A Theory of Desirable Things

Jasper De Bock ISIPTA 2023

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Foundations **L**ab for **i**mprecise **p**robabilities

Ghent University

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A Theory of Desirable Things


Desirable

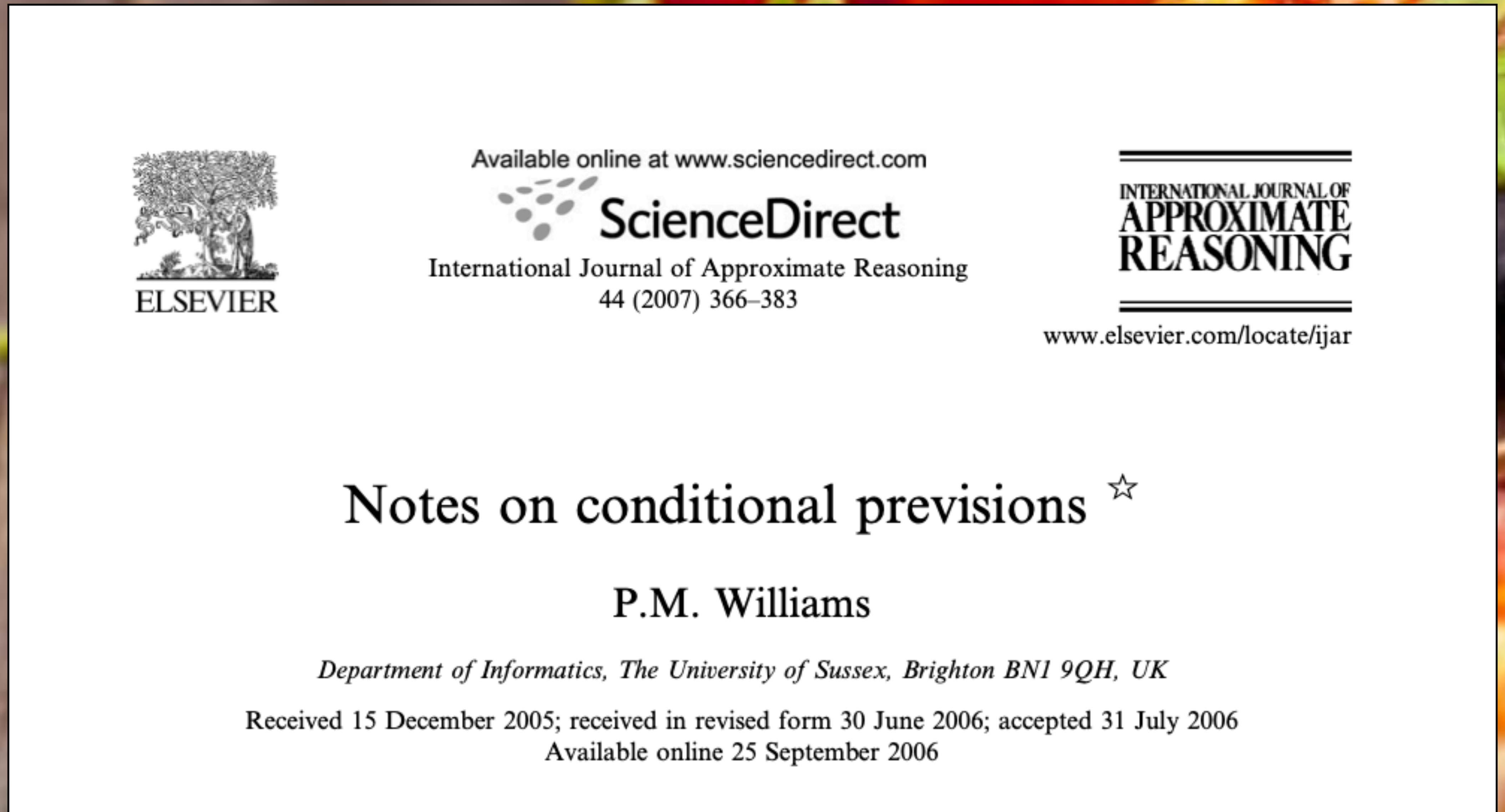
Desirable Gambles

A Theory of Desirable Gambles

3 RULES

1. **Negative gambles** are never desirable
2. **Positive gambles** are always desirable
3. Any **positive linear combination** of desirable gambles is desirable

Sets of Desirable Gambles



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International Journal of Approximate Reasoning 44 (2007) 366–383

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INTERNATIONAL JOURNAL OF APPROXIMATE REASONING

www.elsevier.com/locate/ijar

Notes on conditional previsions ☆

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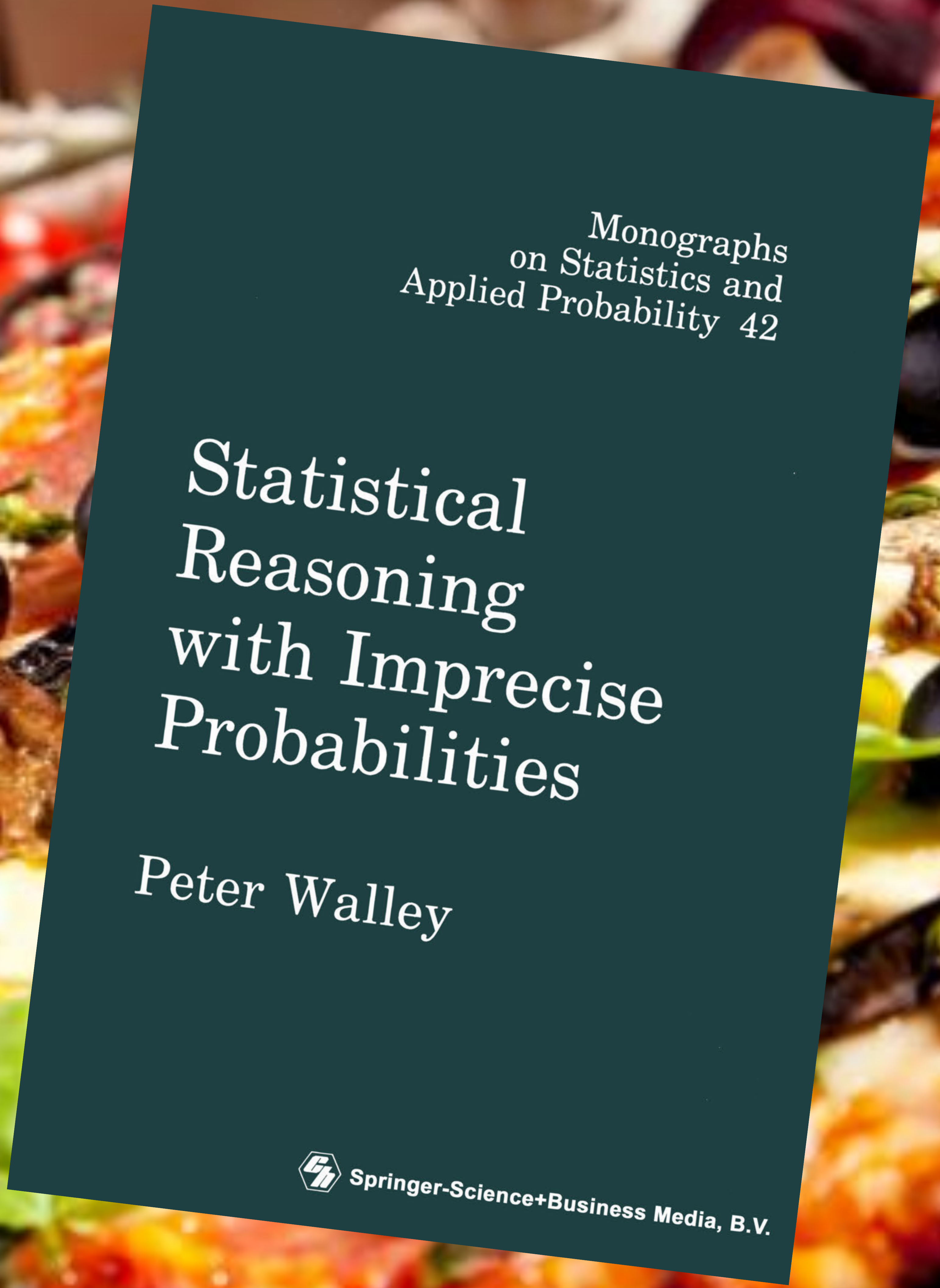
Received 15 December 2005; received in revised form 30 June 2006; accepted 31 July 2006
Available online 25 September 2006




Sets of Desirable Sets of Gambles
Proceedings of Machine Learning Research 103:125–134, 2019
ISIPTA 2019
Interpreting, Axiomatising and Representing Coherent Choice Functions in Terms of Desirability
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Abstract
Choice functions constitute a simple, direct and very general mathematical framework for modelling
representation, non-binary choice models.
1. Introduction
Choice functions provide an elegant unifying mathematical framework for studying set-valued choice: when presented with a set of options, they generally return a subset of them. If this subset is a singleton, it provides a unique optimal choice or decision. But if the answer contains multiple options, these are incomparable and no decision is made between them. Such set-valued choices are a typical feature of decision criteria based on imprecise-probabilistic uncertainty models, which aim to make reliable decisions in the face of severe uncertainty. Maximality and E-admissibility are well-known examples. When working with a choice function, however, it is immaterial whether it is based on such a decision criterion. The primitive objects on this approach are simply the set-valued choices themselves, and the choice function that represents all these choices serves as an uncertainty model in and by itself.
The seminal work by Seidenfeld et al. [17] has shown that a strong advantage of working with choice functions is that they allow us to impose axioms on choices, aimed at characterising what it means for choices to be rational and internally consistent. This is also what we want to do here, but we believe our angle of approach to be novel and unique: rather than think of choice intuitively, we provide it with a
concrete interpretation in terms of desirability [4, 8, 9, 25] or binary preference [15]. Another important feature of our approach is that we consider a very general setting, where the options form an abstract real vector space; horse lotteries and gambles correspond to special cases.
The basic structure of our paper is as follows. We start in Section 2 by introducing choice functions and our interpretation for them. Next, in Section 3, we develop an equivalent way of describing these choice functions: sets of desirable option sets. We use our interpretation to suggest and motivate a number of rationality, or coherence, axioms for such sets of desirable option sets, and show in Section 4 what are the corresponding coherence axioms for choice (or rejection) functions. Section 5 deals with the special case of binary choice, and its relation to the theory of sets of desirable options [4, 8, 9, 25] and binary preference. This is important because our main result in Section 6 shows that any coherent choice model can be represented in terms of sets of such binary choice models. In the remaining Sections 7–9, we consider additional axioms or properties, such as totality, the mixing property, and an Archimedean property, and prove corresponding representation results. This includes representations in terms of sets of strict total orders, sets of lexicographic probability systems, sets of coherent lower previsions and sets of linear previsions.
Proofs have been relegated to the appendix of an extended arXiv version [7].
2. Choice Functions and Their Interpretation
A choice function C is a set-valued operator on sets of options. In particular, for any set of options A, the corresponding value of C is a subset C(A) of A. The options themselves are typically actions amongst which a subject wishes to choose. We here follow a very general approach where these options constitute an abstract real vector space 𝒱 provided with a—so-called background—vector ordering ⪯ and a strict version ≺. The elements u of 𝒱 are called options and 𝒱 is therefore called the option space. We let 𝒱≻0 := {u ∈ 𝒱 : u ≻ 0}. The purpose of a choice function is to represent our subject's choices between such options.
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A Theory of Desirable Gambles

A Theory of Desirable Things

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Things


Things
gambles
polynomials
matrices

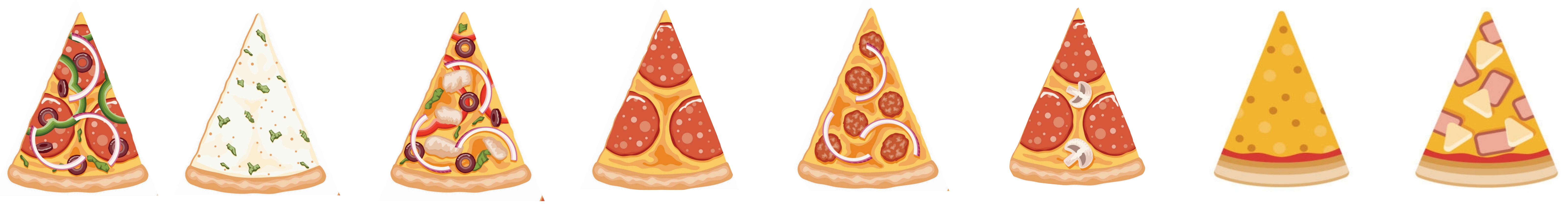

Things
probability measures
horse lotteries
gambles
polynomials
matrices


pizzas
dogs
probability measures
Things
horse lotteries
gambles
polynomials
matrices


pizzas
dogs
probability measures
Things
horse lotteries
gambles
polynomials
matrices
preferences

$\mathcal{T}$ an arbitrary set whose elements we call things

$\mathcal{T}$ an arbitrary set whose elements we call things

$\mathcal{T} = \{$  $\cdots \}$

A Theory of Desirable Things

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Desirable Things

$\mathcal{T}$ an arbitrary set whose elements we call things

DESIRABILITY: a feature that things may have or not


pizzas
dogs
probability measures
Desirable Things
horse lotteries
gambles
polynomials
matrices
preferences


A Theory of Desirable Things
It’s indeed very general, but also very silly… there is no maths at all, and definitely no theory

3 RULES

1. The things in $\mathbf{A_{not}}$ are never desirable
2. The things in $\mathbf{A_{des}}$ are always desirable
3. If the things in A are desirable, then so are the things in **cl**(A)

closure operator **cl**

$A \subseteq \mathbf{cl}(A)$

$A \subseteq B \Rightarrow \mathbf{cl}(A) \subseteq \mathbf{cl}(B)$

$\mathbf{cl}(\mathbf{cl}(A)) = \mathbf{cl}(A)$

$\mathbf{cl}(\varnothing) = \varnothing$

3

If the things in A are desirable, then so are the things in **cl**(A)

3 RULES

1. The things in $\mathbf{A}_{\mathbf{not}}$ are never desirable
2. The things in $\mathbf{A}_{\mathbf{des}}$ are always desirable
3. If the things in A are desirable, then so are the things in **cl**(A)

A Theory of Desirable Things

pizzas

dogs

probability measures

horse lotteries

gambles

polynomials

matrices

preferences


natural extension
conditioning
Sets of Desirable Things
Sets of Desirable Sets of Things
...
choice functions
marginalisation


A Theory of Desirable Things
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Do the FLip quiz?
What replaces the positive hull operator if gambles are replaced by things?
desirable gambles
things
most things still work!
natural extension, conditioning, marginalisation, ...
abstract feature that things may have or not
a set of things is called desirable if it contains at least one desirable thing
arbitrary things (yes, even ...)
$\mathscr{T}$ contains all things
Three rules, with A_{not} and A_{des} subsets of $\mathscr{T}$ and cl a closure operator:
$\mathrm{R}_{\mathrm{not}}$. The things in A_{not} are not desirable.
$\mathrm{R}_{\mathrm{des}}$. The things in A_{des} are desirable.
R_{cl}. If the things in $A \subseteq \mathscr{T}$ are desirable, then so are the things in $\mathrm{cl}(A)$.
A map $\mathrm{cl}\colon \mathscr{P}(\mathscr{T}) \to \mathscr{P}(\mathscr{T})$ is a closure operator if
cl_1. $A \subseteq \mathrm{cl}(A)$
cl_2. if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
cl_3. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
cl_4. $\mathrm{cl}(\emptyset) = \emptyset$
It is unitary if
$\mathrm{cl}(A) = \cup_{t \in A} \mathrm{cl}(\{t\})$
It is finitary if
$\mathrm{cl}(A) = \cup_{B \subseteq A, |B| < \infty} \mathrm{cl}(B)$
It is incremental if
$t \in \mathrm{cl}(A \cup \{a\}) \Rightarrow \exists t_A \in \mathrm{cl}(A)\colon t \in \mathrm{cl}(\{t_A, a\})$
If cl is unitary, then it is clearly also finitary and incremental.
GOAL: model a subject's beliefs about which things in $\mathscr{T}$ are desirable
$\mathscr{P}(\mathscr{T})$: all subsets of $\mathscr{T}$
$\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$: all finite subsets of $\mathscr{T}$
SET OF DESIRABLE THINGS (SDT)
a set $D \subseteq \mathscr{T}$ of things that are all desirable
D is coherent if
D_1. $A_{\mathrm{not}} \cap D = \emptyset$
D_2. $A_{\mathrm{des}} \subseteq D$
D_3. $\mathrm{cl}(D) = D$
$\mathbf{D}$ is the set of all such coherent sets of desirable things.
EXAMPLES: | DESIRABLE GAMBLES | DECISION MAKING | PIZZAS
$\mathscr{T}$ | gambles | preferences $(a \succ b)$ | different types of
desirability | accept that gamble | have that preference | like that
A_{not} | negative gambles | 'irrational' preferences | Hawaii
A_{des} | positive gambles | 'rational' preferences | Margherita
$\mathrm{cl}(A)$ | positive hull of A | transitive closure of A | all in A, also with added crust
$K_D := \{A \in \mathscr{P}(\mathscr{T}) \colon A \cap D \neq \emptyset\}$
$K_D^{\mathrm{fin}} := \{A \in \mathscr{P}_{\mathrm{fin}}(\mathscr{T}) \colon A \cap D \neq \emptyset\}$
Want more? We can also do gambles with nonlinear utility, arbitrary vector spaces, (horse) lotteries, arbitrary convex spaces, choice functions, ...
SET OF DESIRABLE SETS OF THINGS (SDS)
a set $K \subseteq \mathscr{P}(\mathscr{T})$ of sets of things that are all desirable
We have several coherence notions for K. In each of them, the purple parts can be included or not. If they are, we obtain notions of coherence for $K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$.
K is coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ if
$\mathrm{K}_1^{\mathrm{fin}}$. $\emptyset \notin K$
$\mathrm{K}_2^{\mathrm{fin}}$. if $A \subseteq B \in \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ and $A \in K$, then also $B \in K$
$\mathrm{K}_3^{\mathrm{fin}}$. if $A \in K$ then also $A \setminus A_{\mathrm{not}} \in K$
$\mathrm{K}_4^{\mathrm{fin}}$. $\{t\} \in K$ for all $t \in A_{\mathrm{des}}$
$\mathrm{K}_5^{\mathrm{fin}}$. if $\emptyset \neq \mathscr{A} \subseteq K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ and $t_S \in \mathrm{cl}(S)$ for all $S \in \mathscr{S}_{\mathscr{A}}$, ? then $\{t_S \colon S \in \mathscr{S}_{\mathscr{A}}\} \in K$ if $\{t_S \colon S \in \mathscr{S}_{\mathscr{A}}\} \in \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$
K is finitely coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ if K_{1-4} and
$\mathrm{K}_{5\mathrm{fin}}^{\mathrm{fin}}$. if $\emptyset \neq \mathscr{A} \subseteq K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ is finite and $t_S \in \mathrm{cl}(S)$ for all $S \in \mathscr{S}_{\mathscr{A}}$, ? then $\{t_S \colon S \in \mathscr{S}_{\mathscr{A}}\} \in K$.
K is 2-coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ if K_{1-4} and
$\mathrm{K}_{5\mathrm{bin}}^{\mathrm{fin}}$. if $A, B \in K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ and $t_{a,b} \in \mathrm{cl}(\{a,b\})$ for all $a \in A$ and $b \in B$, then $\{t_{a,b} \colon a \in A, b \in B\} \in K$.
K is 1-coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ if K_{1-4} and
$\mathrm{K}_{5\mathrm{un}}^{\mathrm{fin}}$. if $A \in K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ and $t_a \in \mathrm{cl}(\{a\})$ for all $a \in A$, then $\{t_a \colon a \in A\} \in K$
? = all selections from $\mathscr{A}$:
$\mathscr{S}_{\mathscr{A}} := \{\{t_A \colon A \in \mathscr{A}\} \colon t_A \in A \text{ for all } A \in \mathscr{A}\}$
$(\exists \mathscr{D} \subseteq \mathbf{D})\ K = \bigcap_{D \in \mathscr{D}} K_D$
$(\exists \mathscr{D} \subseteq \mathbf{D})\ K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T}) = \bigcap_{D \in \mathscr{D}} K_D^{\mathrm{fin}}$
$(\exists \mathscr{D} \subseteq \mathbf{D})\ \mathrm{fin}(K) = \bigcap_{D \in \mathscr{D}} K_D$
all supersets of the sets in $K \cap \mathscr{P}_{\mathrm{fin}}(\mathscr{T})$
K coherent ⇒ K coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ ⇐ $\mathrm{fin}(K)$ coherent
finitary cl
unitary cl
K finitely coherent ⇒ K finitely coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ ⇔ $\mathrm{fin}(K)$ finitely coherent
finitary cl
unitary cl
incremental cl
K 2-coherent ⇒ K 2-coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ ⇔ $\mathrm{fin}(K)$ 2-coherent
unitary cl
unitary cl
K 1-coherent ⇒ K 1-coherent in $\mathscr{P}_{\mathrm{fin}}(\mathscr{T})$ ⇔ $\mathrm{fin}(K)$ 1-coherent