Average Behaviour of Imprecise Markov Chains: A Single Pointwise Ergodic Theorem for Six Different Models ① ② ③ ④ ⑤ ⑥ 2 3

LOCAL MODELS

GLOBAL MODELS

IMPRECISE MARKOV CHAINS

$\mathscr{X}$ finite state space

X_0 $\quad x_n = X_n \quad X_{n+1}$

$\mathscr{P}_{x_{1:0}}$ $\quad \mathscr{P}_{x_{1:n}} = \mathscr{P}_{x_n}$

local sets of probability mass functions

imprecise Markov assumption

local upper expectations $\overline{\mathrm{E}}_{x_{1:n}} \colon \mathbb{R}^{\mathscr{X}} \to \mathbb{R}$

① ② ③ ④ ⑤ ⑥

Ω set of paths

$\omega = x_1 x_2 x_3 \dots x_n \dots$

global upper expectations $\overline{\mathrm{E}}(\cdot|x_{1:n}) \colon \overline{\mathbb{R}}^{\Omega} \to \overline{\mathbb{R}}$

upper probability of a set of paths $A \subseteq \Omega$

$\overline{\mathrm{P}}(A) := \overline{\mathrm{E}}(\mathbb{I}_A)$

$$\overline{T}h(x) := \sup_{p \in \mathscr{P}_x} \sum_{x \in \mathscr{X}} p(x)h(x)$$

$$\overline{T}_f h(x) := f(x) + \overline{T}h(x)$$

$$\mathbb{I}_A(\omega) := \begin{cases} 1 \text{ if } \omega \in A \\ 0 \text{ if } \omega \notin A \end{cases}$$

WEAK ERGODICITY 1 1

for all $f \in \mathbb{R}^{\mathscr{X}}$

$$\overline{\mathrm{E}}_{\mathrm{av},\overline{T}}(f) := \lim_{n \to +\infty} \frac{1}{n} \big[\overline{T}_f^{\,n-1} f\big](x)$$

exists and is equal for all $x \in \mathscr{X}$

!

for all $f \in \mathbb{R}^{\mathscr{X}}$

$$\overline{\mathrm{E}}_{\mathrm{av},\infty}(f) := \lim_{n \to +\infty} \overline{\mathrm{E}}\Big(\tfrac{1}{n}\textstyle\sum_{i=1}^n f(X_i) \,\Big|\, x_1\Big)$$

exists and is equal for all $x_1 \in \mathscr{X}$

$$\overline{\mathrm{E}}_{\mathrm{av},\overline{T}} = \overline{\mathrm{E}}_{\mathrm{av},\infty}$$

!

$$-\overline{\mathrm{E}}_{\mathrm{av},\infty}(-f) \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(X_i) \leq \limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f(X_i) \leq \overline{\mathrm{E}}_{\mathrm{av},\infty}(f)$$

$$-\overline{\mathrm{E}}_{\mathrm{av},\infty}(-f) = \underline{\mathrm{E}}_{\mathrm{av},\infty}(f)$$

2 2

(strictly) almost surely, for any $f \in \mathbb{R}^{\mathscr{X}}$

There is a non-negative real supermartingale that tends to infinity on all paths where this is not true.

The set of paths for which this is not true has upper probability zero.

3 Measure-theoretic models

The local models $\mathscr{P}_{x_{1:n}}$ are used to define a set $\mathbb{P}$ of compatible global probability measures. $\overline{\mathrm{E}}(h|x_{1:n})$ is the supremum over the expectations $E_P(h|x_{1:n})$ of the measures P in $\mathbb{P}$.

A probability measure P is compatible with $\mathscr{P}_{x_{1:n}}$ if $P(X_{n+1}|x_{1:n}) \in \mathscr{P}_{x_{1:n}}$

1. $\mathbb{P}$ is the set of all compatible probability measures that are time-homogeneous Markov chains
2. $\mathbb{P}$ is the set of all compatible probability measures that are Markov chains
3. $\mathbb{P}$ is the set of all compatible probability measures

Game-theoretic models

The local models $\overline{\mathrm{E}}_{x_{1:n}}$ are used to define a set $\mathbb{M}$ of compatible gambling strategies, called supermartingales. $\overline{\mathrm{E}}(h|x_{1:n})$ is the infimum capital needed at $x_{1:n}$ so as to hedge h with these gambling strategies.

A supermartingale $\mathscr{M}$ maps each state sequence $x_{1:n}$ to a capital $\mathscr{M}(x_{1:n})$, in such a way that the capital is expected to decrease: $\overline{\mathrm{E}}_{x_{1:n}}(\mathscr{M}(x_{1:n}X_{n+1})) \leq \mathscr{M}(x_{1:n})$

4. $\mathbb{M}$ is the set of all extended real supermartingales that are bounded below (no unbounded borrowing)
5. $\mathbb{M}$ is the set of all real supermartingales that are bounded below (no unbounded borrowing nor infinite spending)
6. $\mathbb{M}$ is the set of all real supermartingales that are bounded (no unbounded borrowing or spending)

Differences with De Cooman et al. (2016):

1 They require ergodicity, which applies less often

2 Our bounds are tighter

3 They consider only one model ④

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Want more?

1 How do you check this in practice?

2 How different are they?

3 Don't you need measurability?