The meaning of imprecise probabilities: an axiomatic perspective based on choice functions and desirability

EPIMP 2021, Bristol

Jasper De Bock

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Foundations Lab
for imprecise probabilities

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imprecise
randomness

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credal networks

quantum mechanics
with imprecise
probabilities

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imprecise stochastic processes
and
imprecise Markov chains

choice functions

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choice functions

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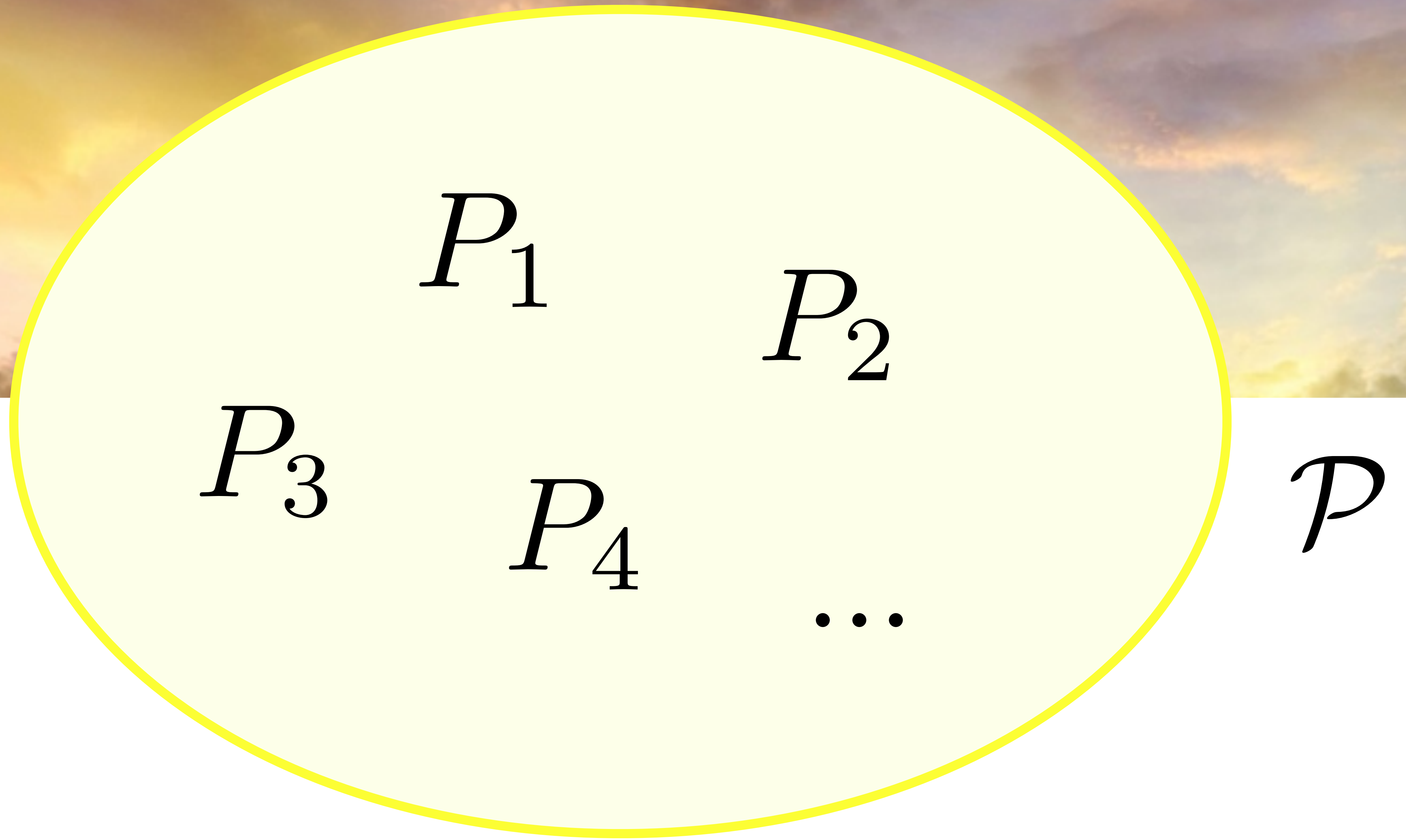
P_1
P_2
P_3
P_4
...
$\mathcal{P}$


meaning?
P_1
P_2
P_3
P_4
...
$\mathcal{P}$

meaning?



popular:

The “true” P belongs to $\mathcal{P}$

independence?

decision making?

P_1 P_2 P_3 P_4 ...

$\mathcal{P}$

independence?

How do we model the independence of two events S and H?

P_1 P_2 P_3 P_4 $\dots$

$\mathcal{P}$

popular:

complete independence

$$P(S \cap H) = P(S)P(H) \text{ for all } P \in \mathcal{P}$$

WHY?

How to choose from a set of options O, if each $o \in O$ has a utility function $u_o : \mathcal{X} \to \mathbb{R}$?

decision making?

P_1 P_2 P_3 P_4 ...

$\mathcal{P}$

popular:

E-admissibility

choose $o \in O$ if it maximizes $E_P(u_o)$ for at least one $P \in \mathcal{P}$

WHY?

The meaning of imprecise probabilities: an axiomatic perspective based on choice functions and desirability

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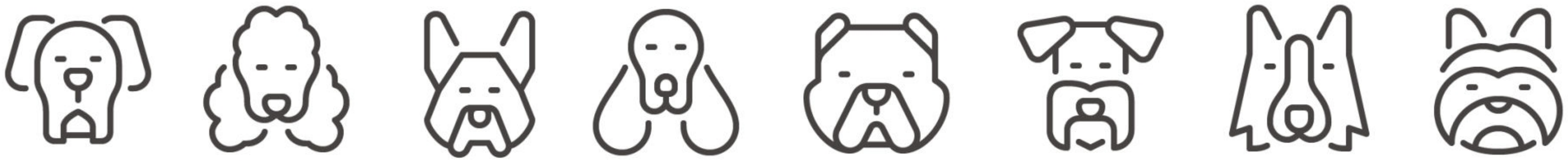
DESIRABILITY

A set of "things" $\mathcal{T}$

Which "things" are "desirable"?

DESIRABILITY

$\mathcal{T} := \{$  $\cdots \}$

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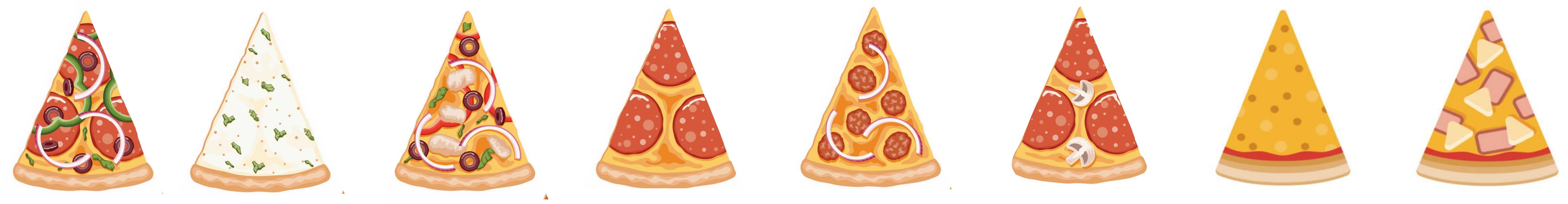
$\mathcal{T} := \{$  $\cdots \}$

A set of “things” $\mathcal{T}$

Which “things” are “desirable”?

DESIRABILITY

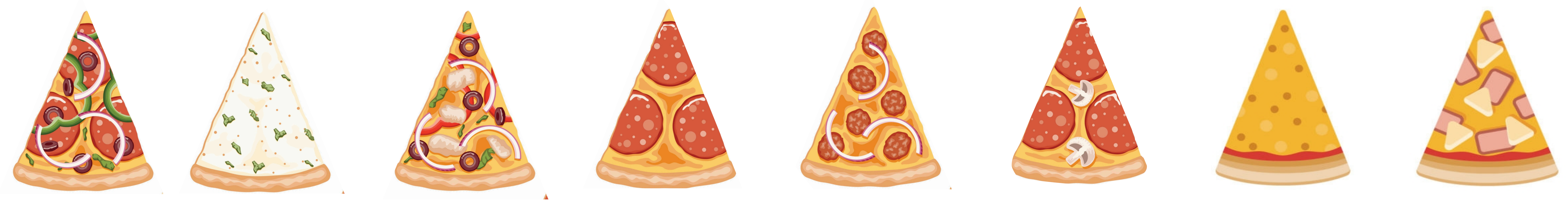
$\mathcal{T} :=$ (a subset of) a vector space

A set of “things” $\mathcal{T}$

Which “things” are “desirable”?

How do we model this?

DESIRABILITY

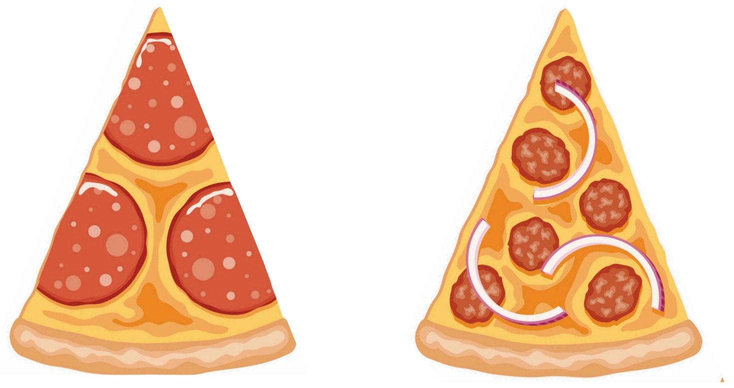
$\mathcal{T} := \{$  $\cdots \}$

A set of "things" $\mathcal{T}$

Which "things" are "desirable"?

How do we model this?

set of desirable things $D \subseteq \mathcal{T}$

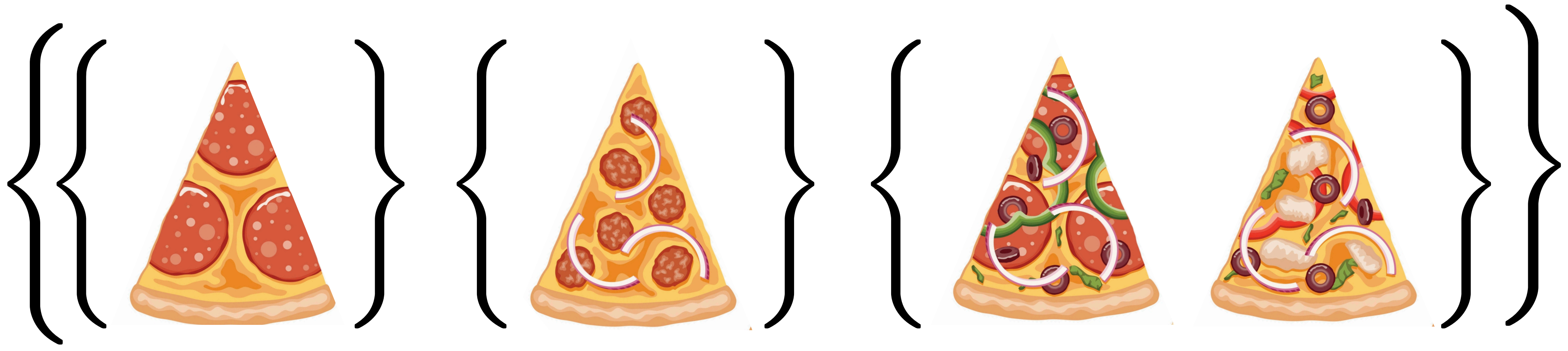
$D = \{$  $\}$

A set of "things" $\mathcal{T}$

Which "things" are "desirable"?

How do we model this?

set of desirable sets (of things) $K \subseteq \mathcal{P}(\mathcal{T})$

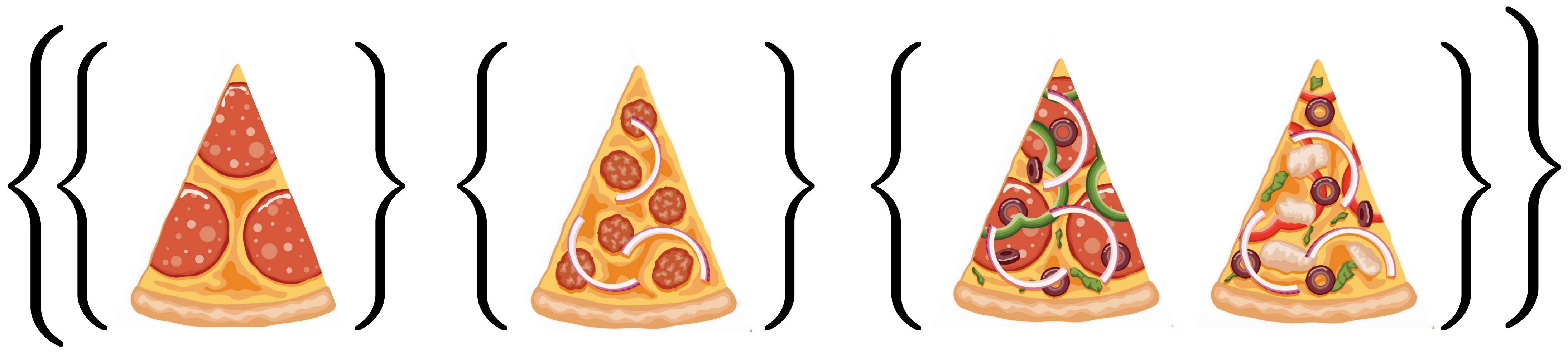
$K =$ 

Axioms for K

K1 $\quad \emptyset \notin K$

K2 $\quad A \in K \text{ and } A \subseteq B \Rightarrow B \in K$

set of desirable sets (of things) $K \subseteq \mathcal{P}(\mathcal{T})$

$K =$ 

one
set of desirable sets
(of things)

$$K$$

$\Leftrightarrow$

a set of
sets of desirable
things

$$D \in \mathbb{D}$$

! K satisfies K1 and K2 if and only if there is a set $\mathbb{D}$ of sets of desirable things such that !

$$A \in K \Leftrightarrow (\forall D \in \mathbb{D}) A \cap D \neq \emptyset$$

optional additional “rationality” criteria

the things in A_{not} are never desirable

$A_{\text{not}} = \{$  $\}$

optional additional "rationality" criteria

the things in A_{not} are never desirable

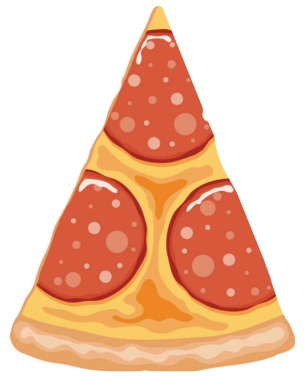
the things in A are all desirable
$\Rightarrow$ the things in $\mathrm{cl}(A)$ are all desirable

closure operator cl

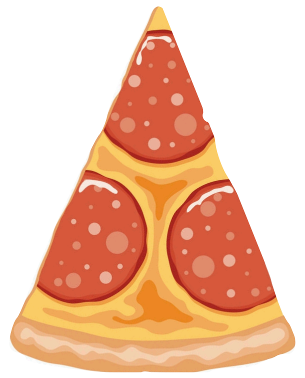
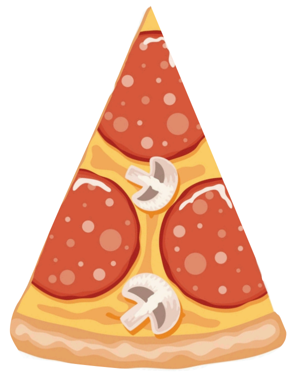
$A \subseteq \mathrm{cl}(A)$

$A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$

$\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$

 and  $\Rightarrow$ 

$\mathrm{cl}(A) :=$

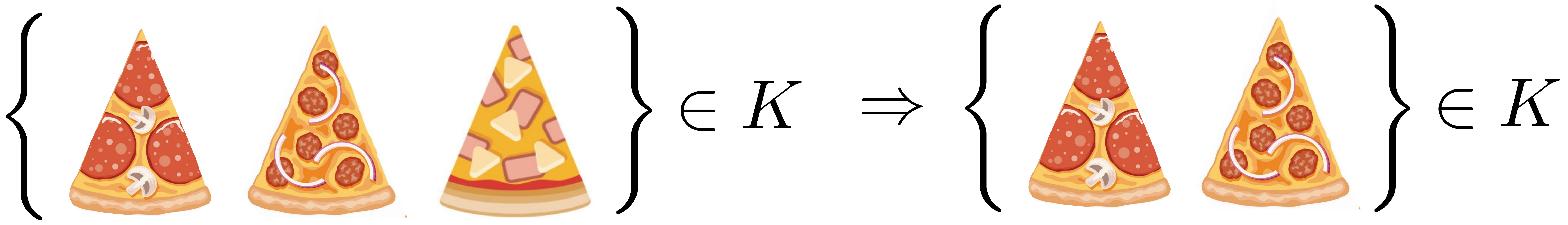
- $A \cup \{$  $\}$ if $\{$   $\} \subseteq A$
- A otherwise

Axioms for K

K1 $\emptyset \notin K$

K2 $A \in K$ and $A \subseteq B \Rightarrow B \in K$

K3 $A \in K \Rightarrow A \setminus A_{\text{not}} \in K$



Axions for K

$$\mathcal{S}_{\mathcal{A}} := \{\{t_A : A \in \mathcal{A}\} : t_A \in A \text{ for all } A \in \mathcal{A}\}$$

K1 $\emptyset \notin K$

K2 $A \in K$ and $A \subseteq B \Rightarrow B \in K$

K3 $A \in K \Rightarrow A \setminus A_{\text{not}} \in K$

K4 choose $\mathcal{A} \subseteq K$ and, for every selection $S \in \mathcal{S}_{\mathcal{A}}$, some $t_S \in \text{cl}(S)$

$$\Rightarrow \{t_S : S \in \mathcal{S}_{\mathcal{A}}\} \in K$$

$$\mathcal{S}_{\mathcal{A}} \coloneqq \{\{t_A \colon A \in \mathcal{A}\} \colon t_A \in A \text{ for all } A \in \mathcal{A}\}$$

K4 choose $\mathcal{A} \subseteq K$ and, for every selection $S \in \mathcal{S}_{\mathcal{A}}$, some $t_S \in \mathrm{cl}(S)$

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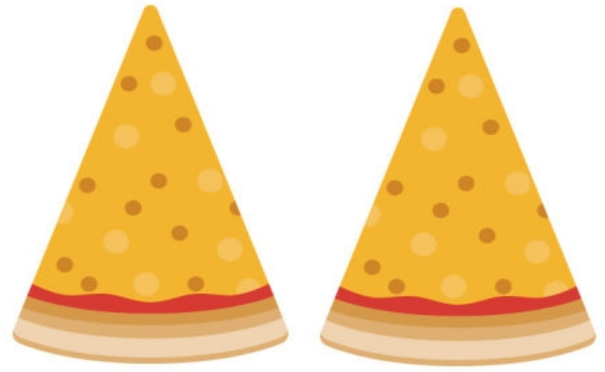
$\{$  $\} \in K$

$\{$  $\} \in K$

$\mathcal{S}_{\mathcal{A}} = \{S_1, S_2, S_3, S_4\}$

$\mathcal{S}_{\mathcal{A}} := \{\{t_A : A \in \mathcal{A}\} : t_A \in A \text{ for all } A \in \mathcal{A}\}$

$S_1 = \{$  $\}$ $S_2 = \{$  $\}$ $S_3 = \{$  $\}$ $S_4 = \{$  $\}$

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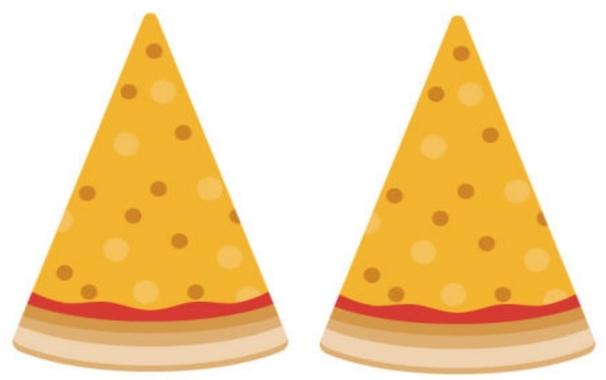
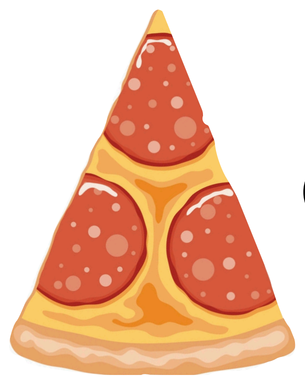
$\mathcal{S}_{\mathcal{A}} = \{S_1, S_2, S_3, S_4\}$ $\qquad$ $\mathcal{S}_{\mathcal{A}} := \{\{t_A : A \in \mathcal{A}\} : t_A \in A \text{ for all } A \in \mathcal{A}\}$

$S_1 = \{$  $\}$ $S_2 = \{$  $\}$ $S_3 = \{$  $\}$ $S_4 = \{$  $\}$

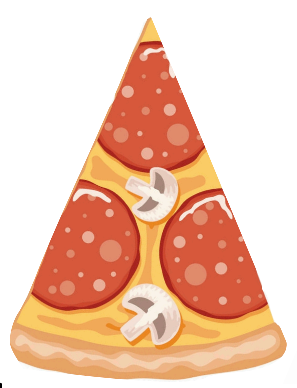
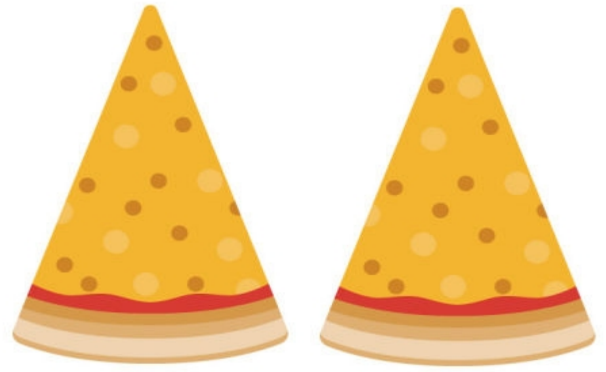
K4 $\quad$ choose $\mathcal{A} \subseteq K$ and, for every selection $S \in \mathcal{S}_{\mathcal{A}}$, some $t_S \in \mathrm{cl}(S)$

$\Rightarrow \{t_S : S \in \mathcal{S}_{\mathcal{A}}\} \in K$

$\{$  $\} \in K$ $\qquad$  $\in \mathrm{cl}(S_1)$  $\in \mathrm{cl}(S_2)$

$\{$  $\} \in K$ $\qquad$  $\in \mathrm{cl}(S_3)$  $\in \mathrm{cl}(S_4)$

$\mathcal{S}_{\mathcal{A}} = \{S_1, S_2, S_3, S_4\}$ $\qquad$ $\mathcal{S}_{\mathcal{A}} := \{\{t_A : A \in \mathcal{A}\} : t_A \in A \text{ for all } A \in \mathcal{A}\}$

$S_1 = \Big\{$  $\Big\}$ $S_2 = \Big\{$  $\Big\}$ $S_3 = \Big\{$  $\Big\}$ $S_4 = \Big\{$  $\Big\}$

K4 choose $\mathcal{A} \subseteq K$ and, for every selection $S \in \mathcal{S}_{\mathcal{A}}$, some $t_S \in \mathrm{cl}(S)$

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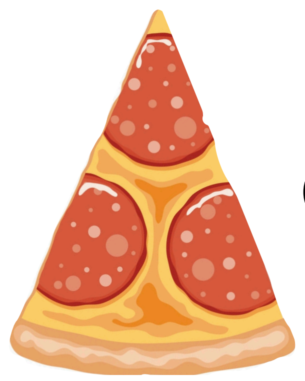
$\Big\{$  $\Big\} \in K$ $\qquad$  $\in \mathrm{cl}(S_1)$  $\in \mathrm{cl}(S_2)$

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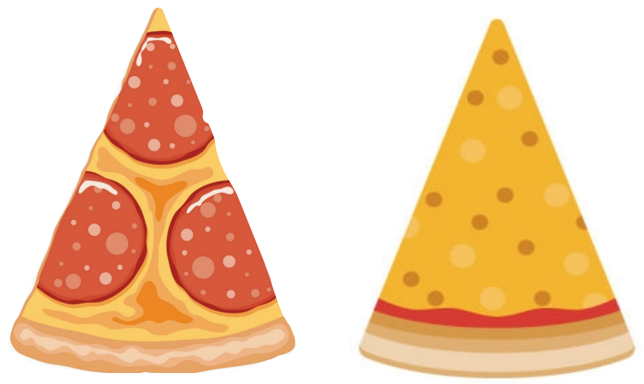
$\Rightarrow \Big\{$  $\Big\} \in K$

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one
set of desirable sets
(of things)

$\Leftrightarrow$

a set of
sets of desirable
things

$$K \qquad\qquad D \in \mathbb{D}$$

! K satisfies K1 to K4 if and only if there is a set $\mathbb{D}$ of sets of desirable things such that

$$A \in K \Leftrightarrow (\forall D \in \mathbb{D}) A \cap D \neq \emptyset$$

and every $D \in \mathcal{D}$ satisfies $A_{\mathrm{not}} \cap D = \emptyset$ and $\mathrm{cl}(D) = D$!

A set of “things” $\mathcal{T}$

Which “things” are “desirable”?

DESIRABILITY

$\mathcal{T} :=$ the vector space of all gambles (bounded real functions) on $\mathcal{X}$

A set of “things” $\mathcal{T}$

Which “things” are “desirable”?

$$A_{\text{not}} = \{f : f \leq 0\}$$

$$\text{cl}(A) = \text{posi}(A) := \left\{ \sum_{i=1}^{n} \lambda_i f_i : \lambda_i > 0, f_i \in A, n \in \mathbb{N} \right\}$$

$$\{f\} \in K \text{ if } f \gneq 0$$

$\mathcal{T}$:= the vector space of all gambles (bounded real functions) on $\mathcal{X}$

one
set of desirable sets
(of things)

$\Leftrightarrow$

a set of
sets of desirable
things

$$K \qquad D \in \mathbb{D}$$

! K satisfies K1 to K4 if and only if there is a set $\mathbb{D}$ of sets of desirable things such that !

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and every $D \in \mathcal{D}$ satisfies $A_{\text{not}} \cap D = \emptyset$ and $\text{cl}(D) = D$

one set of desirable sets (of gambles)	$\Leftrightarrow$	a set of sets of desirable gambles
K	!	$D \in \mathbb{D}$
that is coherent		that are coherent

A set of “things” $\mathcal{T} := \{o_1 \succ o_2 : o_1, o_2 \in \mathcal{O}\}$

Which “things” are “desirable”?

DESIRABILITY

$\mathcal{T} := \Big\{$

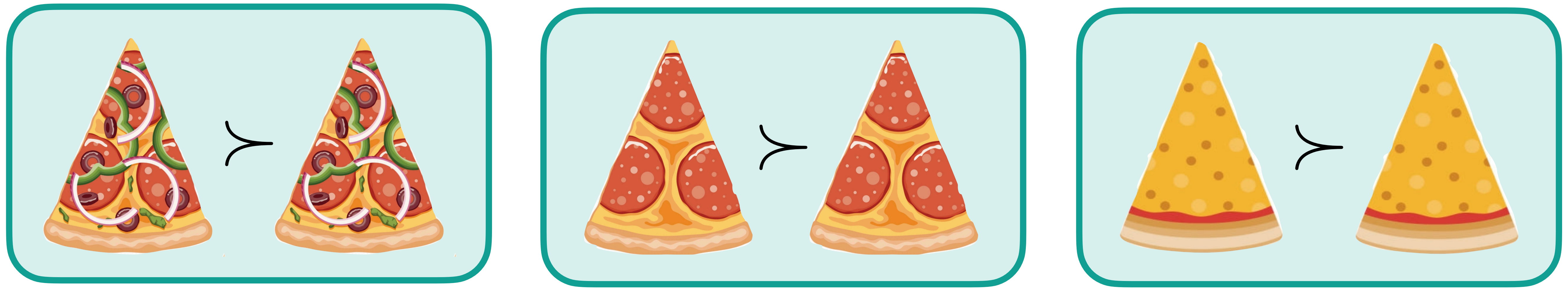
$\cdots \Big\}$

A set of “things” $\mathcal{T} := \{o_1 \succ o_2 \colon o_1, o_2 \in \mathcal{O}\}$

Which “things” are “desirable”?

$$A_{\text{not}} = \{o \succ o \colon o \in \mathcal{O}\}$$

$$= \Bigg\{$$



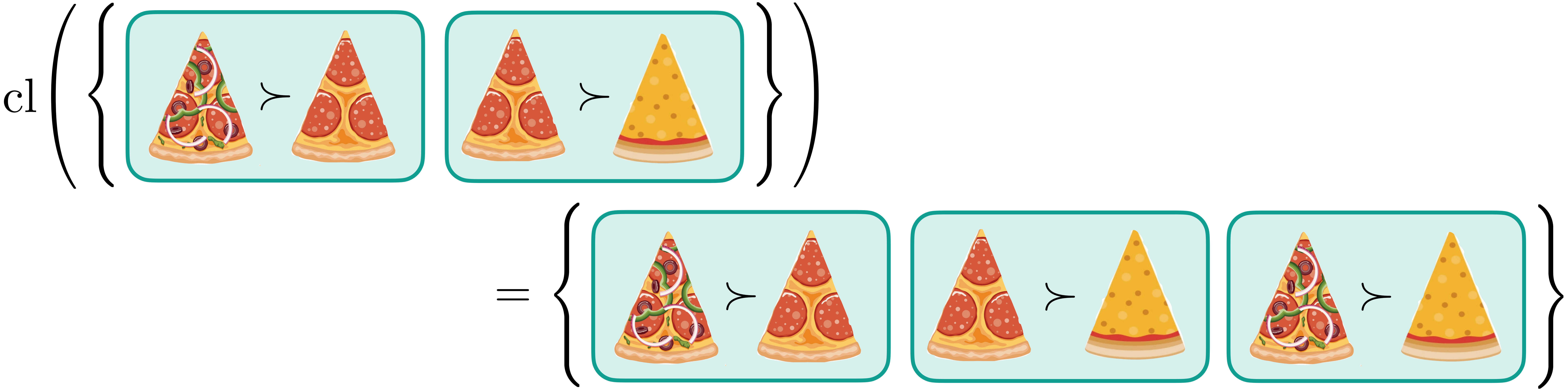
$$\cdots \Bigg\}$$

A set of "things" $\mathcal{T} := \{o_1 \succ o_2 \colon o_1, o_2 \in \mathcal{O}\}$

Which "things" are "desirable"?

$$A_{\text{not}} = \{o \succ o \colon o \in \mathcal{O}\}$$

Let $\mathrm{cl}(A)$ be the set of all preferences that can be obtained from A by transitivity



A set of "things" $\mathcal{T} := \{o_1 \succ o_2 \colon o_1, o_2 \in \mathcal{O}\}$

Which "things" are "desirable"?

$A_{\text{not}} = \{o \succ o \colon o \in \mathcal{O}\}$

Let $\text{cl}(A)$ be the set of all preferences that can be obtained from A by transitivity

$A_{\text{not}} \cap D = \emptyset$ ⟷ the preferences in D are irreflexive

$\text{cl}(D) = D$ ⟷ the preferences in D are transitive

$\Rightarrow$ D represents a strict partial order

one
set of desirable sets
(of things)

K

$\Leftrightarrow$

a set of
sets of desirable
things

$D \in \mathbb{D}$

! K satisfies K1 to K4 if and only if there is a set $\mathbb{D}$ of sets of desirable things such that

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one
set of desirable sets
(of preferences)

K

$\Leftrightarrow$

a set of
sets of desirable
preferences

$D \in \mathbb{D}$

$\Updownarrow$

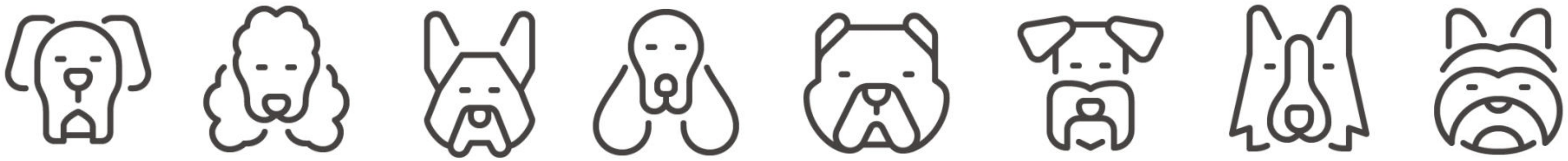
a set of
strict partial orders

CHOICE FUNCTIONS

A set of “options” $\mathcal{O}$

Given a subset of “options” $O \subseteq \mathcal{O}$, which option(s) do we “choose” ?

CHOICE FUNCTIONS

$\mathcal{O} := \{$  $\cdots \}$

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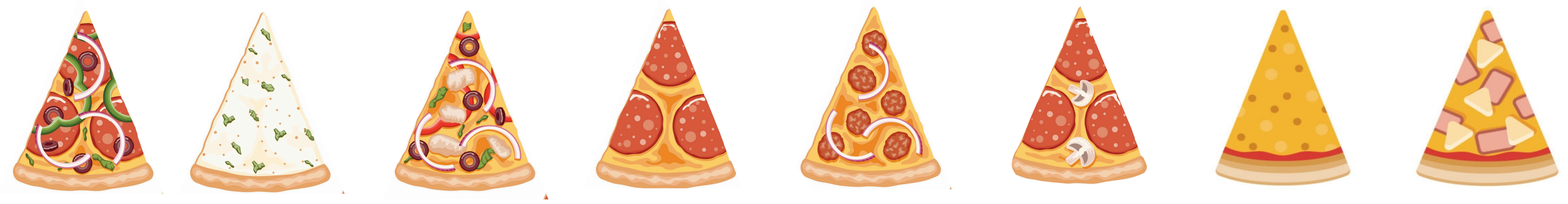
$\mathcal{O} := \{$  $\cdots \}$

A set of “options” $\mathcal{O}$

Given a subset of “options” $O \subseteq \mathcal{O}$, which option(s) do we “choose” ?

CHOICE FUNCTIONS

$\mathcal{O} :=$ (a subset of) a vector space

A set of “options” $\mathcal{O}$

Given a subset of “options” $O \subseteq \mathcal{O}$, which option(s) do we “choose” ?

CHOICE FUNCTIONS

$\mathcal{O} := \{$  $\cdots \}$

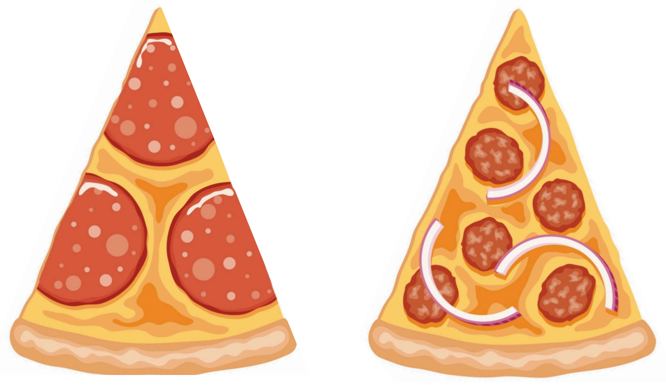
A set of “options” $\mathcal{O}$

Given a subset of “options” $O \subseteq \mathcal{O}$, which option(s) do we “choose” ?

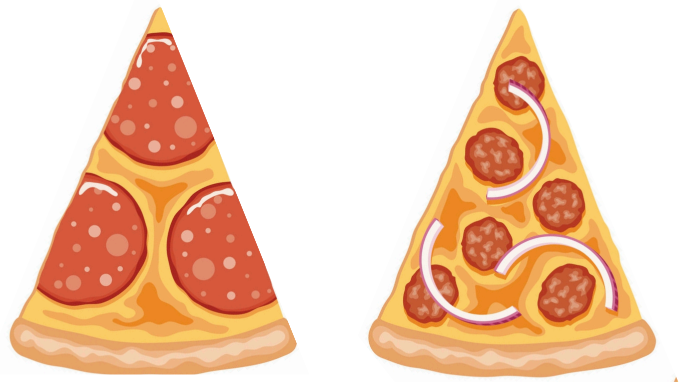
CHOICE FUNCTION C: the options in $C(O) \subseteq O$ are chosen

$\Updownarrow$ $\Updownarrow$

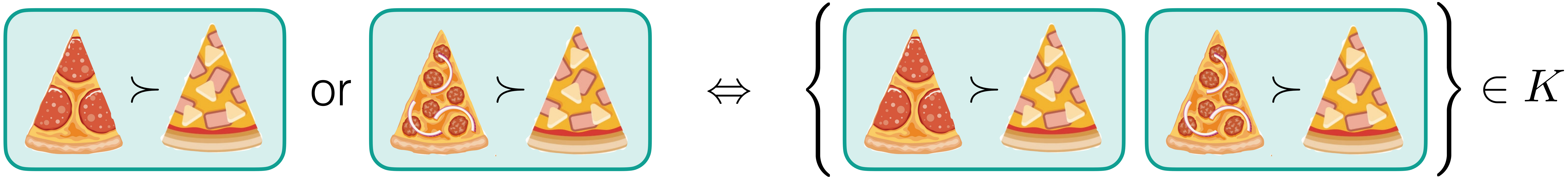
REJECTION FUNCTION R: the options in $R(O) := O \setminus C(O)$ are rejected

$O = \{$  $\}$ $C(O) = \{$  $\}$ $\Leftrightarrow$ $R(O) = \{$  $\}$

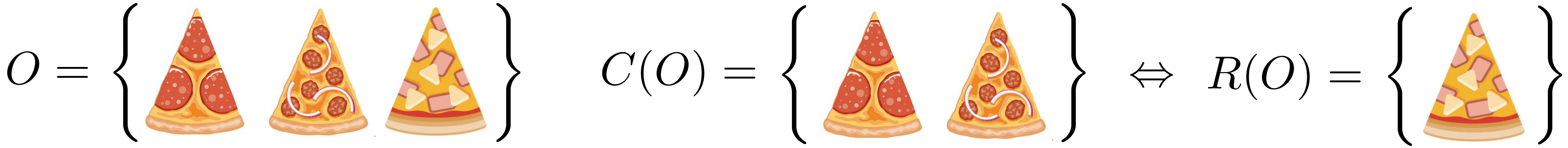
REJECTION FUNCTION R : the options in $R(O) := O \setminus C(O)$ are rejected

$O = \{$  $\}$ $C(O) = \{$  $\}$ $\Leftrightarrow$ $R(O) = \{$  $\}$

an option is rejected if there is some other option that is better



REJECTION FUNCTION R : the options in $R(O) := O \setminus C(O)$ are rejected



an option is rejected if there is some other option that is better

$$o \in R(O) \Leftrightarrow \{\tilde{o} \succ o \colon \tilde{o} \in \mathcal{O}\} \in K$$

REJECTION FUNCTION R : the options in $R(O) \coloneqq O \setminus C(O)$ are rejected

! choice function $\Leftrightarrow$ rejection function $\Leftrightarrow$ set of desirable sets of preferences !

one
set of desirable sets
(of preferences)

$\Leftrightarrow$

a set of
sets of desirable
preferences

K

$D \in \mathbb{D}$

$\Updownarrow$

$\Updownarrow$

choice
function C

a set of
strict partial orders

Let $\mathcal{O}$ be the set of all gambles/utility functions

special case!

a set of probabilities

↓ expected utility

a set of strict partial orders

E-admissibility

choice function C $\Longleftarrow$

Let $\mathcal{O}$ be the set of all gambles/utility functions

special case!

a set of probabilities

expected utility

a set of strict partial orders

desirability theory + extra axioms $\Rightarrow$

E-admissibility

choice function C $\Leftarrow$

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$

E-admissibility

choice function C

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$

$\Leftrightarrow 0 \in R(\{o - u \colon o \in O\})$

$\Leftrightarrow \{o - u \succ 0 \colon o \in O\} \in K$

E-admissibility

choice function C

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$

$\Leftrightarrow 0 \in R(\{o - u \colon o \in O\})$

$\Leftrightarrow \{o - u \succ 0 \colon o \in O\} \in K$

preference to 0 = desirability: $u \in R(O) \Leftrightarrow \{o - u \colon o \in O\} \in K$

E-admissibility

choice function C

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$

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E1 to E4 = K1 to K4 with $A_{\text{not}} = \{u \colon u \leq 0\}$

$\text{cl}(A) = \text{posi}(A)$

E-admissibility

choice function C

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v : o \in O\})$

$\Leftrightarrow 0 \in R(\{o - u : o \in O\})$

$\Leftrightarrow \{o - u \succ 0 : o \in O\} \in K$

preference to 0 = desirability: $u \in R(O) \Leftrightarrow \{o - u : o \in O\} \in K$

E1 to E4 = K1 to K4 with $A_{\text{not}} = \{u : u \leq 0\}$

$\text{cl}(A) = \text{posi}(A)$

E-admissibility

E5 $\inf u > 0 \Rightarrow \{u\} \in K$

choice function C

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$
$\Leftrightarrow 0 \in R(\{o - u \colon o \in O\})$
$\Leftrightarrow \{o - u \succ 0 \colon o \in O\} \in K$

preference to 0 = desirability: $u \in R(O) \Leftrightarrow \{o - u \colon o \in O\} \in K$

E1 to E4 = K1 to K4 with $A_{\text{not}} = \{u \colon u \leq 0\}$
$\text{cl}(A) = \text{posi}(A)$

E-admissibility

choice function C

E5 $\inf u > 0 \Rightarrow \{u\} \in K$

E6 $A \in K \Rightarrow \{u - \epsilon \colon u \in A, \epsilon > 0\} \in K$

Let $\mathcal{O}$ be the set of all gambles/utility functions

E0 $u \in R(O) \Leftrightarrow u + v \in R(\{o + v \colon o \in O\})$
$\Leftrightarrow 0 \in R(\{o - u \colon o \in O\})$
$\Leftrightarrow \{o - u \succ 0 \colon o \in O\} \in K$

preference to 0 = desirability: $u \in R(O) \Leftrightarrow \{o - u \colon o \in O\} \in K$

E1 to E4 = K1 to K4 with $A_{\text{not}} = \{u \colon u \leq 0\}$
$\text{cl}(A) = \text{posi}(A)$

E-admissibility

choice function C

E5 $\inf u > 0 \Rightarrow \{u\} \in K$

E6 $A \in K \Rightarrow \{u - \epsilon \colon u \in A, \epsilon > 0\} \in K$

E7 $-\epsilon \notin C(\{u, -u, -\epsilon\})$ for all $\epsilon > 0$

E0 to E7

choice function C

$\Leftrightarrow$

E-admissibility

a set of probabilities

independent information has no value

E8 $\mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon \notin C\big(\{g_F, h_F, \mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon\}\big)$

When two events, E and F, are 'independent' then it is not reasonable to *spend resources* in order to use the state of one, E versus E^c, to decide between two gambles that depend solely on the other event, F versus F^c.

independent information has no value

E8 $$\mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon \notin C\big(\{g_F, h_F, \mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon\}\big)$$

When two events, E and F, are ‘independent’ then it is not reasonable to *spend resources* in order to use the state of one, E versus E^c, to decide between two gambles that depend solely on the other event, F versus F^c.

E8

E0 to E7

choice function C

$\Leftrightarrow$

complete independence

E-admissibility

a set of probabilities

independent information has no value

E8 $\mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon \notin C\big(\{g_F, h_F, \mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon\}\big)$

E9 $\{\mathbb{I}_E - \epsilon \colon \epsilon > 0\} \in K$ and $\{\mathbb{I}_{E^c} - \epsilon \colon \epsilon > 0\} \in K$

E8 and E9 for gambles that depend on F

E0 to E6 E-admissibility

choice function C $\Rightarrow$ a set of probabilities

independent information has no value

E8 $\mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon \notin C\big(\{g_F, h_F, \mathbb{I}_E g_F + \mathbb{I}_{E^c} h_F - \epsilon\}\big)$

When two events, E and F, are 'independent' then it is not reasonable to *spend resources* in order to use the state of one, E versus E^c, to decide between two gambles that depend solely on the other event, F versus F^c.

E0

$$\Leftrightarrow \quad -\epsilon \notin C\big(\{\mathbb{I}_{E^c}(g_F - h_F), \mathbb{I}_E(h_F - g_F), -\epsilon\}\big)$$

$$\Leftrightarrow \quad -\epsilon \notin C\big(\{\mathbb{I}_{E^c} f_F, -\mathbb{I}_E f_F, -\epsilon\}\big)$$

SIPTA

The Society for Imprecise Probabilities: Theories and Applications

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Welcome

We are SIPTA, the *Society for Imprecise Probabilities: Theories and Applications*, and we are convinced that there is more to uncertainty than probabilities. There is much more, in fact. Would you like to use probabilities but don't know ... alues? Or would you like to model uncertainty without any proba... at there are numerous mathematical models th... hout sharp numerical probabiliti...

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