

How to Make your Markov Chains Robust

An Imprecise Probabilities Perspective



Jasper De Bock & Alexander Erreygers

21 September 2023

Eindhoven



FLip

Foundations Lab for
imprecise probabilities

Jasper De Bock & Alexander Erreygers



GHENT
UNIVERSITY



FLip

Foundations Lab for
imprecise probabilities



GHENT
UNIVERSITY



FLip

Foundations Lab for
imprecise probabilities



GHENT
UNIVERSITY

So what are imprecise
probabilities?

MODELLING UNCERTAINTY

belief functions

probability measures

sets of
desirable gambles

sets of probability
measures

logic

set theory

imprecise probabilities

random
sets

So what are imprecise
probabilities?

choice
functions

lower and upper
expectations

p-boxes

probability intervals

MODELLING UNCERTAINTY

belief functions

probability measures

sets of
desirable gambles

**sets of probability
measures**

logic

set theory

imprecise probabilities

random
sets

So what are imprecise
probabilities?

choice
functions

lower and upper
expectations

p-boxes

probability intervals

$$P \in \mathcal{P}$$

**sets of probability
measures**

imprecise probabilities

choice
functions

lower and upper
expectations

probability intervals

$$P \in \mathcal{P}$$

sets of probability
measures

imprecise probabilities

$$\overline{P}(A) = \sup_{P \in \mathcal{P}} P(A)$$

$$\underline{P}(A) = \inf_{P \in \mathcal{P}} P(A)$$

choice
functions

lower and upper
expectations

probability intervals

$$P \in \mathcal{P}$$

sets of probability
measures

imprecise probabilities

$$\bar{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$$

choice
functions

**lower and upper
expectations**

$$\underline{E}(f) = \inf_{P \in \mathcal{P}} E_P(f) = -\bar{E}(-f)$$

probability intervals

PRECISE DECISION MAKING

choose the one with the
highest probability

$$P \in \mathcal{P}$$

**sets of probability
measures**

IMPRECISE DECISION MAKING

choose the one(s) with the

- highest lower probability
- highest upper probability
- highest probability for at least one $P \in \mathcal{P}$
- ...

imprecise probabilities

**choice
functions**

lower and upper
expectations

probability intervals

MODELLING UNCERTAINTY

belief functions

probability measures

sets of
desirable gambles

sets of probability
measures

logic

set theory

imprecise probabilities

random
sets

So what are imprecise
probabilities?

choice
functions

lower and upper
expectations

p-boxes

probability intervals

MODELLING UNCERTAINTY

belief functions

probability measures

sets of
desirable gambles

sets of probability
measures

logic

set theory

imprecise probabilities

random
sets

Why would I use this?

choice
functions

lower and upper
expectations

p-boxes

probability intervals

MODELLING UNCERTAINTY

belief functions

probability measures

sets of
desirable gambles

**sets of probability
measures**

logic

set theory

imprecise probabilities

random
sets

Why would I use this?

choice
functions

lower and upper
expectations

p-boxes

probability intervals

MODELLING UNCERTAINTY

beliefs of groups

complete uncertainty

computational
pragmatism

solve curse
of dimensionality

partial knowledge

small data sets

imprecise probabilities

Why would I use this?

global sensitivity analysis

structural
sensitivity analysis

SIPTA.org

society for

imprecise probabilities

theories and applications



FLip

Foundations Lab for
imprecise probabilities



GHENT
UNIVERSITY



FLip

Foundations Lab for
imprecise probabilities

Jasper De Bock & Alexander Erreygers



GHENT
UNIVERSITY

How to Make your Markov Chains Robust

An Imprecise Probabilities Perspective



Jasper De Bock & Alexander Erreygers

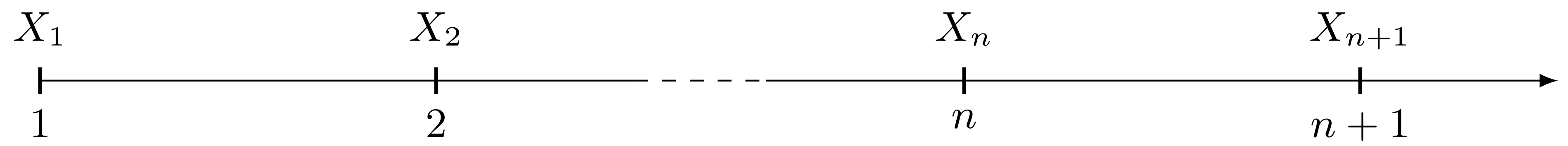
21 September 2023

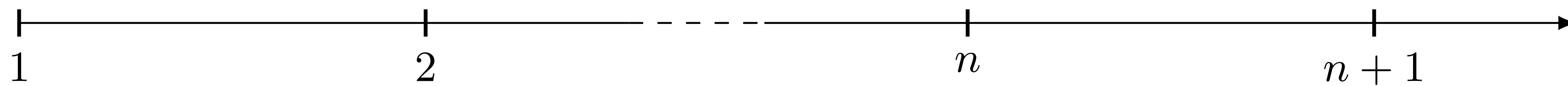
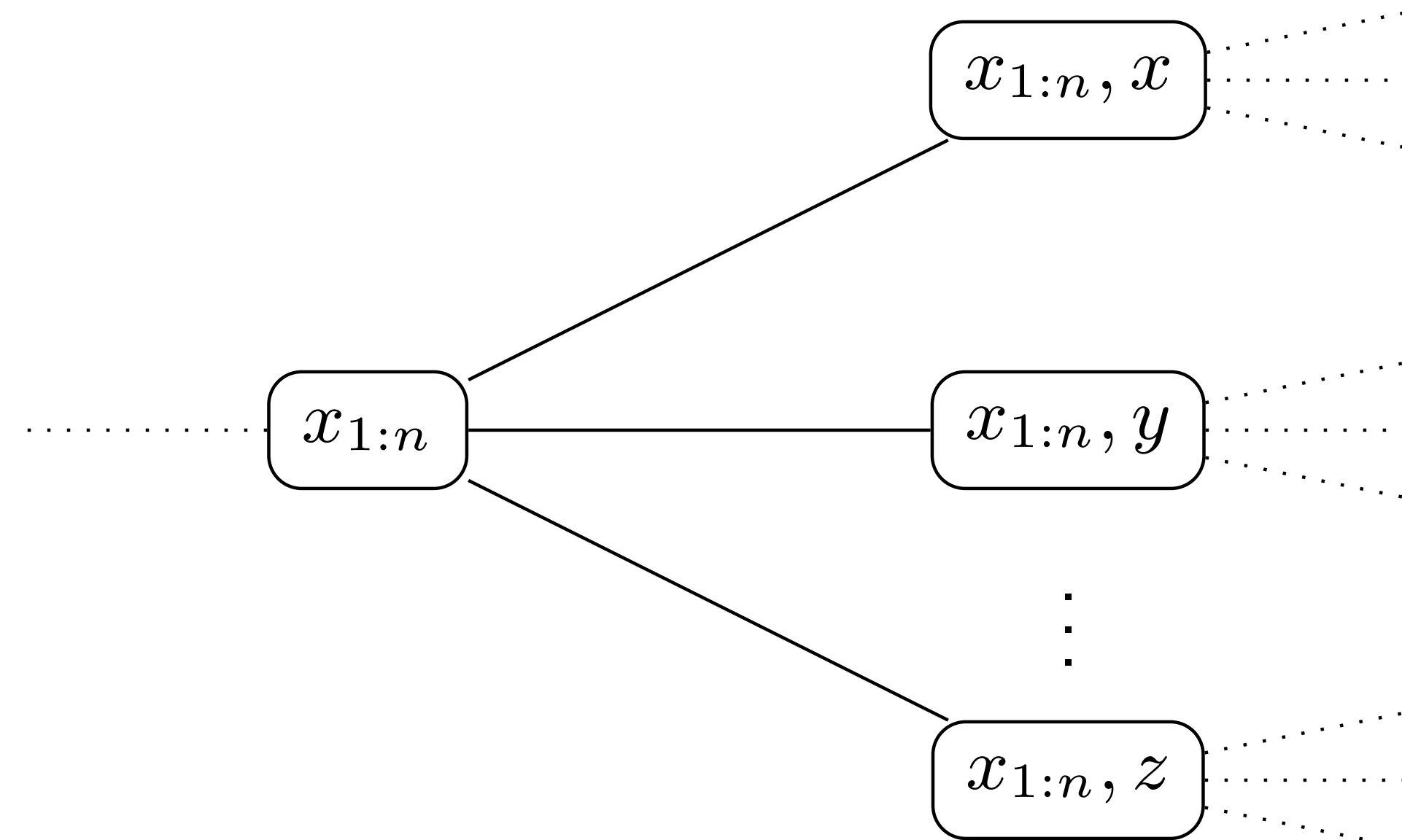
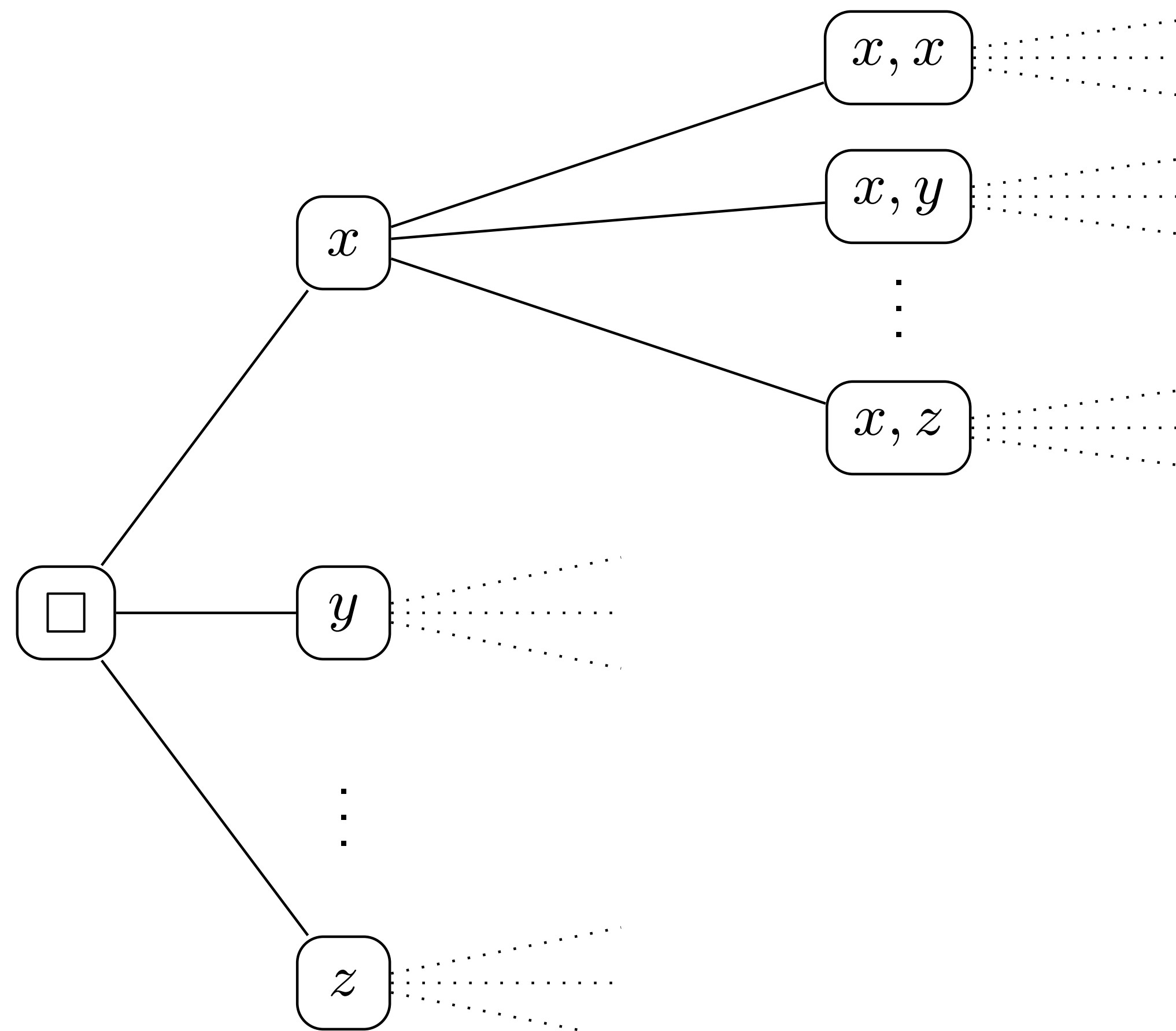
Eindhoven

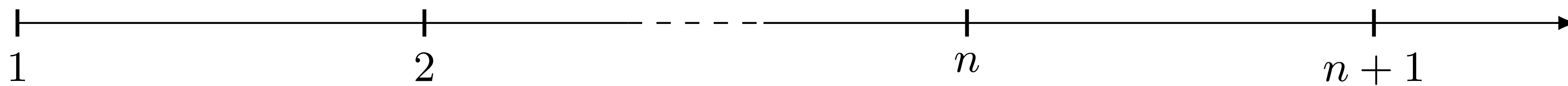
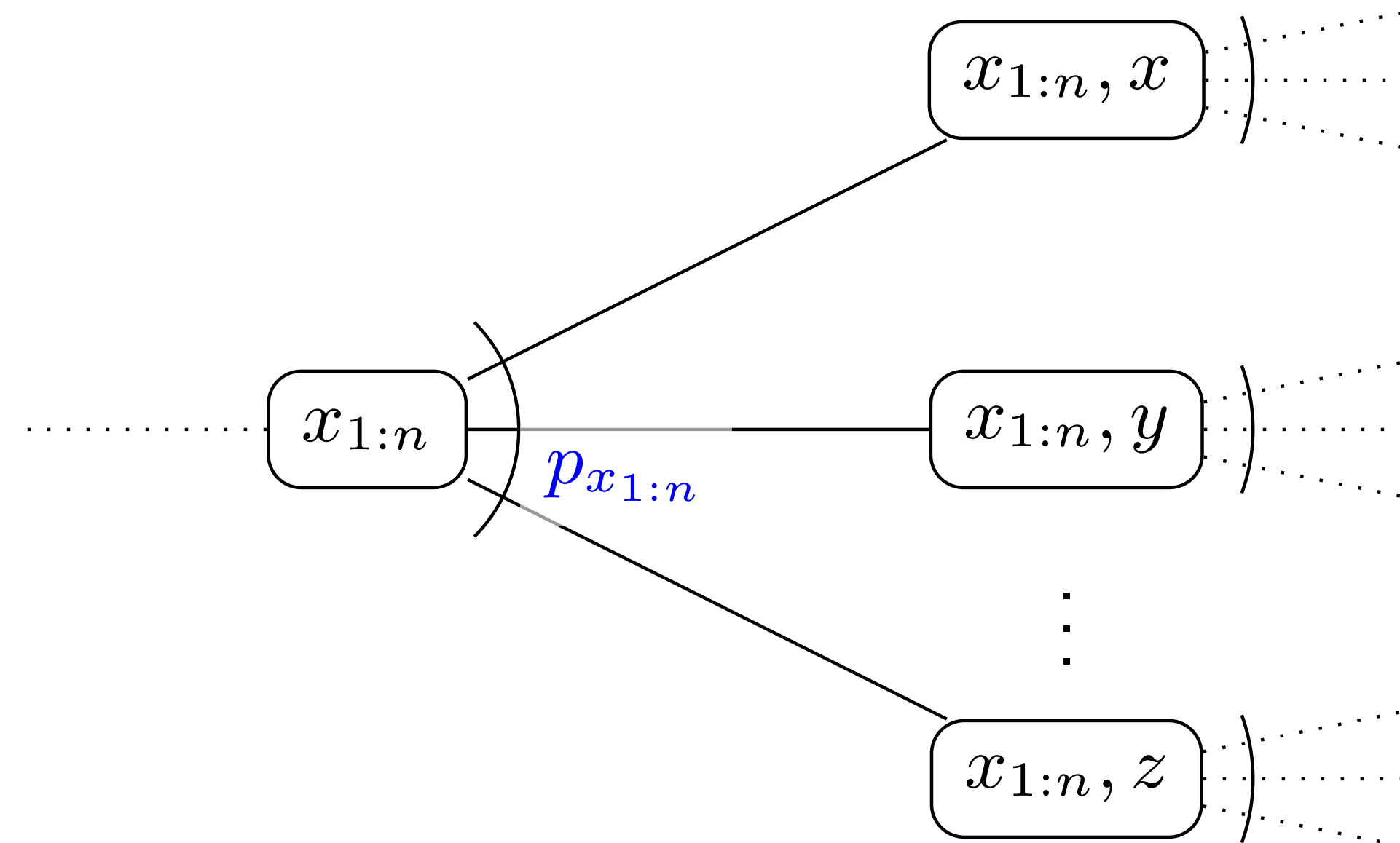
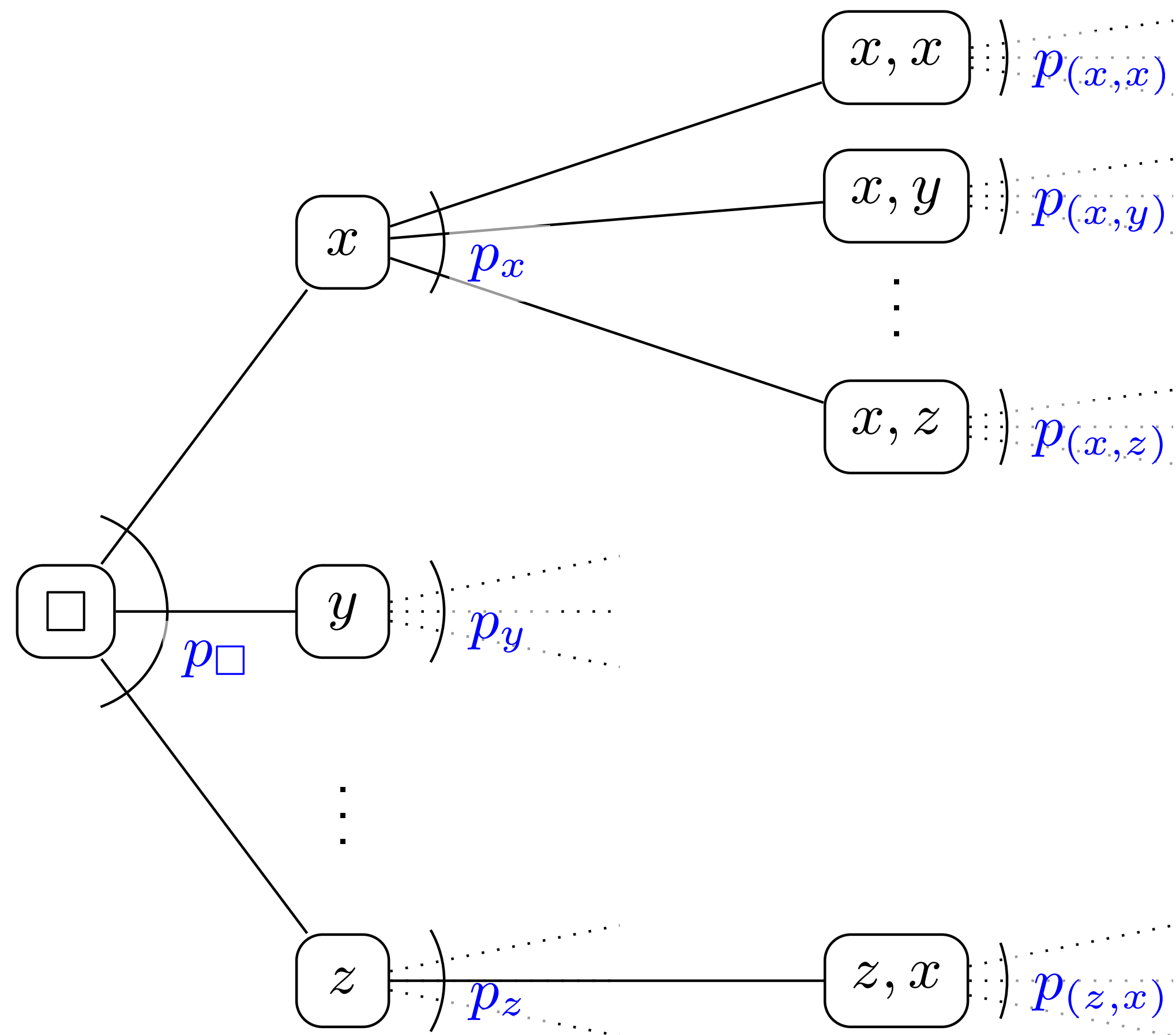
Uncertain processes

We consider a system whose uncertain state X_t
takes values in the state space $\mathcal{X}$ and
changes over time $t \in \mathbb{T}$.

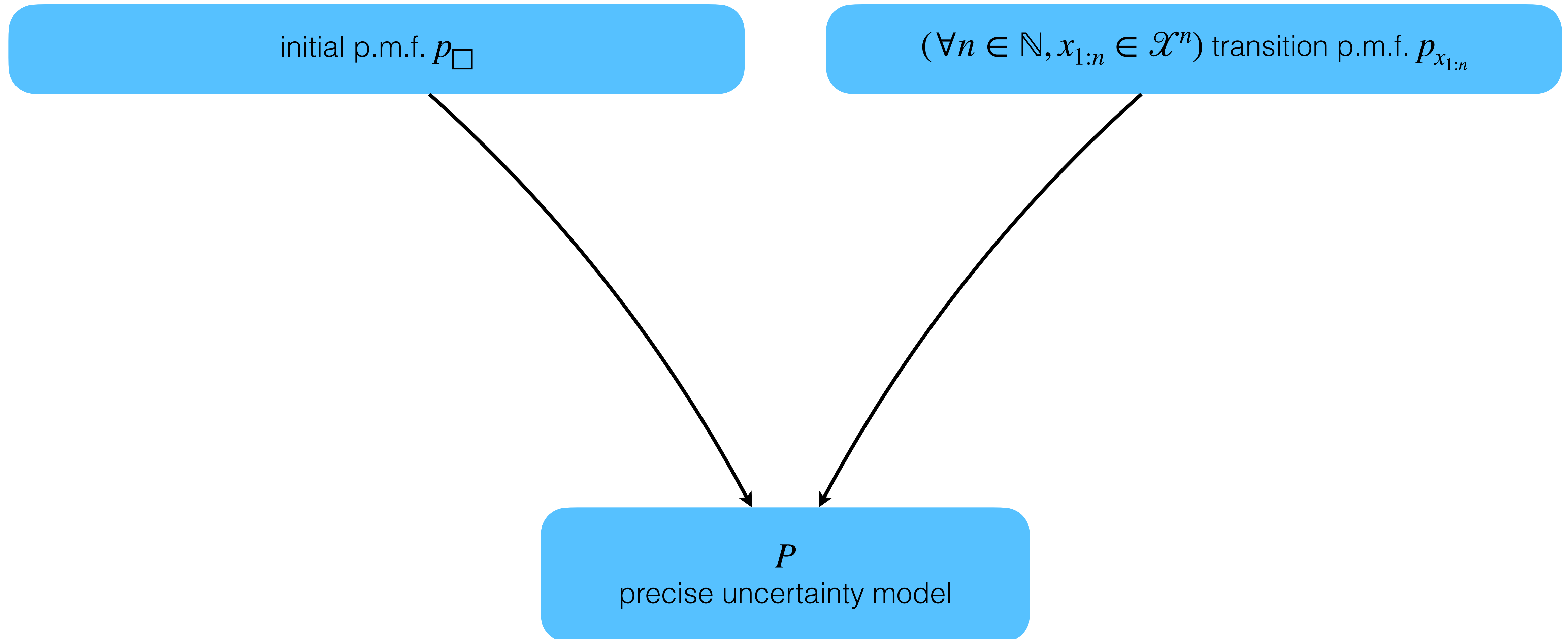
Let us focus on finite-state discrete-time uncertain processes, so with
 $\mathcal{X}$ finite and $\mathbb{T} = \mathbb{N} = \{1, 2, 3, \dots\}$.

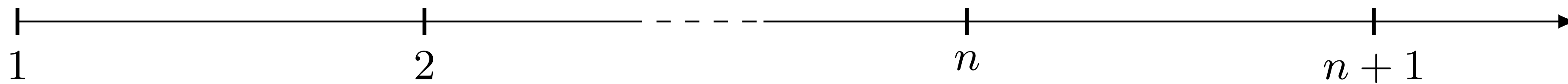
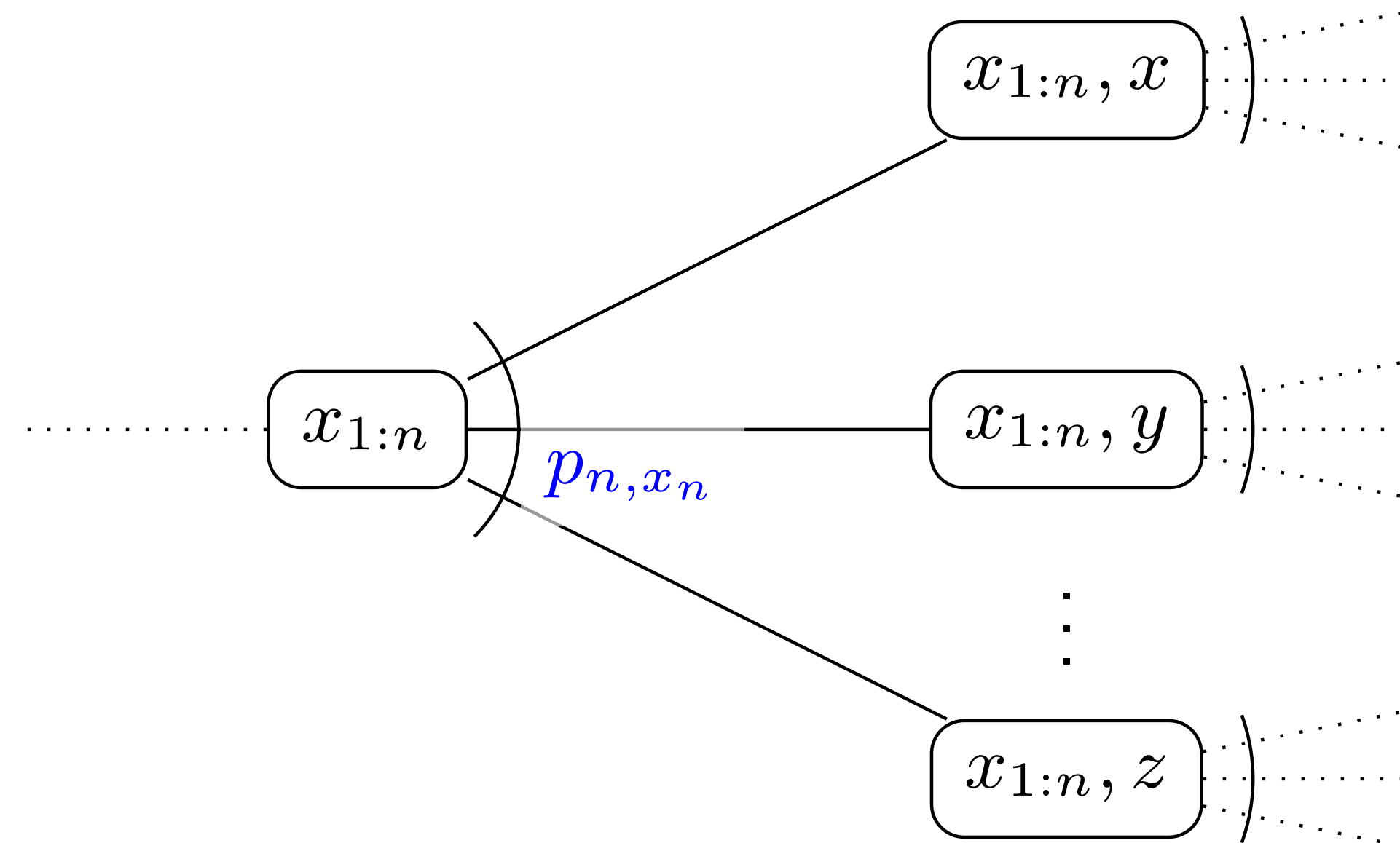
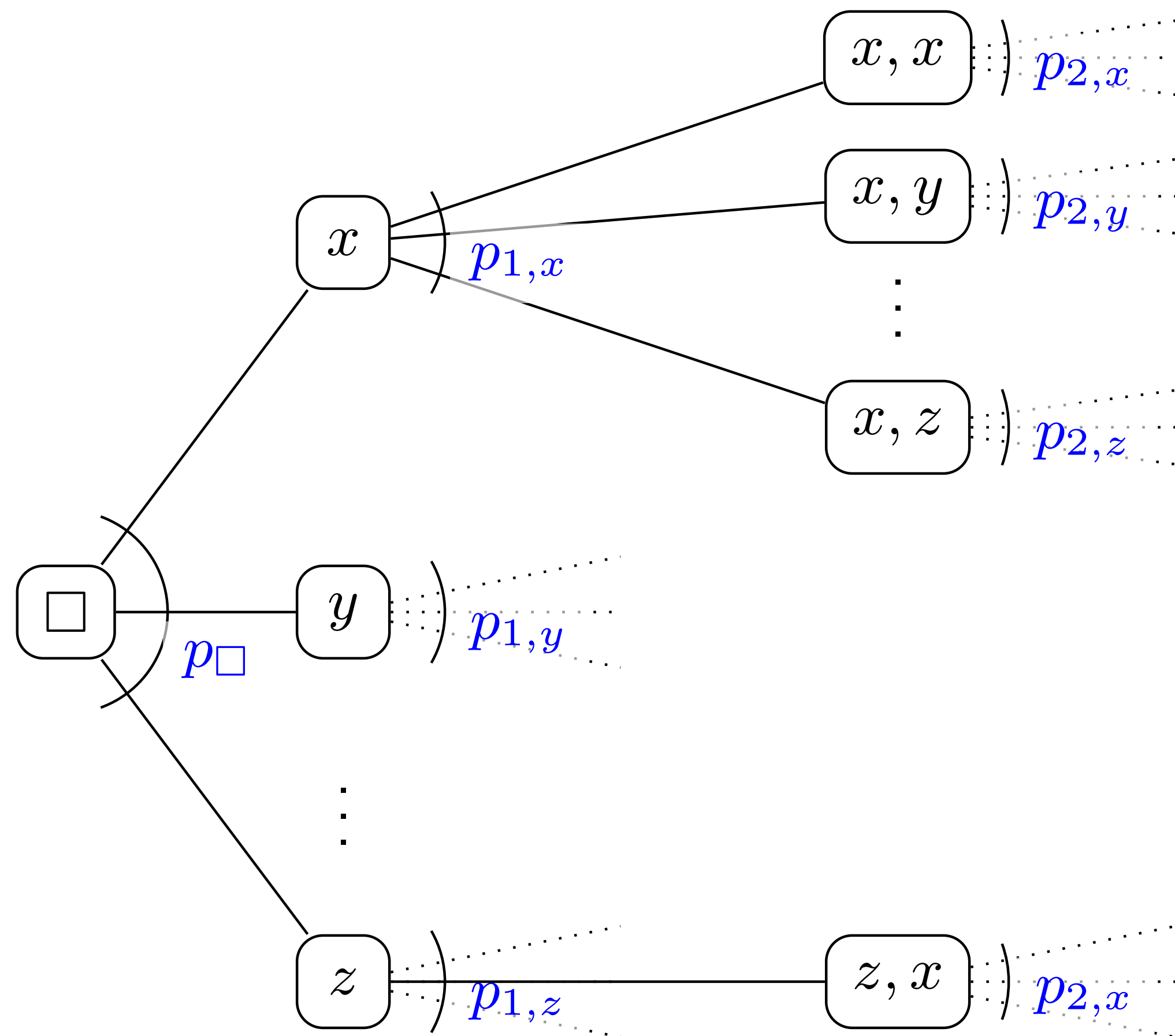




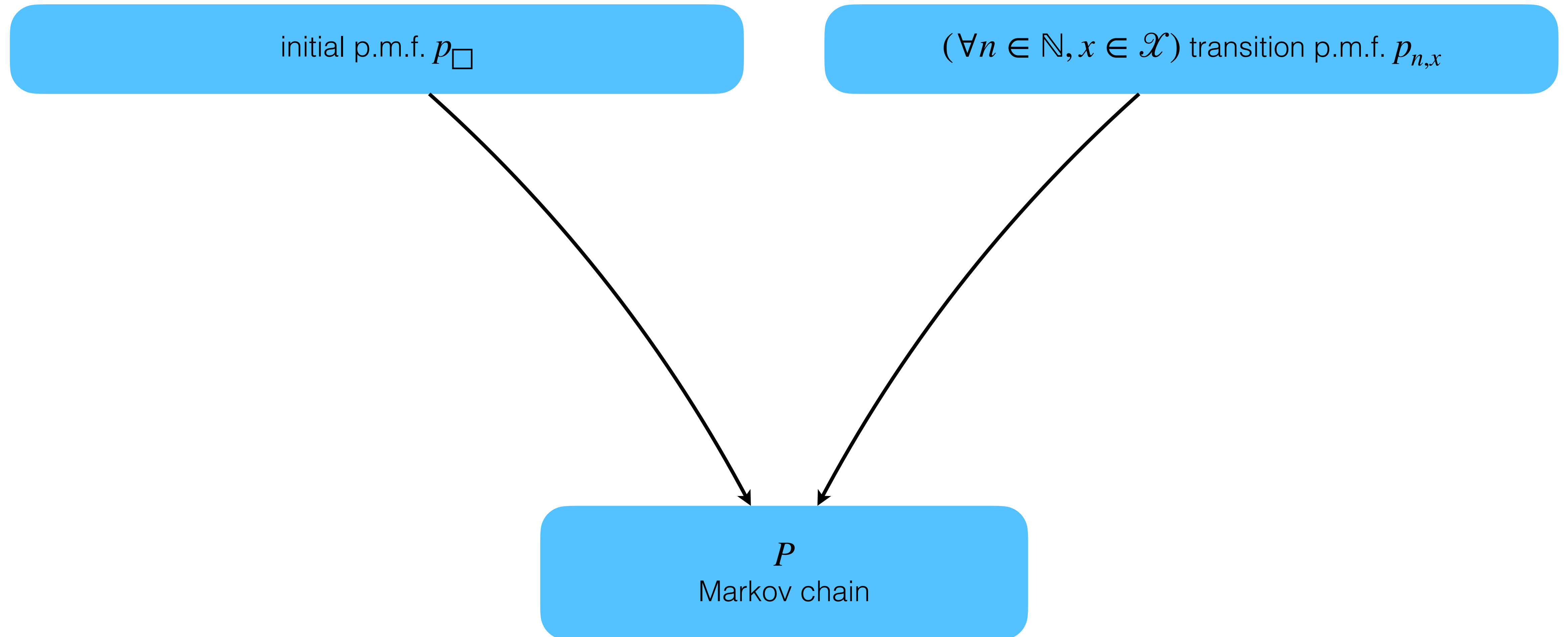


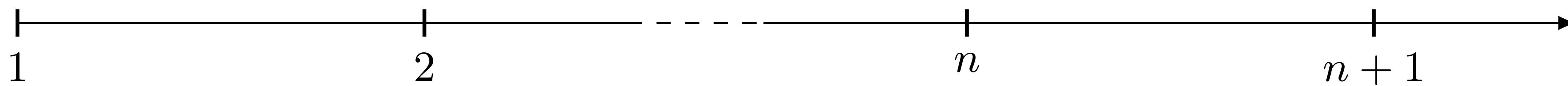
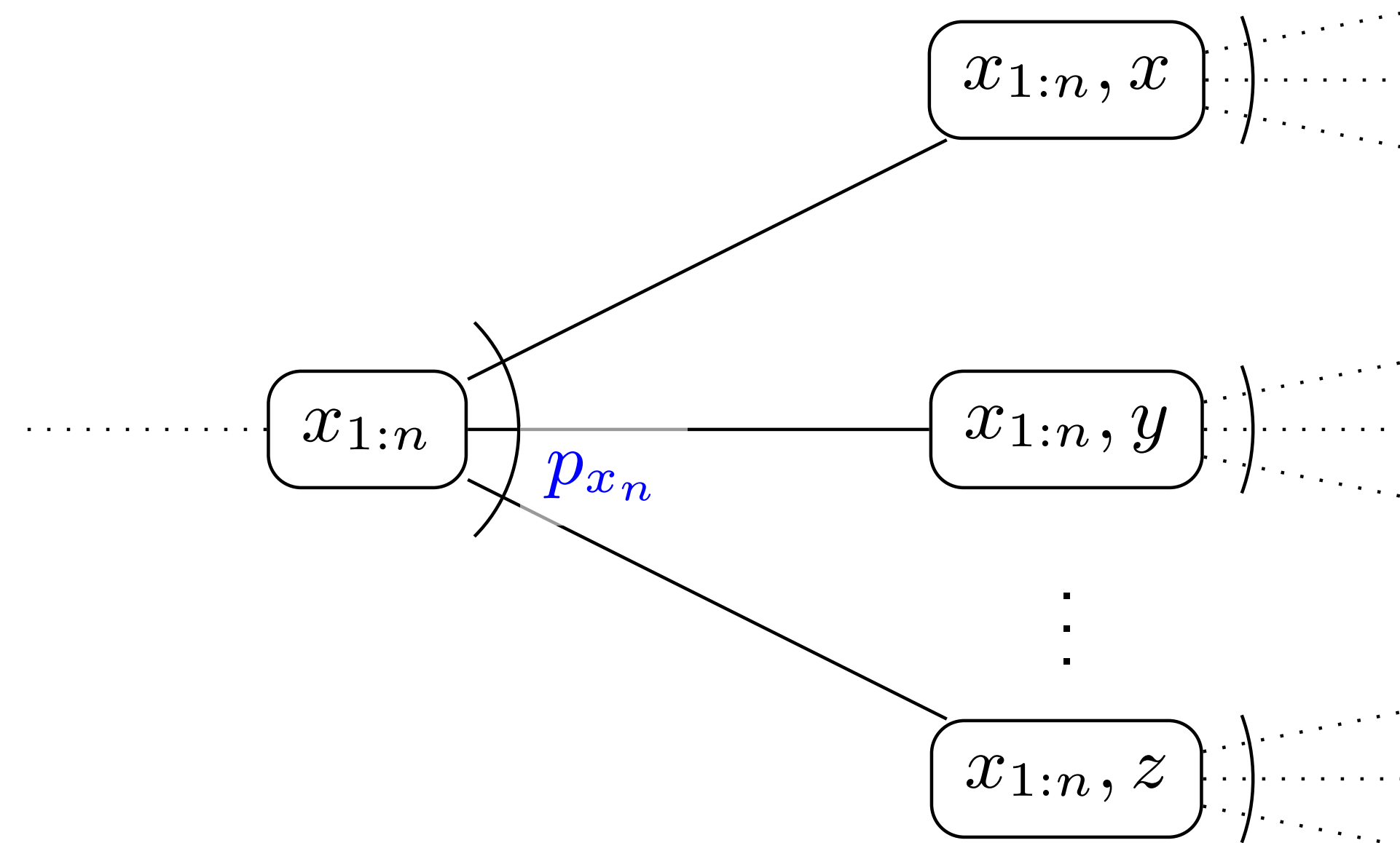
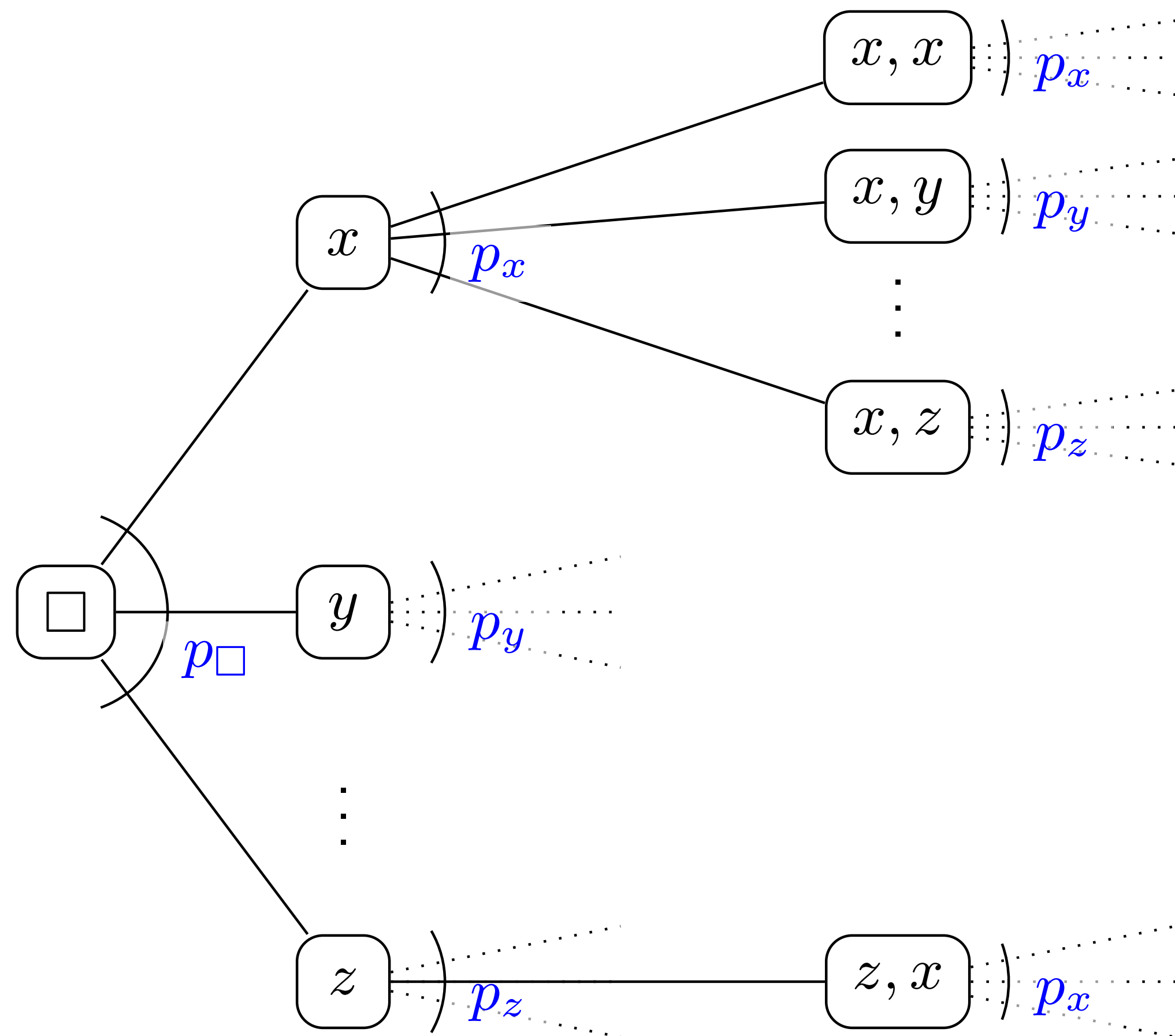
Precise uncertainty model



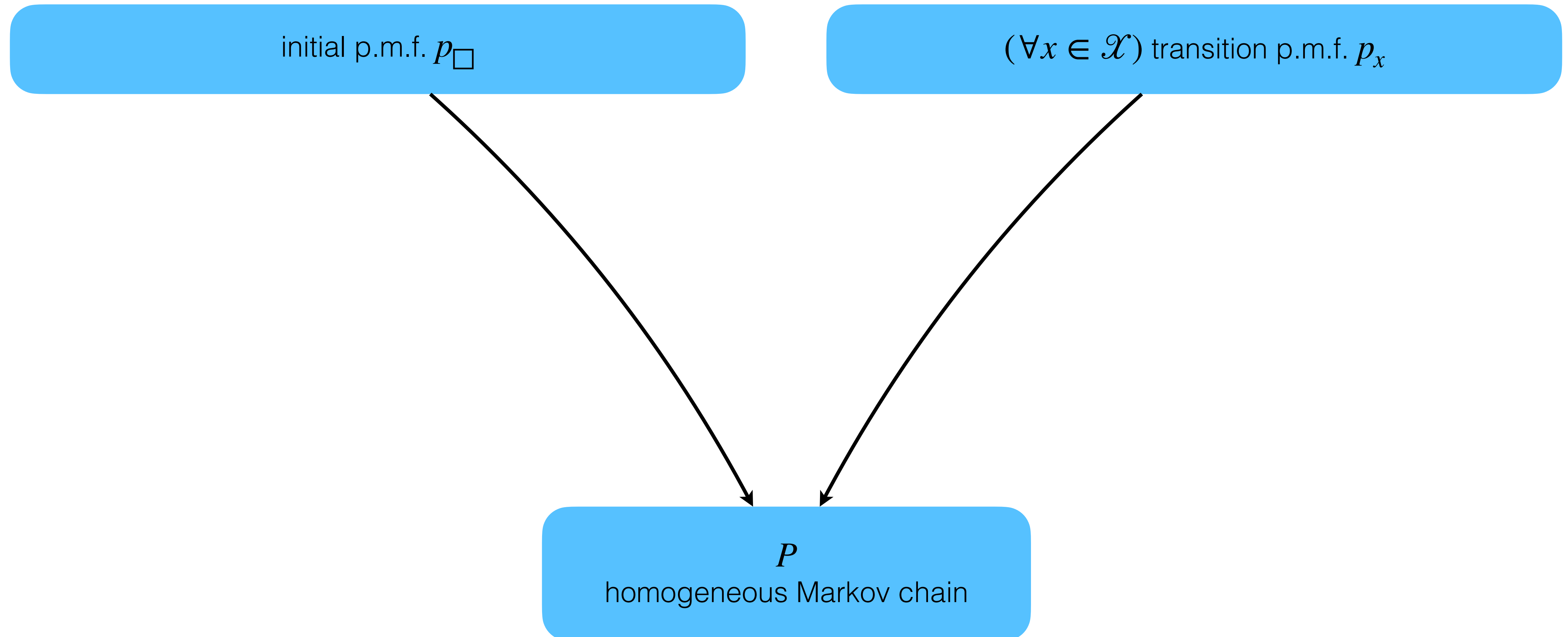


Markov chain





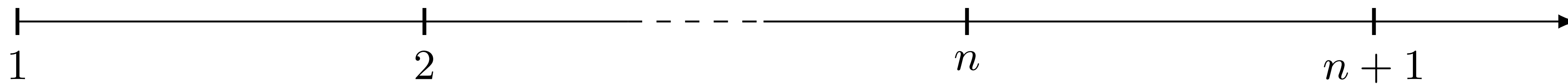
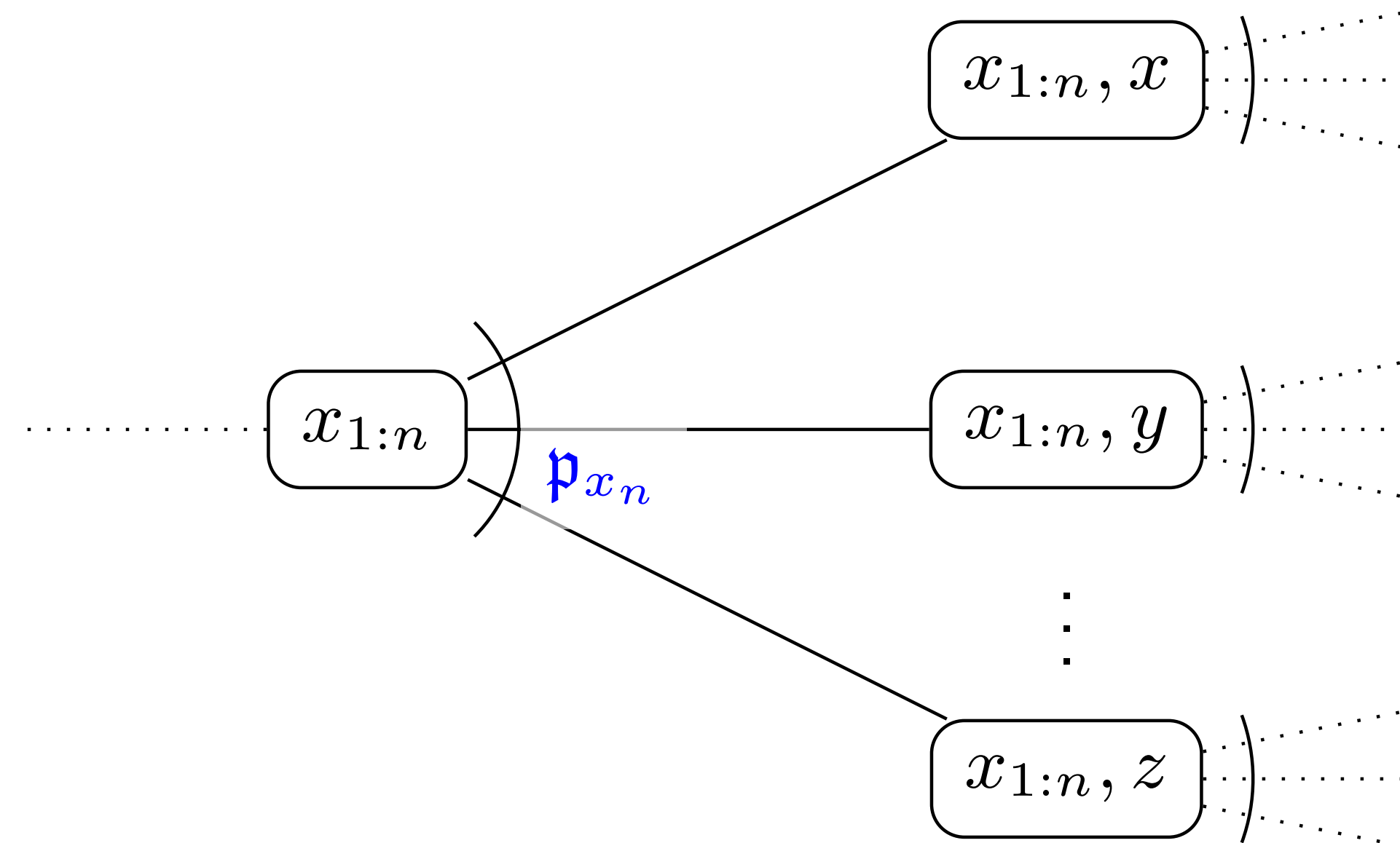
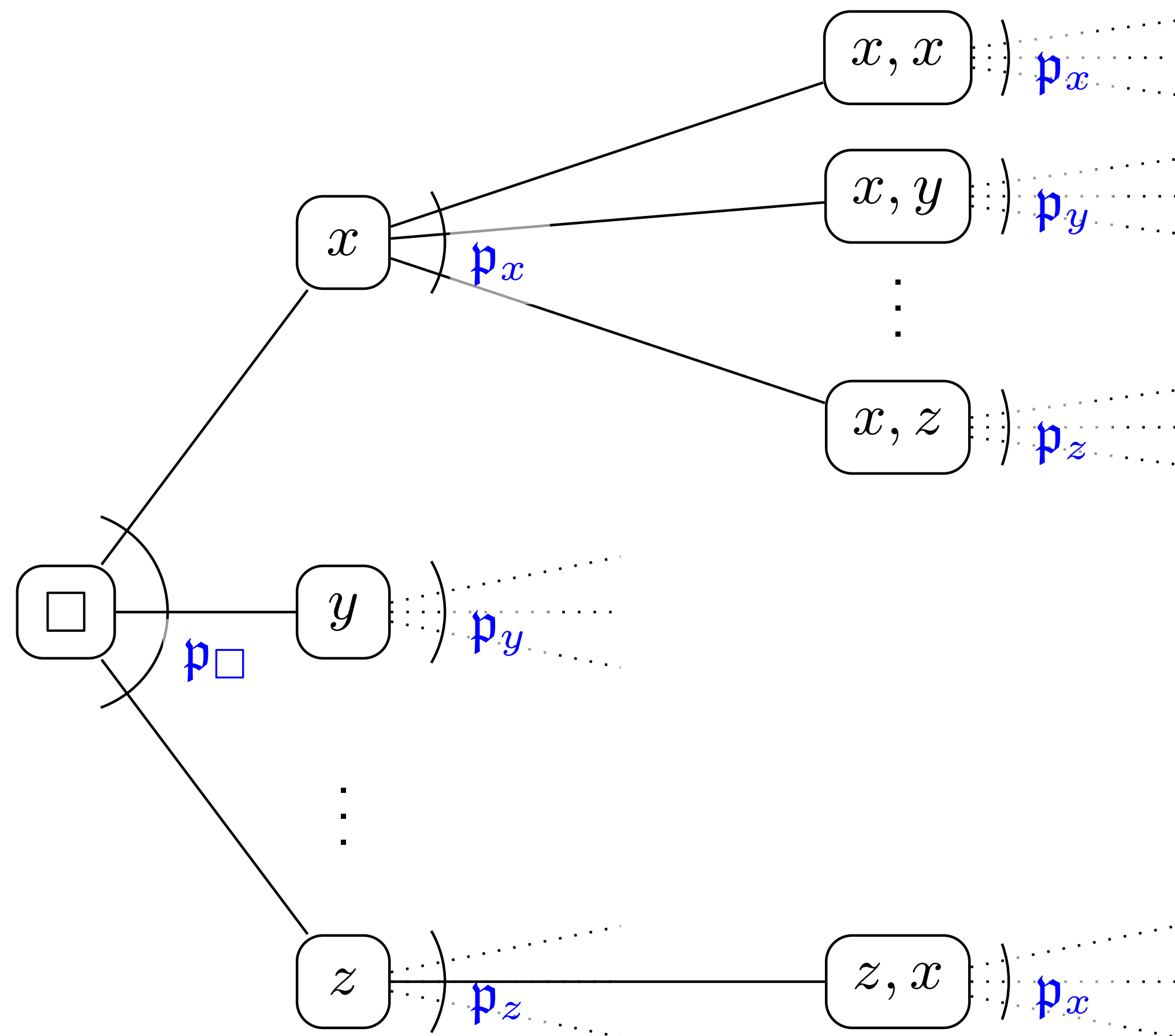
Homogeneous Markov chain



Imprecise Markov chain

set of initial p.m.f.s $\mathfrak{p}_{\square}$

$(\forall x \in \mathcal{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$



Imprecise Markov chain

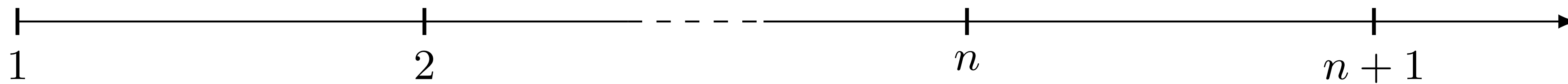
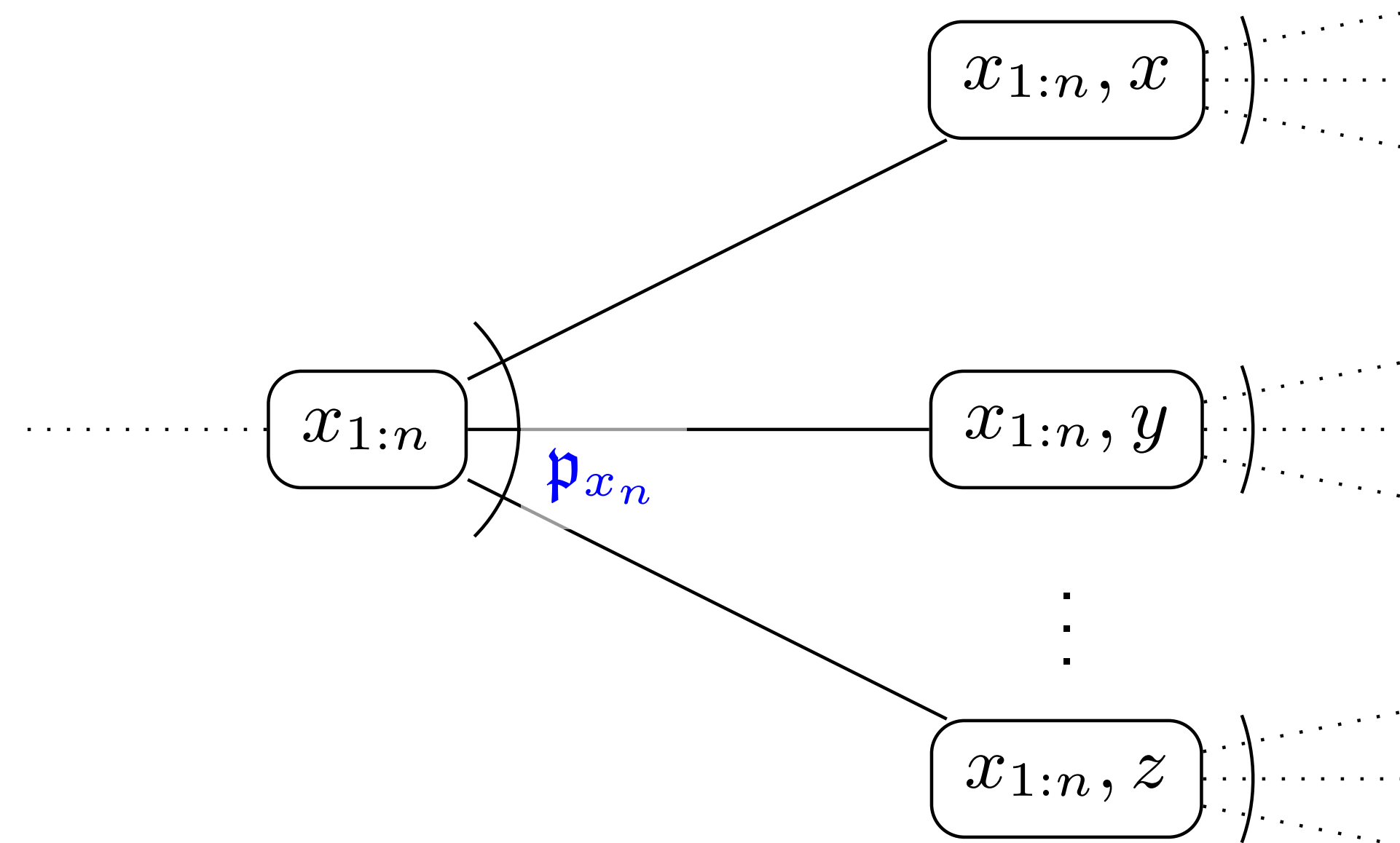
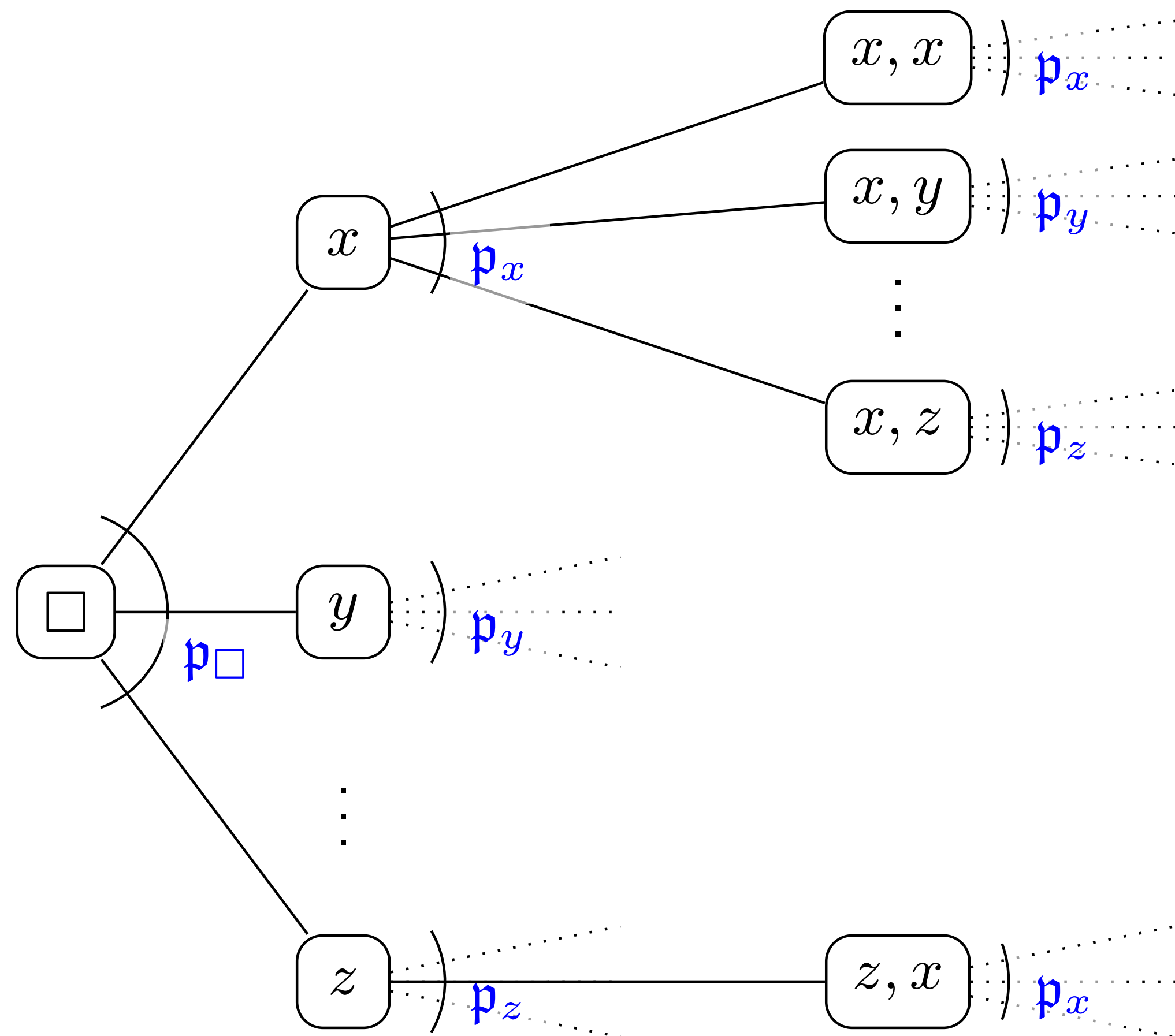
set of initial p.m.f.s $\mathfrak{p}_{\square}$

$(\forall x \in \mathcal{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$

homogeneous Markov chain P is **compatible** if $p_{\square} \in \mathfrak{p}_{\square}$ and $(\forall x \in \mathcal{X}) p_x \in \mathfrak{p}_x$

$\mathcal{P}^{\text{HM}}$

set of compatible
homogeneous Markov chains



Imprecise Markov chains

set of initial p.m.f.s $\mathfrak{p}_{\square}$

$(\forall x \in \mathcal{X})$ a set of transition p.m.f.s $\mathfrak{p}_x$

P is **compatible** if $p_{\square} \in \mathfrak{p}_{\square}$ and $(\forall n \in \mathbb{N}, x_{1:n} \in \mathcal{X}^n) p_{x_{1:n}} \in \mathfrak{p}_{x_n}$

$\mathcal{P}$

set of compatible
precise uncertainty models

$\supseteq$

$\mathcal{P}^M$

set of compatible
(inhomogeneous) Markov chains

$\supseteq$

$\mathcal{P}^{HM}$

set of compatible
homogeneous Markov chains

Lumping

detailed state space $\mathcal{X}$



P homogeneous & Markov

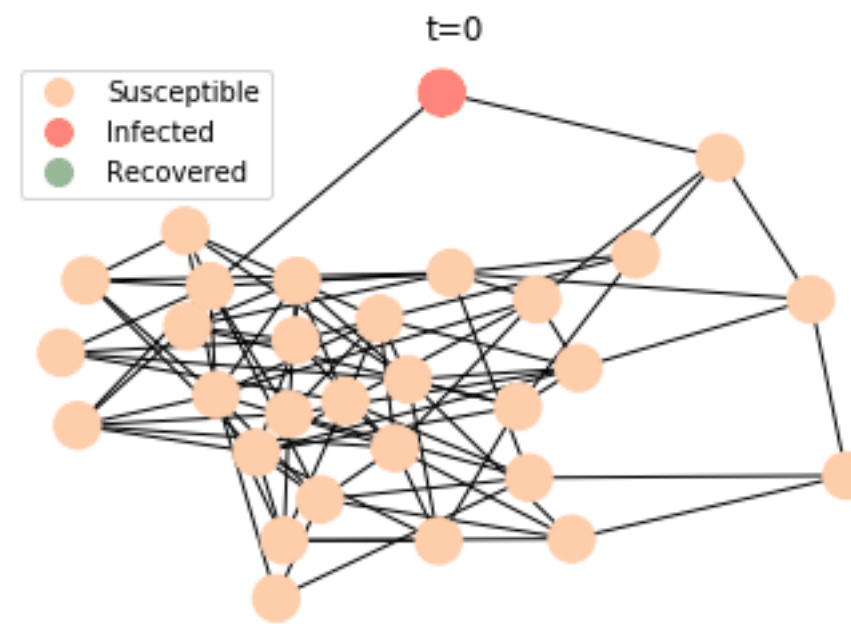
Lumping

S/I/R per individual

detailed state space $\mathcal{X}$



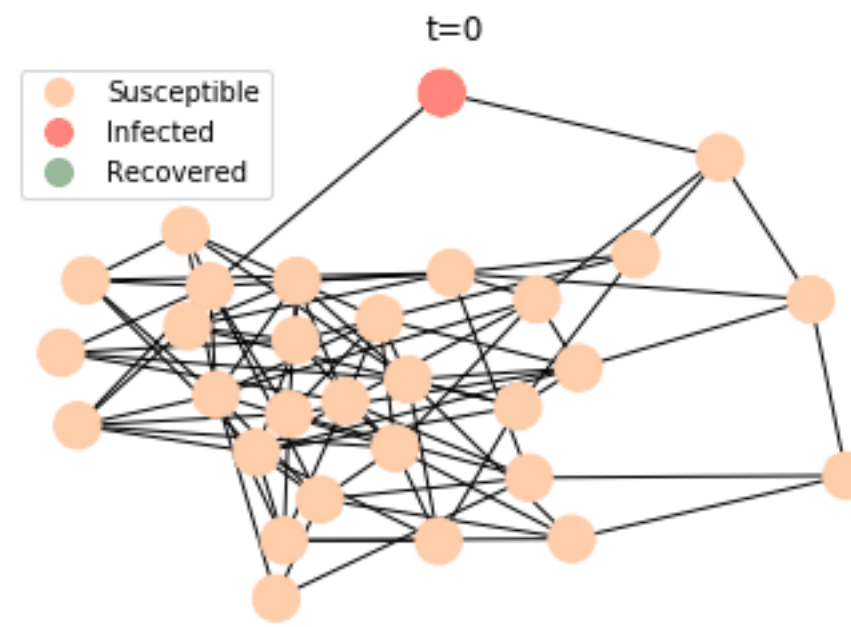
P homogeneous & Markov



<https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/>

Lumping

S/I/R per individual



<https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/>

detailed state space $\mathcal{X}$



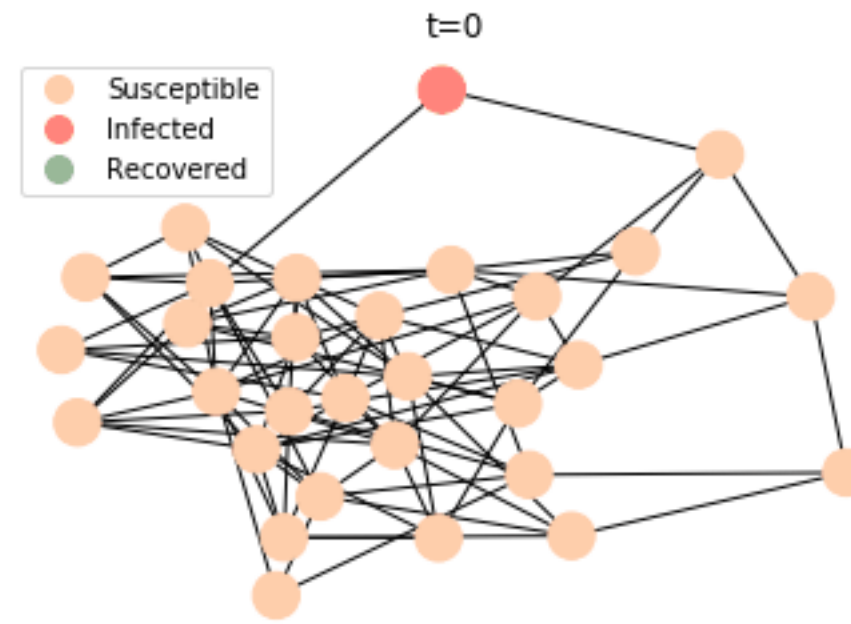
P homogeneous & Markov



smaller lumped state space $\hat{\mathcal{X}}$

Lumping

S/I/R per individual



individuals S/I/R

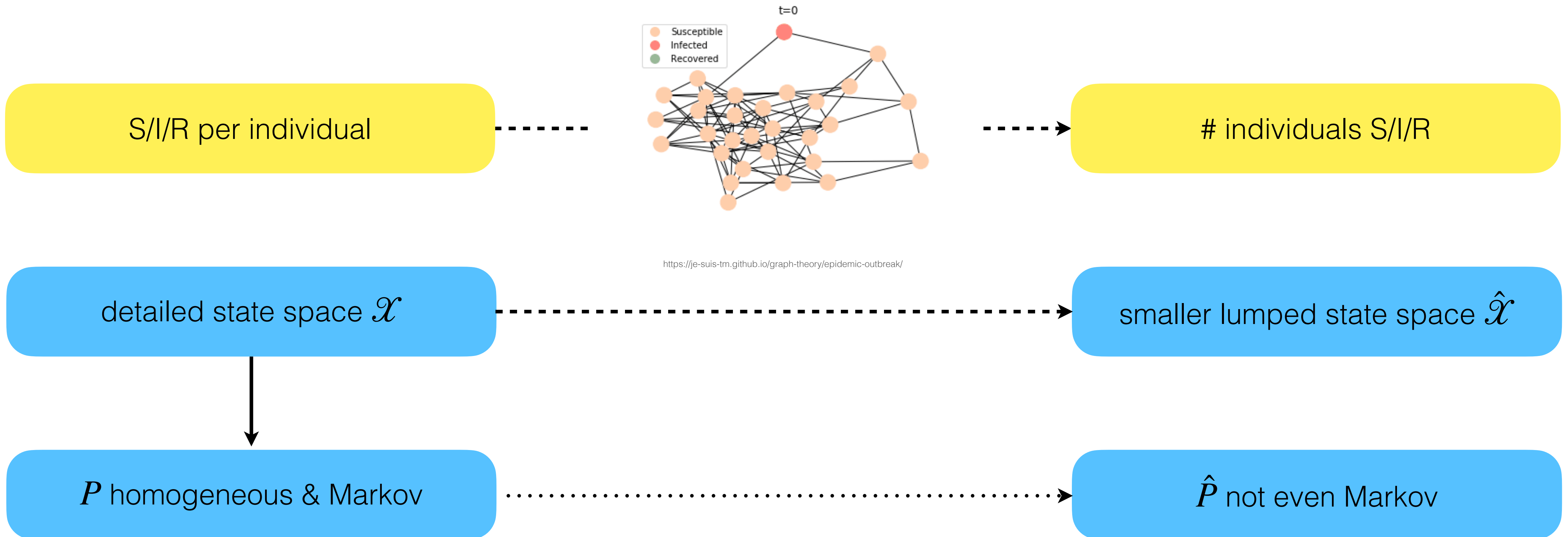
detailed state space $\mathcal{X}$

smaller lumped state space $\hat{\mathcal{X}}$

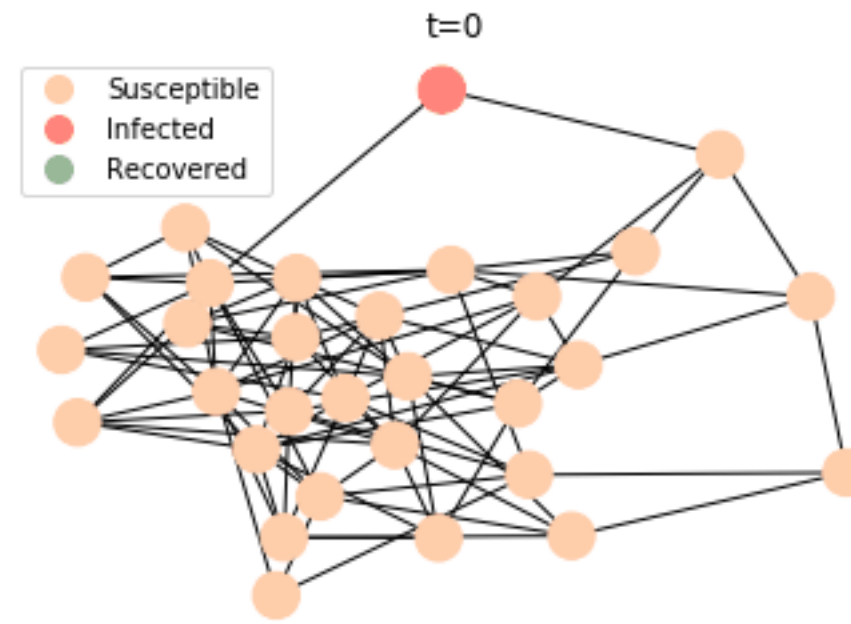
P homogeneous & Markov

<https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/>

Lumping



Lumping



<https://je-suis-tm.github.io/graph-theory/epidemic-outbreak/>

S/I/R per individual

individuals S/I/R

detailed state space $\mathcal{X}$

smaller lumped state space $\hat{\mathcal{X}}$

P homogeneous & Markov

$\hat{P}$ not even Markov

$p_{\square} \rightarrow \hat{p}_{\square}$
 $p_x, p_y, \dots \rightarrow \hat{p}_{\hat{x}}$

$\hat{\mathcal{P}}$

Inference for imprecise Markov chains

consider a (measurable) variable $f: \mathcal{X}^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}$

$$\overline{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$$

$$\geq$$

$$\overline{E}^{\text{M}}(f) = \sup_{P \in \mathcal{P}^{\text{M}}} E_P(f)$$

$$\geq$$

$$\overline{E}^{\text{HM}}(f) = \sup_{P \in \mathcal{P}^{\text{HM}}} E_P(f)$$

Inference for imprecise Markov chains

consider a (measurable) variable $f: \mathcal{X}^{\mathbb{N}} \rightarrow \overline{\mathbb{R}}$

$$\overline{E}(f) = \sup_{P \in \mathcal{P}} E_P(f)$$

$$\geq$$

$$\overline{E}^{\mathbf{M}}(f) = \sup_{P \in \mathcal{P}^{\mathbf{M}}} E_P(f)$$

$$\geq$$

$$\overline{E}^{\mathbf{HM}}(f) = \sup_{P \in \mathcal{P}^{\mathbf{HM}}} E_P(f)$$

finitary variables

$$g(X_{1:n})$$

non-finitary variables

$$\tau_G: \mathcal{X}^{\mathbb{N}} \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}: (x_n)_{n \in \mathbb{N}} \mapsto \inf\{n \in \mathbb{N}: x_n \in G\}$$

Inference for $\mathcal{P}^{\text{HM}}$

Compute $\bar{E}^{\text{HM}}(f)$?

Inference for $\mathcal{P}^{\text{HM}}$

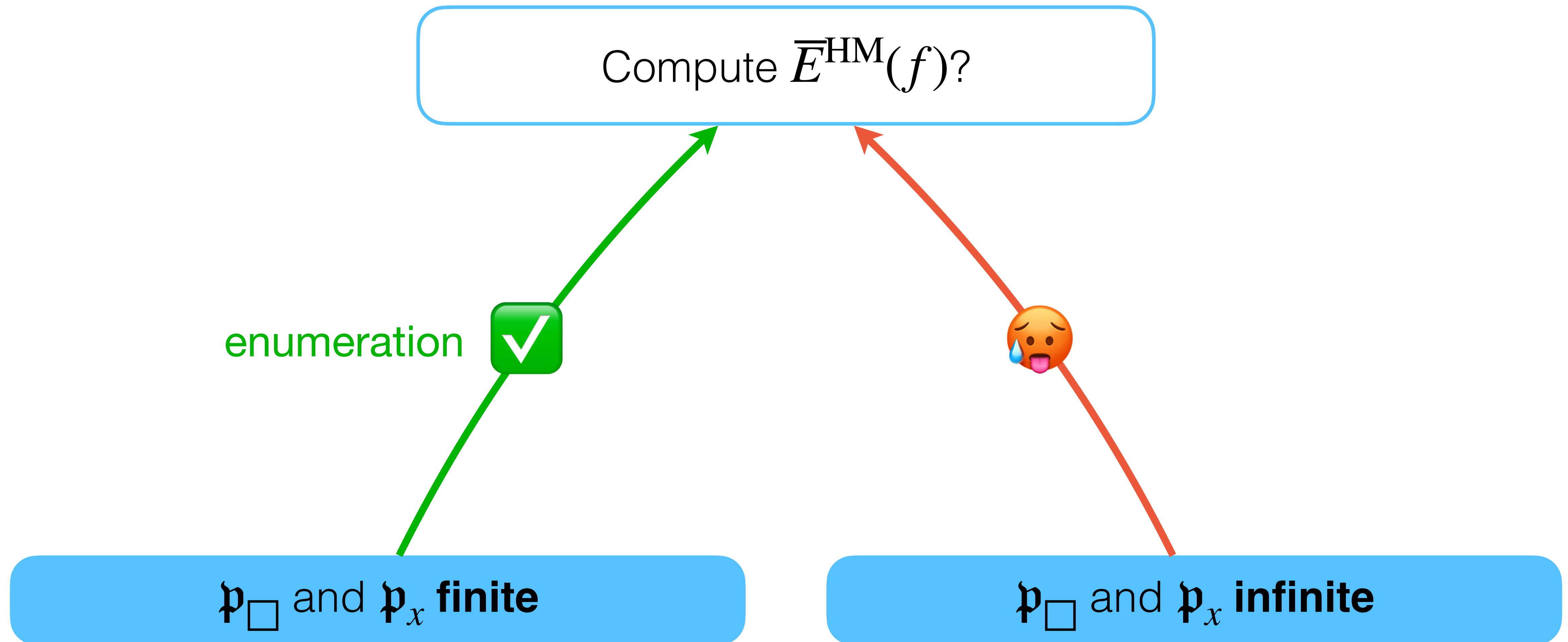
Compute $\bar{E}^{\text{HM}}(f)$?

enumeration



$\mathfrak{p}_{\square}$ and $\mathfrak{p}_x$ **finite**

Inference for $\mathcal{P}^{\text{HM}}$



Inference for $\mathcal{P}^M$

Compute $\bar{E}^M(f)$?

Inference for $\mathcal{P}^M$

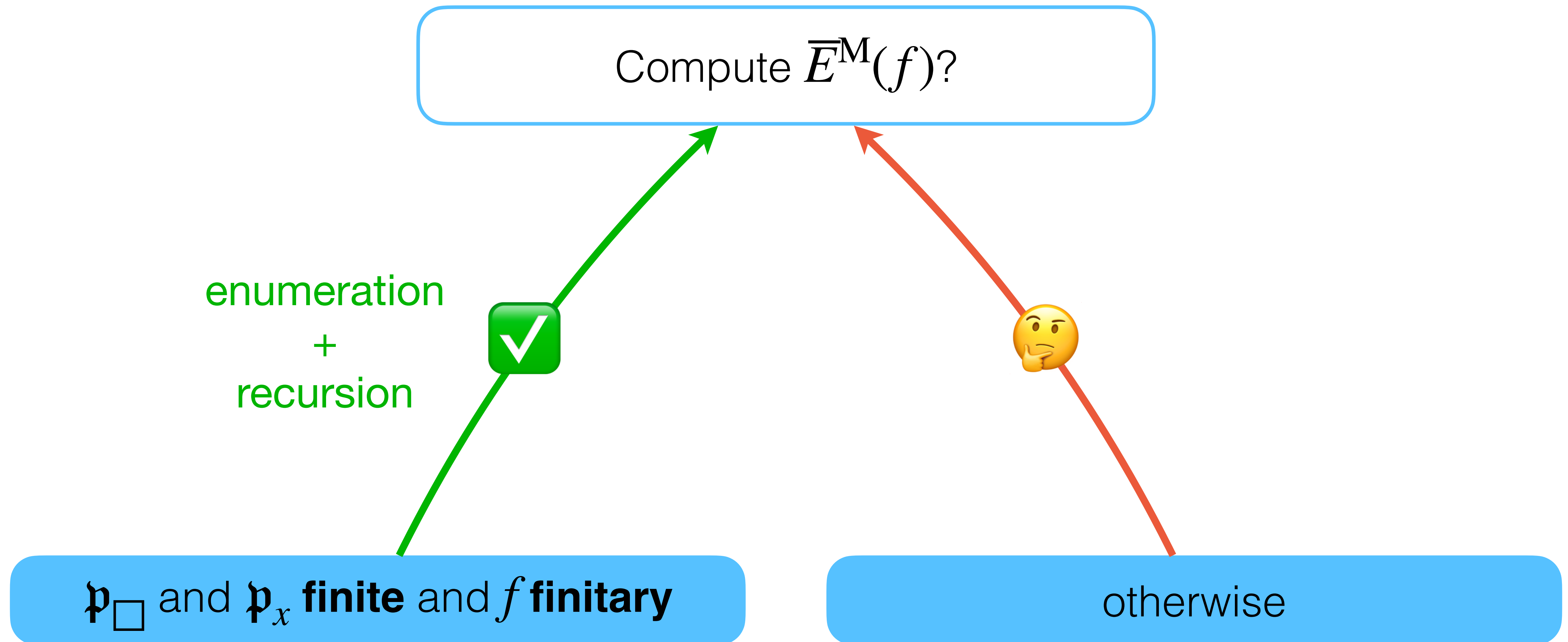
Compute $\bar{E}^M(f)$?

enumeration
+
recursion



$\mathfrak{p}_{\square}$ and $\mathfrak{p}_x$ **finite** and f **finitary**

Inference for $\mathcal{P}^M$



Inference for $\mathcal{P}$

Compute $\bar{E}(f)$?

Inference for $\mathcal{P}$

Compute $\bar{E}(f)$?

computing $\bar{E}(f)$ requires

- * 1 evaluation of $\bar{E}_{\mathfrak{p}_{\square}}$
- * $\mathcal{O}(|\mathcal{X}|^{n-1})$ evaluations of $\bar{E}_{\mathfrak{p}}$.

$$f = g(X_{1:n})$$

f has a **sum-product decomposition** if

$$f = g_1(X_1) + g_2(X_2)h_1(X_1) + \cdots + g_n(X_n)h_{n-1}(X_{n-1})\cdots h_1(X_1) = \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

f has a **sum-product decomposition** if

$$f = g_1(X_1) + g_2(X_2)h_1(X_1) + \cdots + g_n(X_n)h_{n-1}(X_{n-1})\cdots h_1(X_1) = \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$

$$g(X_n)$$

$$\text{finite time average } \frac{1}{n} \sum_{k=1}^n g(X_k)$$

$$\text{finite horizon reward } \sum_{k=1}^n r_k(X_k)$$

$$\mathbb{I}_{\{S\mathcal{U}^{\leq n}G\}} \text{ for the time-bounded until event } \{S\mathcal{U}^{\leq n}G\} = \{k \leq n, X_k \in G \text{ and } X_1, \dots, X_{k-1} \in S\}$$

Inference for $\mathcal{P}$

Compute $\bar{E}(f)$?

computing $\bar{E}(f)$ requires

- * 1 evaluation of $\bar{E}_{\mathfrak{p}_{\square}}$
- * $\mathcal{O}(|\mathcal{X}|^{n-1})$ evaluations of $\bar{E}_{\mathfrak{p}}$.

$$f = g(X_{1:n})$$

computing $\bar{E}(f)$ requires

- * 1 evaluation of $\bar{E}_{\mathfrak{p}_{\square}}$
- * $(n-1)|\mathcal{X}|$ evaluations of $\bar{E}_{\mathfrak{p}}$.

$$f = \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_{\ell}(X_{\ell})$$

Inference for $\mathcal{P}$

Compute $\bar{E}(f)$?

computing $\bar{E}(f)$ requires

- * 1 evaluation of $\bar{E}_{\mathbf{p}_{\square}}$
- * $\mathcal{O}(|\mathcal{X}|^{n-1})$ evaluations of $\bar{E}_{\mathbf{p}}$.

$$f = g(X_{1:n})$$

computing $\bar{E}(f)$ requires

$$h_1, \dots, h_{n-1} \geq 0$$

$$\Downarrow$$

$$\bar{E}(f) = \bar{E}^M(f)$$

$$f = \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_{\ell}(X_{\ell})$$

$$f = \lim_{n \rightarrow +\infty} \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_{\ell}(X_{\ell})$$



$$\bar{E}(f) = \lim_{n \rightarrow +\infty} \bar{E} \left(\sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_{\ell}(X_{\ell}) \right)$$

$$f = \lim_{n \rightarrow +\infty} \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$



$$\bar{E}(f) = \lim_{n \rightarrow +\infty} \bar{E} \left(\sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell) \right)$$

time average $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n g(X_k)$

discounted reward $\lim_{n \rightarrow +\infty} \sum_{k=1}^n \lambda^{k-1} r(X_k)$

$\mathbb{I}_{\{S\mathcal{U}G\}}$ for the until event $\{S\mathcal{U}G\} = \{X_k \in G \text{ and } X_1, \dots, X_{k-1} \in S\}$

hitting time $\tau_G = \inf\{n \in \mathbb{N} : X_n \in G\}$

$$f = \lim_{n \rightarrow +\infty} \sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell)$$



$$\bar{E}(f) = \lim_{n \rightarrow +\infty} \bar{E} \left(\sum_{k=1}^n g_k(X_k) \prod_{\ell=1}^{k-1} h_\ell(X_\ell) \right)$$

time average $\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{k=1}^n g(X_k)$

space average $\lim_{n \rightarrow +\infty} \sum_{k=1}^n \lambda^{k-1} r(X_k)$

$h_1, h_2, \dots \geq 0$
 $\Downarrow$
 $\bar{E}(f) \stackrel{\text{grad}}{=} \bar{E}^M(f) \stackrel{\text{gear}}{=} \bar{E}^{\text{HM}}(f)$

$\mathbb{I}_{\{S \mathcal{U} G\}}$ for

$\{X_k \in G \text{ and } X_1, \dots, X_{k-1} \in S\}$

hitting time $\tau_G = \inf\{n \in \mathbb{N} : X_n \in G\}$

How to Make your Markov Chains Robust

An Imprecise Probabilities Perspective



Jasper De Bock & Alexander Erreygers

21 September 2023

Eindhoven

Continuous-time imprecise Markov chains

$$p_{x_{1:n}}: \mathcal{X} \rightarrow [0,1]: y \mapsto \cancel{P(X_{n+1} = y \mid X_1 = x_1, \dots, X_n = x_n)}$$

$$q_{x_{1:n}}^{t_{1:n}}: \mathcal{X} \rightarrow \mathbb{R}: y \mapsto \left. \frac{\partial^+}{\partial \Delta} P(X_{t_n+\Delta} = y \mid X_{t_1} = x_1, \dots, X_{t_n} = x_n) \right|_{\Delta=0}$$

P is **compatible** if $p_{\square} \in \mathfrak{p}_{\square}$ and $(\forall n \in \mathbb{N}, t_{1:n} \in \mathbb{R}_{\geq 0}^n, x_{1:n} \in \mathcal{X}^n) q_{x_{1:n}}^{t_{1:n}} \in \mathfrak{q}_{x_n}$

$\mathcal{P}$

set of compatible
precise uncertainty models

$\mathcal{P}^M$

set of compatible
(inhomogeneous) Markov chains

$\mathcal{P}^{HM}$

set of compatible
homogeneous Markov chains

Limit distributions

Ergodicity

$$\bar{E}_{\infty}(f) = \lim_{n \rightarrow +\infty} \bar{E}(f(X_n) | X_1 = x)$$

II

$$\bar{E}_{\infty}^M(f) = \lim_{n \rightarrow +\infty} \bar{E}^M(f(X_n) | X_1 = x)$$

IV

$$\bar{E}_{\infty}^{\text{HM}}(f) = \lim_{n \rightarrow +\infty} \bar{E}^{\text{HM}}(f(X_n) | X_1 = x)$$

~~$$\pi_{\infty}(y) = \lim_{n \rightarrow +\infty} P(X_n = y | X_1 = x)$$~~

~~$$E_{\infty}(f) = \sum_{y \in \mathcal{X}} \pi_{\infty}(y) f(y)$$~~

Limit distributions

Ergodicity

$$\bar{E}_{\infty}(f) = \lim_{n \rightarrow +\infty} \bar{E}(f(X_n) | X_1 = x)$$

II

$$\bar{E}_{\infty}^M(f) = \lim_{n \rightarrow +\infty} \bar{E}^M(f(X_n) | X_1 = x)$$

IV

$$\bar{E}_{\infty}^{\text{HM}}(f) = \lim_{n \rightarrow +\infty} \bar{E}^{\text{HM}}(f(X_n) | X_1 = x)$$

Weak ergodicity

$$\geq \bar{E}_{\text{av},\infty}(f) = \lim_{n \rightarrow +\infty} \bar{E}\left(\frac{1}{n} \sum_{i=1}^n f(X_i) \mid X_1 = x\right)$$

II

$$\geq \bar{E}_{\text{av},\infty}^M(f) = \lim_{n \rightarrow +\infty} \bar{E}^M\left(\frac{1}{n} \sum_{i=1}^n f(X_i) \mid X_1 = x\right)$$

II

$$= \bar{E}_{\text{av},\infty}^{\text{HM}}(f) = \lim_{n \rightarrow +\infty} \bar{E}^{\text{HM}}\left(\frac{1}{n} \sum_{i=1}^n f(X_i) \mid X_1 = x\right)$$

Pointwise ergodic theorem

$$\lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n f(X_i) = E_{\infty}(f) = \sum_{x \in \mathcal{X}} \pi_{\infty}(x) f(x) \quad \text{almost surely (with probability one)}$$

For **all three types** of imprecise Markov chains:

$$\underline{E}_{\infty}(f) \leq \underline{E}_{\text{av},\infty}(f) \leq \liminf_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n f(X_i) \leq \limsup_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=1}^n f(X_i) \leq \bar{E}_{\text{av},\infty}(f) \leq \bar{E}_{\infty}(f)$$

almost surely (with lower probability one)

How to Make your Markov Chains Robust

An Imprecise Probabilities Perspective



Jasper De Bock & Alexander Erreygers

21 September 2023

Eindhoven

