A Theory of Desirable Things

Jasper.De Bock@UGent.be

Do the FLip quiz?

What replaces the positive hull operator if gambles are replaced by things?

desirable ~~gambles~~ things — most things still work!

natural extension, conditioning, marginalisation, ...

abstract feature that things may have or not

a set of things is called desirable if it contains at least one desirable thing

arbitrary things (yes, even ...) $\mathscr{T}$ contains all things

Three rules, with A_{not} and A_{des} subsets of $\mathscr{T}$ and cl a closure operator:

R_{not}. The things in A_{not} are not desirable.
R_{des}. The things in A_{des} are desirable.
R_{cl}. If the things in $A \subseteq \mathscr{T}$ are desirable, then so are the things in $\mathrm{cl}(A)$.

A map $\mathrm{cl}\colon \mathscr{P}(\mathscr{T}) \to \mathscr{P}(\mathscr{T})$ is a **closure operator** if

cl_1. $A \subseteq \mathrm{cl}(A)$
cl_2. if $A \subseteq B$, then $\mathrm{cl}(A) \subseteq \mathrm{cl}(B)$
cl_3. $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$
cl_4. $\mathrm{cl}(\emptyset) = \emptyset$

It is **unitary** if

$$\mathrm{cl}(A) = \cup_{t \in A}\mathrm{cl}(\{t\})$$

It is **finitary** if

$$\mathrm{cl}(A) = \cup_{B \subseteq A, |B| < \infty}\mathrm{cl}(B)$$

It is **incremental** if

$$t \in \mathrm{cl}(A \cup \{a\}) \Rightarrow \exists t_A \in \mathrm{cl}(A)\colon t \in \mathrm{cl}(\{t_A, a\})$$

If cl is unitary, then it is clearly also finitary and incremental.

GOAL:

model a subject's beliefs about which things in $\mathscr{T}$ are desirable

$\mathscr{P}(\mathscr{T})$: all subsets of $\mathscr{T}$
$\mathscr{P}_{\text{fin}}(\mathscr{T})$: all finite subsets of $\mathscr{T}$

SET OF DESIRABLE THINGS (SDT)

a set $D \subseteq \mathscr{T}$ of things that are all desirable

D is **coherent** if

D_1. $A_{\text{not}} \cap D = \emptyset$
D_2. $A_{\text{des}} \subseteq D$
D_3. $\mathrm{cl}(D) = D$

$\mathbf{D}$ is the set of all such coherent sets of desirable things.

EXAMPLES:	DESIRABLE GAMBLES	DECISION MAKING	PIZZAS
$\mathscr{T}$	gambles	preferences $(a \succ b)$	different types of
desirability	accept that gamble	have that preference	like that
A_{not}	negative gambles	'irrational' preferences	Hawaii
A_{des}	positive gambles	'rational' preferences	Margherita
$\mathrm{cl}(A)$	positive hull of A	transitive closure of A	all in A, also with added crust

$$K_D := \{A \in \mathscr{P}(\mathscr{T})\colon A \cap D \neq \emptyset\}$$
$$K_D^{\text{fin}} := \{A \in \mathscr{P}_{\text{fin}}(\mathscr{T})\colon A \cap D \neq \emptyset\}$$

Want more? We can also do gambles with nonlinear utility, arbitrary vector spaces, (horse) lotteries, arbitrary convex spaces, choice functions, ...

SET OF DESIRABLE SETS OF THINGS (SDS)

a set $K \subseteq \mathscr{P}(\mathscr{T})$ of sets of things that are all desirable

We have several coherence notions for K. In each of them, the purple parts can be included or not. If they are, we obtain notions of coherence for $K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$.

K is **coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$** if

$\mathrm{K}_1^{\text{fin}}$. $\emptyset \notin K$
$\mathrm{K}_2^{\text{fin}}$. if $A \subseteq B \in \mathscr{P}_{\text{fin}}(\mathscr{T})$ and $A \in K$, then also $B \in K$
$\mathrm{K}_3^{\text{fin}}$. if $A \in K$ then also $A \setminus A_{\text{not}} \in K$
$\mathrm{K}_4^{\text{fin}}$. $\{t\} \in K$ for all $t \in A_{\text{des}}$
$\mathrm{K}_5^{\text{fin}}$. if $\emptyset \neq \mathscr{A} \subseteq K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$ and $t_S \in \mathrm{cl}(S)$ for all $S \in \mathscr{S}_{\mathscr{A}}$, (?) then $\{t_S\colon S \in \mathscr{S}_{\mathscr{A}}\} \in K$ if $\{t_S\colon S \in \mathscr{S}_{\mathscr{A}}\} \in \mathscr{P}_{\text{fin}}(\mathscr{T})$

K is **finitely coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$** if K_{1-4} and

$\mathrm{K}_{5\text{fin}}^{\text{fin}}$. if $\emptyset \neq \mathscr{A} \subseteq K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$ is finite and $t_S \in \mathrm{cl}(S)$ for all $S \in \mathscr{S}_{\mathscr{A}}$, (?) then $\{t_S\colon S \in \mathscr{S}_{\mathscr{A}}\} \in K$.

K is **2-coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$** if K_{1-4} and

$\mathrm{K}_{5\text{bin}}^{\text{fin}}$. if $A, B \in K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$ and $t_{a,b} \in \mathrm{cl}(\{a,b\})$ for all $a \in A$ and $b \in B$, then $\{t_{a,b}\colon a \in A, b \in B\} \in K$.

K is **1-coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$** if K_{1-4} and

$\mathrm{K}_{5\text{un}}^{\text{fin}}$. if $A \in K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$ and $t_a \in \mathrm{cl}(\{a\})$ for all $a \in A$, then $\{t_a\colon a \in A\} \in K$

(?) = all selections from $\mathscr{A}$:

$$\mathscr{S}_{\mathscr{A}} := \{\{t_A\colon A \in \mathscr{A}\}\colon t_A \in A \text{ for all } A \in \mathscr{A}\}$$

$$\mathscr{A} = \{\{\cdot,\cdot\},\{\cdot,\cdot\}\}$$

$$\mathscr{S}_{\mathscr{A}} = \{\{\cdot,\cdot\},\{\cdot,\cdot\},\{\cdot,\cdot\},\{\cdot,\cdot\}\}$$

all supersets of the sets in $K \cap \mathscr{P}_{\text{fin}}(\mathscr{T})$

$$(\exists \mathscr{D} \subseteq \mathbf{D})\ K = \bigcap_{D \in \mathscr{D}} K_D \qquad (\exists \mathscr{D} \subseteq \mathbf{D})\ K \cap \mathscr{P}_{\text{fin}}(\mathscr{T}) = \bigcap_{D \in \mathscr{D}} K_D^{\text{fin}} \Leftarrow (\exists \mathscr{D} \subseteq \mathbf{D})\ \mathrm{fin}(K) = \bigcap_{D \in \mathscr{D}} K_D$$

K coherent (⇕ with top)	⇒	K coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$ (⇕ finitary cl; ⇓ from top)	⇐	$\mathrm{fin}(K)$ coherent (⇕ with top)
⇓⇑ unitary cl		⇓⇑ finitary cl	⇗ finitary cl	
K finitely coherent	⇒	K finitely coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$	⇔	$\mathrm{fin}(K)$ finitely coherent
⇓⇑ unitary cl		⇓⇑ incremental cl		
K 2-coherent	⇒	K 2-coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$	⇔	$\mathrm{fin}(K)$ 2-coherent
⇓⇑ unitary cl		⇓⇑ unitary cl		
K 1-coherent	⇒	K 1-coherent in $\mathscr{P}_{\text{fin}}(\mathscr{T})$	⇔	$\mathrm{fin}(K)$ 1-coherent