Average behaviour of imprecise Markov chains

A single pointwise ergodic theorem for six different models

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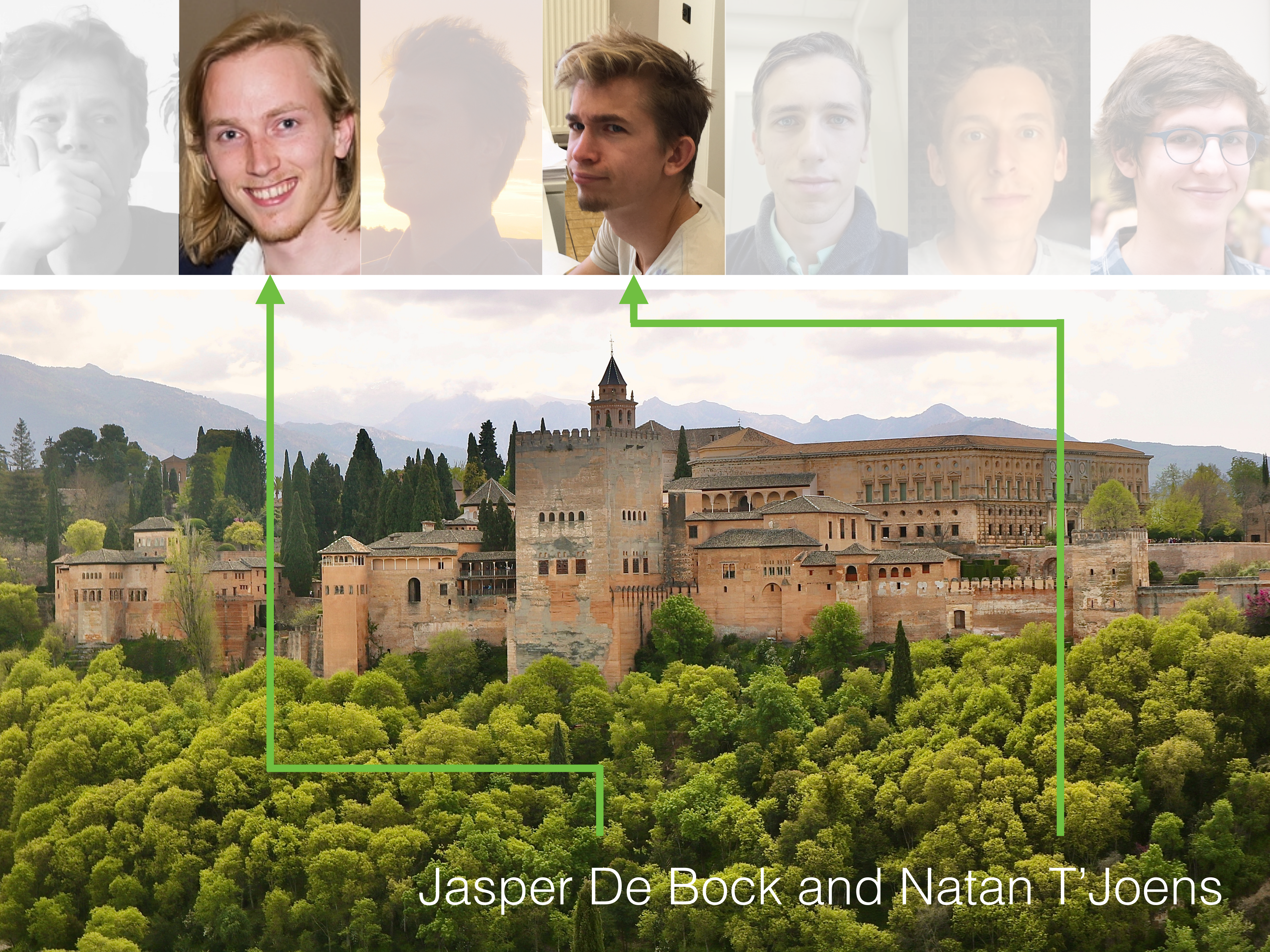
Jasper De Bock and Natan T'Joens


Jasper De Bock and Natan T'Joens

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Foundations Lab
for imprecise probabilities

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credal networks

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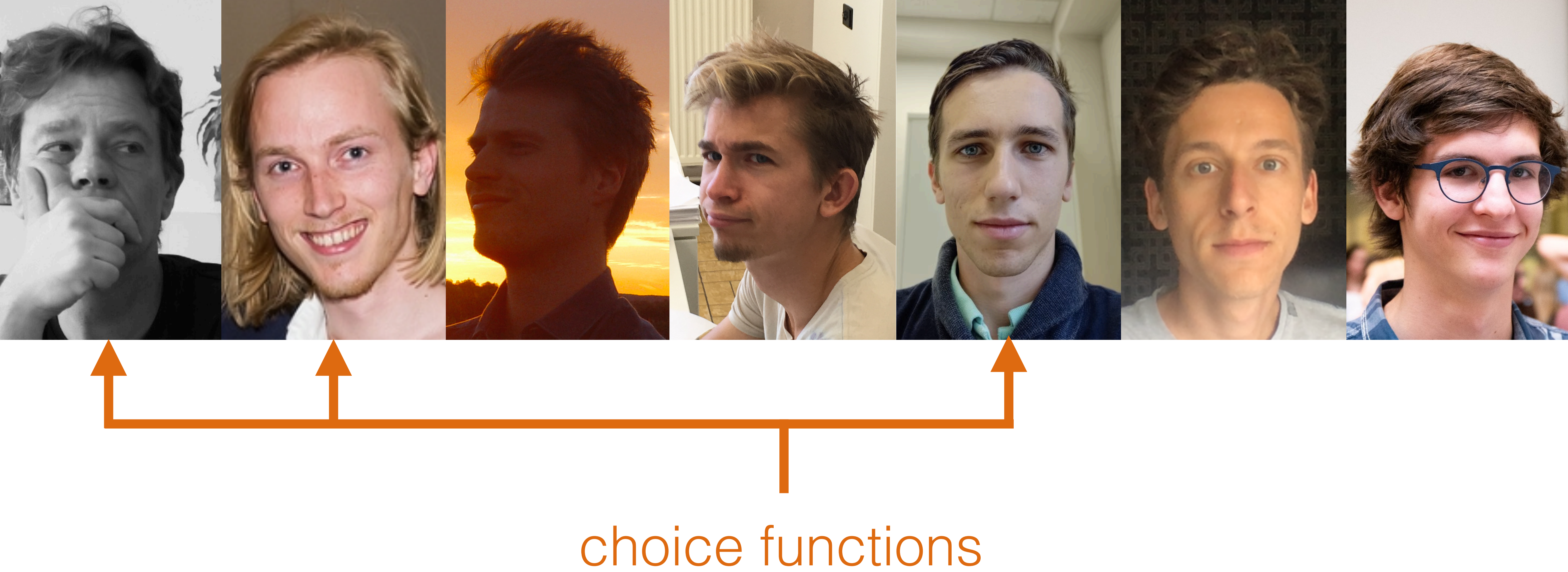

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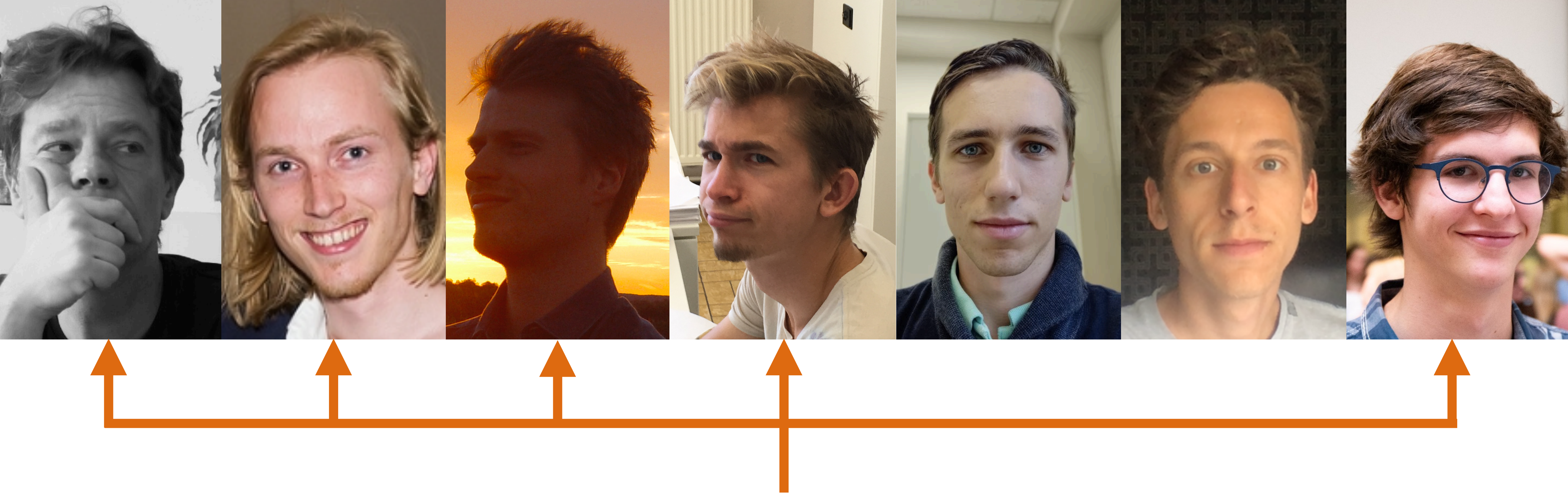
Foundations Lab
for imprecise probabilities



quantum mechanics with imprecise probabilities

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imprecise stochastic processes
and imprecise Markov chains

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$P($$) = {}^{1}/_{2}$

$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$

$\lim_{n\to\infty}$ #  / (#  + # ) $= {}^1/_2$

$P($$) = {}^1/_2$

Strong law of large numbers

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

identically distributed with $P($  $) = {}^1\!/_2$

+ independence assumption

⇒ $\lim_{n\to\infty}$ $\#$  / ($\#$  $+ \; \#$ ) $= {}^1\!/_2$

almost surely !

Strong law of large numbers

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

$\in \mathcal{X}$ finite

identically distributed with mass function P

+ independence assumption

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = E(f)$$

almost surely

Strong law of large numbers

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

identically distributed with mass function P

+ independence assumption

$$\Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = E(f) \quad ?$$

almost surely

Markov chains

$$x$$
$$\|$$
$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

~~identically distributed~~

~~+ independence~~

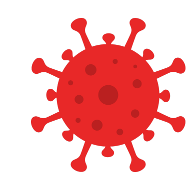
$$P_x(X_{n+1})$$

Markov assumption

Markov chains

- ✓ **Reliability engineering** (failure probabilities, …)
- ✓ **Queuing theory** (waiting in line …)
  - dimensioning of call centers
  - router queues on the internet

- ✓ **Chemical reactions** (time-evolution …)



- ✓ **Epidemiology** (time until threshold…)
- ✓ **Pagerank**

Google

- ✓ **…**

Strong law of large numbers ?

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

transition probabilities P_x for each x in $\mathcal{X}$

+ Markov assumption

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \boxed{E(f)} \; ?$$

almost surely

Markov chains

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad \underset{}{\overset{x}{\overset{\|}{X_n}}} \quad X_{n+1} \quad \cdots$$

~~identically distributed~~

~~+ independence~~

Markov assumption

$$P_x(X_{n+1})$$

ergodicity

$$\cancel{E(f)} \Rightarrow \boxed{E_\infty(f)} := \lim_{n\to\infty} E(f(X_n)|X_1 \cancel{=} x)$$

Pointwise ergodic theorem

$$X_1 \;\; X_2 \;\; X_3 \;\; X_4 \;\; \cdots \;\; X_n \;\; X_{n+1} \;\; \cdots$$

transition probabilities P_x for each x in $\mathcal{X}$

+ Markov assumption + ergodicity

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = E_\infty(f)$$

almost surely

Pointwise ergodic theorem

$$X_1 \;\; X_2 \;\; X_3 \;\; X_4 \;\; \cdots \;\; X_n \;\; X_{n+1} \;\; \cdots$$

transition ~~probabilities~~ P_x for each x in $\mathcal{X}$

+ Markov assumption + ergodicity

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = E_\infty(f)$$?

almost surely

Imprecise Markov chains

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

$X_n = x$

~~identically distributed~~

~~+ independence~~

$\mathcal{P}_x$ set of transition probabilities

Imprecise Markov assumption

Pointwise ergodic theorem ?

$$X_1 \;\; X_2 \;\; X_3 \;\; X_4 \;\; \cdots \;\; X_n \;\; X_{n+1} \;\; \cdots$$

set of transition probabilities $\mathcal{P}_x$ for each x in $\mathcal{X}$

+ imprecise Markov assumption

$$\Rightarrow \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) = \boxed{E_\infty(f)} \; ?$$

almost surely

Imprecise Markov chains

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad \underset{}{X_n} \quad X_{n+1} \quad \cdots$$

$$X_n = x$$

$\mathcal{P}_x$ set of transition probabilities

~~independence~~

Imprecise Markov assumption

ergodicity

$$\cancel{E_\infty(f)} \;\Rightarrow\; \boxed{\underline{E}_\infty(f)} := \lim_{n \to \infty} \underline{E}(f(X_n) | X_1 = \cancel{x})$$

$$\boxed{\overline{E}_\infty(f)} := \lim_{n \to \infty} \overline{E}(f(X_n) | X_1 = \cancel{x})$$

Imprecise pointwise ergodic theorem

$$X_1 \quad X_2 \quad X_3 \quad X_4 \quad \cdots \quad X_n \quad X_{n+1} \quad \cdots$$

set of transition probabilities $\mathcal{P}_x$ for each x in $\mathcal{X}$

+ imprecise Markov assumption + ergodicity

$$\underline{E}_\infty(f) \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i)$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(X_i) \leq \overline{E}_\infty(f)$$

almost surely !

De Cooman et al. (2016)

Why is this new version better?

1. ~~ergodicity~~ $\Rightarrow$ weak ergodicity

2. our bounds are tighter:

$$\underline{E}_{\infty}(f) \leq \underline{E}_{\mathrm{av},\infty}(f) \leq \overline{E}_{\mathrm{av},\infty}(f) \leq \overline{E}_{\infty}(f)$$

3. our result applies to 6 different types of imprecise Markov chains

Average Behaviour of Imprecise Markov Chains: A Single Pointwise Ergodic Theorem for Six Different Models ① ② ③ ④ ⑤ ⑥ 2 3

IMPRECISE MARKOV CHAINS

LOCAL MODELS

$\mathscr{X}$ finite state space

$X_0 \quad x_n = X_n \quad X_{n+1}$ (with $x_n \in \mathscr{X}$)

$\mathscr{P}_{x_{1:0}} \quad \mathscr{P}_{x_{1:n}} = \mathscr{P}_{x_n}$

local sets of probability mass functions

imprecise Markov assumption

local upper expectations $\overline{\mathrm{E}}_{x_{1:n}} \colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}$

GLOBAL MODELS

Ω set of paths

$\omega = x_1 x_2 x_3 \dots x_n \dots$ (with $\omega \in \Omega$)

global upper expectations $\overline{\mathrm{E}}(\cdot|x_{1:n}) \colon \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

upper probabilty of a set of paths $A \subseteq \Omega$

$\overline{\mathrm{P}}(A) := \overline{\mathrm{E}}(\mathbb{I}_A)$

$$\overline{T}h(x) := \sup_{p \in \mathscr{P}_x} \sum_{x \in \mathscr{X}} p(x)h(x)$$

$$\overline{T}_f h(x) := f(x) + \overline{T}h(x)$$

$$\mathbb{I}_A(\omega) := \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$$

1 WEAK ERGODICITY 1

for all $f \in \mathbb{R}^{\mathscr{X}}$

$$\overline{\mathrm{E}}_{\mathrm{av},\overline{T}}(f) := \lim_{n \to +\infty} \frac{1}{n} \left[\overline{T}_f^{\,n-1} f\right](x)$$

exists and is equal for all $x \in \mathscr{X}$

for all $f \in \mathbb{R}^{\mathscr{X}}$

$$\overline{\mathrm{E}}_{\mathrm{av},\infty}(f) := \lim_{n \to +\infty} \overline{\mathrm{E}}\left(\tfrac{1}{n}\textstyle\sum_{i=1}^n f(X_i) \,\middle|\, x_1\right)$$

exists and is equal for all $x_1 \in \mathscr{X}$

$$\overline{\mathrm{E}}_{\mathrm{av},\overline{T}} = \overline{\mathrm{E}}_{\mathrm{av},\infty}$$

$$-\overline{\mathrm{E}}_{\mathrm{av},\infty}(-f) \leq \liminf_{n \to \infty} \frac{1}{n} \sum_{i=1}^n f(X_i) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{i=1}^n f(X_i) \leq \overline{\mathrm{E}}_{\mathrm{av},\infty}(f)$$

where $-\overline{\mathrm{E}}_{\mathrm{av},\infty}(-f) = \underline{\mathrm{E}}_{\mathrm{av},\infty}(f)$ 2 2

(strictly) almost surely, for any $f \in \mathbb{R}^{\mathscr{X}}$

There is a non-negative real supermartingale that tends to infinity on all paths where this is not true.

The set of paths for which this is not true has upper probability zero.

3 Measure-theoretic models

The local models $\mathscr{P}_{x_{1:n}}$ are used to define a set $\mathbb{P}$ of compatible global probability measures. $\overline{\mathrm{E}}(h|x_{1:n})$ is the supremum over the expectations $E_P(h|x_{1:n})$ of the measures P in $\mathbb{P}$.

A probability measure P is compatible with $\mathscr{P}_{x_{1:n}}$ if $P(X_{n+1}|x_{1:n}) \in \mathscr{P}_{x_{1:n}}$

① $\mathbb{P}$ is the set of all compatible probability measures that are time-homogeneous Markov chains

② $\mathbb{P}$ is the set of all compatible probability measures that are Markov chains

③ $\mathbb{P}$ is the set of all compatible probability measures

Game-theoretic models

The local models $\overline{\mathrm{E}}_{x_{1:n}}$ are used to define a set $\mathbb{M}$ of compatible gambling strategies, called supermartingales. $\overline{\mathrm{E}}(h|x_{1:n})$ is the infimum capital needed at $x_{1:n}$ so as to hedge h with these gambling strategies.

A supermartingale $\mathscr{M}$ maps each state sequence $x_{1:n}$ to a capital $\mathscr{M}(x_{1:n})$, in such a way that the capital is expected to decrease: $\overline{\mathrm{E}}_{x_{1:n}}(\mathscr{M}(x_{1:n}X_{n+1})) \leq \mathscr{M}(x_{1:n})$

④ $\mathbb{M}$ is the set of all extended real supermartingales that are bounded below (no unbounded borrowing)

⑤ $\mathbb{M}$ is the set of all real supermartingales that are bounded below (no unbounded borrowing nor infinite spending)

⑥ $\mathbb{M}$ is the set of all real supermartingales that are bounded (no unbounded borrowing or spending)

Differences with De Cooman et al. (2016):

1 They require ergodicity, which applies less often

2 Our bounds are tighter

3 They consider only one model ④

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Want more?

1 How do you check this in practice?

2 How different are they?

3 Don't you need measurability?