A convenient characterisation of convergent upper transition operators

Let $\mathcal{X}$ be a finite set. An **upper transition operator** $\overline{T}$ is a map from $\mathbb{R}^{\mathcal{X}}$ to $\mathbb{R}^{\mathcal{X}}$ such that

T1. $\overline{T}f \leq \max f$ for all $f \in \mathbb{R}^{\mathcal{X}}$;
T2. $\overline{T}(f+g) \leq \overline{T}f + \overline{T}g$ for all $f, g \in \mathbb{R}^{\mathcal{X}}$;
T3. $\overline{T}(\lambda f) = \lambda \overline{T}f$ for all $\lambda \in \mathbb{R}_{>0}$, $f \in \mathbb{R}^{\mathcal{X}}$.

It's conjugate **lower transition operator** is

$$\underline{T}\colon \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}\colon f \mapsto -\overline{T}(-f).$$

An upper transition operator $\overline{T}$ is **convergent** if

$$\lim_{n\to+\infty} \overline{T}^n f \text{ exists for all } f \in \mathbb{R}^{\mathcal{X}},$$

and **ergodic** if this limit is constant for all $f \in \mathbb{R}^{\mathcal{X}}$.

Every upper transition operator $\overline{T}$ is induced by some family $(\mathcal{P}^x)_{x\in\mathcal{X}}$ of *closed* sets of probability mass functions:

$$\overline{T}f(x) = \max\{\mathrm{E}_p(f)\colon p \in \mathcal{P}^x\} \text{ for all } f \in \mathbb{R}^{\mathcal{X}}, x \in \mathcal{X},$$

and hence

$$\underline{T}f(x) = \min\{\mathrm{E}_p(f)\colon p \in \mathcal{P}^x\} \text{ for all } f \in \mathbb{R}^{\mathcal{X}}, x \in \mathcal{X}.$$

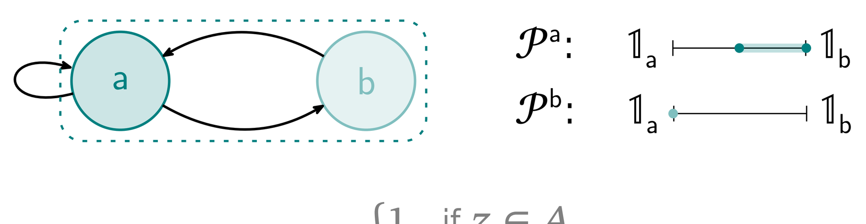
An upper transition operator $\overline{T}$ induces the graph $\overline{\mathcal{G}}(\overline{T})$, with $\mathcal{X}$ as set of nodes and a directed edge from x to y if $\overline{T}\mathbb{1}_y(x) = \max\{p(y)\colon p \in \mathcal{P}^x\} > 0$.

Two states **communicate** if there is a directed path from x to y and vice versa. This equivalence relation partitions $\mathcal{X}$ into (equivalence/communication) classes.

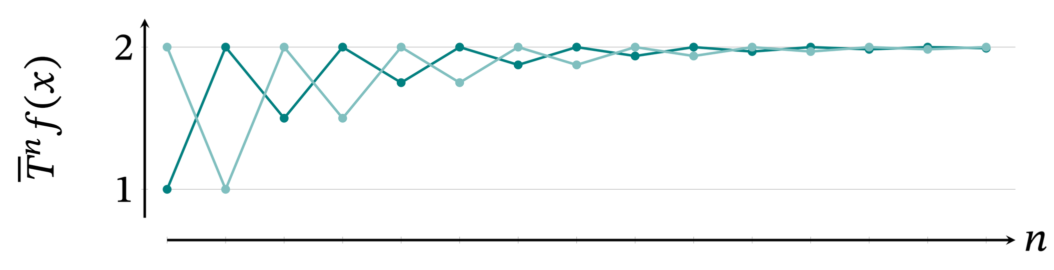
The class C is **maximal** if it doesn't have outgoing edges, and **aperiodic** if its period p_C, the greatest common divisor of the lengths of the directed cycles in C, is 1.

$\overline{\mathcal{G}}(\overline{T})$ has only one class

$\overline{T}$ is convergent $\Leftrightarrow$ $\overline{T}$ is ergodic
$\Updownarrow$ $\Updownarrow$
the single maximal class of $\overline{\mathcal{G}}(\overline{T})$ is aperiodic



$\mathbb{1}_A\colon \mathcal{X} \to \mathbb{R}\colon z \mapsto \begin{cases} 1 & \text{if } z \in A \\ 0 & \text{otherwise} \end{cases}$ $\quad \mathbb{1}_{\{x\}} =: \mathbb{1}_x$

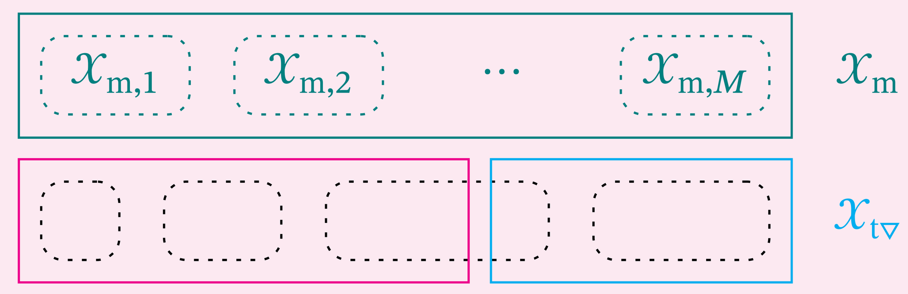


Enumerate the maximal classes of $\overline{\mathcal{G}}(\overline{T})$ as $\mathcal{X}_{\mathrm{m},1}, \dots, \mathcal{X}_{\mathrm{m},M}$, and let $\mathcal{X}_{\mathrm{m}} := \mathcal{X}_{\mathrm{m},1} \cup \dots \cup \mathcal{X}_{\mathrm{m},M}$.

$\mathcal{X}_{\mathrm{m}}$ is **lower reachable** from a transient state $x \in \mathcal{X} \setminus \mathcal{X}_{\mathrm{m}}$ if $\underline{T}^n \mathbb{1}_{\mathcal{X}_{\mathrm{m}}}(x) > 0$ for some $n \in \mathbb{N}$.

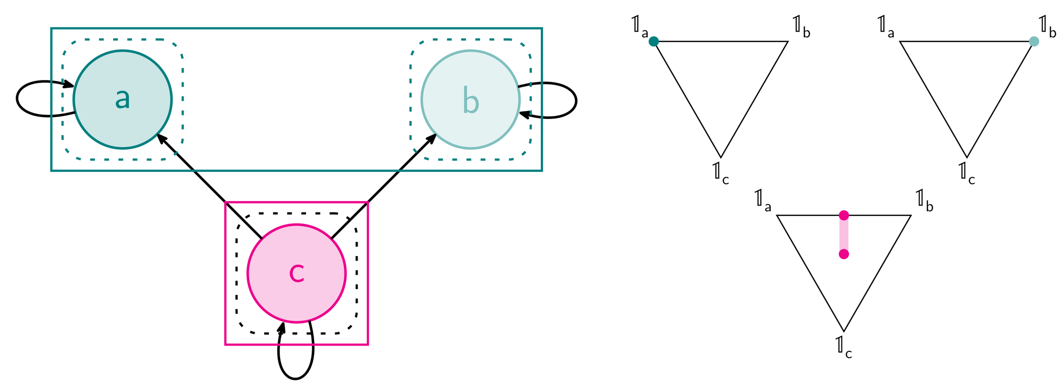
$\mathcal{X}_{\mathrm{t}\blacktriangledown}$ collects the transient states from which $\mathcal{X}_{\mathrm{m}}$ is lower reachable, and can be determined recursively.

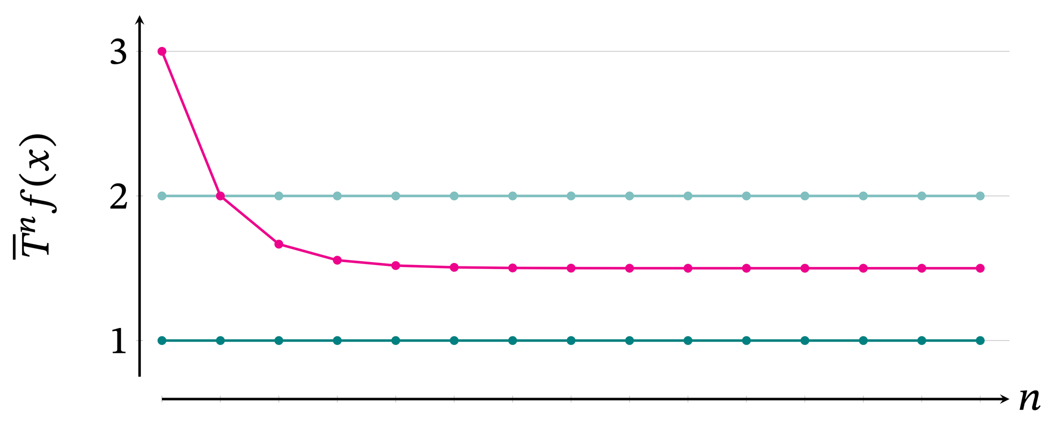
Letting $\mathcal{X}_{\mathrm{t}\triangledown} := \mathcal{X} \setminus (\mathcal{X}_{\mathrm{m}} \cup \mathcal{X}_{\mathrm{t}\blacktriangledown})$, we obtain the following partition of $\mathcal{X}$:

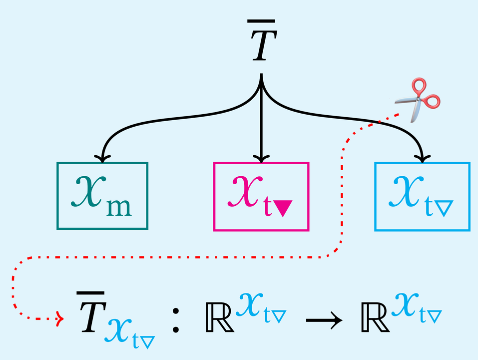


$\mathcal{X}_{\mathrm{t}\triangledown} = \emptyset$

$\overline{T}$ is convergent
$\Updownarrow$
every maximal class $\mathcal{X}_{\mathrm{m},k}$ of $\overline{\mathcal{G}}(\overline{T})$ is aperiodic

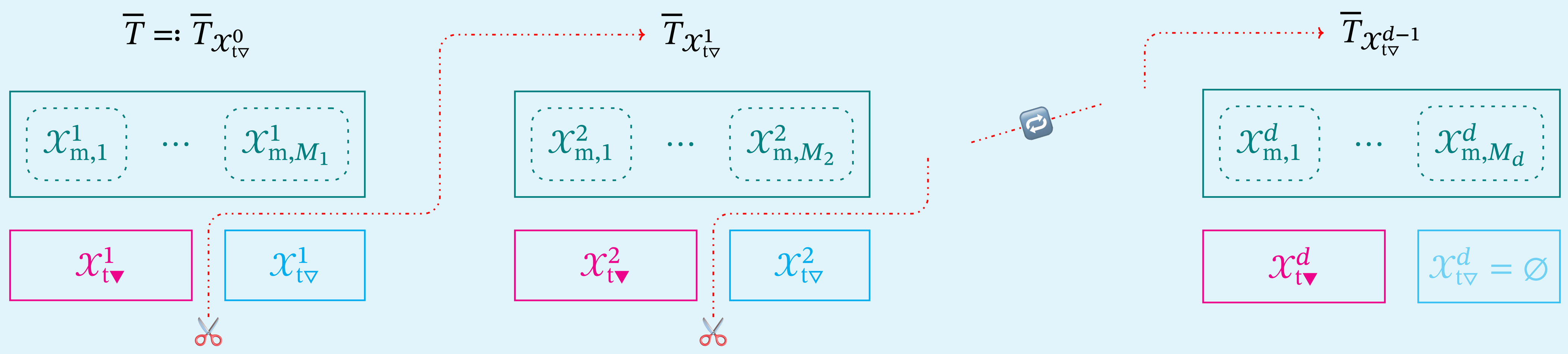






$\overline{T}_{\mathcal{X}_{\mathrm{t}\triangledown}}\colon \mathbb{R}^{\mathcal{X}_{\mathrm{t}\triangledown}} \to \mathbb{R}^{\mathcal{X}_{\mathrm{t}\triangledown}}$

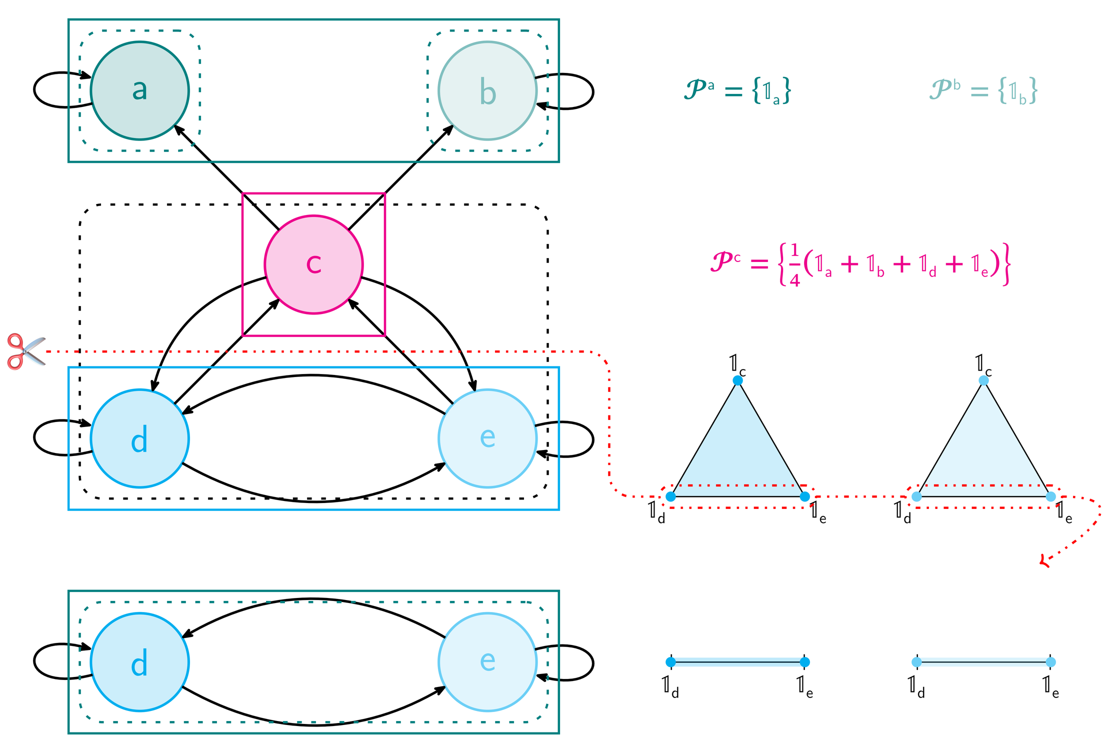
$\overline{T}_{\mathcal{X}_{\mathrm{t}\triangledown}} g(x) := \max\{\mathrm{E}_{p\downarrow\mathcal{X}_{\mathrm{t}\triangledown}}(g)\colon p \in \mathcal{P}^x, \sum_{y\in\mathcal{X}_{\mathrm{t}\triangledown}} p(y) = 1\}$

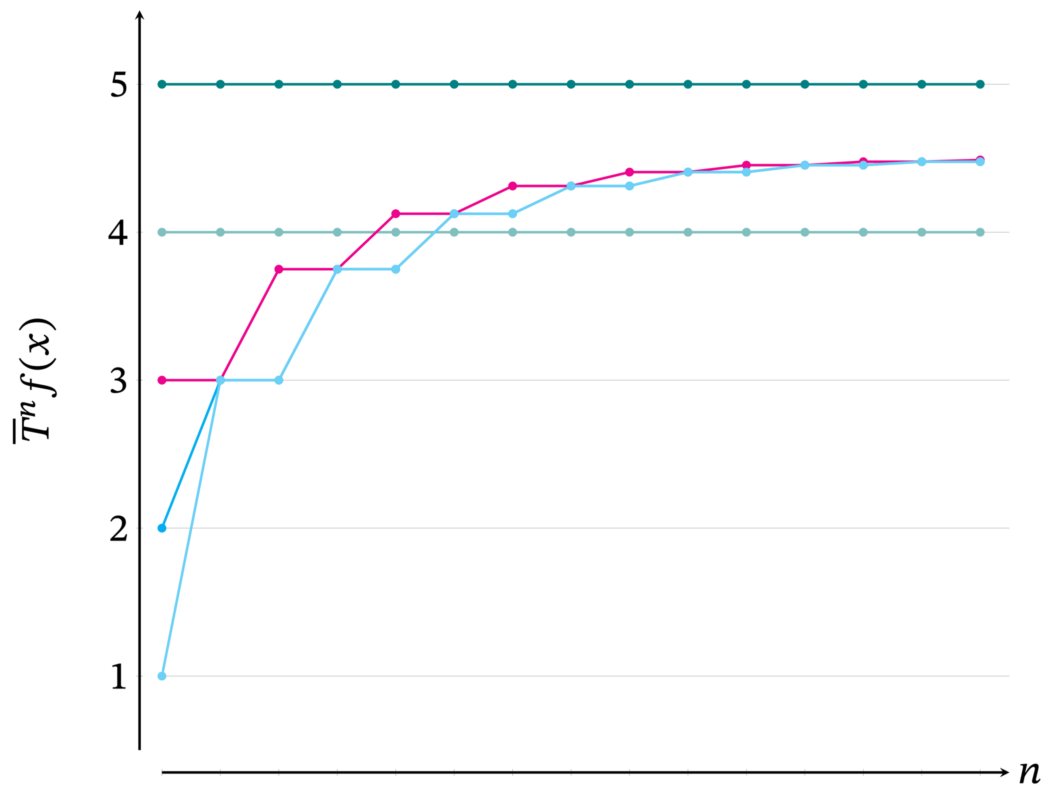


$\mathcal{P}^x$ finite(ly generated) for all $x \in \mathcal{X}_{\mathrm{t}\triangledown}$

$\overline{T}$ is convergent
$\Updownarrow$
in every level $\ell \in \{1, \dots, d\}$,
every maximal class $\mathcal{X}^{\ell}_{\mathrm{m},k}$ of $\overline{\mathcal{G}}(\overline{T}_{\mathcal{X}^{\ell-1}_{\mathrm{t}\triangledown}})$ is aperiodic

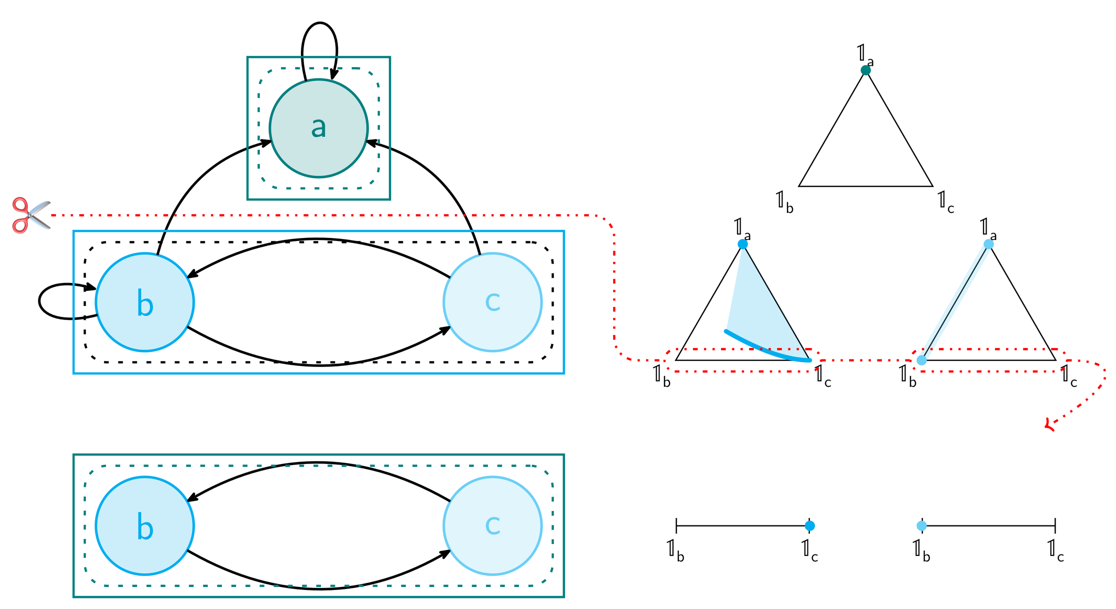
$\overline{T}$ is **ergodic**
$\Updownarrow$ (Hermans & De Cooman, 2012)
$\mathcal{X}_{\mathrm{t}\triangledown} = \emptyset$ and $\overline{\mathcal{G}}(\overline{T})$ has *one* maximal class
which is furthermore aperiodic

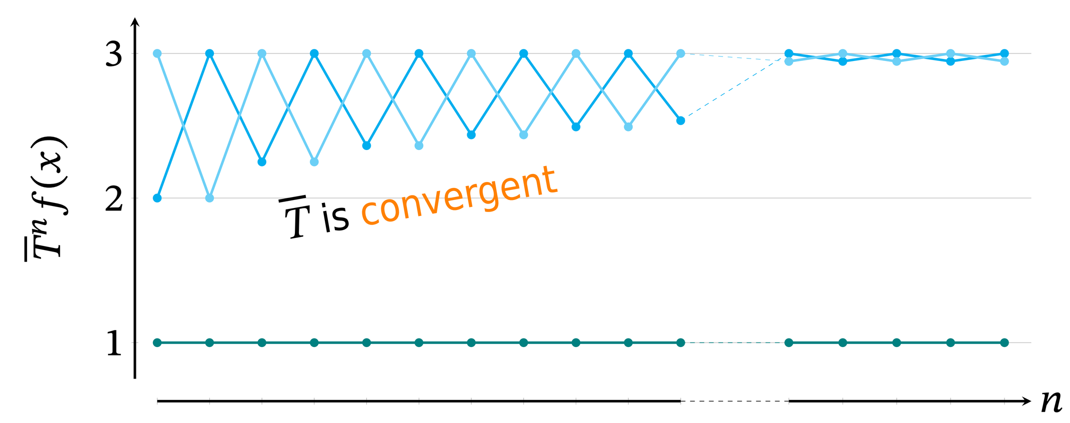




$\overline{T}$ is convergent
$\Uparrow$ $\Downarrow$ ✗
in every level $\ell \in \{1, \dots, d\}$,
every maximal class $\mathcal{X}^{\ell}_{\mathrm{m},k}$ of $\overline{\mathcal{G}}(\overline{T}_{\mathcal{X}^{\ell-1}_{\mathrm{t}\triangledown}})$ is aperiodic

Example of ✗





the sole maximal class $\mathcal{X}^1_{\mathrm{m},1} = \{a\}$ of $\overline{\mathcal{G}}(\overline{T}_{\mathcal{X}^0_{\mathrm{t}\triangledown}})$ is aperiodic
the sole maximal class $\mathcal{X}^2_{\mathrm{m},1} = \{b, c\}$ of $\overline{\mathcal{G}}(\overline{T}_{\mathcal{X}^1_{\mathrm{t}\triangledown}})$ is periodic

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