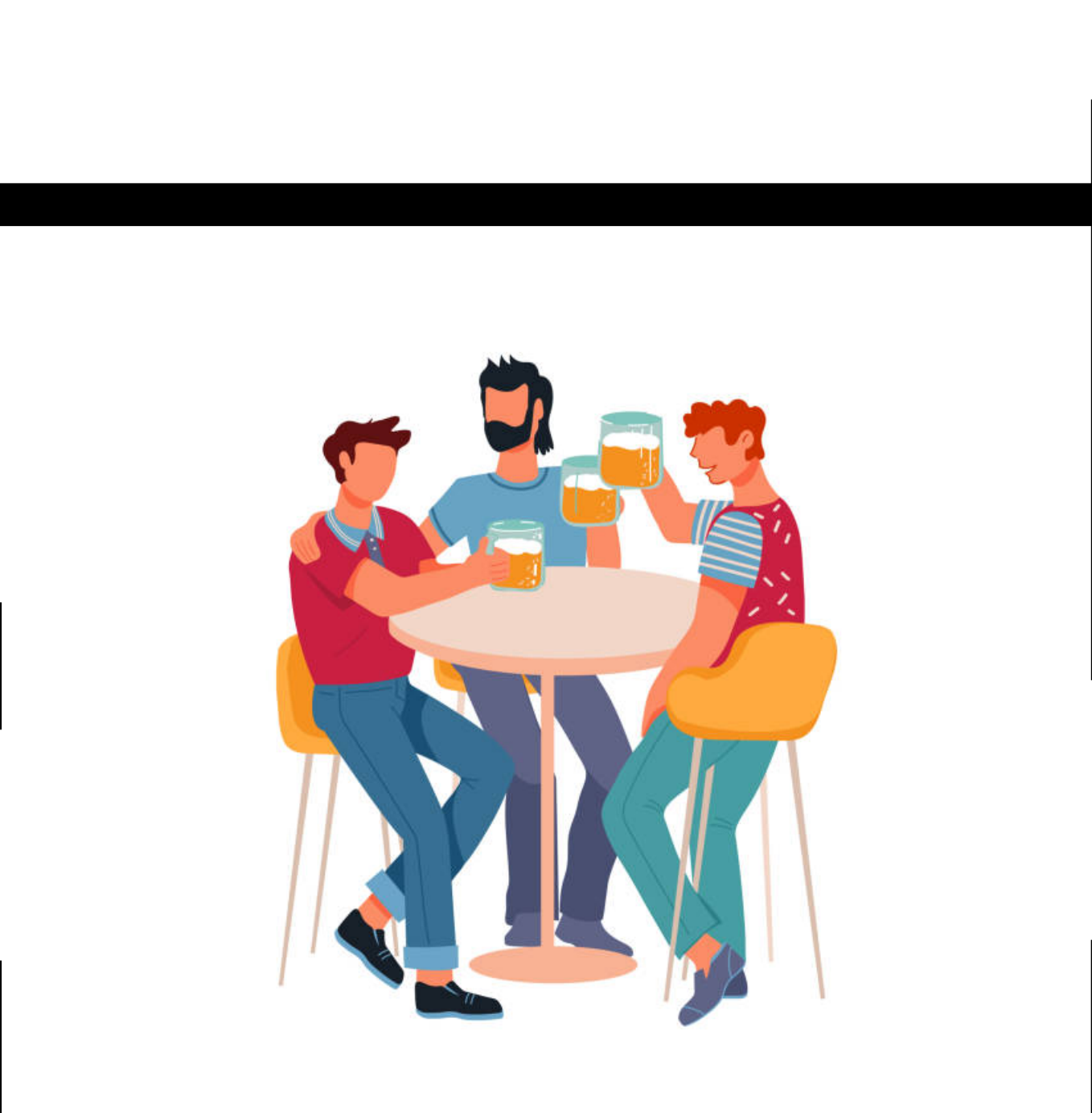
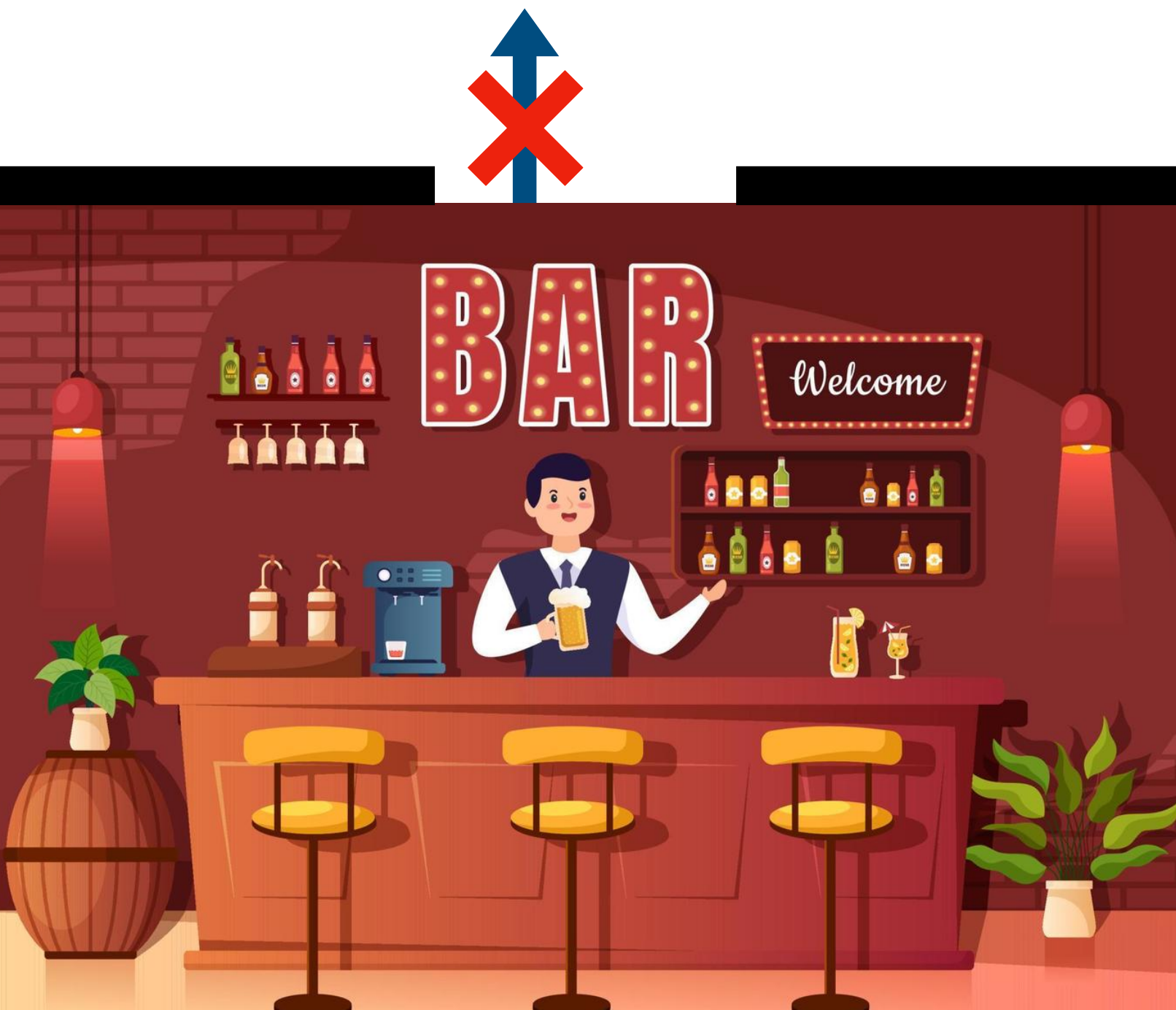
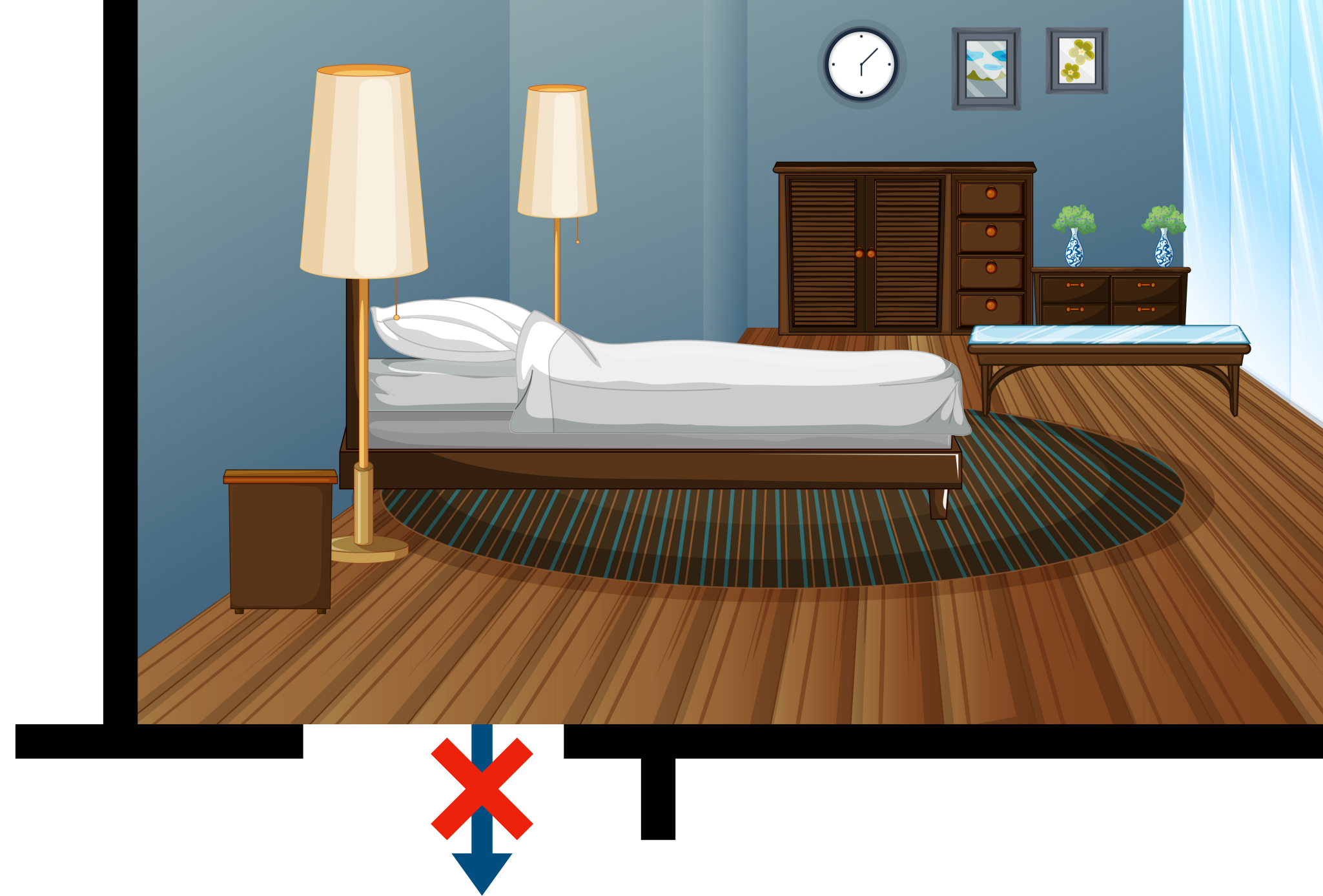
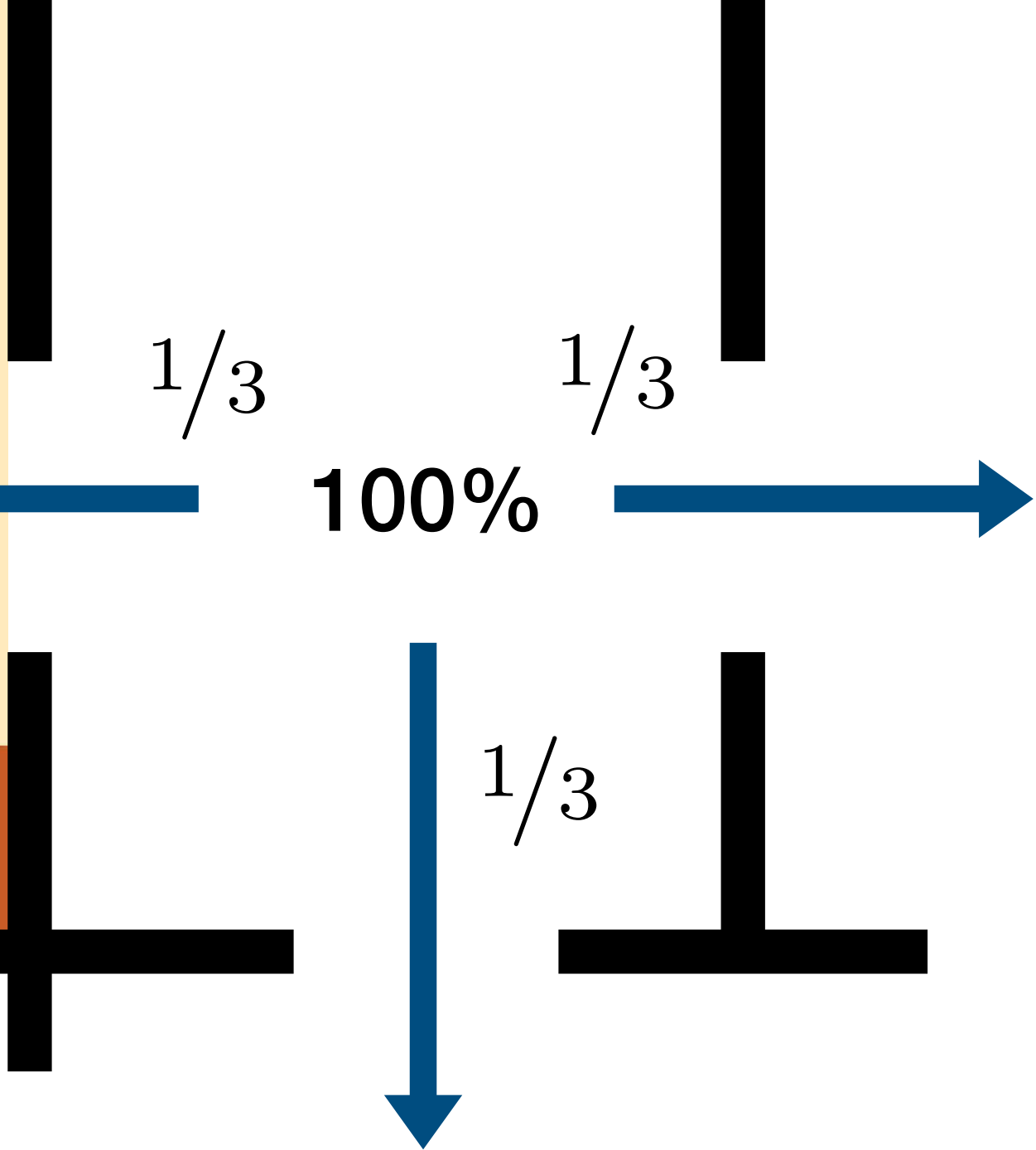
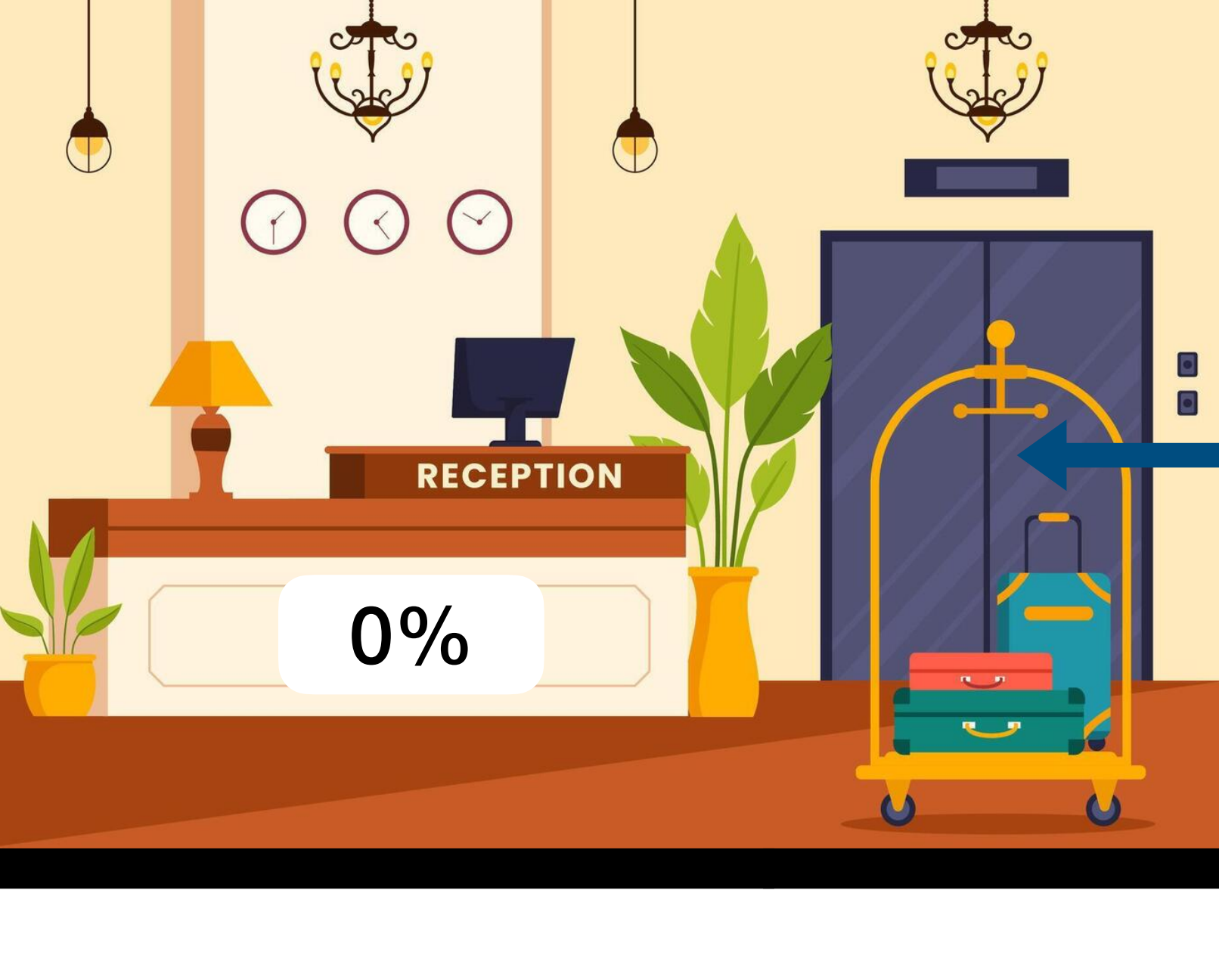




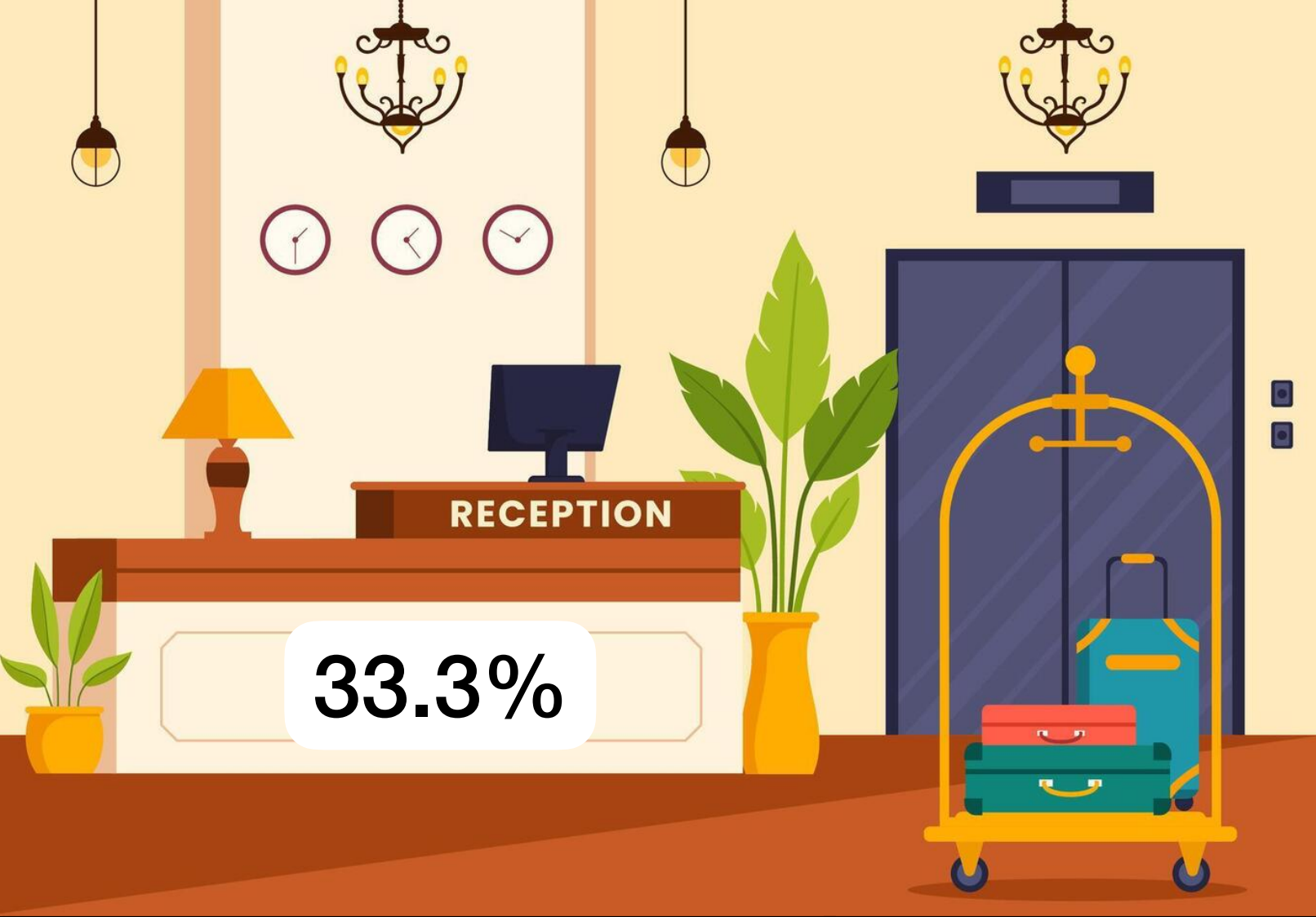
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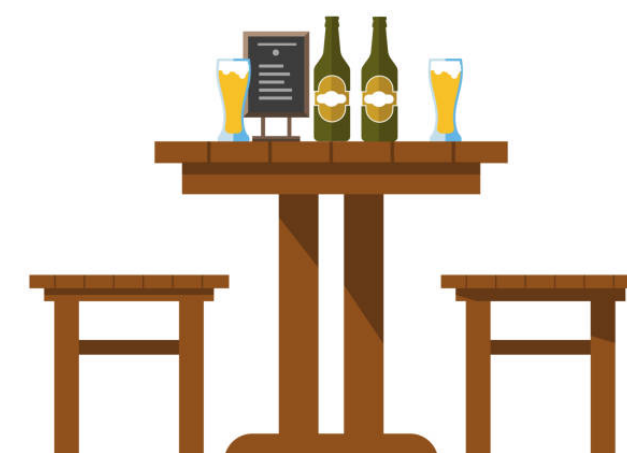


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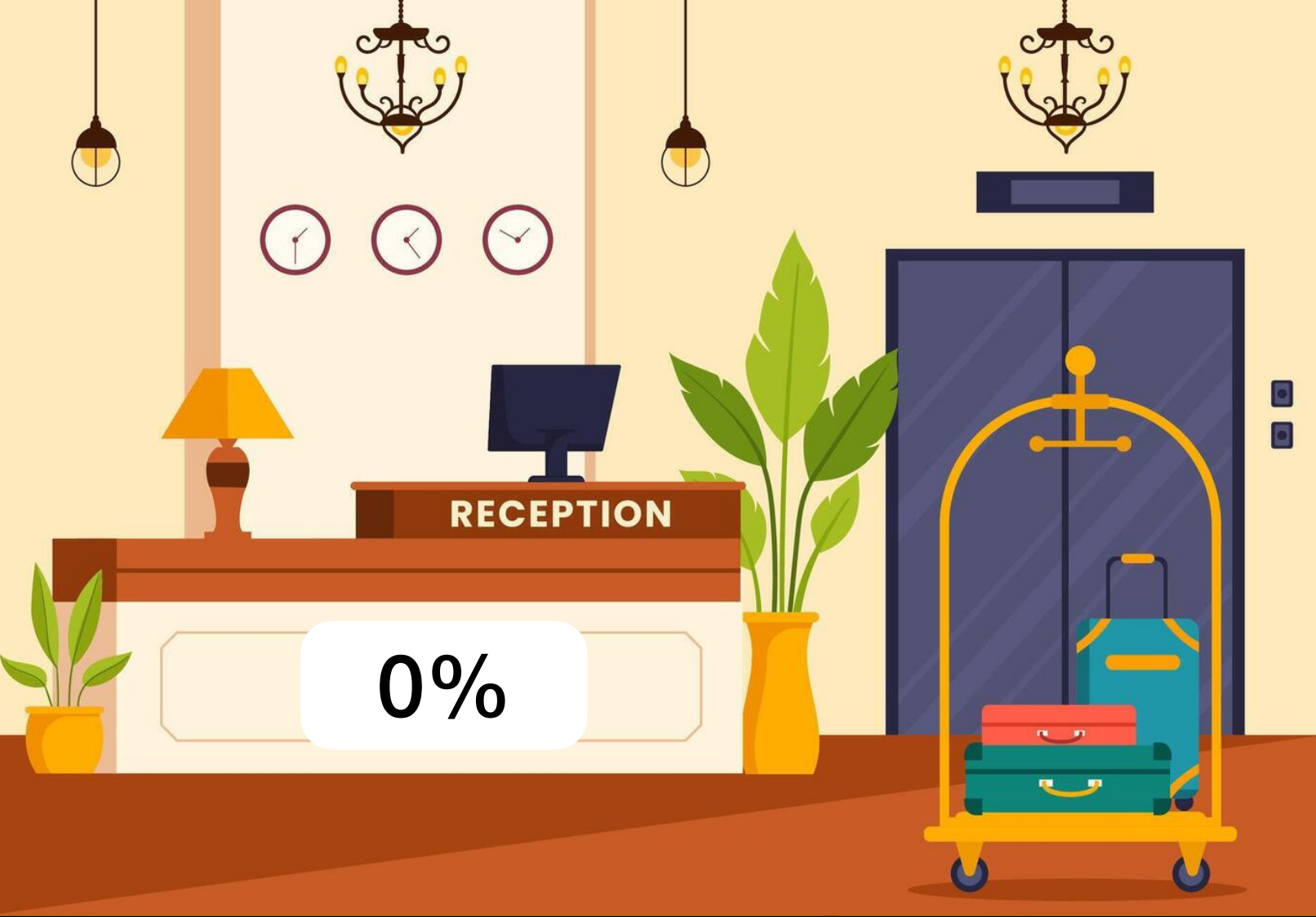
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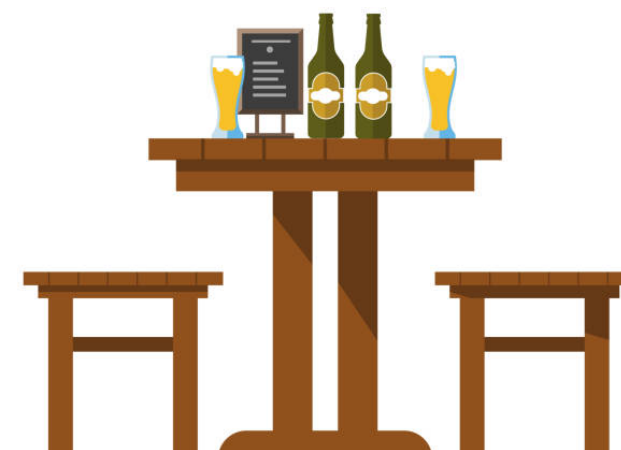
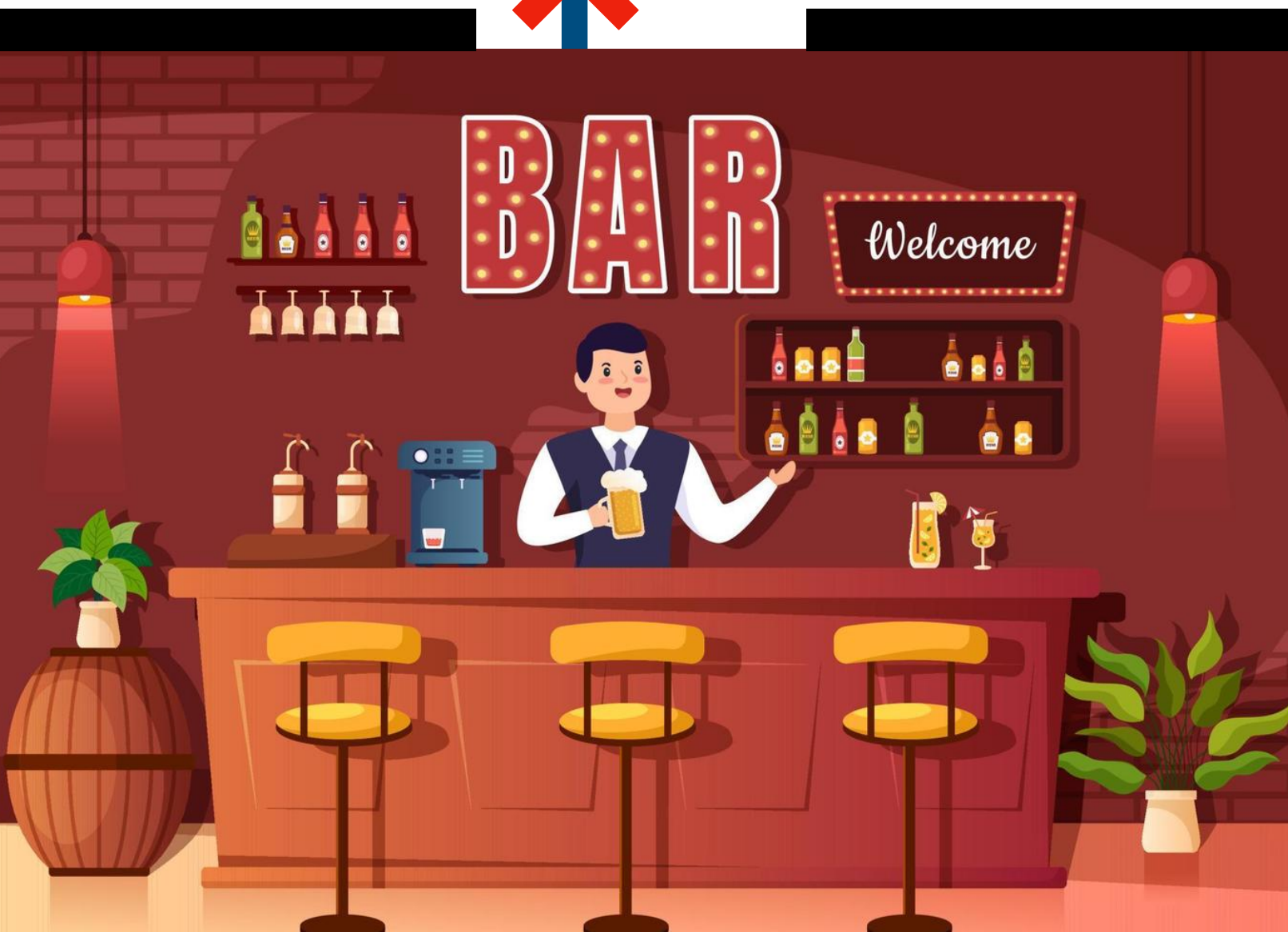
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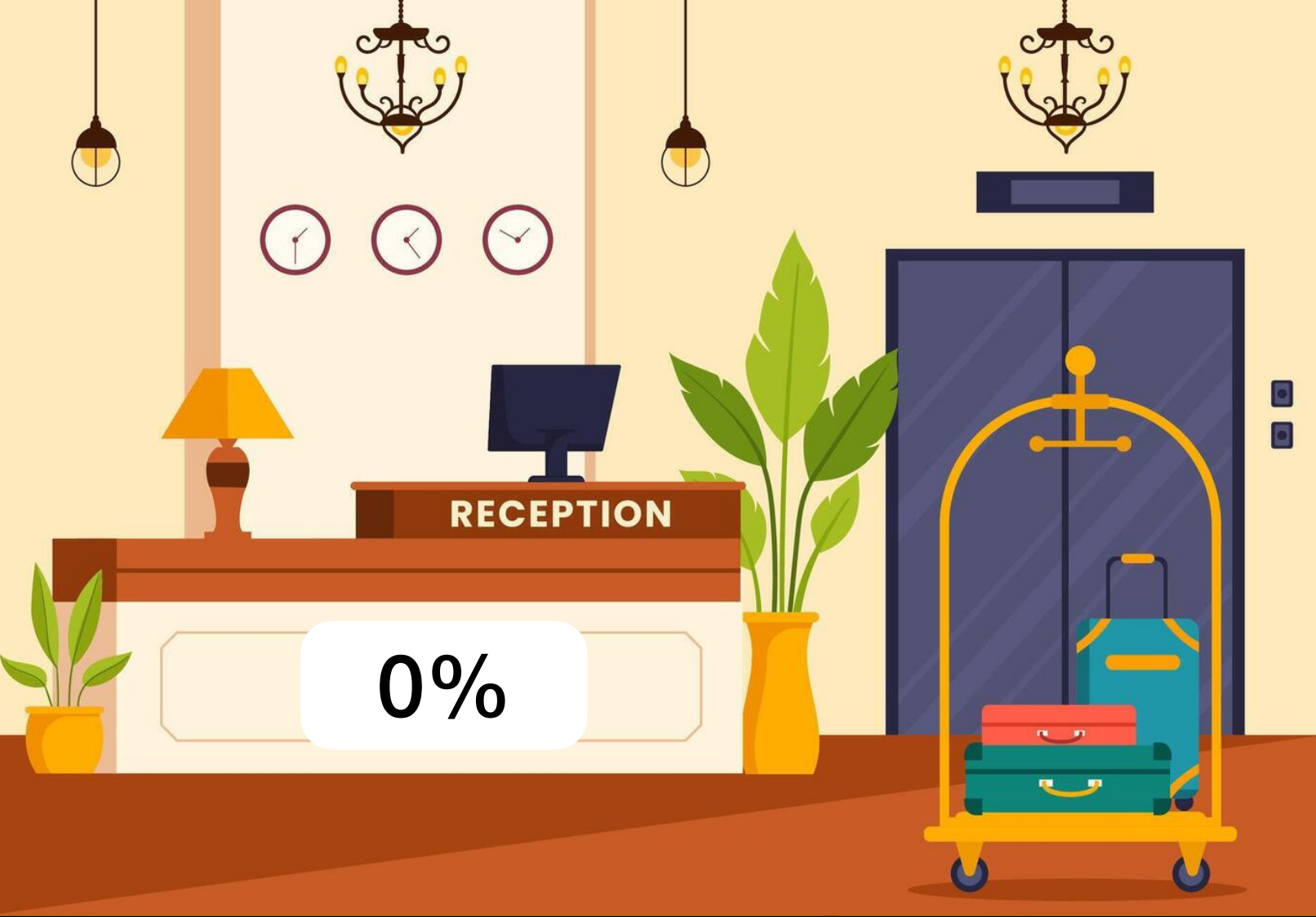
What  
happens if he  
keeps on  
walking?

BAR

Welcome







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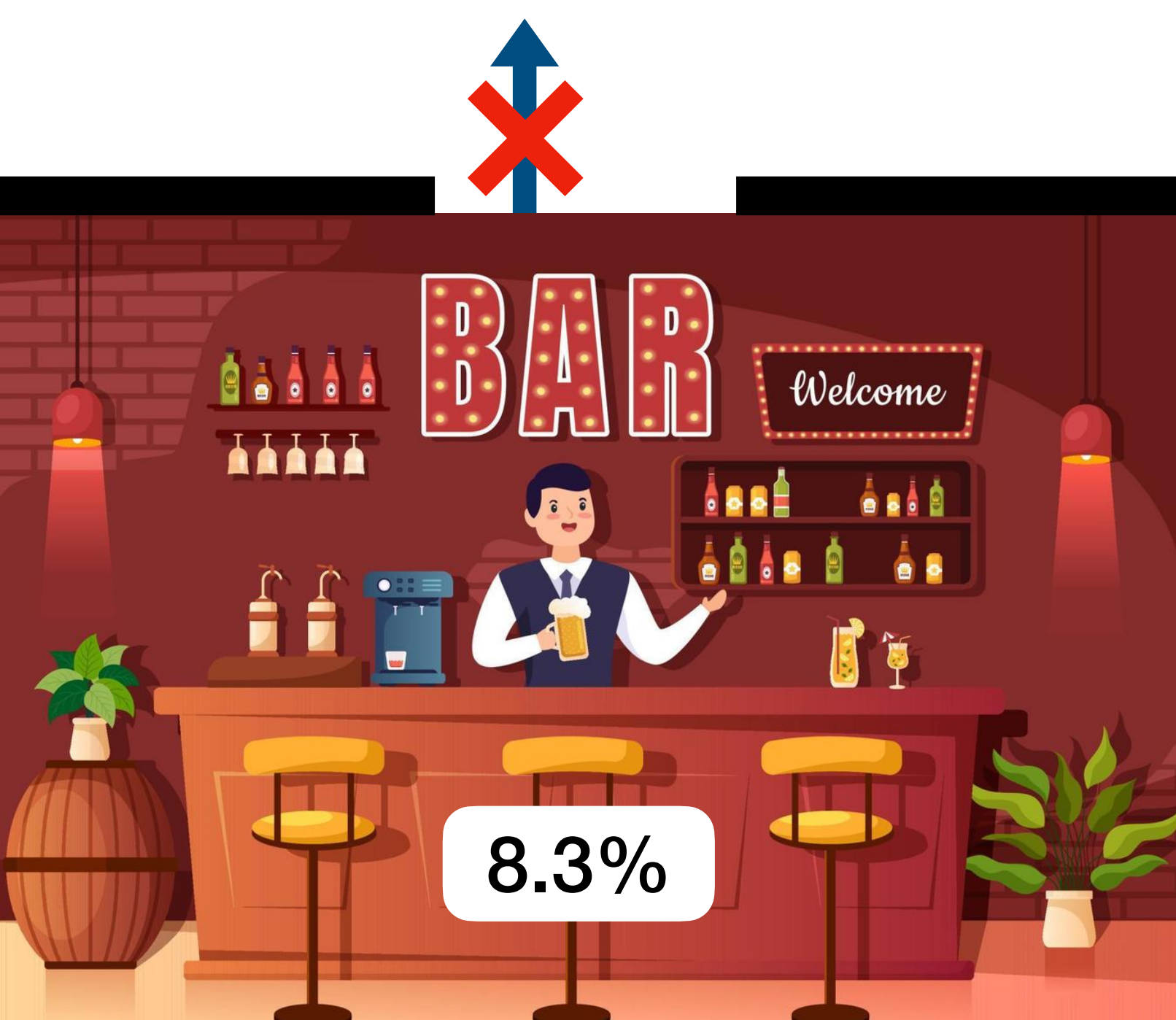


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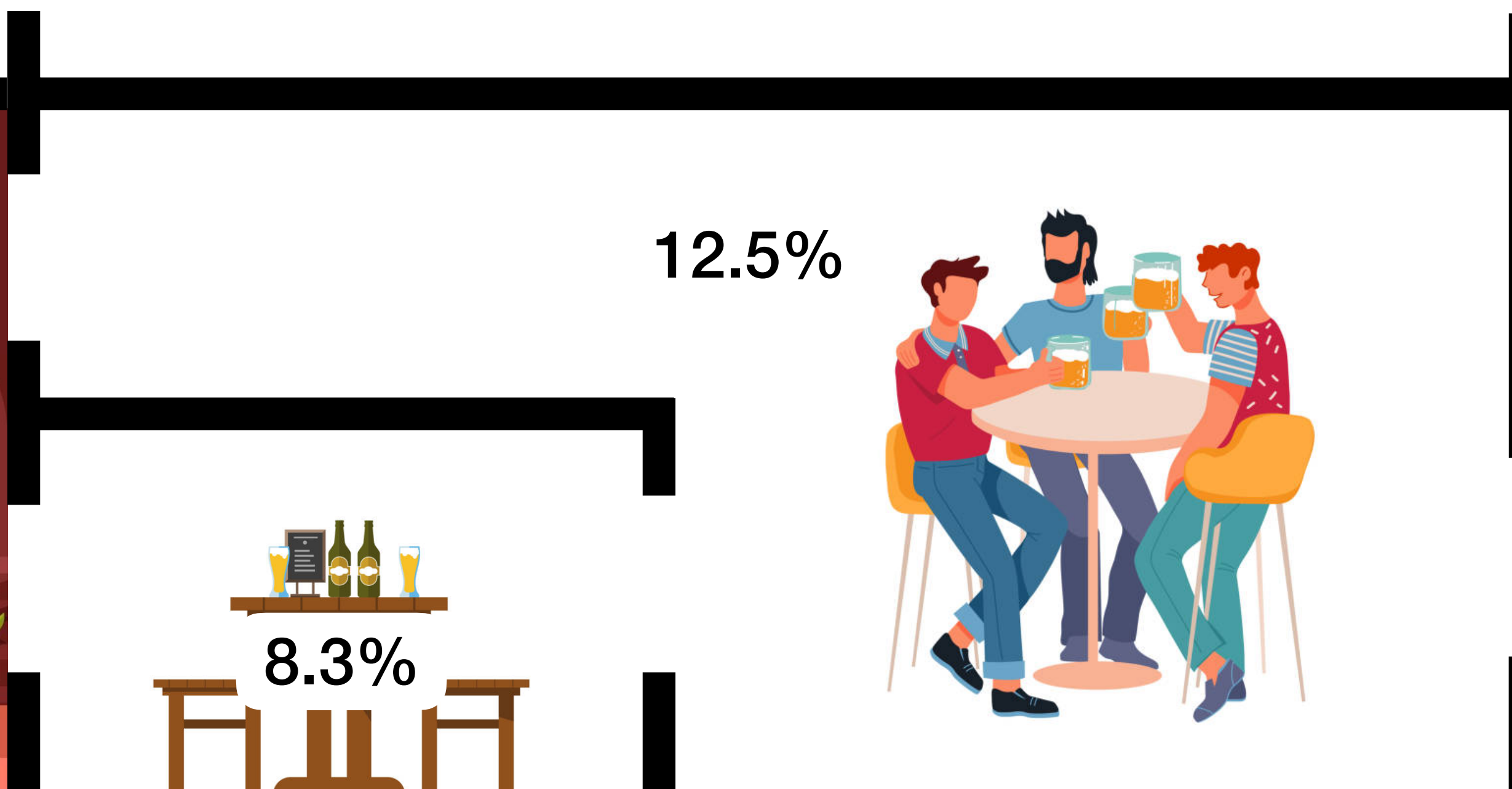


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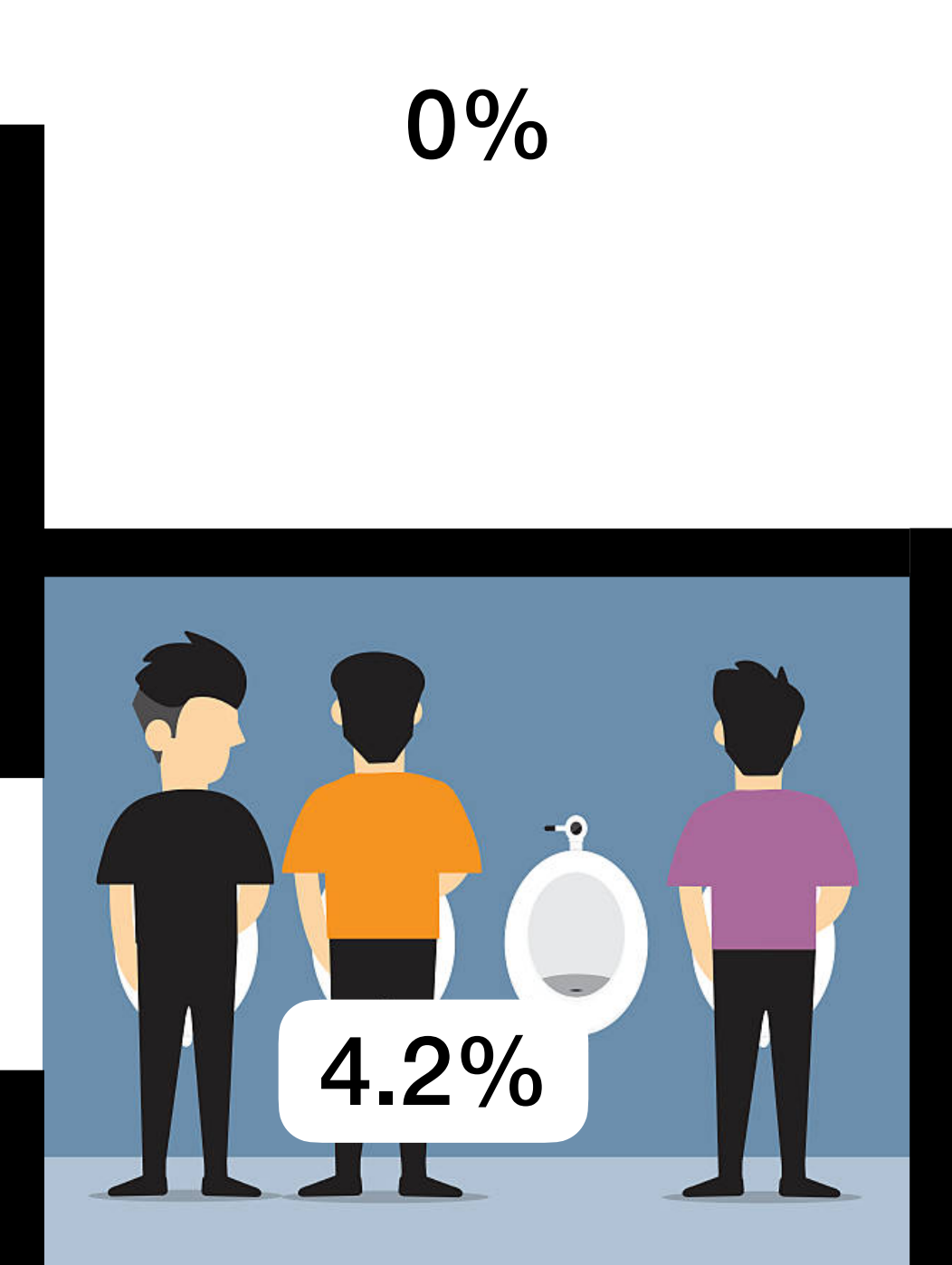


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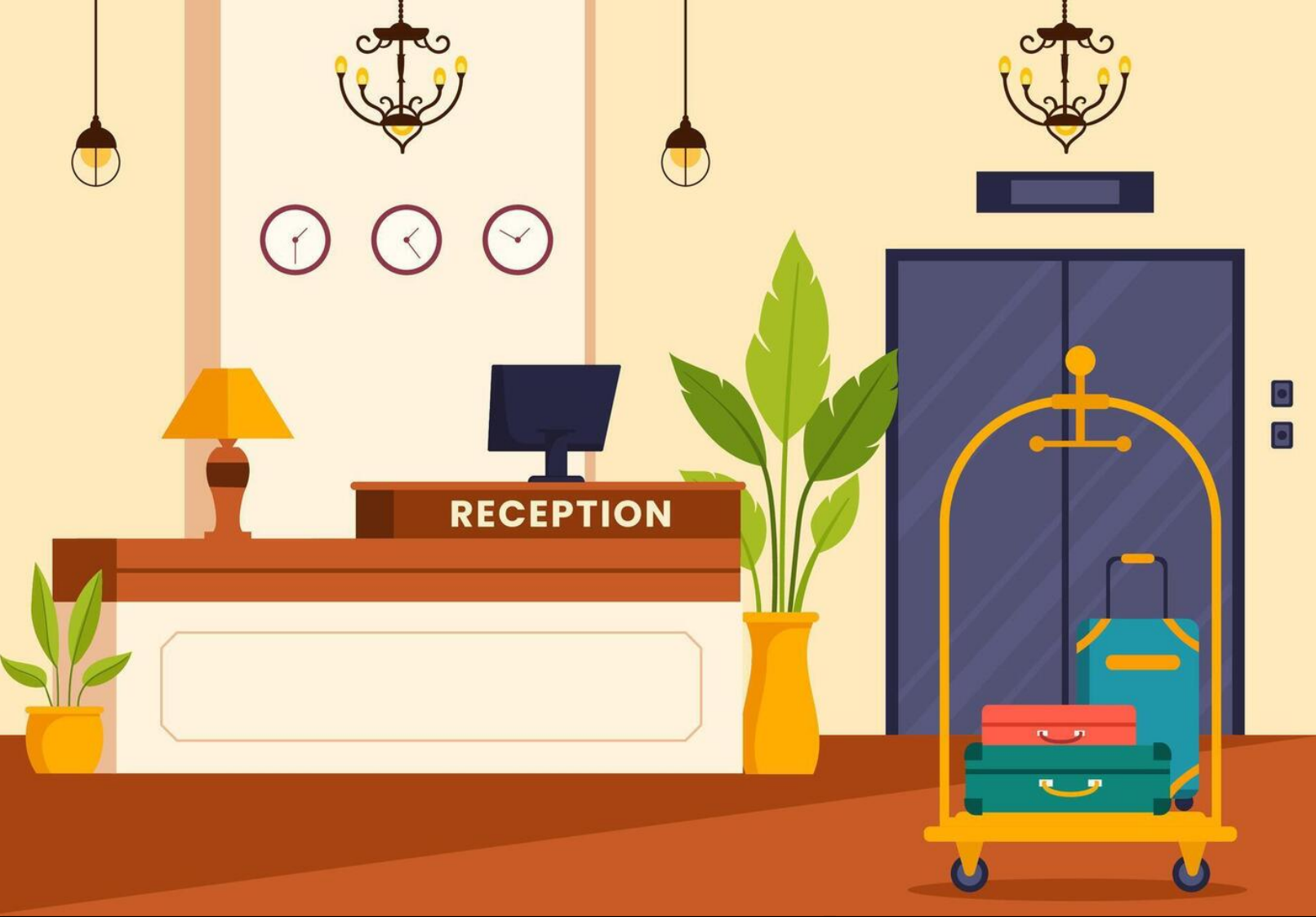
Does this  
change if the  
**START** is  
different?

BAR

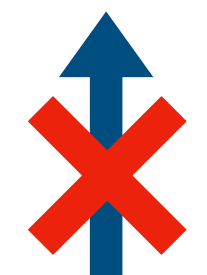
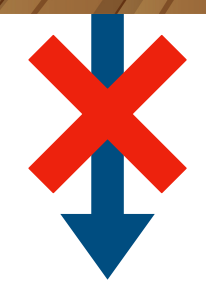
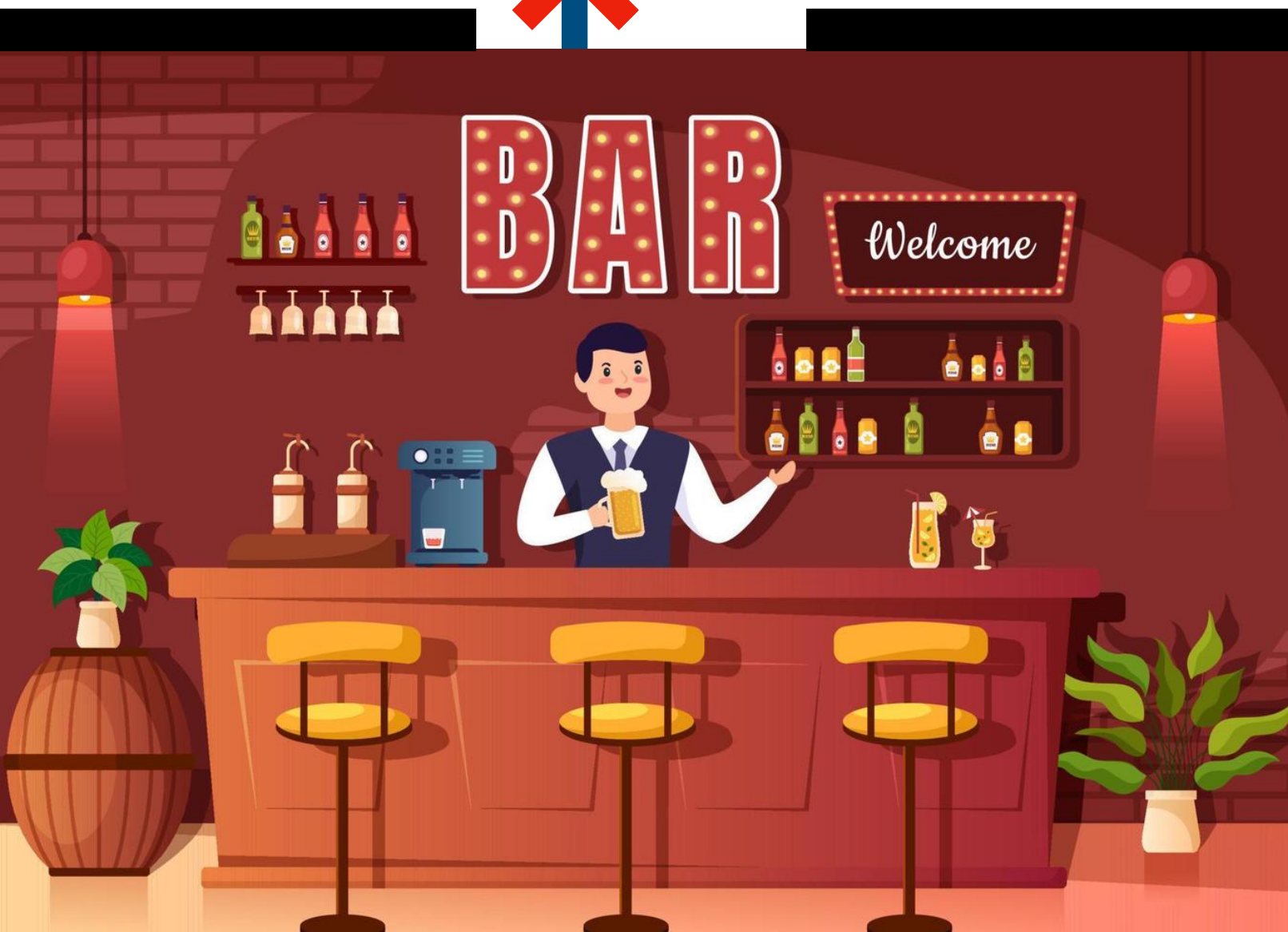
Welcome



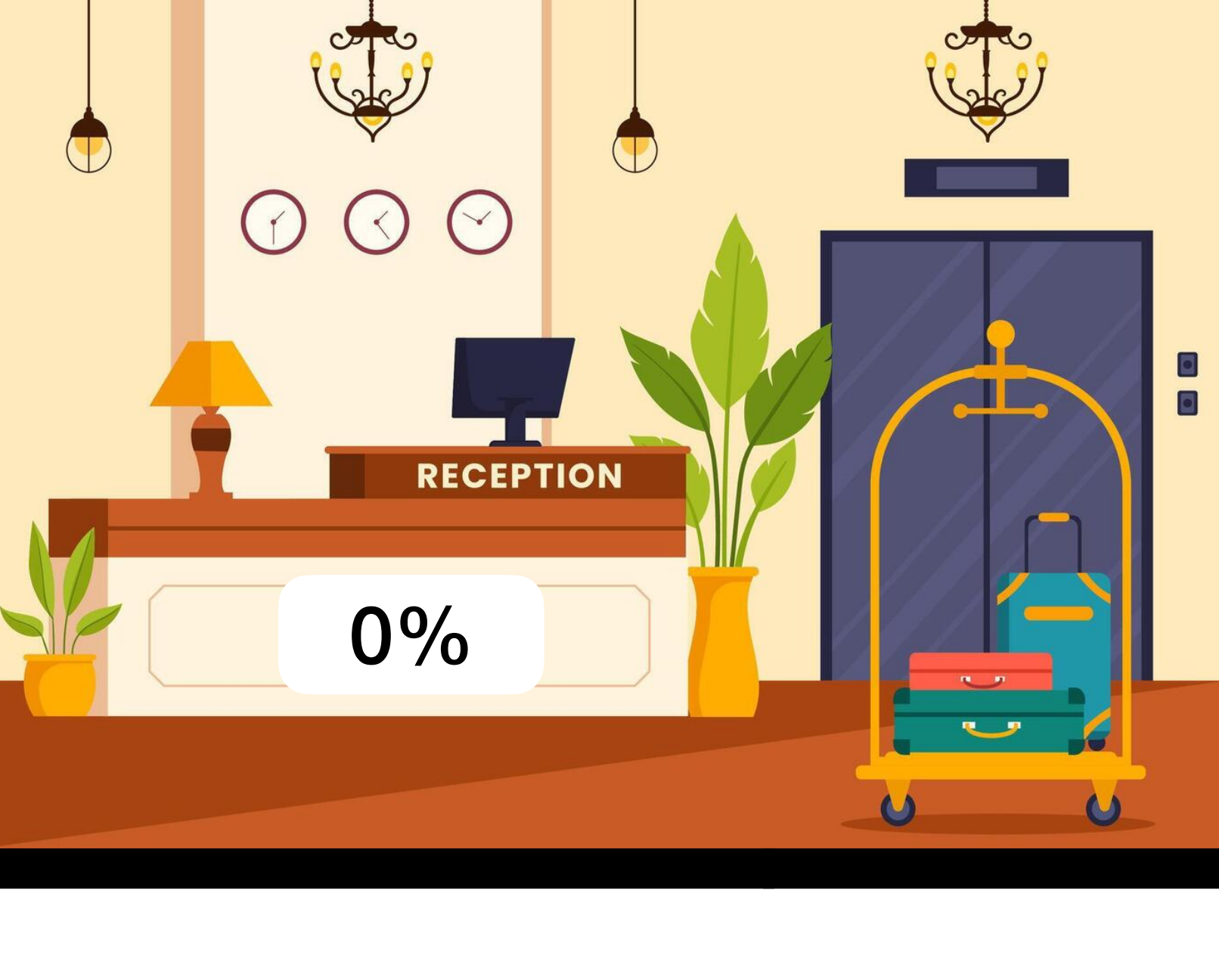




START





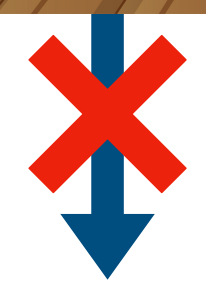


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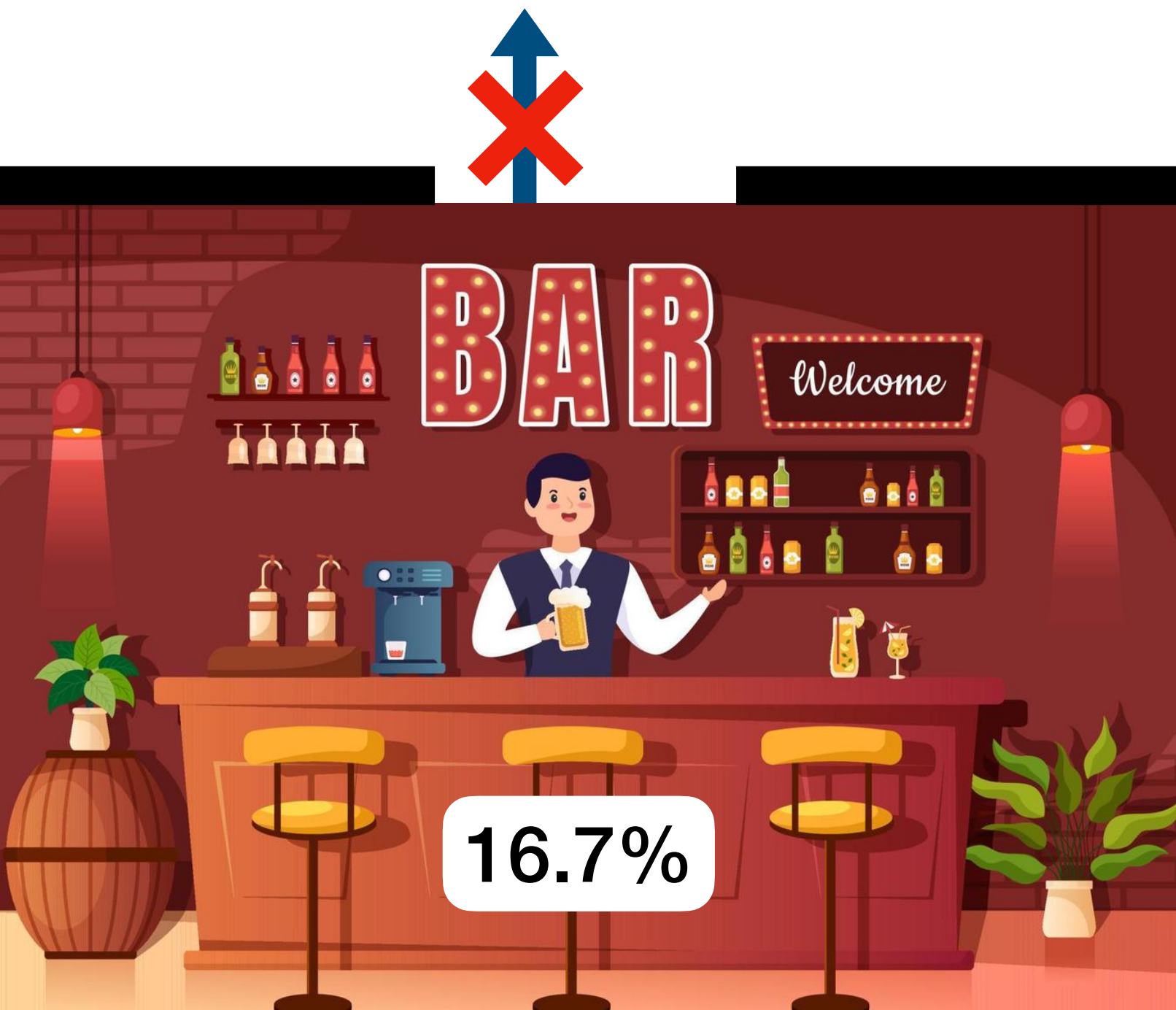


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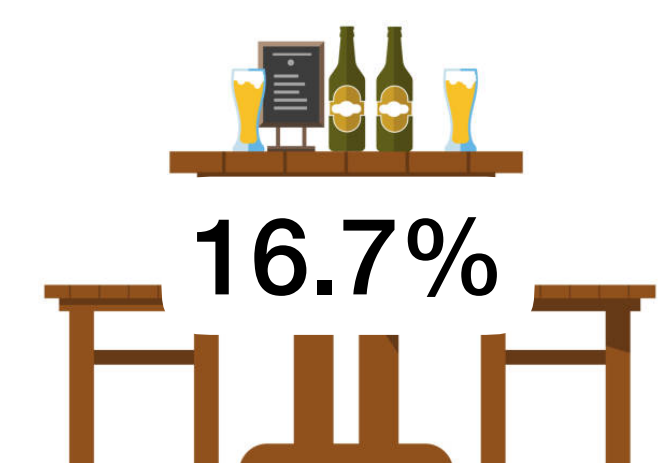


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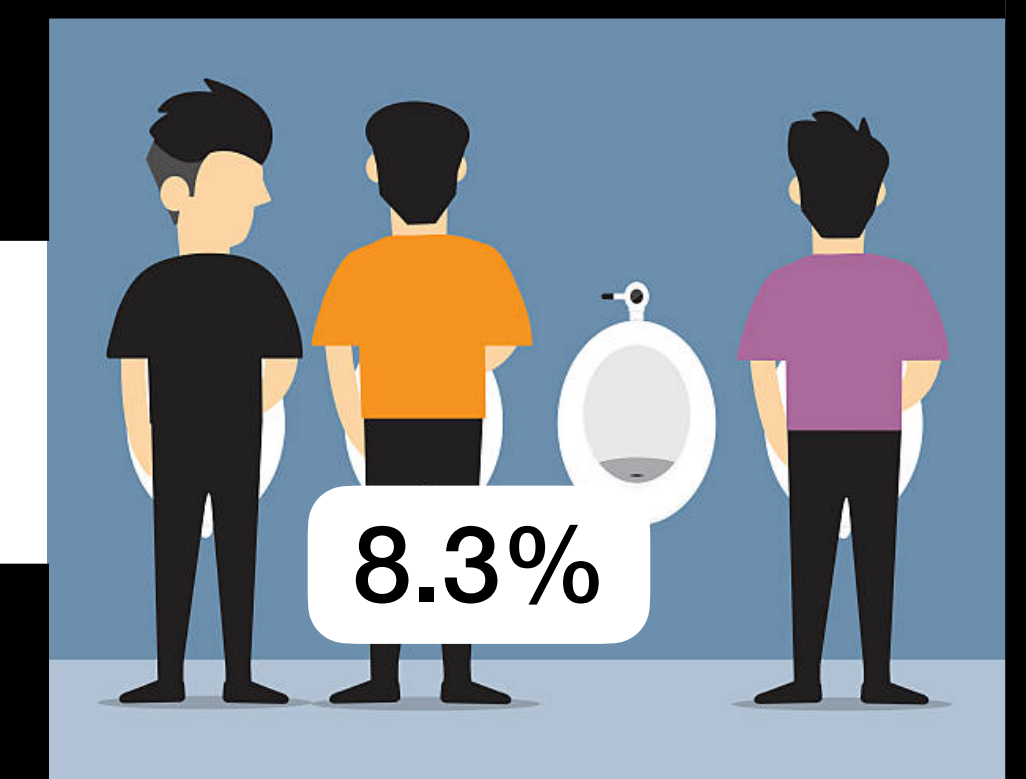


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8.3%





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# THEOREM

A precise Markov Chain is  
convergent if and only if  
all its maximal communication  
classes are aperiodic



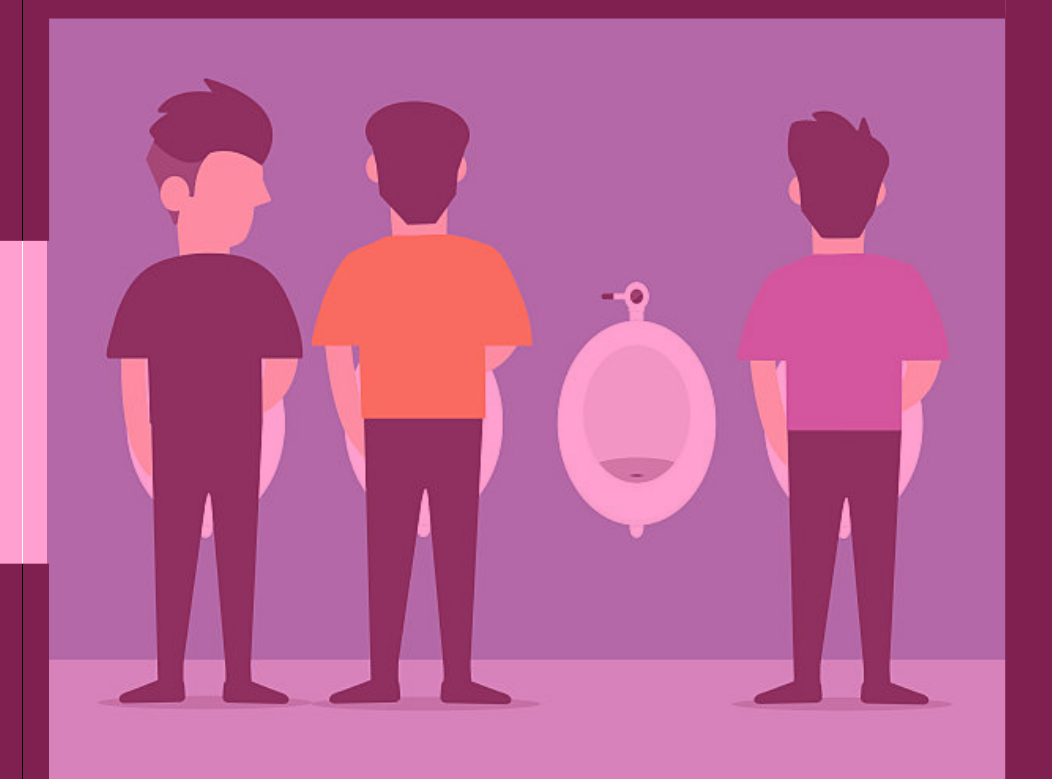
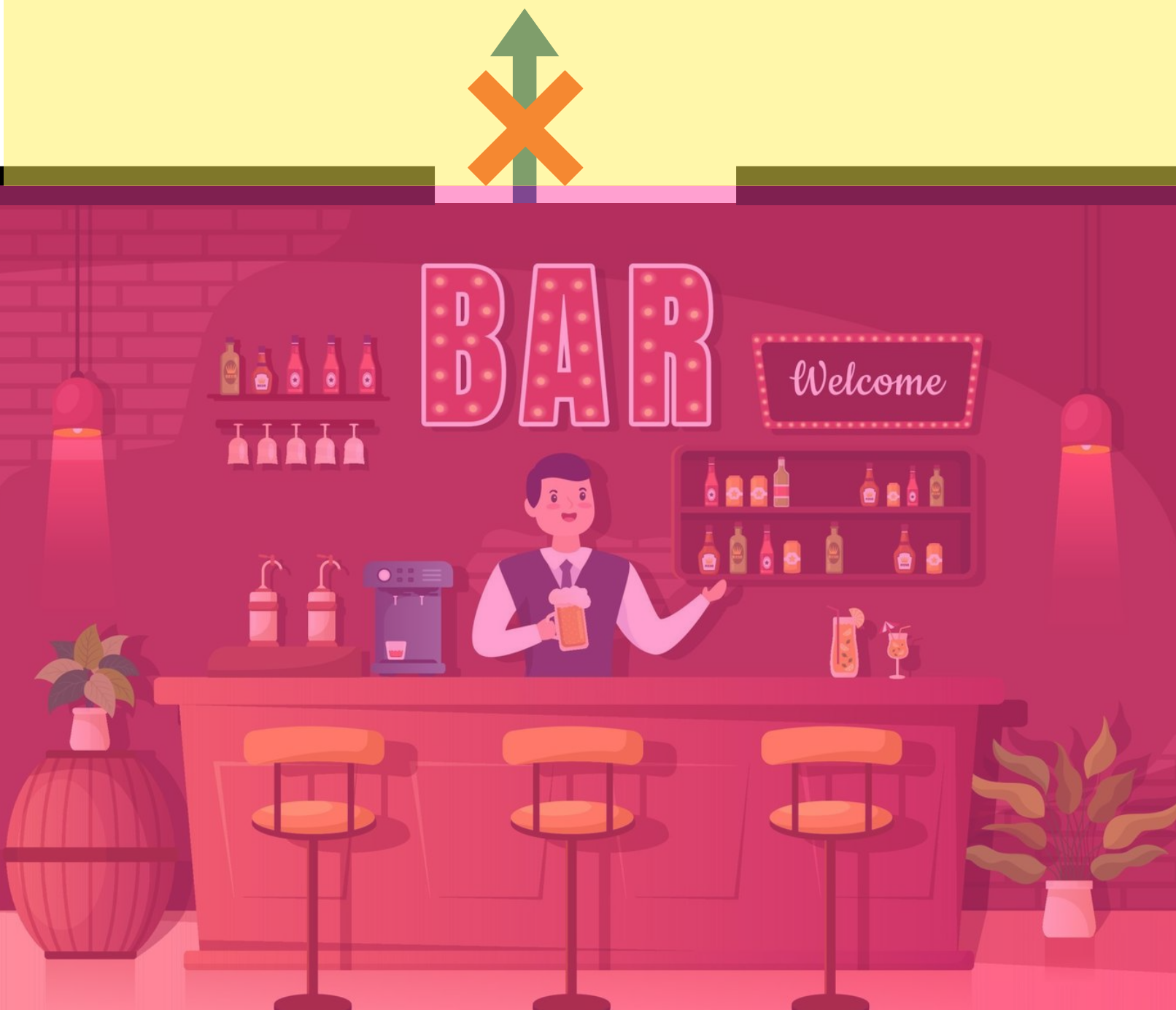


# THEOREM

A precise Markov Chain is convergent if and only if all its maximal communication classes are aperiodic









# THEOREM

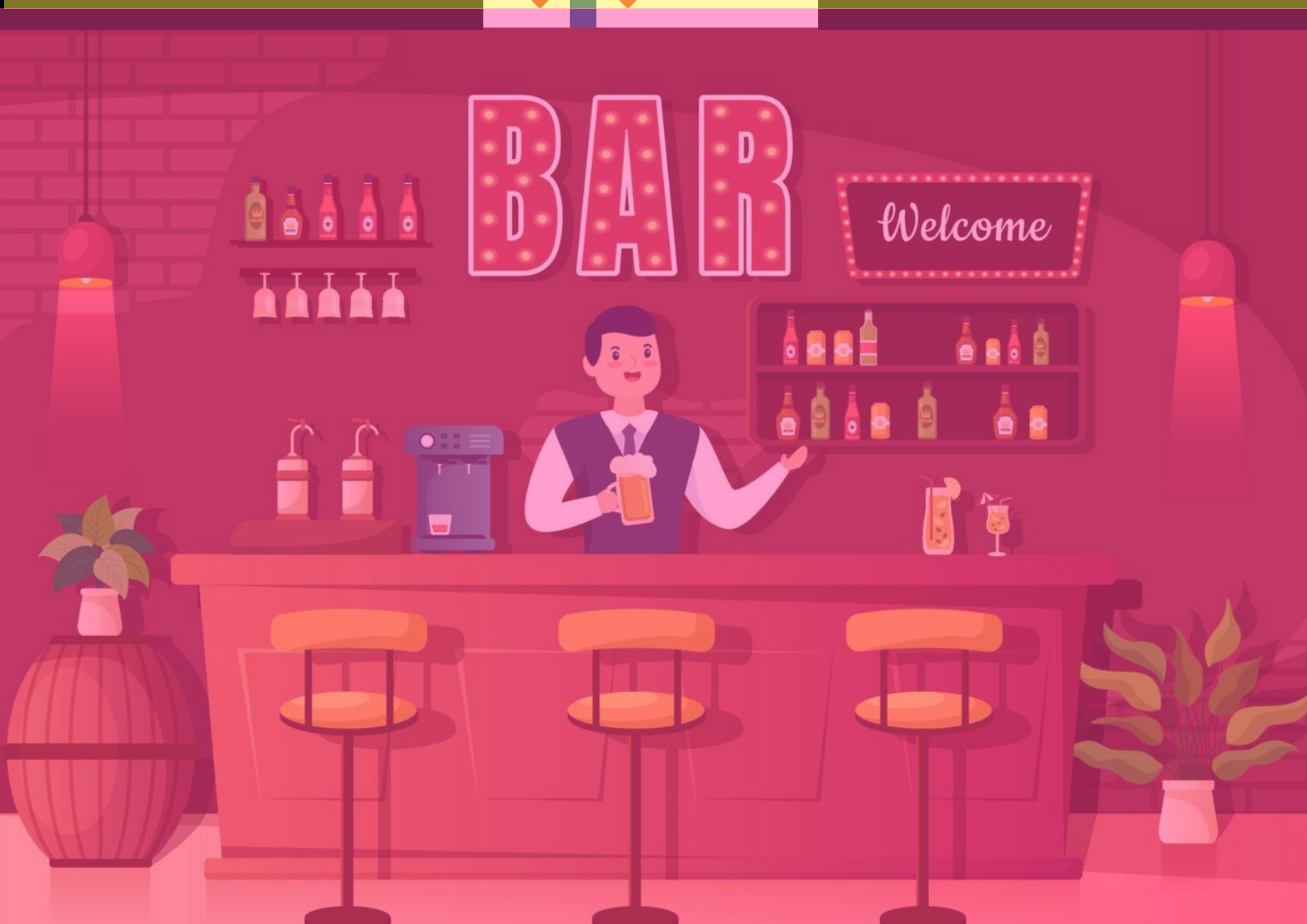
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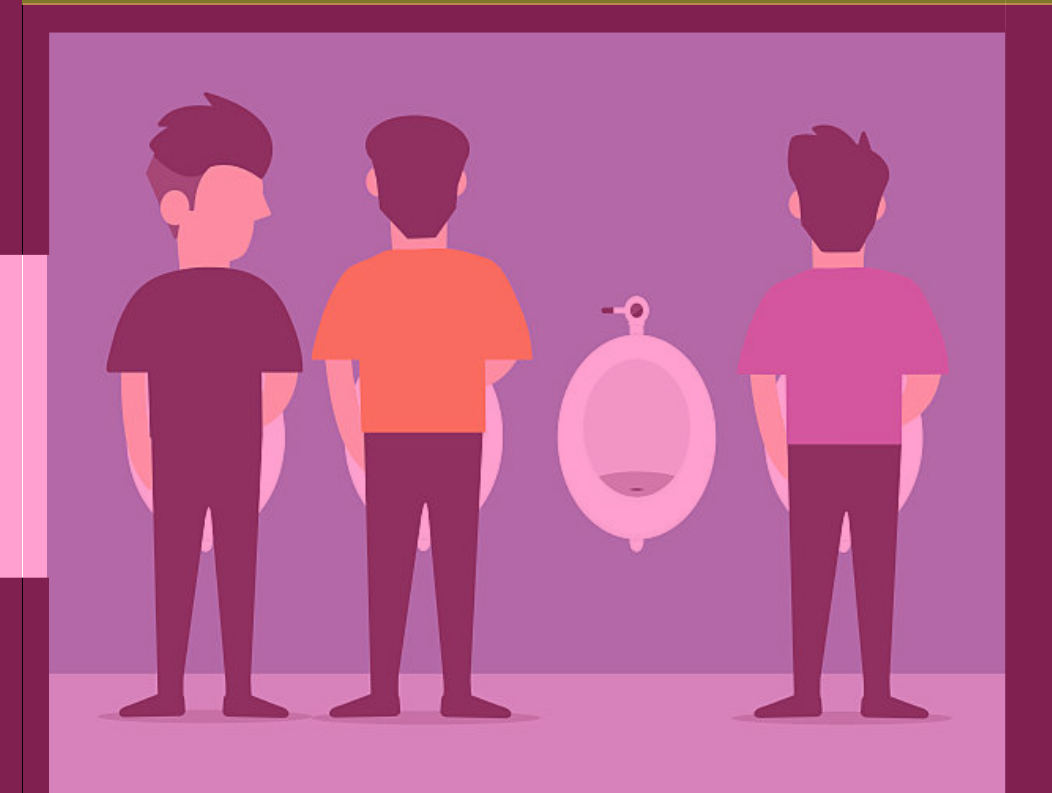




MAXIMAL  
COMMUNICATION  
CLAS



MAXIMAL  
COMMUNICATION  
CLAS





# THEOREM

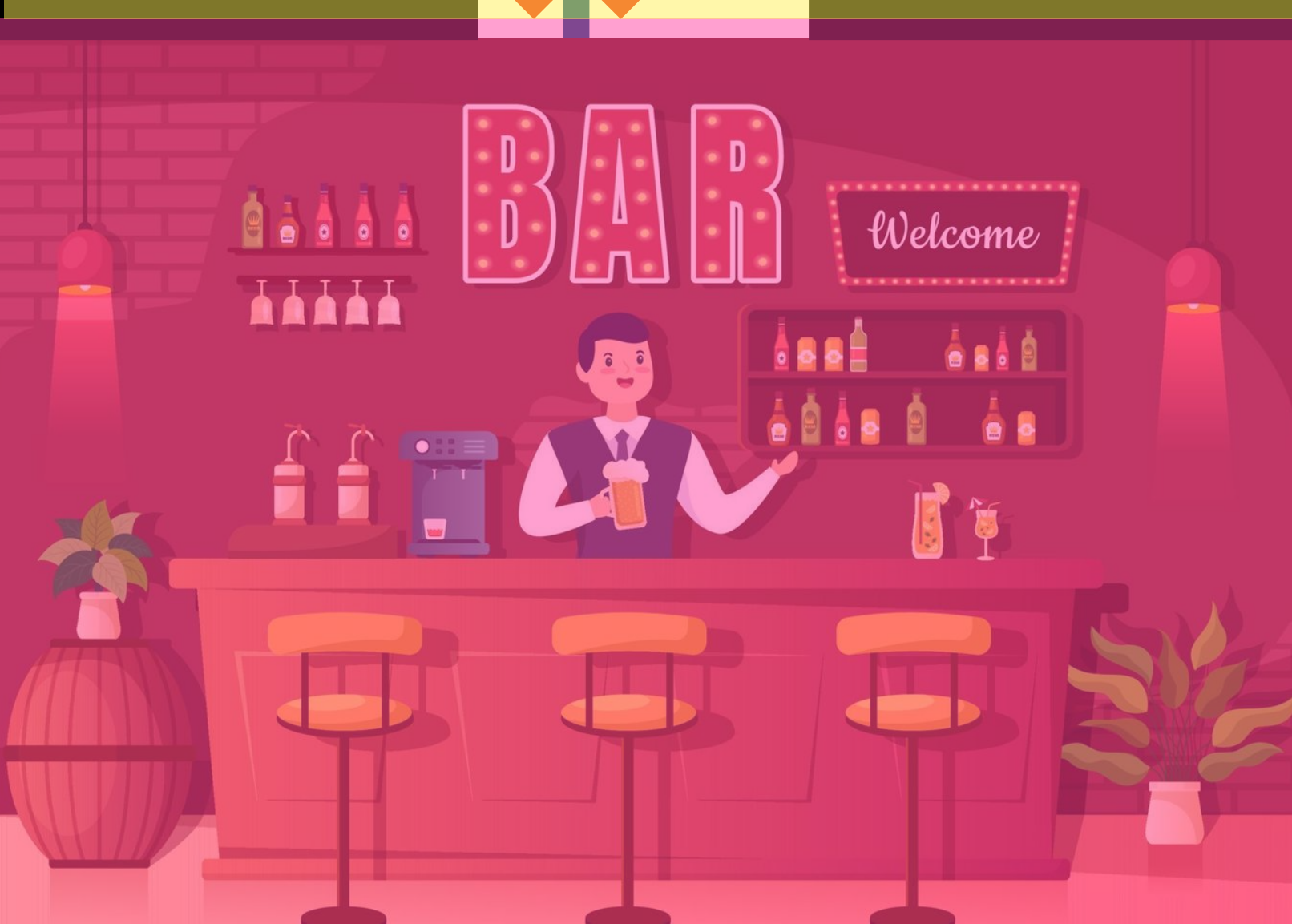
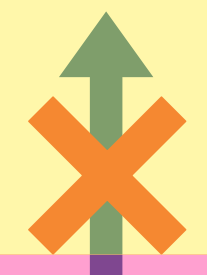
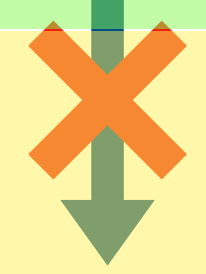
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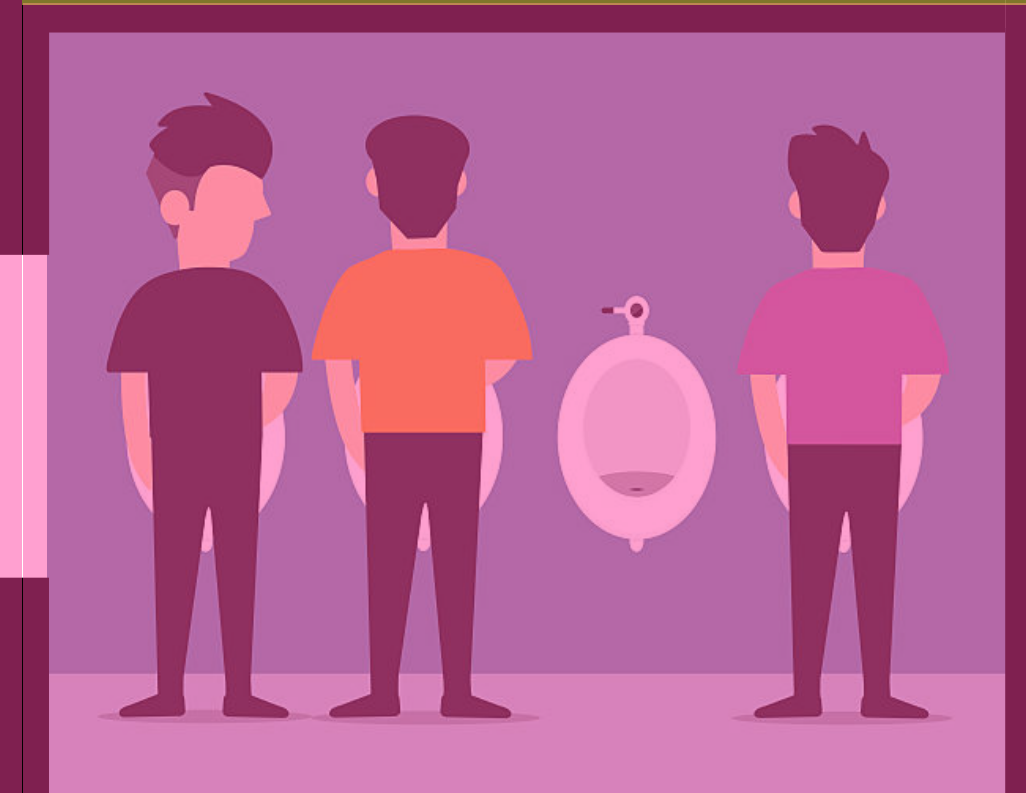




MAXIMAL  
COMMUNICATION  
CLAS



MAXIMAL  
COMMUNICATION  
CLAS



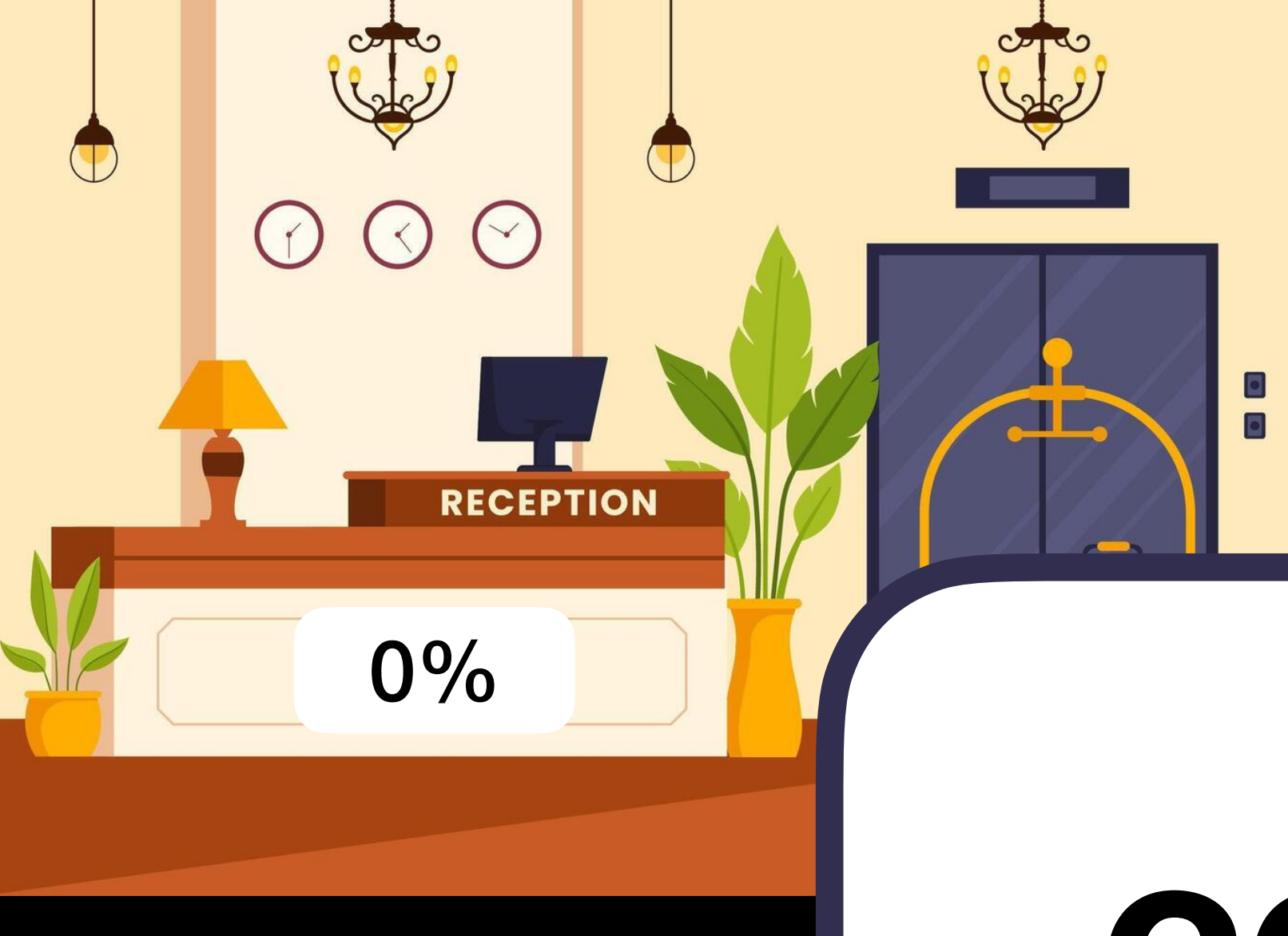


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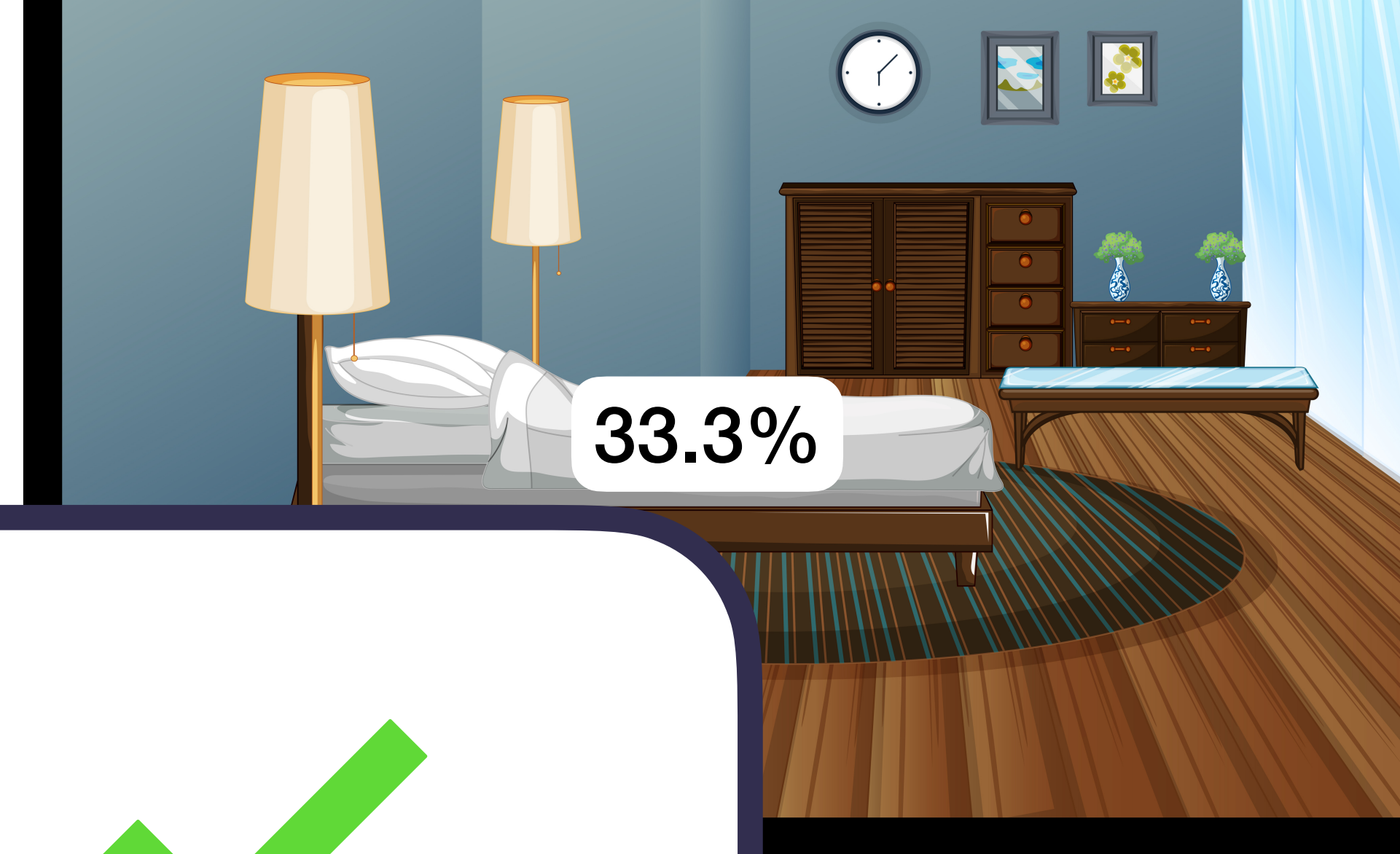






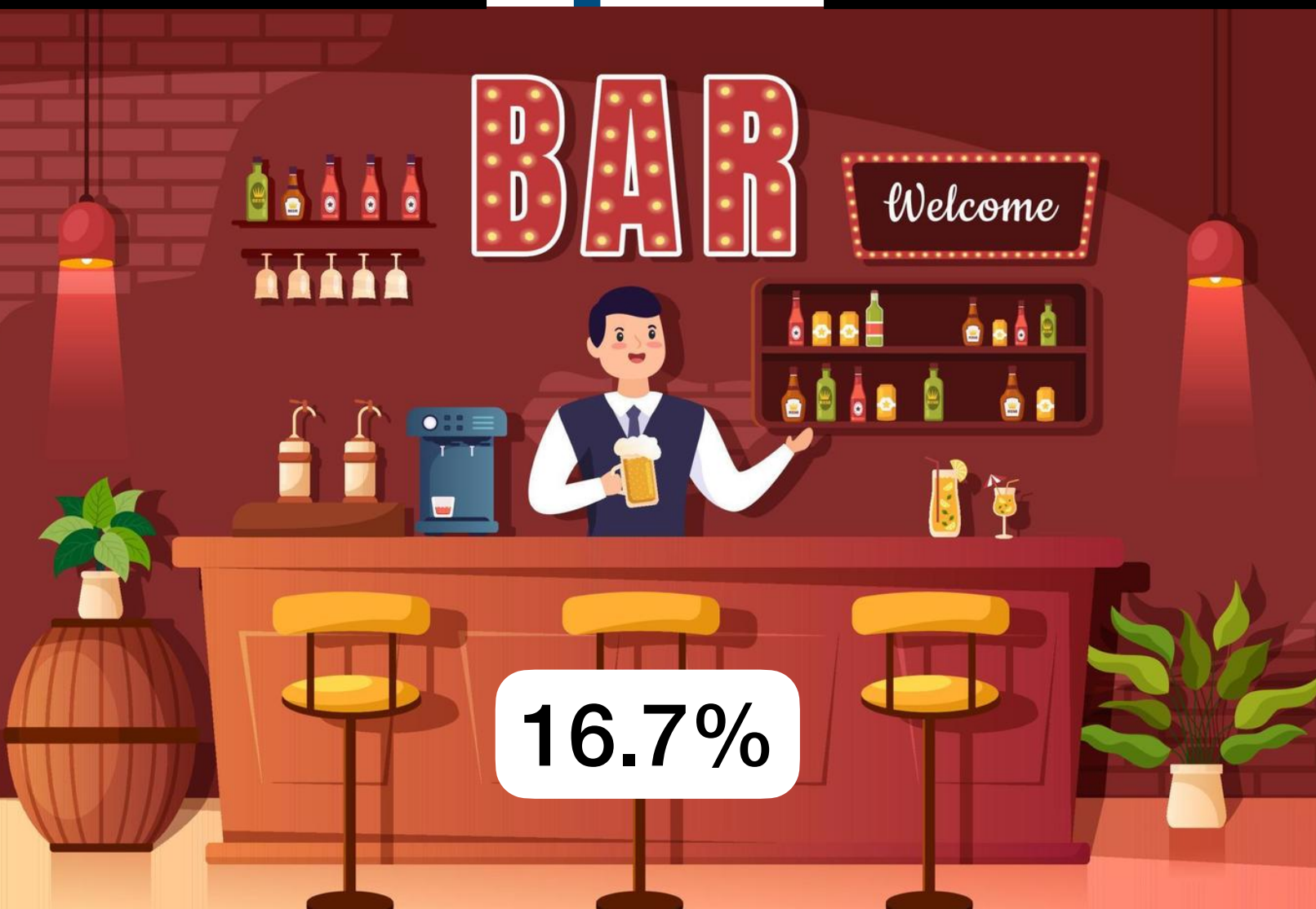
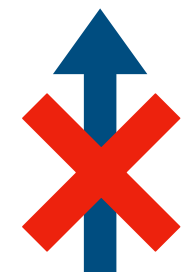
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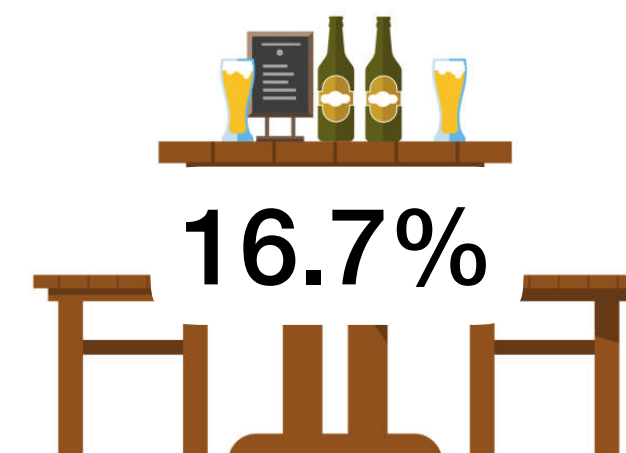


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convergent



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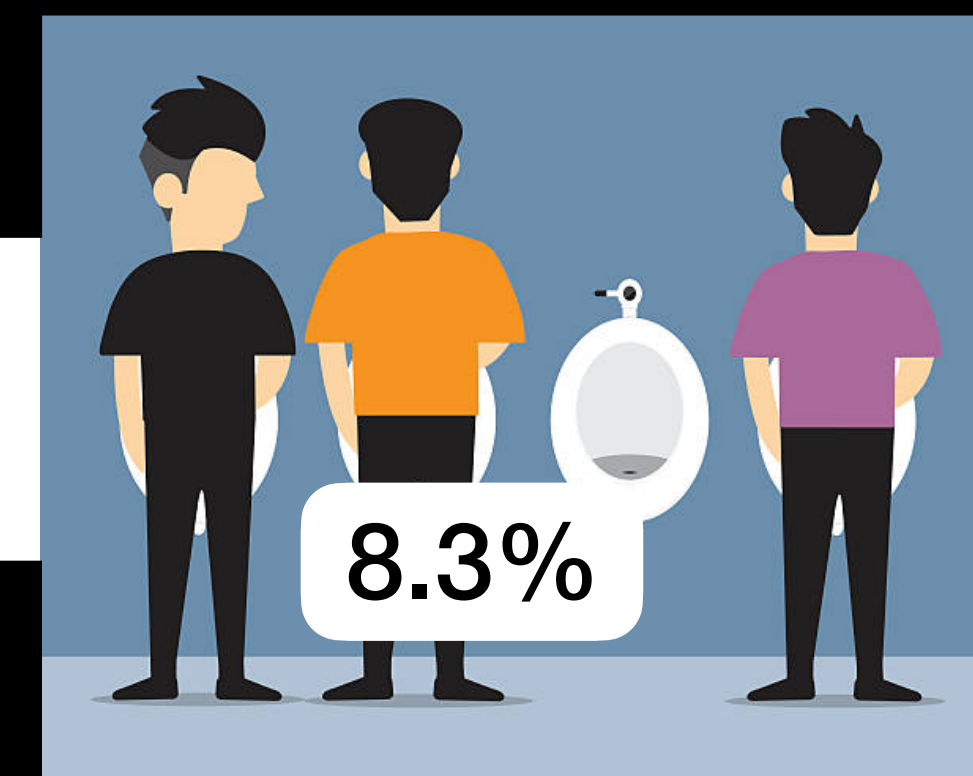


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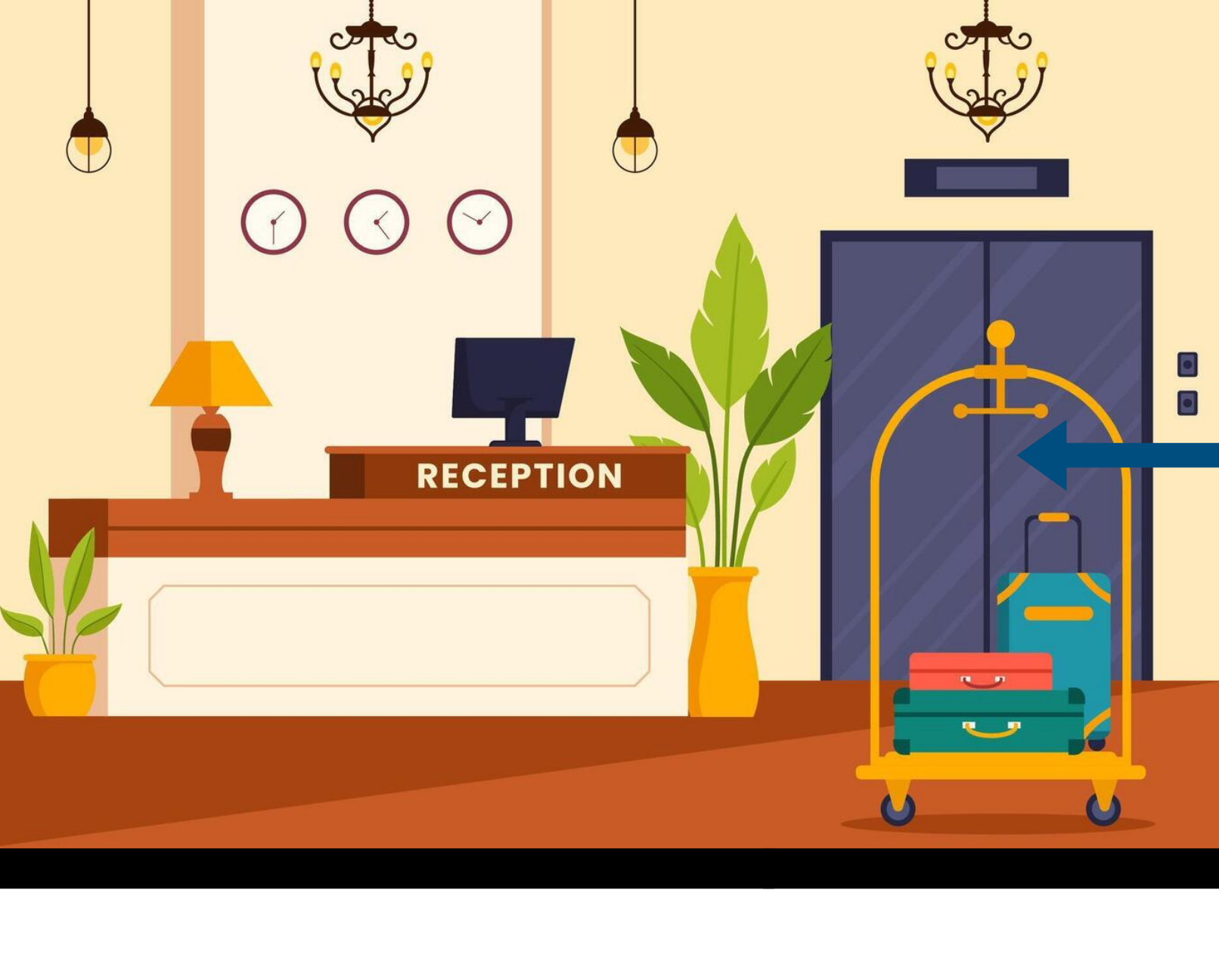




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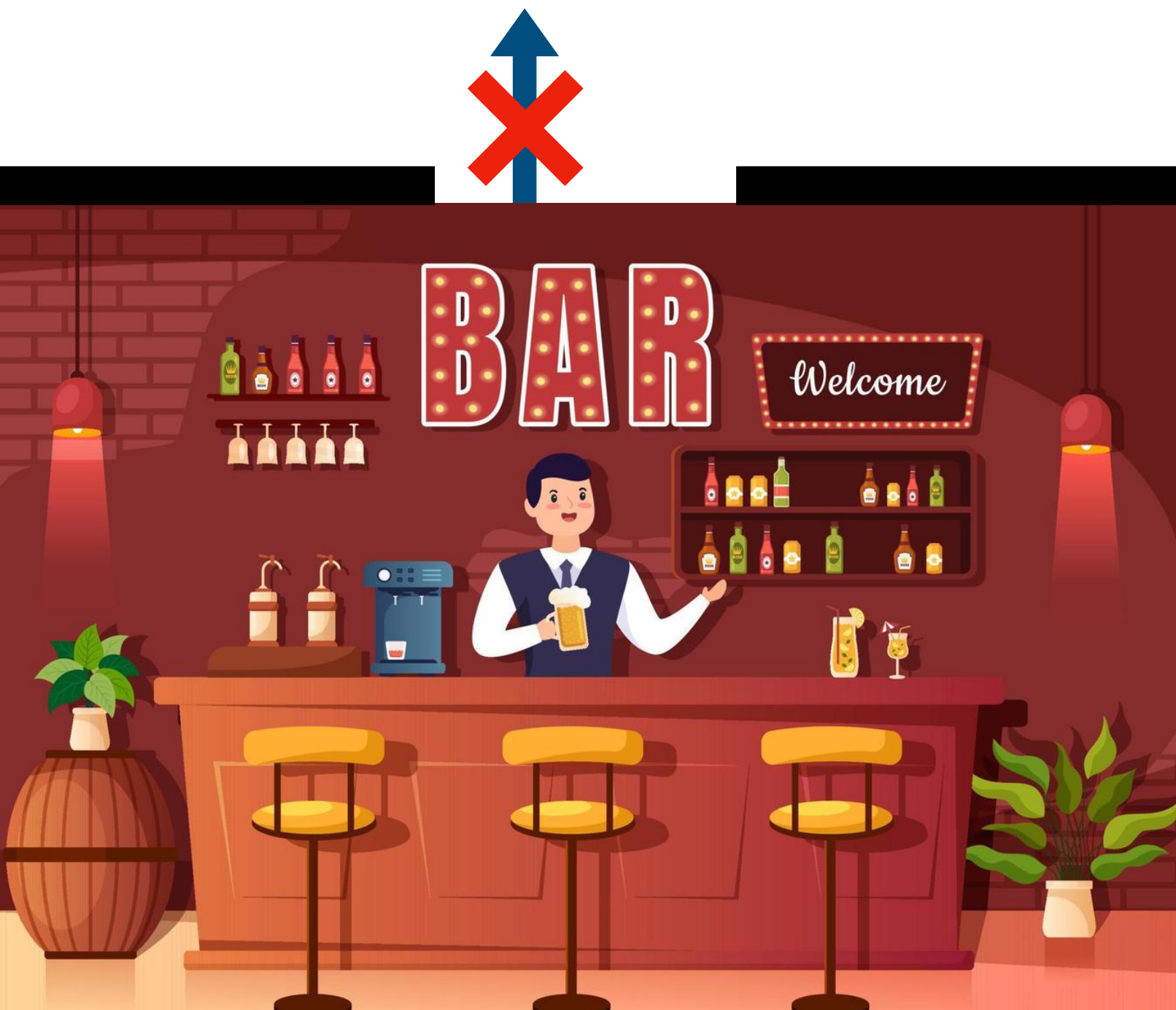




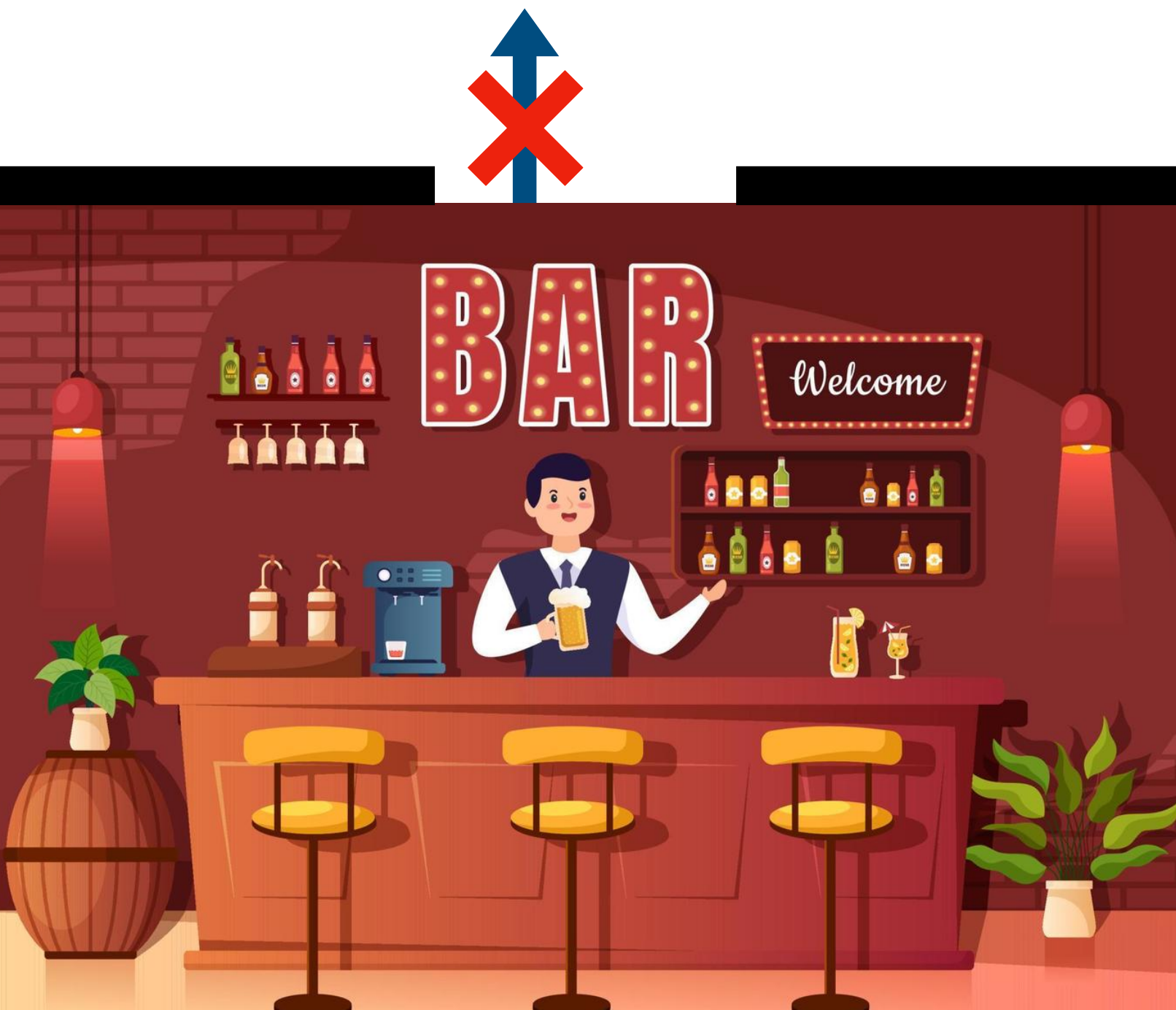
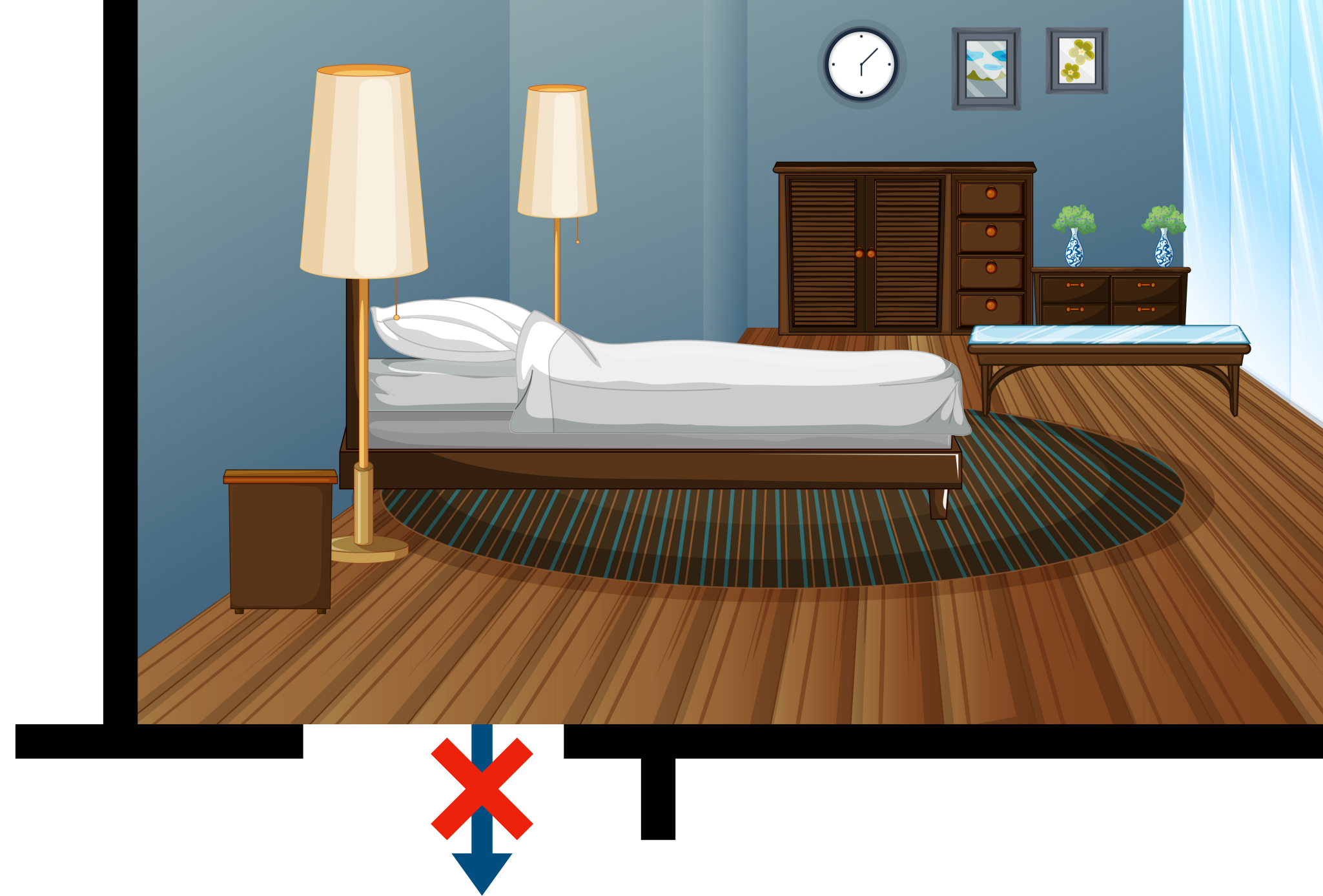
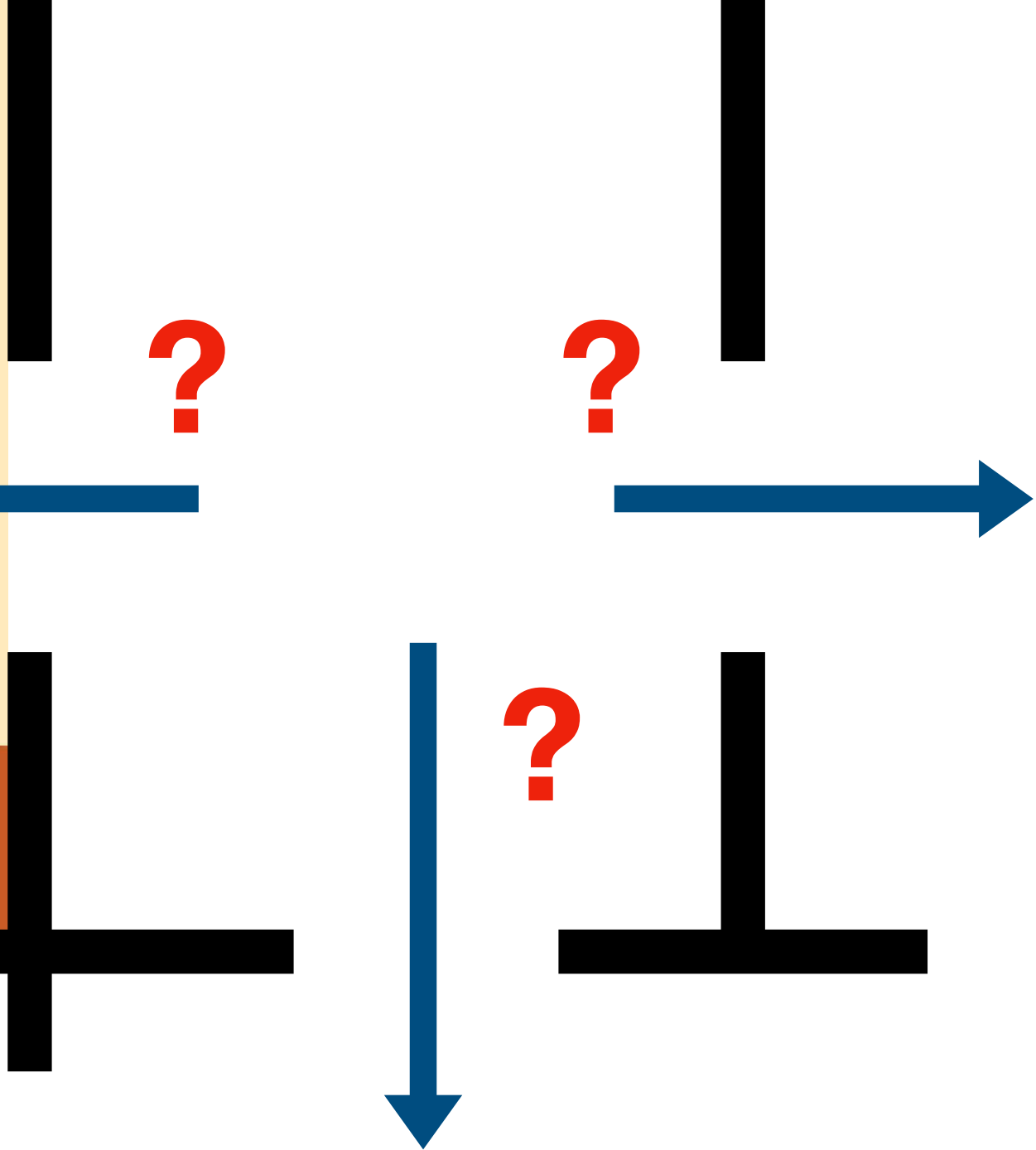
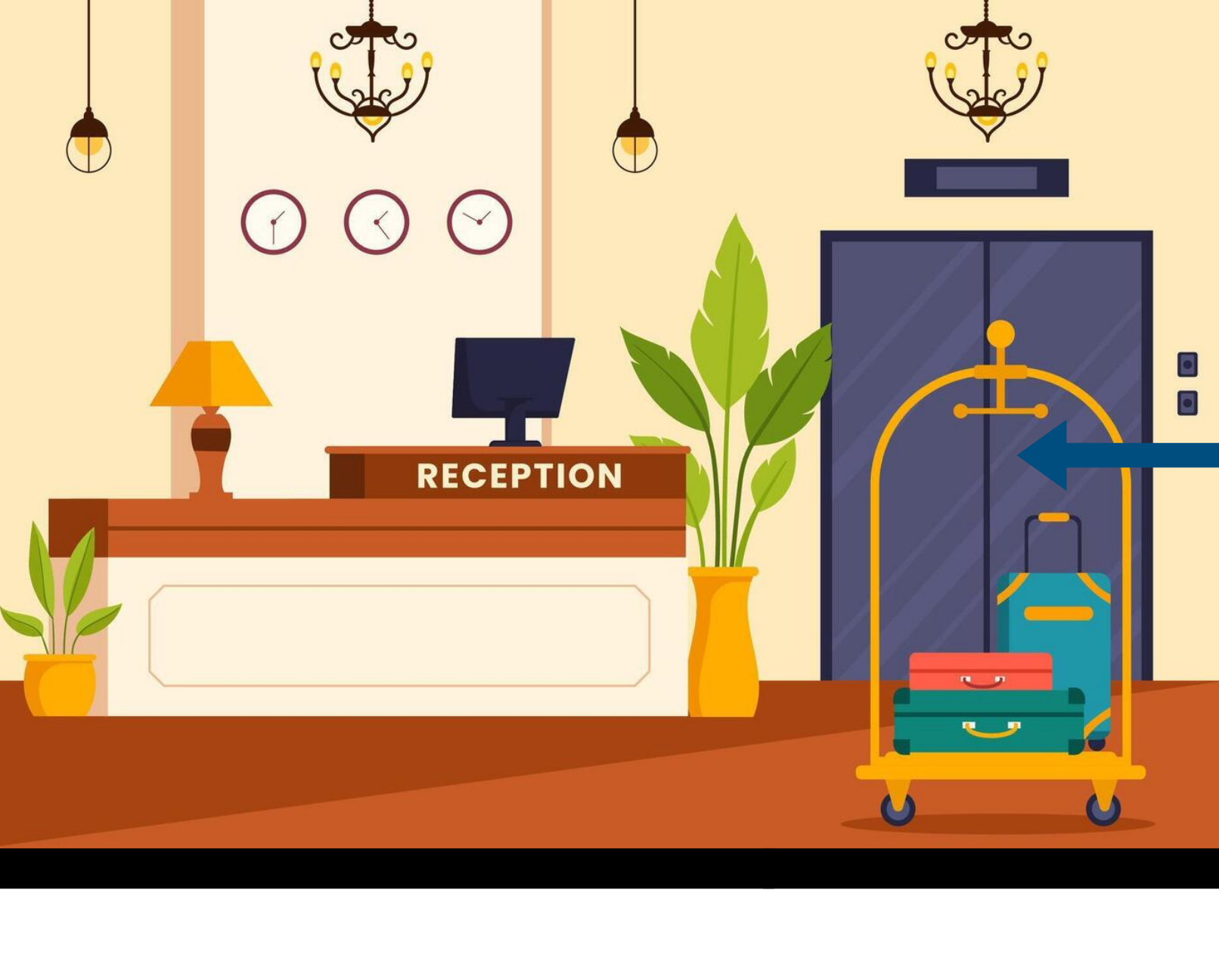
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$\frac{1}{3}$

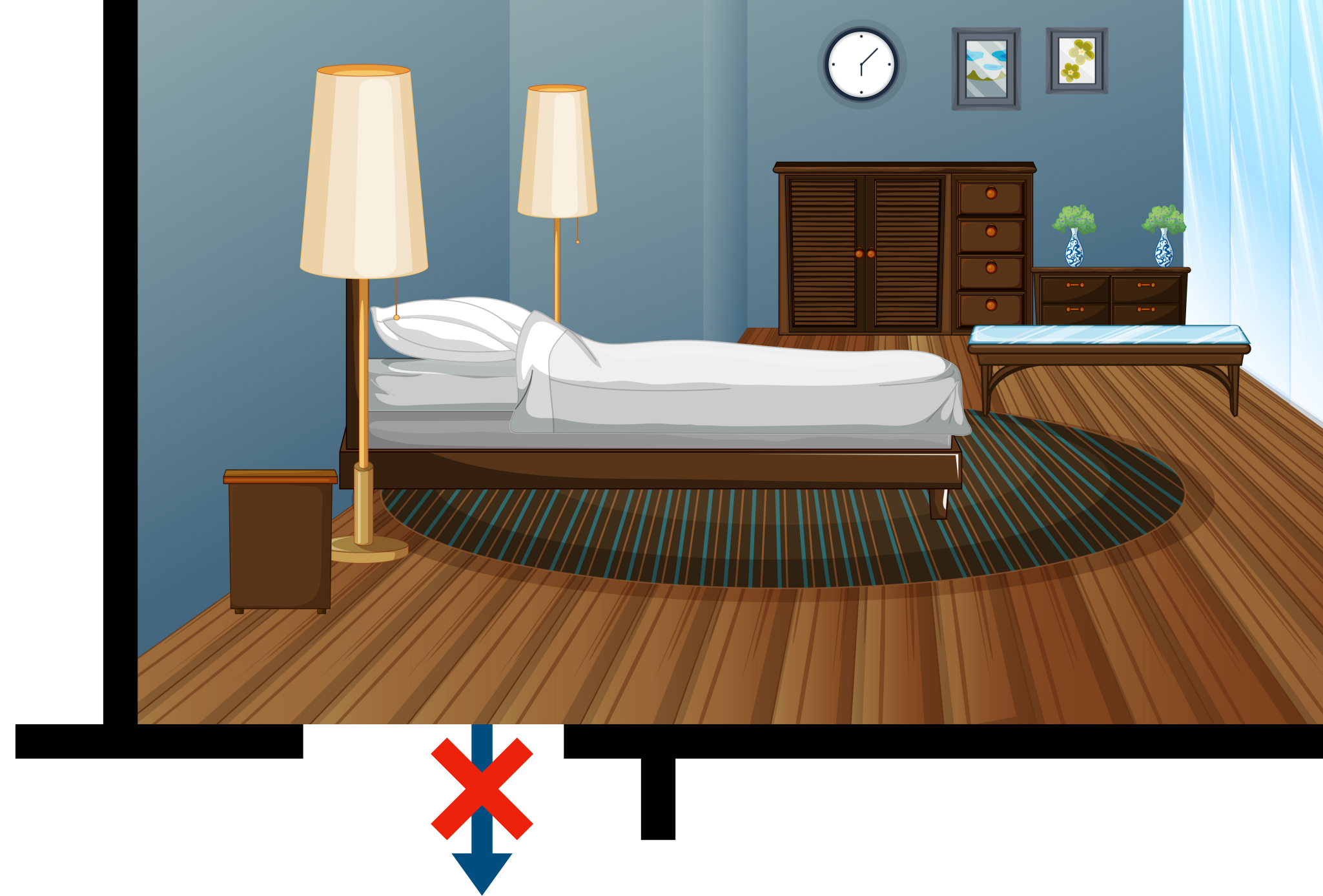
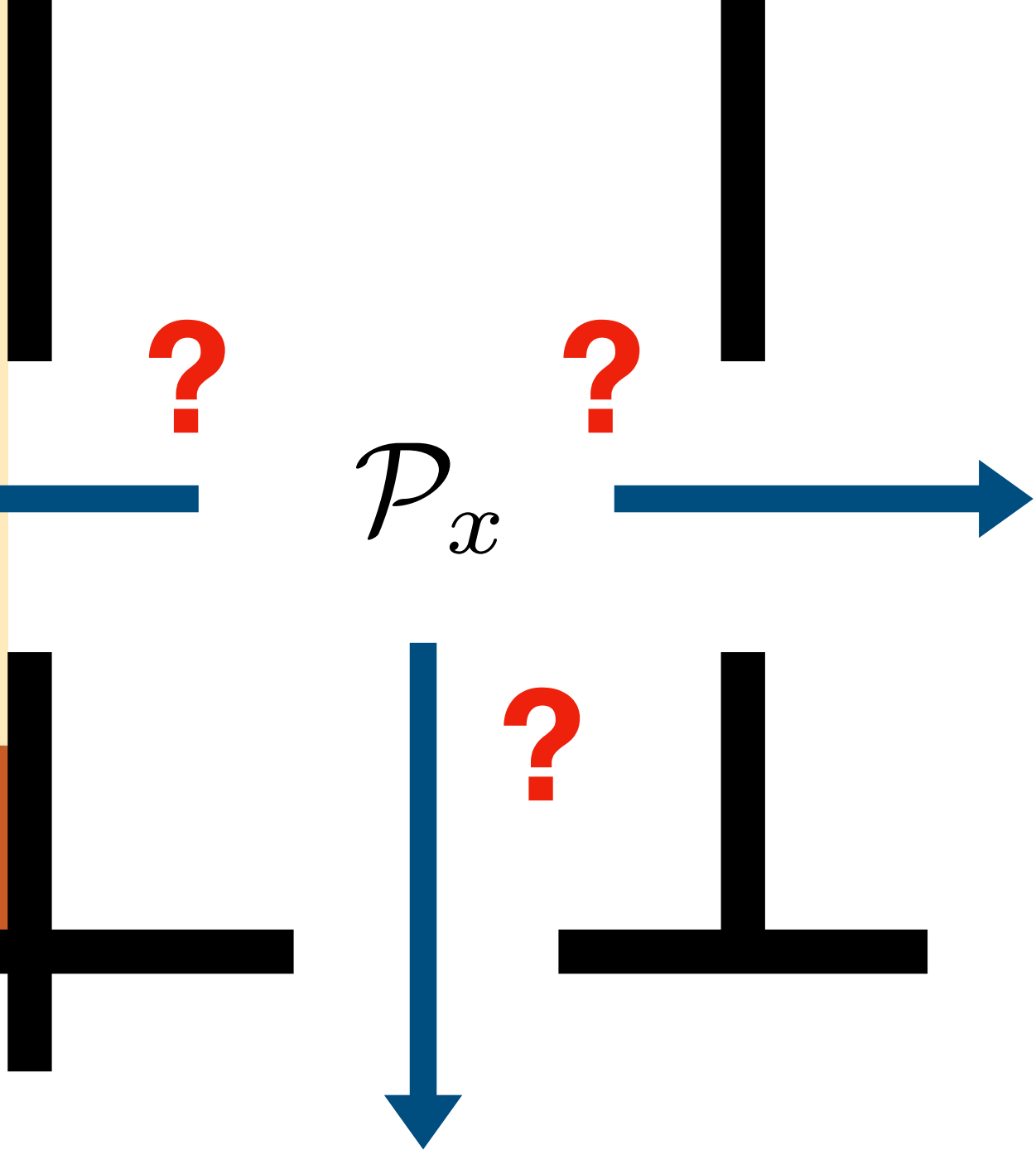
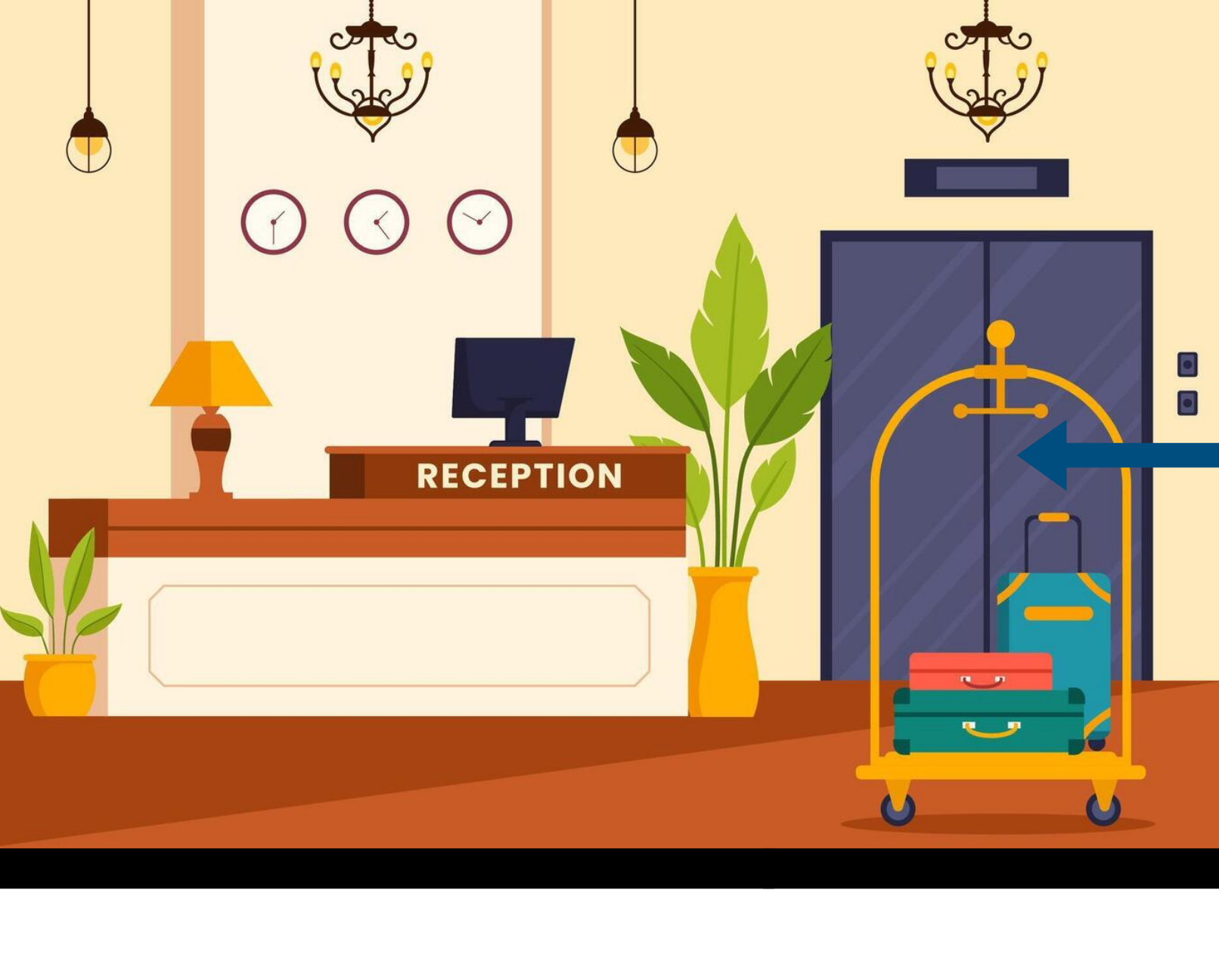
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## upper transition operator

$$\bar{T}: \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}} : f \rightarrow \bar{T}f$$

$$\text{T1. } \bar{T}(f + g) \leq \bar{T}f + \bar{T}g$$

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$$\text{T3. } \bar{T}f \leq \max f$$





## upper transition operator

$$\bar{T}: \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}}: f \rightarrow \bar{T}f$$

$$\updownarrow \quad \bar{T}f(x) = \max_{p \in \mathcal{P}_x} E_p(f)$$

$\mathcal{P}_x$  for all  $x \in \mathcal{X}$





# convergence

for all  $f \in \mathbb{R}^{\mathcal{X}}$

$(\overline{T}^n f)_{n \in \mathbb{N}}$  converges

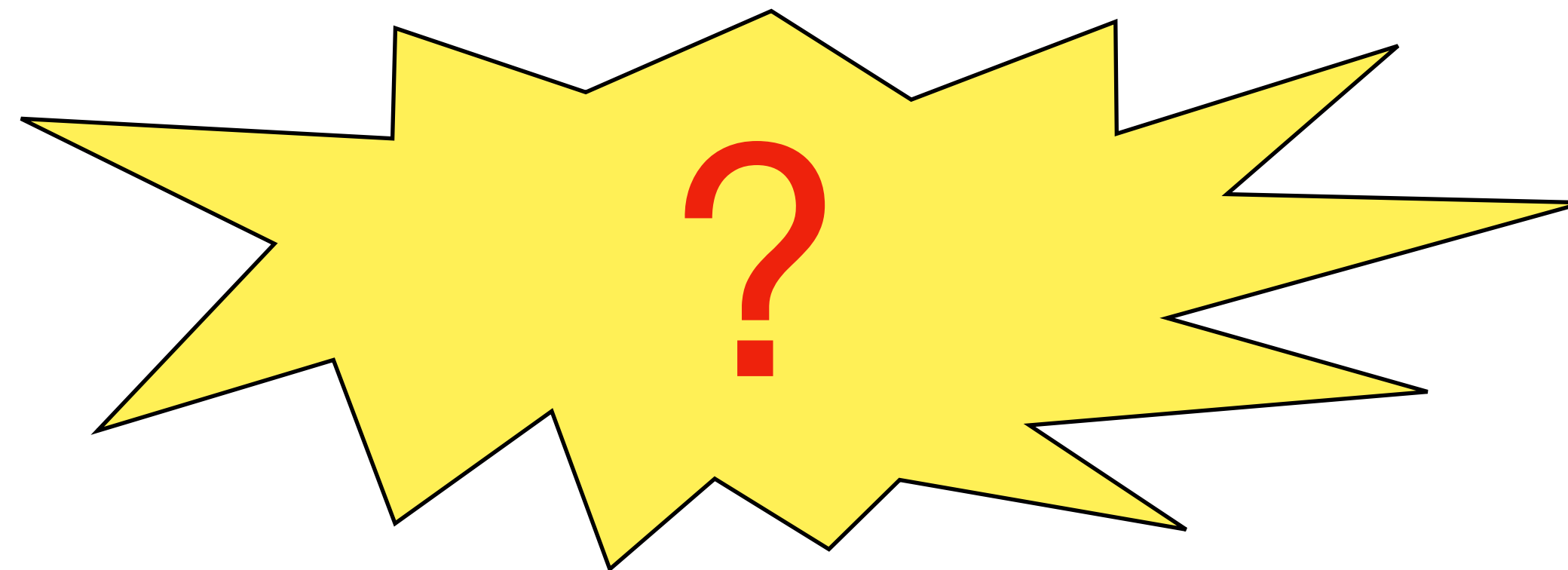
~~to a constant~~





# THEOREM

An upper transition operator  
is **convergent** if and only if ...







## A convenient characterisation of convergent upper transition operators

Let  $\mathcal{X}$  be a finite set. An **upper transition operator**  $\bar{T}$  is a map from  $\mathbb{R}^{\mathcal{X}}$  to  $\mathbb{R}^{\mathcal{X}}$  such that

- $\bar{T}f \leq \max f$  for all  $f \in \mathbb{R}^{\mathcal{X}}$ ;
- $\bar{T}(f+g) \leq \bar{T}f + \bar{T}g$  for all  $f, g \in \mathbb{R}^{\mathcal{X}}$ ;
- $\bar{T}(\lambda f) = \lambda \bar{T}f$  for all  $\lambda \in \mathbb{R}_{>0}$ ,  $f \in \mathbb{R}^{\mathcal{X}}$ .

It's conjugate **lower transition operator** is

$$\underline{T}: \mathbb{R}^{\mathcal{X}} \rightarrow \mathbb{R}^{\mathcal{X}}: f \mapsto -\bar{T}(-f).$$

An upper transition operator  $\bar{T}$  is **convergent** if

$$\lim_{n \rightarrow +\infty} \bar{T}^n f \text{ exists for all } f \in \mathbb{R}^{\mathcal{X}},$$

and **ergodic** if this limit is constant for all  $f \in \mathbb{R}^{\mathcal{X}}$ .

Every upper transition operator  $\bar{T}$  is induced by some family  $(\mathcal{P}^x)_{x \in \mathcal{X}}$  of closed sets of probability mass functions:

$$\bar{T}f(x) = \max\{E_p(f) : p \in \mathcal{P}^x\} \text{ for all } f \in \mathbb{R}^{\mathcal{X}}, x \in \mathcal{X},$$

and hence

$$\underline{T}f(x) = \min\{E_p(f) : p \in \mathcal{P}^x\} \text{ for all } f \in \mathbb{R}^{\mathcal{X}}, x \in \mathcal{X}.$$

An upper transition operator  $\bar{T}$  induces the graph  $\mathcal{G}(\bar{T})$ , with  $\mathcal{X}$  as set of nodes and a directed edge from  $x$  to  $y$  if  $\bar{T}1_y(x) = \max\{p(y) : p \in \mathcal{P}^x\} > 0$ .

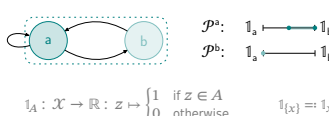
Two states **communicate** if there is a directed path from  $x$  to  $y$  and vice versa. This equivalence relation partitions  $\mathcal{X}$  into (equivalence/communication) classes:

The class  $C$  is **maximal** if it doesn't have outgoing edges, and **aperiodic** if its period  $p_C$ , the greatest common divisor of the lengths of the directed cycles in  $C$ , is 1.

$\mathcal{G}(\bar{T})$  has only **one class**

$\bar{T}$  is **convergent**  $\iff$   $\bar{T}$  is **ergodic**

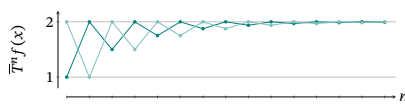
$\iff$   
the single **maximal class** of  $\mathcal{G}(\bar{T})$  is **aperiodic**



$$\mathcal{P}^a: \begin{array}{c} 1_a \xrightarrow{\quad} 1_b \\ 1_b \xrightarrow{\quad} 1_a \end{array}$$

$$1_A: \mathcal{X} \rightarrow \mathbb{R}: z \mapsto \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{otherwise} \end{cases}$$

$$1_{\{a\}} = 1_A$$

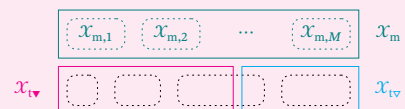


Enumerate the maximal classes of  $\mathcal{G}(\bar{T})$  as  $\mathcal{X}_{m,1}, \dots, \mathcal{X}_{m,M}$ , and let  $\mathcal{X}_m := \mathcal{X}_{m,1} \cup \dots \cup \mathcal{X}_{m,M}$ .

$\mathcal{X}_m$  is **lower reachable** from a transient state  $x \in \mathcal{X} \setminus \mathcal{X}_m$  if  $\bar{T}^n 1_{\mathcal{X}_m}(x) > 0$  for some  $n \in \mathbb{N}$ .

$\mathcal{X}_m$  collects the transient states from which  $\mathcal{X}_m$  is **lower reachable**, and can be determined recursively.

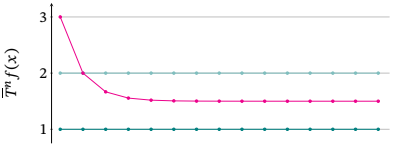
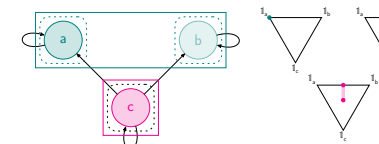
Letting  $\mathcal{X}_{tr} := \mathcal{X} \setminus (\mathcal{X}_m \cup \mathcal{X}_{tr})$ , we obtain the following [partition] of  $\mathcal{X}$ :



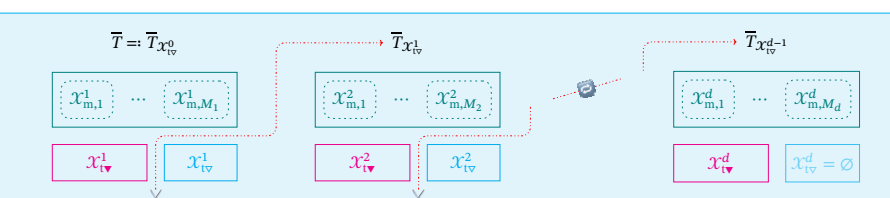
$\mathcal{X}_{tr} = \emptyset$

$\bar{T}$  is **convergent**

$\iff$   
every **maximal class**  $\mathcal{X}_{m,k}$  of  $\mathcal{G}(\bar{T})$  is **aperiodic**



$$\bar{T}_{\mathcal{X}_m} g(x) = \max\{E_p(1_{\mathcal{X}_m}(g)) : p \in \mathcal{P}^x, \sum_{y \in \mathcal{X}_m} p(y) = 1\}$$



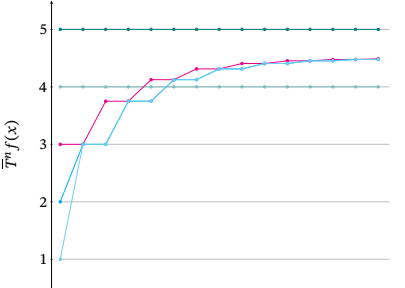
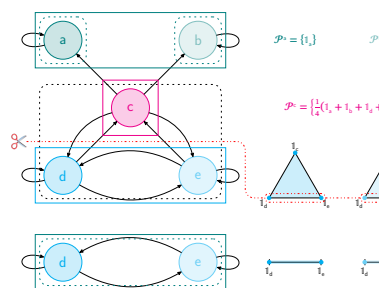
$\mathcal{P}^x$  **finite**(ly generated) for all  $x \in \mathcal{X}_{tr}$

$\bar{T}$  is **convergent**

$\iff$   
in every level  $\ell \in \{1, \dots, d\}$ ,  
every **maximal class**  $\mathcal{X}_{m,k}^\ell$  of  $\mathcal{G}(\bar{T}_{\mathcal{X}_{tr}^{\ell-1}})$  is **aperiodic**

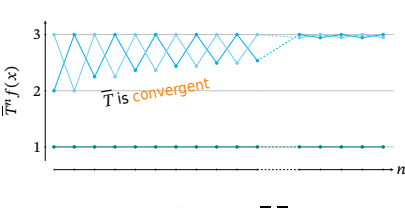
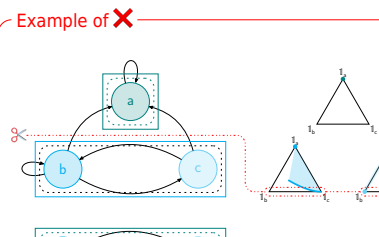
$\bar{T}$  is **ergodic**

$\iff$  (Hermans & De Cooman, 2012)  
 $\mathcal{X}_{tr} = \emptyset$  and  $\mathcal{G}(\bar{T})$  has one **maximal class**  
which is furthermore **aperiodic**



$\bar{T}$  is **convergent**

$\iff$   
in every level  $\ell \in \{1, \dots, d\}$ ,  
every **maximal class**  $\mathcal{X}_{m,k}^\ell$  of  $\mathcal{G}(\bar{T}_{\mathcal{X}_{tr}^{\ell-1}})$  is **aperiodic**



Jasper De Bock  
Alexander Erreygers  
Floris Persiau

GHENT  
UNIVERSITY

Carnegie  
Mellon  
University

the sole maximal class  $\mathcal{X}_{m,1}^1 = \{a\}$  of  $\mathcal{G}(\bar{T}_{\mathcal{X}_{tr}^0})$  is **aperiodic**  
the sole maximal class  $\mathcal{X}_{m,1}^2 = \{b, c\}$  of  $\mathcal{G}(\bar{T}_{\mathcal{X}_{tr}^1})$  is **periodic**

# SEE YOU AT THE POSTER!