

Interpreting, axiomatising and representing
coherent choice functions
in terms of desirability

Jasper De Bock & Gert de Cooman
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Interpreting, axiomatising and representing

choice functions

terms of desirability

?



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$$C(\textcircled{1} \textcircled{2} \textcircled{3} \textcircled{4}) = ?$$

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$$C(\text{beer}, \text{martini}, \text{water}, \text{orange juice}) = ?$$

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$$C \left(\text{beer}, \text{martini} \text{ /}, \text{water} \text{ /}, \text{smoothie} \text{ /} \right) = \text{beer}$$


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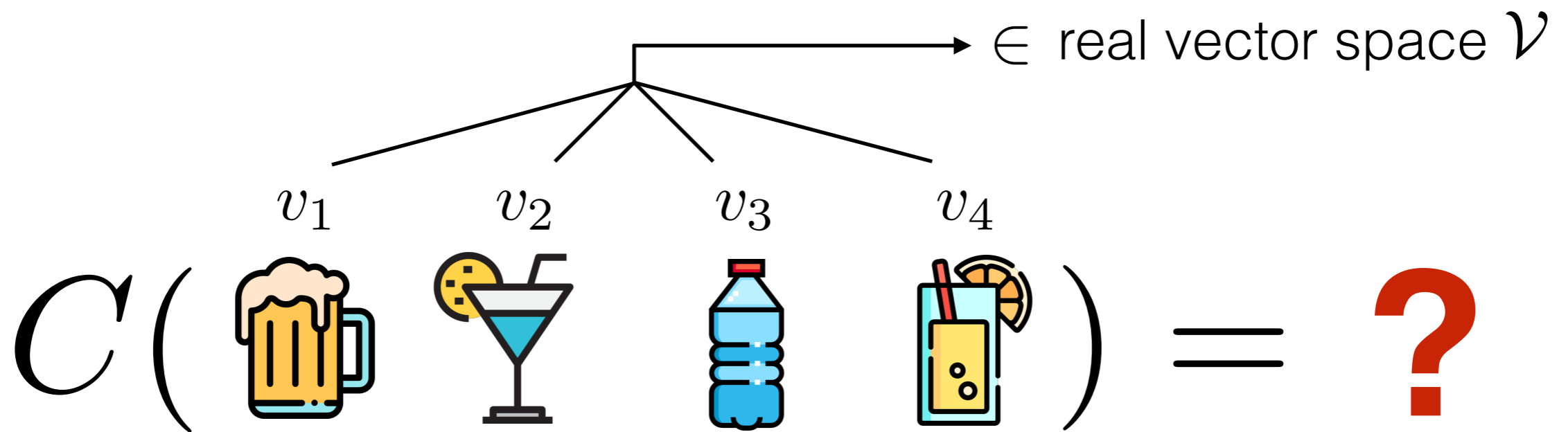
$$C(\text{~~beer~~, \text{martini}, \text{water}, \text{~~smoothie~~}) = \left\{ \text{water}, \text{martini} \right\}$$

The diagram illustrates a choice function C applied to a set of four items: a beer mug, a martini glass, a water bottle, and a smoothie cup. The beer mug and smoothie cup are crossed out with red diagonal lines, indicating they are not chosen. The result of the choice function is a set containing the water bottle and the martini glass, shown within large curly braces.

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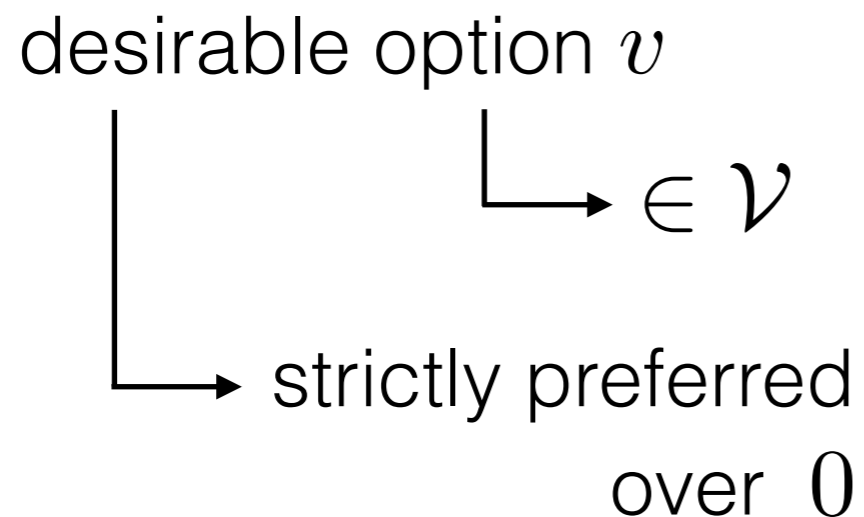
$$C \left(\begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \img alt="beer mug" data-bbox="225 765 300 880" & \img alt="martini glass" data-bbox="330 765 405 885" & \img alt="water bottle" data-bbox="475 765 520 885" & \img alt="orange juice" data-bbox="585 765 650 885" \end{array} \right) = ?$$

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Interpreting, axiomatising and representing
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in terms of desirability

elementary concept



Interpreting, axiomatising and representing coherent choice functions in terms of desirability

elementary concept

desirable option v
↳ $\in \mathcal{V}$
↳ strictly preferred
over 0

derived notion 1

set of desirable options D
↳ $\subseteq \mathcal{V}$
coherence axioms!



Towards a unified theory of imprecise probability

Peter Walley

Received 1 September 1999; accepted 1 December 1999

Abstract

Coherent upper and lower probabilities, Choquet capacities of order 2, belief functions and possibility measures are amongst the most popular mathematical models for uncertainty and partial ignorance. Examples are given to show that these models are not sufficiently general to represent some common types of uncertainty. In particular, they are not sufficiently informative about expectations and conditional probabilities. Coherent lower previsions and sets of probability measures are considerably more general, but they may not be sufficiently informative for some purposes. Two other models for uncertainty, which involve partial preference orderings and sets of desirable gambles, are discussed. These are more informative and more general than the previous models, and they may provide a suitable mathematical foundation for a unified theory of imprecise probability. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: Choquet capacity; Coherence; Comparative probability; Credal sets; Desirable gambles; Foundations of probability; Interval-valued probability; Lower prevision; Lower probability; Partial preference ordering; Uncertainty measures

1. Introduction

Can there be a unified theory of imprecise probability? At present there are numerous mathematical models, interpretations and applications of imprecise probabilities. The various articles in this volume and in [3,4] give some idea of

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sets of desirable options / partial preference orderings

lower and upper expectations/previsions
closed convex sets of probability measures

lower and upper probabilities

belief and plausibility functions

possibility and necessity measures

probability measures

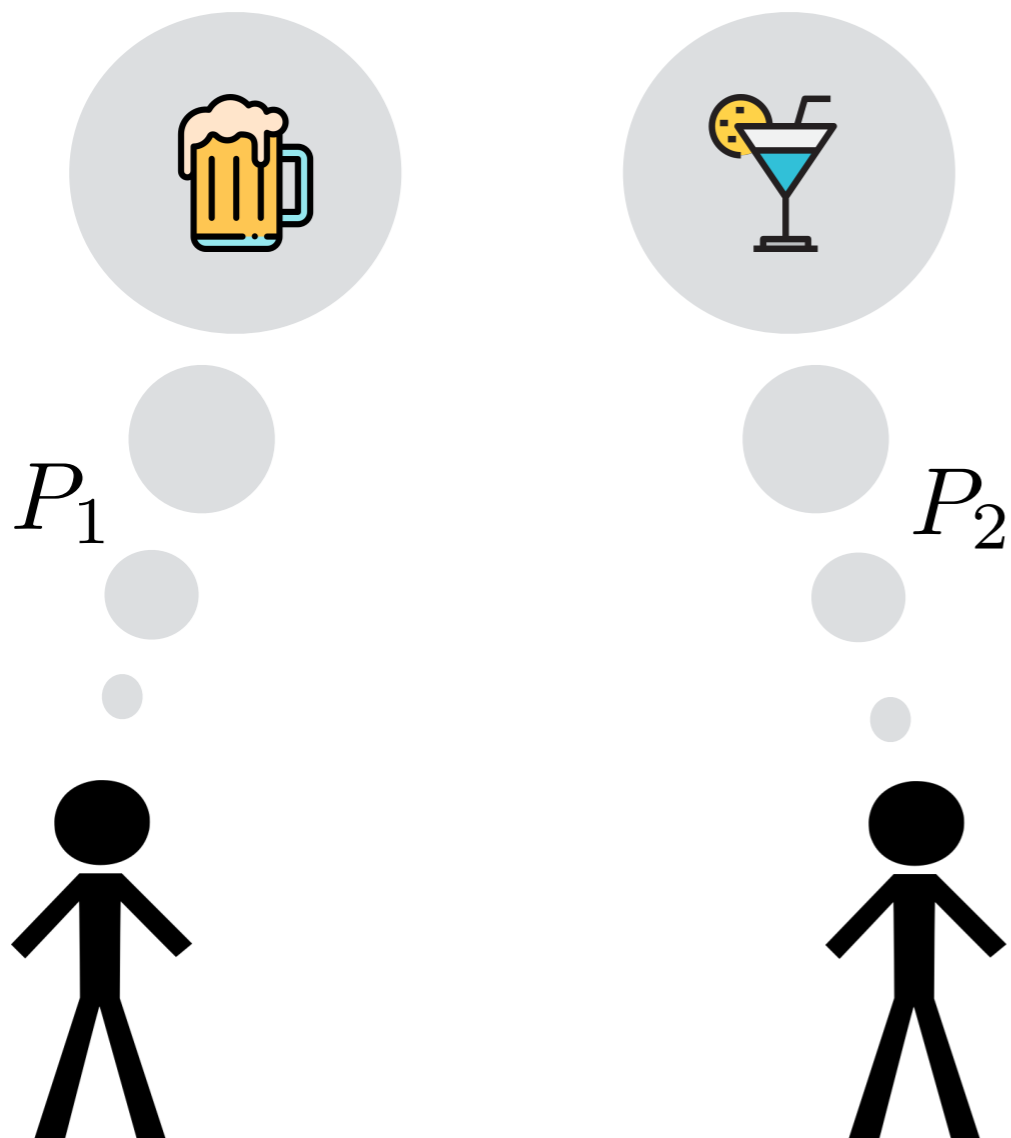
derived notion 1

set of desirable options D

$$\hookrightarrow \subseteq \mathcal{V}$$

coherence axioms!

$$C \left(\text{beer}, \text{martini}, \text{water}, \text{juice} \right) = \left\{ \text{beer}, \text{martini} \right\}$$



sets of desirable options / partial preference orderings

lower and upper expectations/previsions
closed convex sets of probability measures

lower and upper probabilities

belief and plausibility functions

possibility and necessity measures

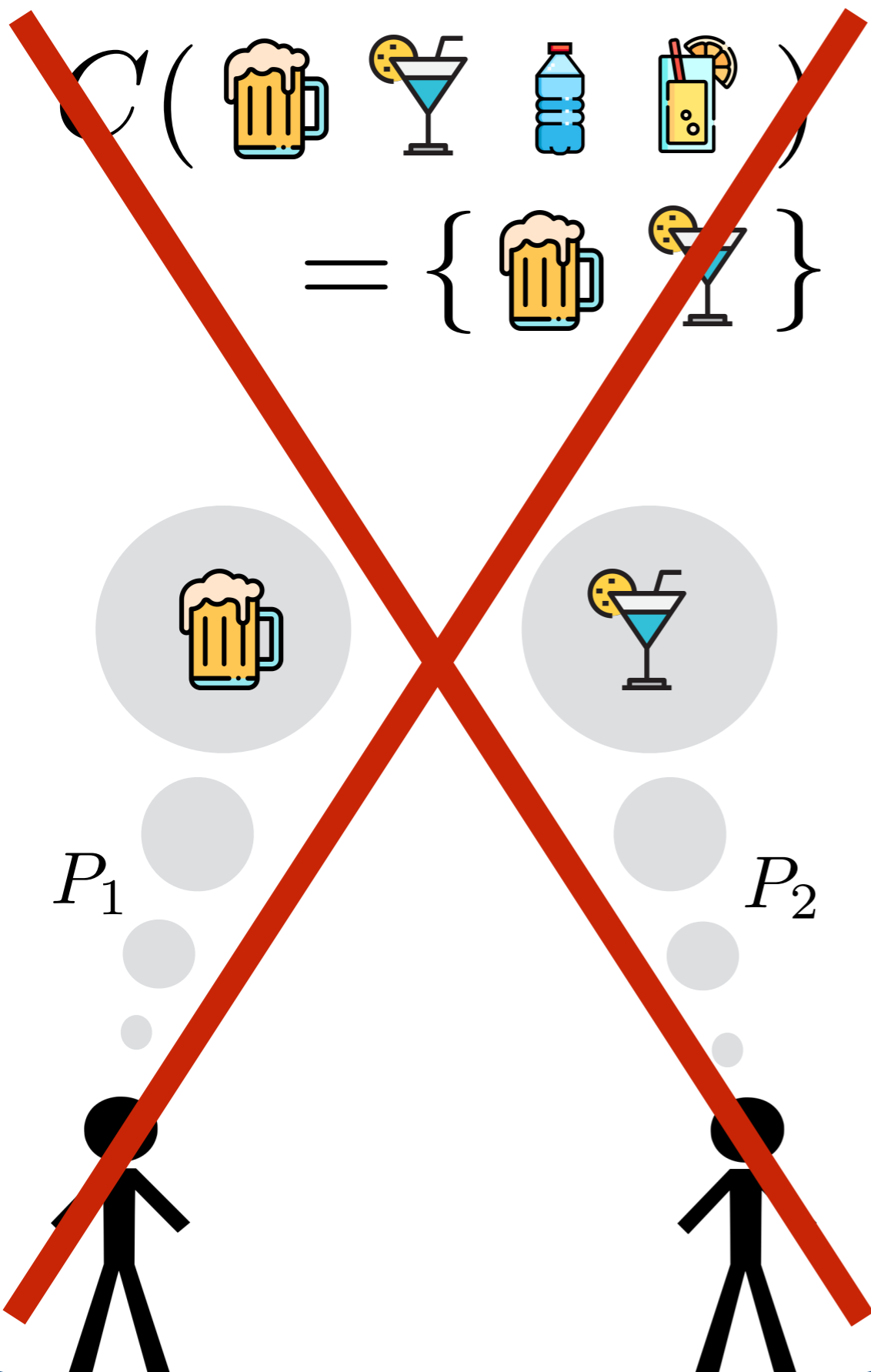
probability measures

derived notion 1

set of desirable options D

$\hookrightarrow \subseteq \mathcal{V}$

coherence axioms!



sets of desirable options / partial preference orderings

lower and upper expectations/previsions
 closed convex sets of probability measures

lower and upper probabilities

belief and plausibility functions

possibility and necessity measures

probability measures

derived notion 1

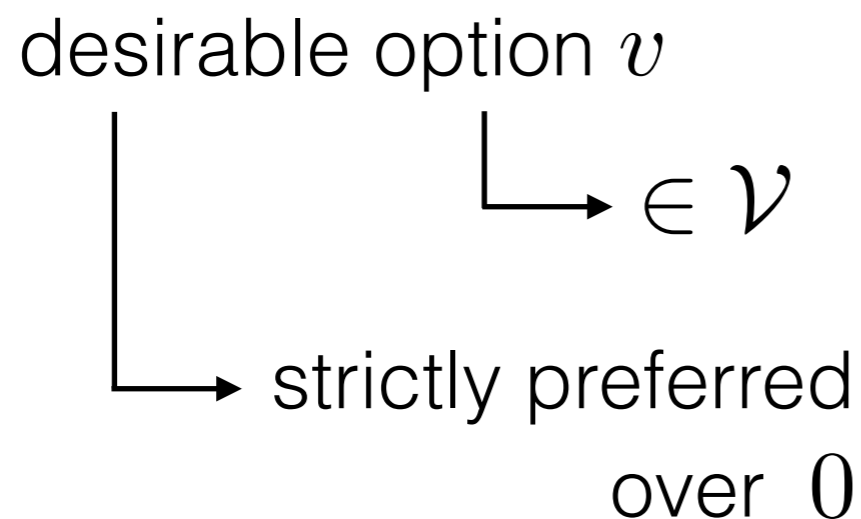
set of desirable options D

$\hookrightarrow \subseteq \mathcal{V}$

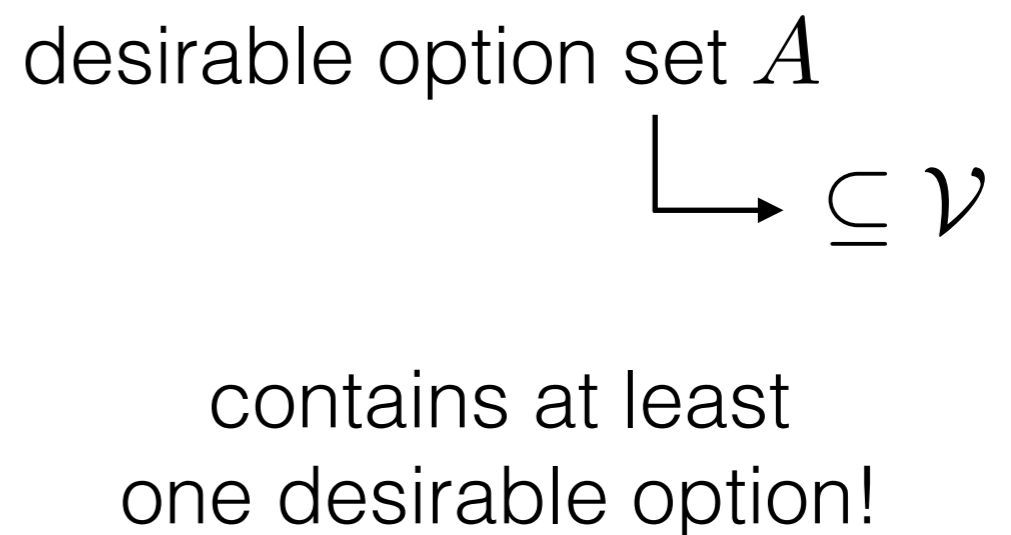
coherence axioms!

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elementary concept



derived notion 2



Interpreting, axiomatising and representing coherent choice functions in terms of desirability

derived notion 3



set of desirable
option sets



derived notion 2

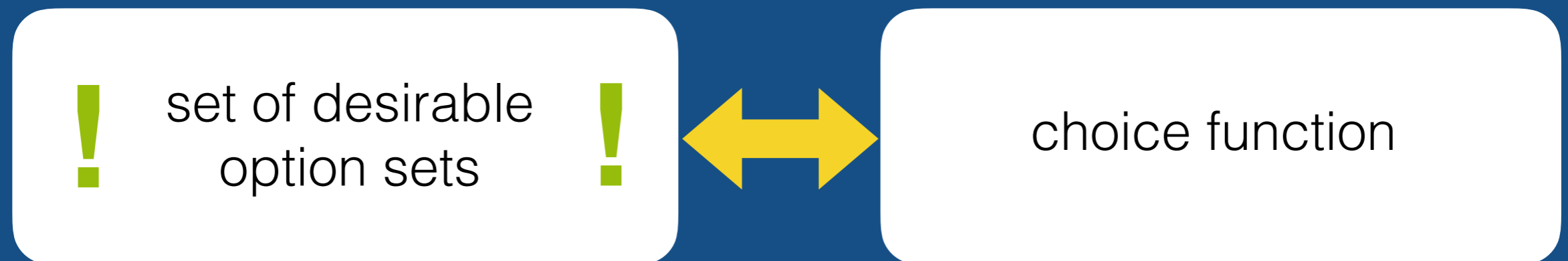
desirable option set A

$\hookrightarrow \subseteq \mathcal{V}$

contains at least
one desirable option!



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sets of desirable options / partial preference orderings

lower and upper expectations/previsions
closed convex sets of probability measures

lower and upper probabilities

belief functions

possibility and necessity measures

probability measures

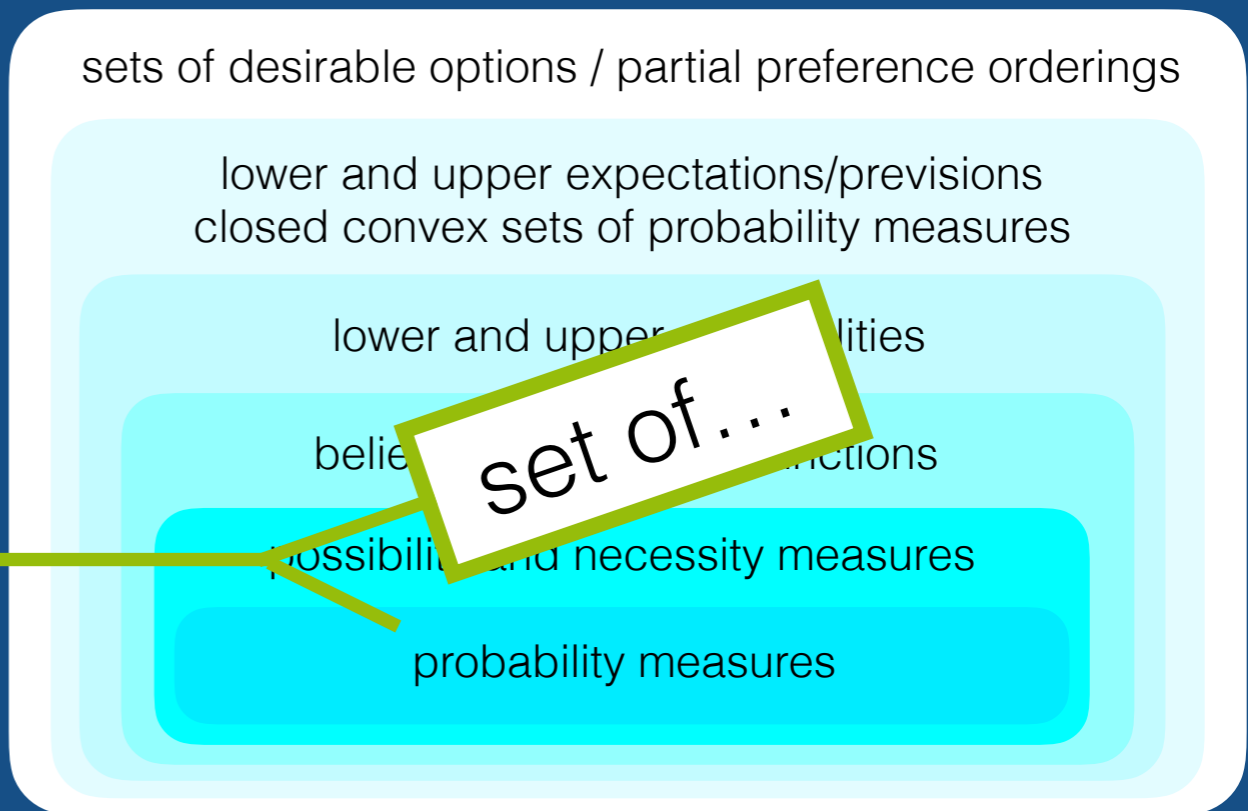
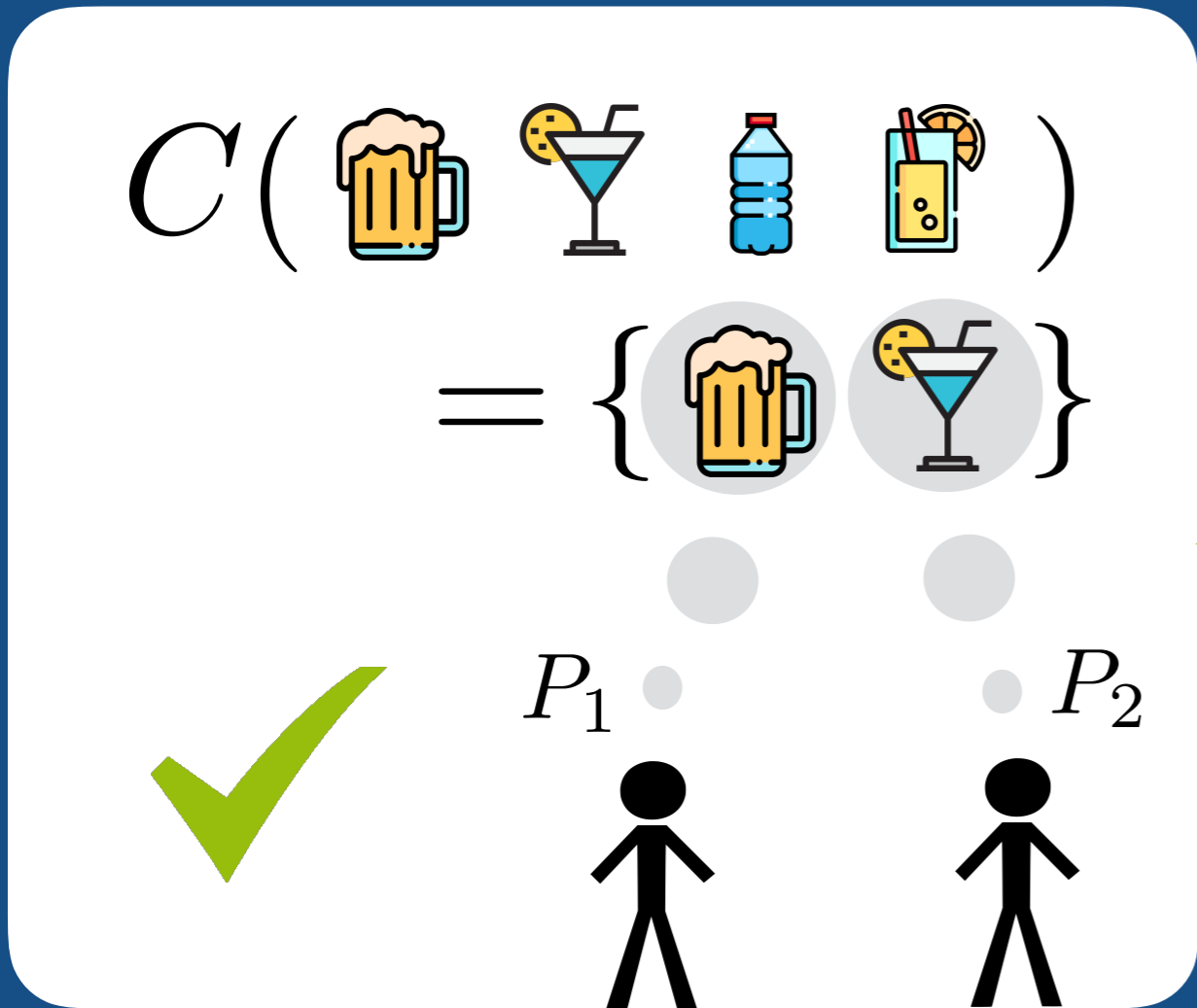
set of...



set of desirable
option sets



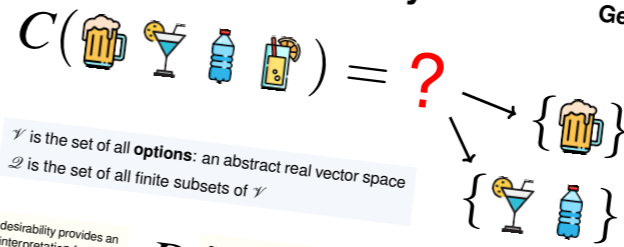
choice function





Interpreting, Axiomatizing and Representing Coherent Choice Functions in Terms of Desirability

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a unifying framework for modelling set-valued choice!

a choice function C is a map from \mathcal{D} to \mathcal{Y} such that $C(A) \subseteq A$ for every $A \in \mathcal{D}$. The corresponding rejection function R is defined by $R(A) := A \setminus C(A)$, for all $A \in \mathcal{D}$.

\mathcal{Y} is the set of all options: an abstract real vector space
 \mathcal{D} is the set of all finite subsets of \mathcal{Y}

desirability provides an interpretation for each of our models

D an option $v \in \mathcal{Y}$ is desirable if it is strictly preferred to zero
Set of desirable options

K an option set $A \in \mathcal{D}$ is desirable if it is thought to contain at least one desirable option
Set of desirable options sets

R/C an option $u \in A$ is rejected from A if at least one option $v \in A$ is strictly preferred over u , in the sense that $v-u$ is desirable
Rejection function / Choice function

- COHERENT**
- D_1 $0 \notin D$
 - D_2 $\mathcal{Y}_{>0} \subseteq D$
 - D_3 if $u, v \in D$ and $(\lambda, \mu) > 0$, then $\lambda u + \mu v \in D$
 - K_0 if $A \in K$ then also $A \setminus \{0\} \in K$, for all $A \in \mathcal{D}$
 - K_1 $\{0\} \notin K$
 - K_2 $\{u\} \in K$, for all $u \in \mathcal{Y}_{>0}$
 - K_3 if $A_1, A_2 \in K$ and if, for all $u \in A_1$ and $v \in A_2$, $(\lambda_{uv}, \mu_{uv}) > 0$, then also $\{\lambda_{uv}u + \mu_{uv}v : u \in A_1, v \in A_2\} \in K$
 - K_4 if $A_1 \in K$ and $A_1 \subseteq A_2 \in \mathcal{D}$, then also $A_2 \in K$
 - R_0 for all $A \in \mathcal{D}$ and $u \in A$: $u \in R(A) \Leftrightarrow 0 \in R(A-u)$
 - R_1 $R(\emptyset) = \emptyset$, and $R(A) \neq A$ for all $A \in \mathcal{D} \setminus \{\emptyset\}$
 - R_2 $0 \in R(\{0, u\})$, for all $u \in \mathcal{Y}_{>0}$
 - R_3 if $A_1, A_2 \in \mathcal{D}$, $0 \in R(A_1 \cup \{0\})$ and $0 \in R(A_2 \cup \{0\})$ and if $(\lambda_{uv}, \mu_{uv}) > 0$ for all $u \in A_1$ and $v \in A_2$, then $0 \in R(\{\lambda_{uv}u + \mu_{uv}v : u \in A_1, v \in A_2\} \cup \{0\})$
 - R_4 if $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$, for all $A_1, A_2 \in \mathcal{D}$

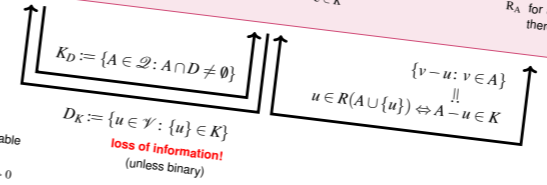
- TOTAL**
- D_T for all $u \in \mathcal{Y} \setminus \{0\}$, either $u \in D$ or $-u \in D$
 - K_T $\{u, -u\} \in K$ for all $u \in \mathcal{Y} \setminus \{0\}$
 - R_T $0 \in R(\{0, u, -u\})$, for all $u \in \mathcal{Y} \setminus \{0\}$

- MIXING**
- D_M if $\text{posi}(A) \cap D \neq \emptyset$, then also $A \cap D \neq \emptyset$, for all $A \in \mathcal{D}$
 - K_M if $B \in K$ and $A \subseteq B \subseteq \text{posi}(A)$, then also $A \in K$, for all $A, B \in \mathcal{D}$
 - R_M if $A \subseteq B \subseteq \text{conv}(A)$ then also $R(B) \cap A \subseteq R(A)$, for all $A, B \in \mathcal{D}$

- ARCHIMEDEAN**
- D_A for all $u \in D$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $u - \varepsilon \in D$
 - K_A for all $A \in K$, there is an $\varepsilon \in \mathbb{R}_{>0}$ such that $A - \varepsilon \in K$
 - R_A for all $A \in \mathcal{D}$ and $u \in \mathcal{Y}$ such that $u \in R(A \cup \{u\})$, there is some $\varepsilon \in \mathbb{R}_{>0}$ such that $u \in R((A - \varepsilon) \cup \{u\})$

OTHER PROPERTIES?

- $\mathcal{Y}_{>0}$ is a convex cone in $\mathcal{Y} \setminus \{0\}$ whose elements must be desirable
- $\lambda \geq 0, \mu \geq 0$ and $\lambda + \mu > 0$
- $\text{posi}(A) := \left\{ \sum_{i=1}^n \lambda_i u_i : n \in \mathbb{N}, \lambda_i \in \mathbb{R}_{>0}, u_i \in A \right\}$
- $\text{conv}(A) := \left\{ \sum_{i=1}^n \lambda_i u_i : n \in \mathbb{N}, \lambda_i \in \mathbb{R}_{>0}, \sum_{i=1}^n \lambda_i = 1, u_i \in A \right\}$
- (so far) only for $\mathcal{Y} = \mathcal{L}(\mathcal{X})$: the set of all bounded real-valued functions (gambles) on some set \mathcal{X} . $\mathcal{Y}_{>0} = \{u \in \mathcal{L}(\mathcal{X}) : \inf u > 0\}$
- \mathcal{D} is closed in the ARCHIMEDEAN cases; how should we modify archimedeanity for \mathcal{D} to not be closed?



an intersection of sets of desirable option sets K amounts to taking the union of the corresponding choice functions C

K is $\star \Leftrightarrow K = \bigcap \{K_D : D \in \mathcal{D}\}$ for some non-empty set \mathcal{D} of \star sets of desirable options

