A desirability-based axiomatisation for coherent choice functions

Jasper De Bock & Gert de Cooman

SMPS/BELIEF 2018

September 20

now :-)

GHENT UNIVERSITY

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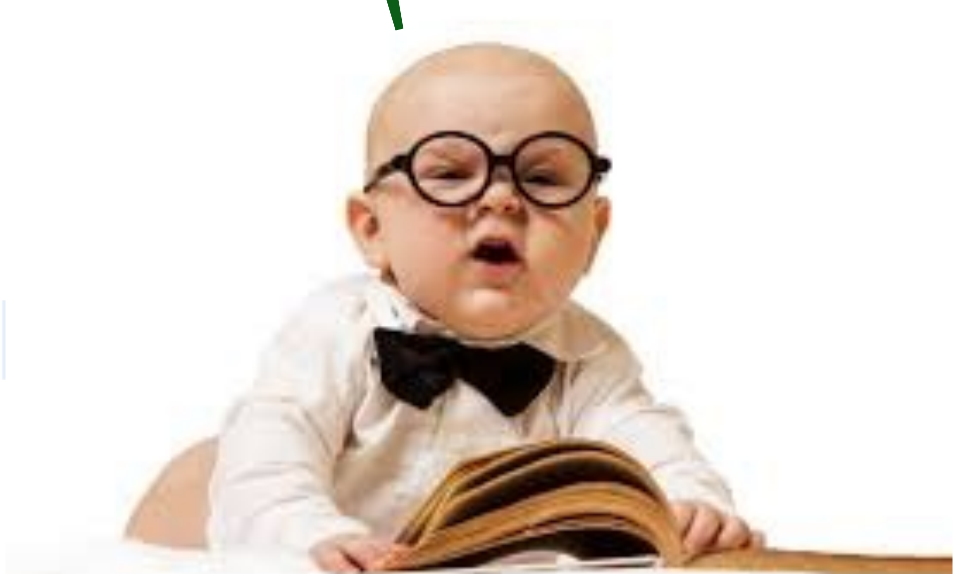
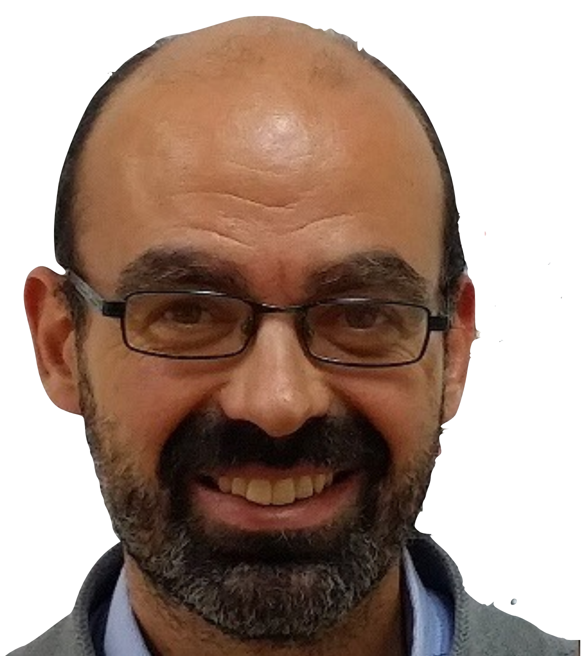
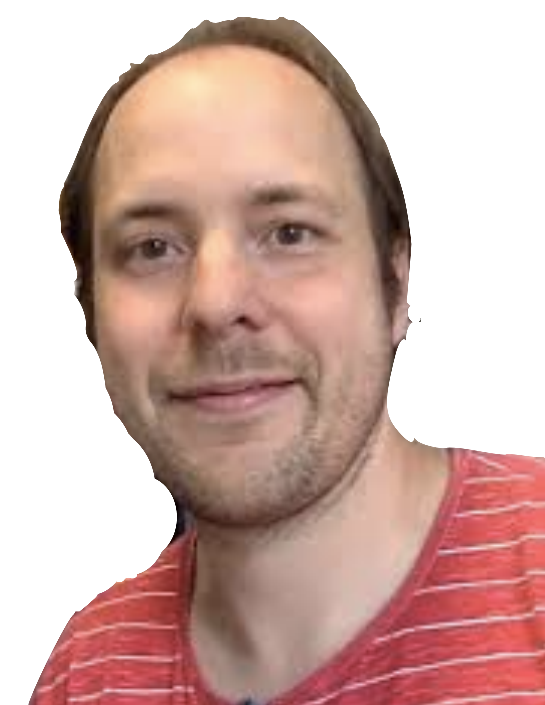


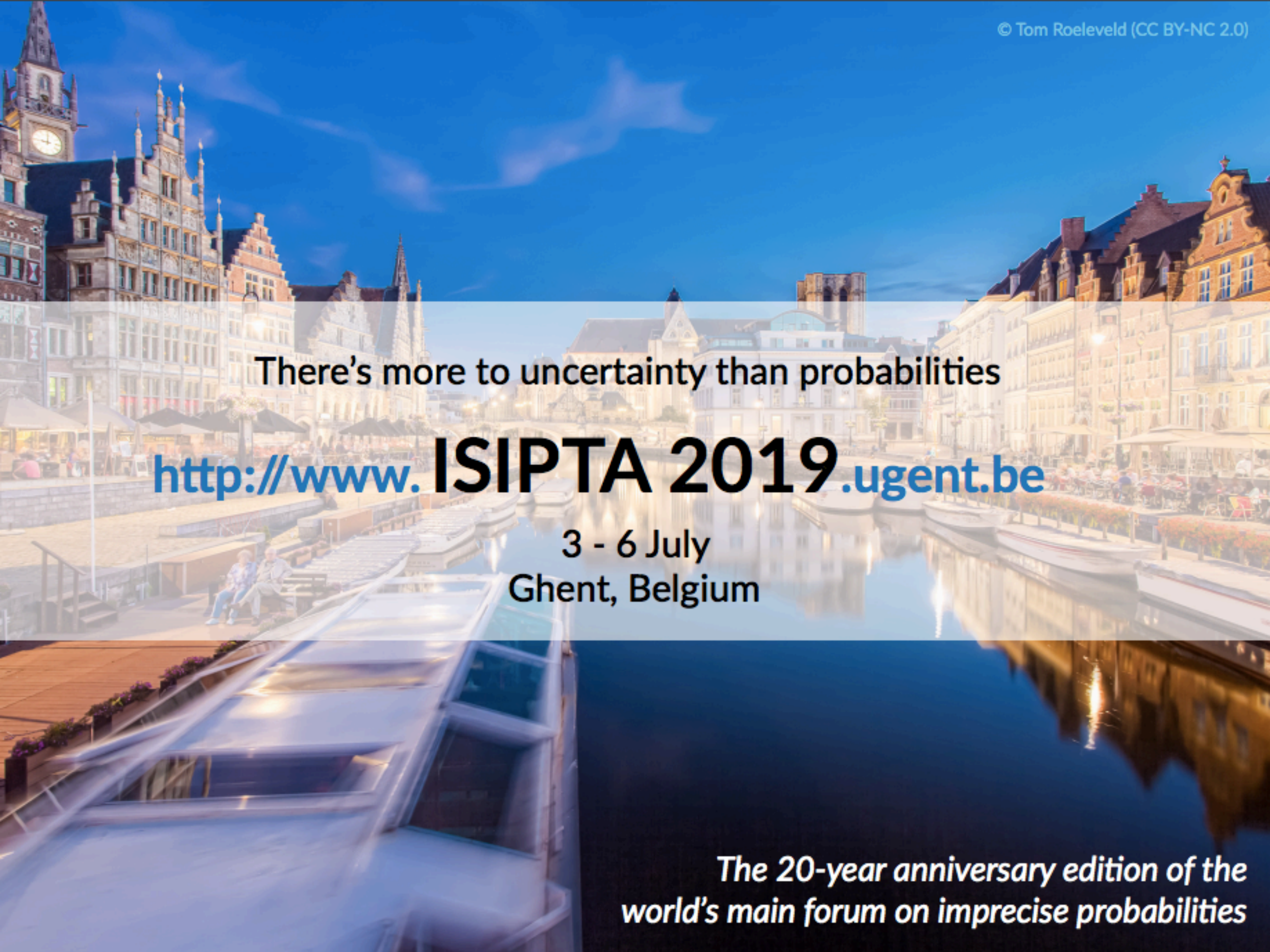
$$C(\textcircled{1}\ \textcircled{2}\ \textcircled{3}\ \textcircled{4}) = \textbf{?}$$

choice functions

$$C(\textcircled{1}\,\textcircled{2}\,\textcircled{3}\,\textcircled{4}) = ?$$



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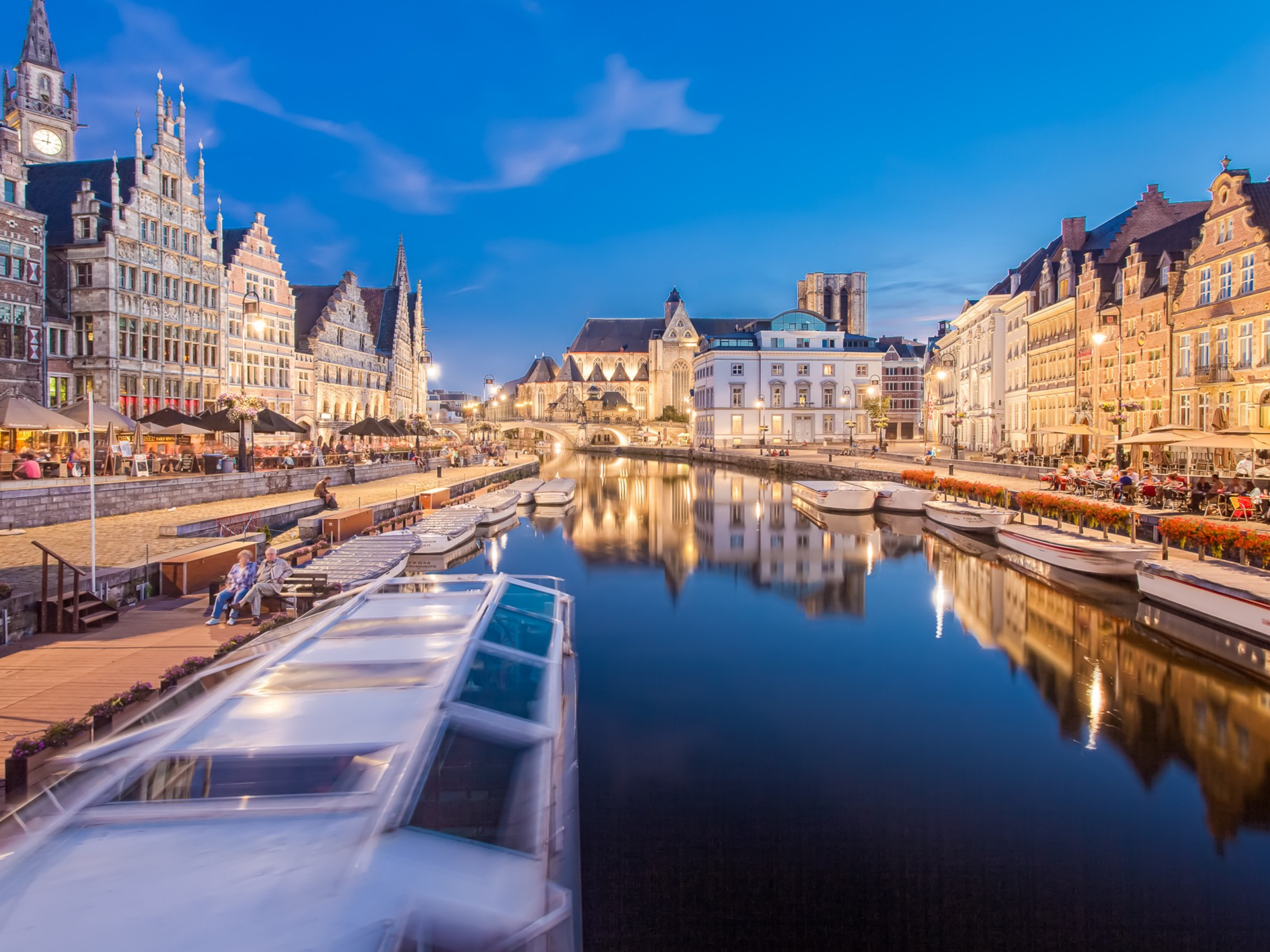

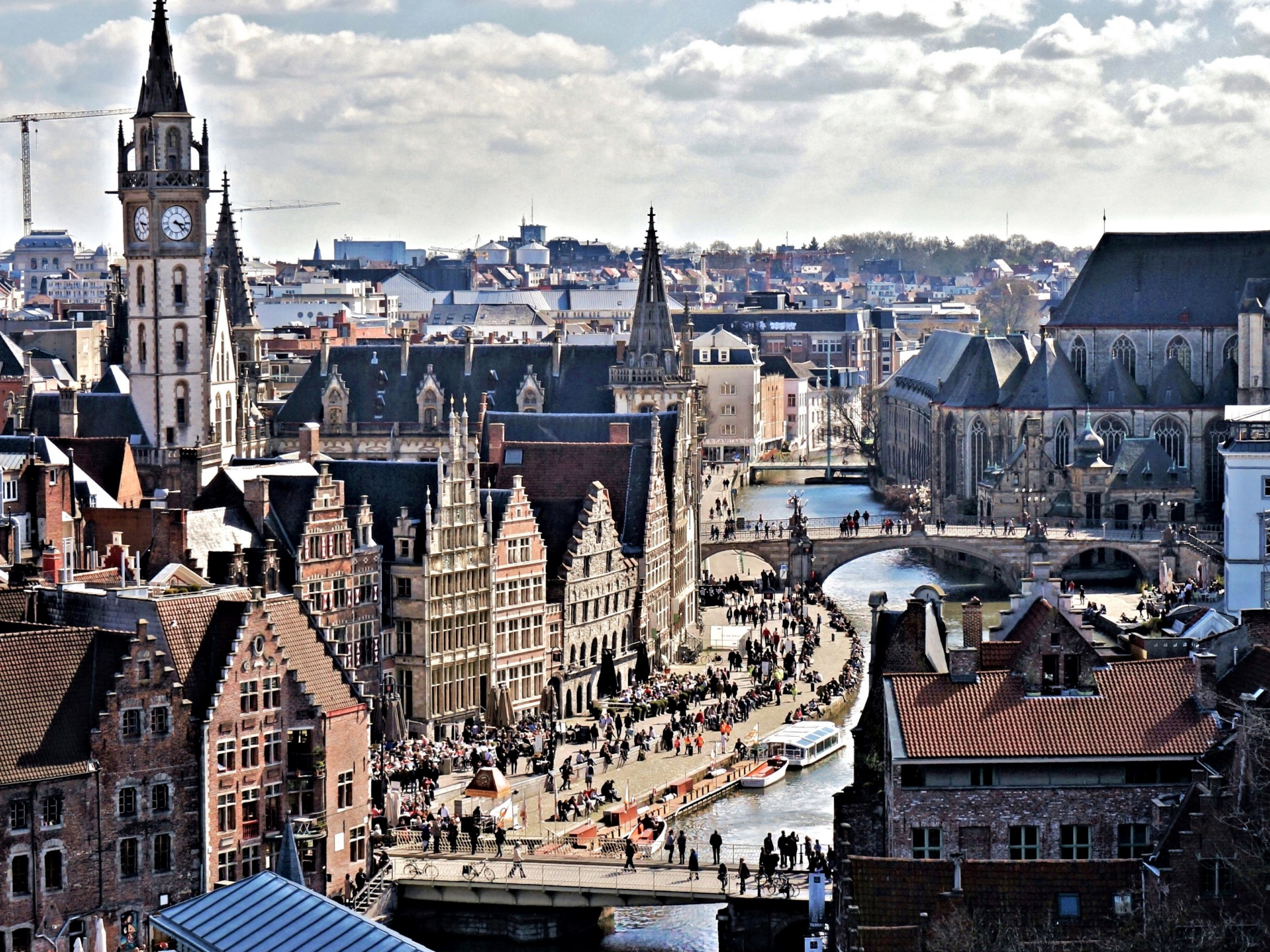

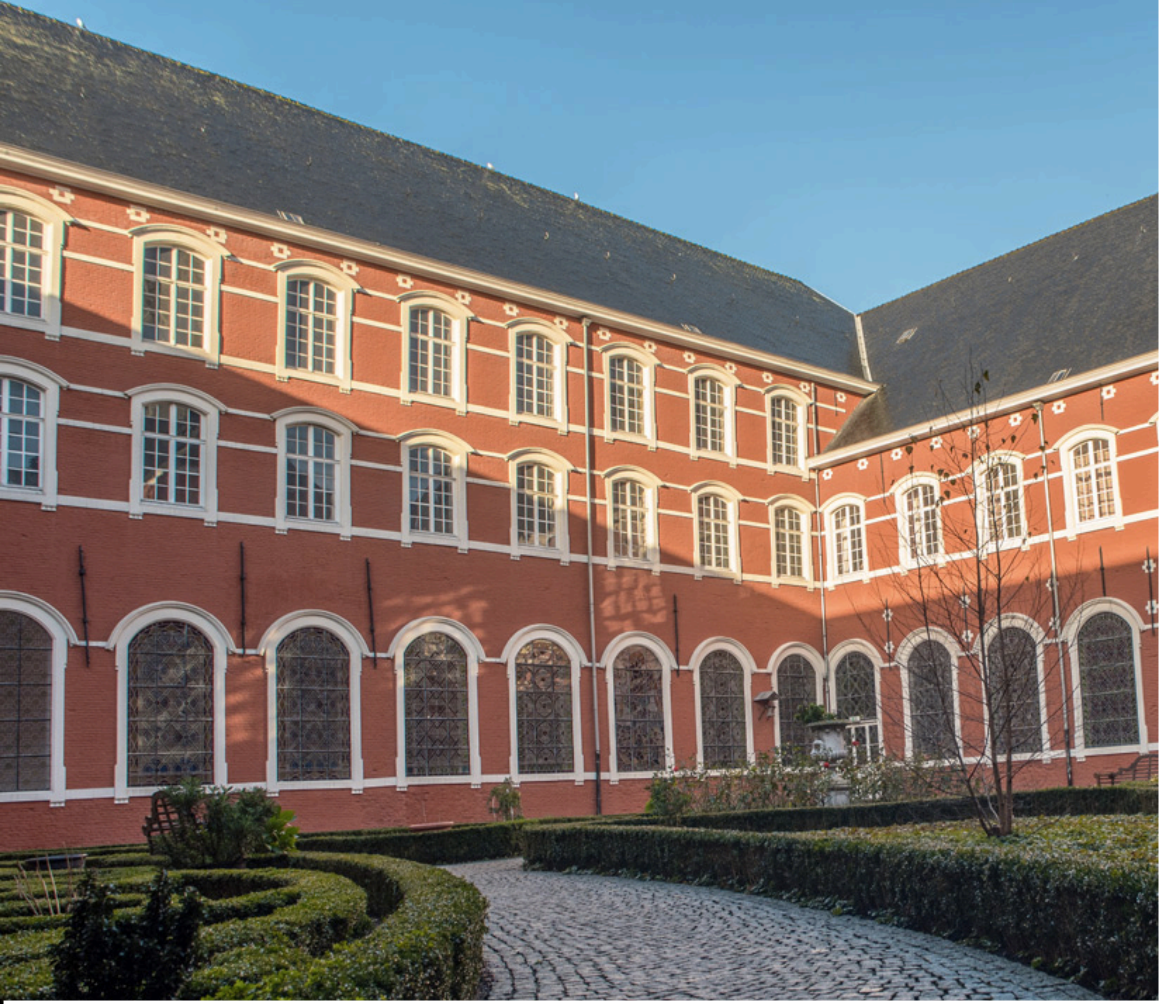

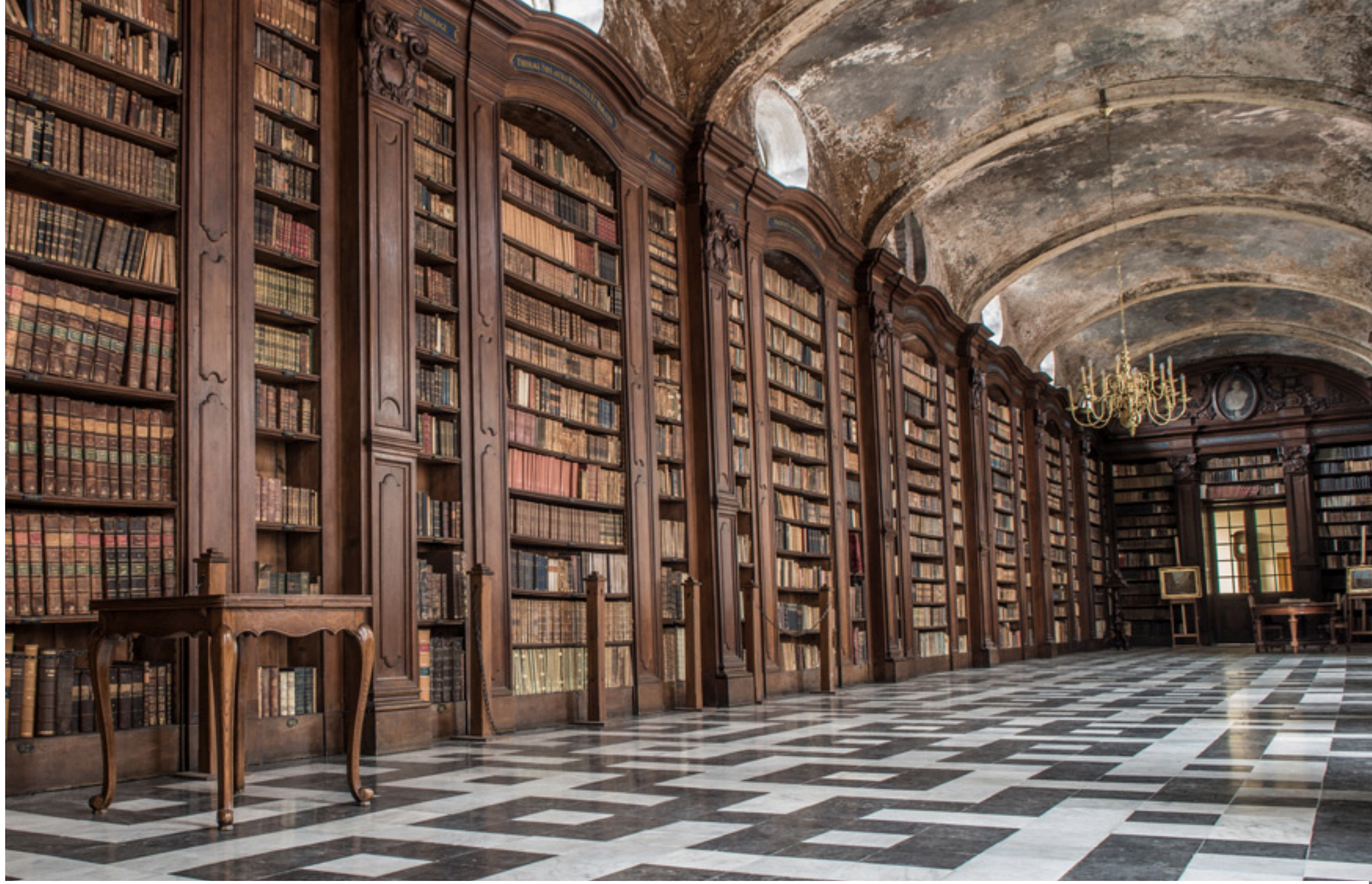

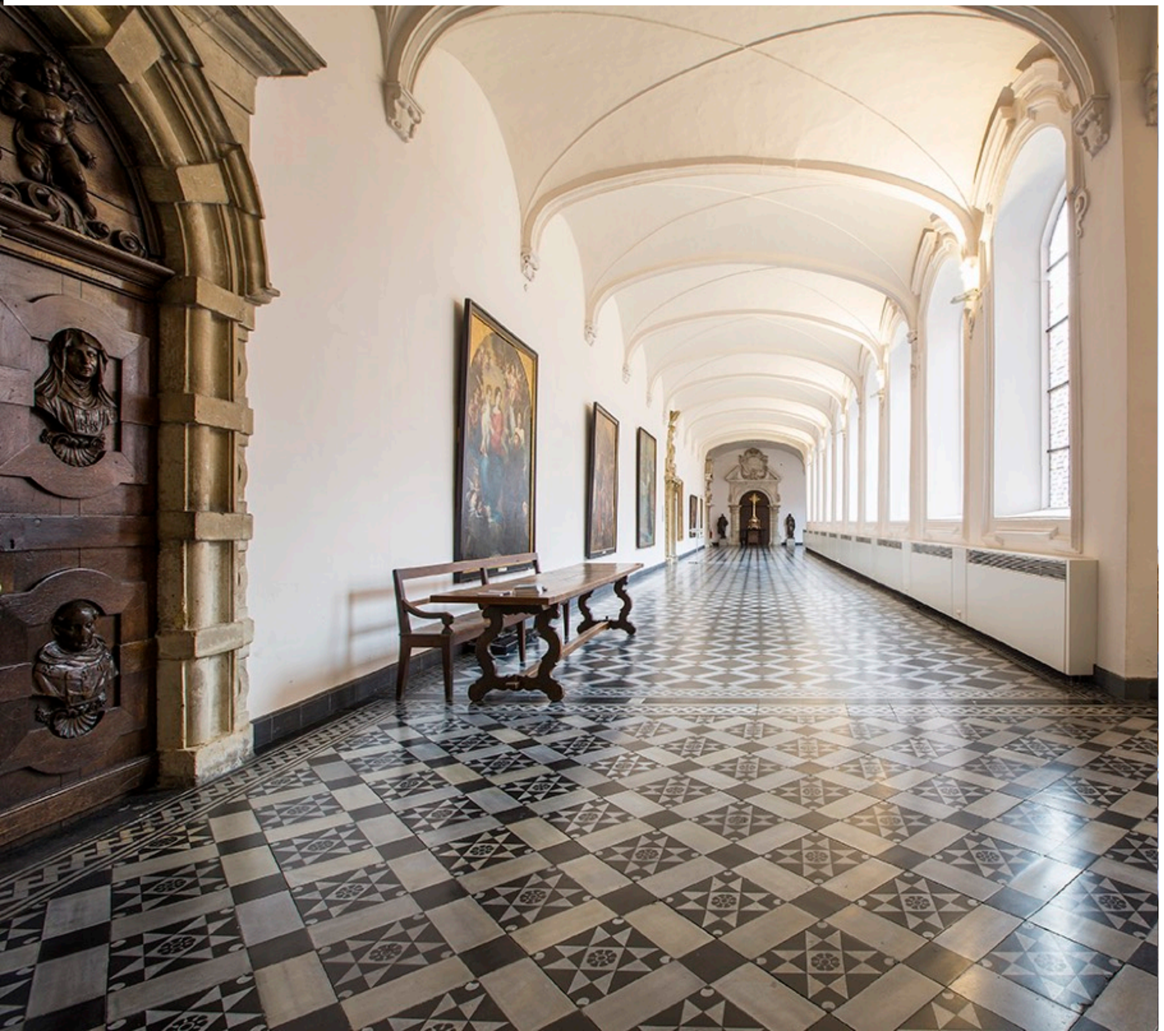

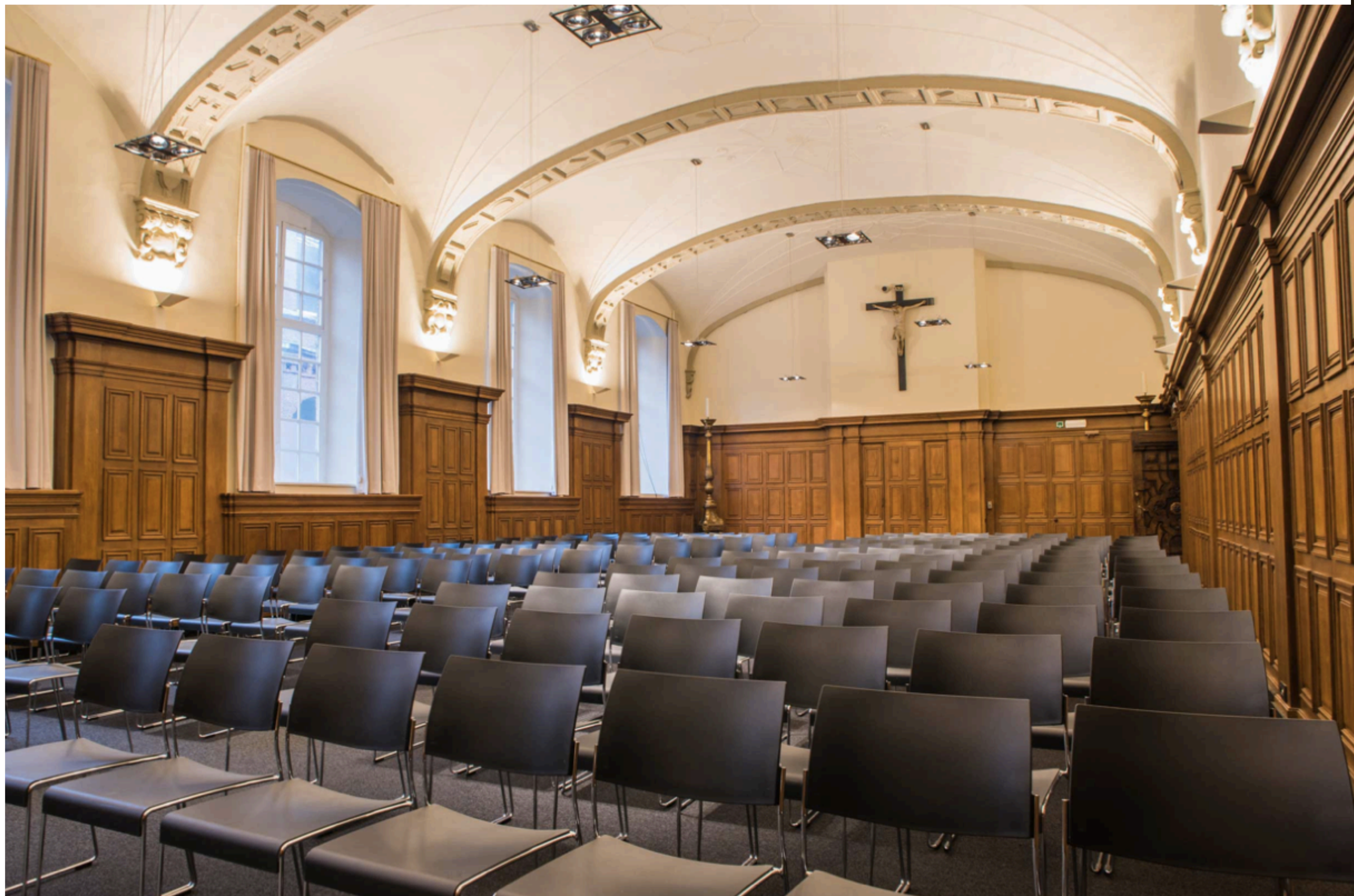

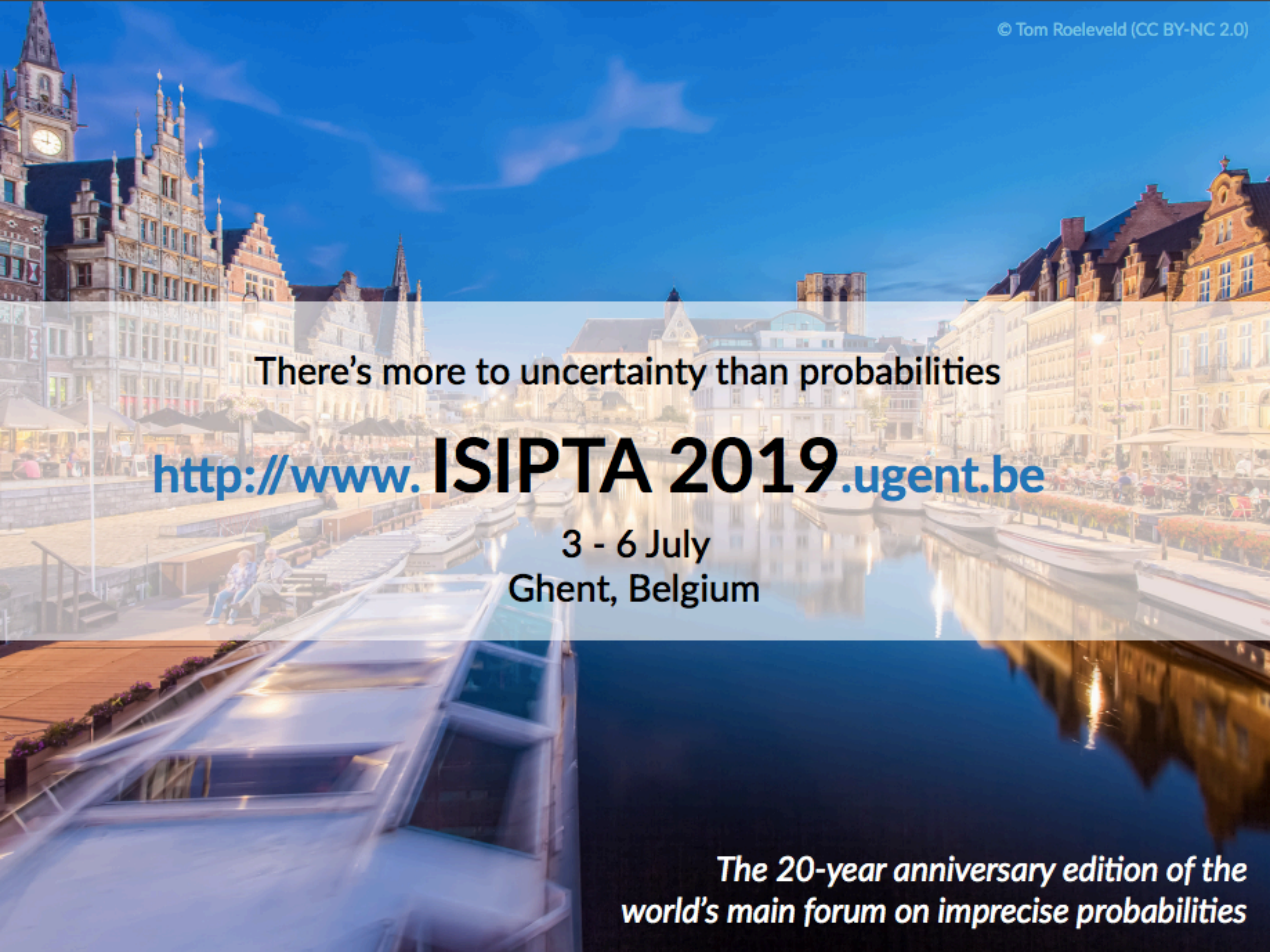
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$$C\left(\textcircled{1}\ \textcircled{2}\ \textcircled{3}\ \textcircled{4}\right) = \textbf{?}$$

$$C(\text{①}\ \text{②}\ \text{③}\ \text{④}) = \textbf{?}$$

u_1 u_2 u_3 u_4

u_i is an uncertain reward: **a gamble**

$\forall x \in \mathcal{X} : u_i(x)$ is the reward that you receive if x happens

$$C\left(\underset{u_1}{\textcircled{1}}\ \underset{u_2}{\textcircled{2}}\ \underset{u_3}{\textcircled{3}}\ \underset{u_4}{\textcircled{4}}\right) = \textbf{?}$$

$$\textcircled{1},\ \textcircled{2},\ \textcircled{3},\ \textcircled{4},$$
$$\left\{\textcircled{1}\ \textcircled{2}\right\}, \left\{\textcircled{2}\ \textcircled{4}\right\},\ \ldots$$

$$C(\textcircled{1}\ \textcircled{2}\ \textcircled{3}\ \textcircled{4}) = ?$$

$u_1 \quad u_2 \quad u_3 \quad u_4$

$$C(\textcircled{1}\ \textcircled{2}\ \textcircled{5}) = ?$$

$u_1 \quad u_2 \quad u_5$

$$C(A) = ?$$

→ any finite subset of $\mathcal{L} = \mathbb{R}^{\mathcal{X}}$

$$C(\underbrace{\overset{u_1}{①}\ \overset{u_2}{②}\ \overset{u_3}{③}\ \overset{u_4}{④}}_{A}) = \{\overset{u_2}{②}\ \overset{u_4}{④}\}$$

$$C(\underbrace{\underset{u_1}{\textcircled{1}}\ \underset{u_2}{\textcircled{2}}\ \underset{u_3}{\textcircled{3}}\ \underset{u_4}{\textcircled{4}}}_{A}) = \{\underset{u_2}{\textcircled{2}}\ \underset{u_4}{\textcircled{4}}\}$$

$$u \in R(A) = \{\underset{u_1}{\textcircled{1}}\ \underset{u_3}{\textcircled{3}}\}$$

f is (a) desirable (gamble)

$\Leftrightarrow$ f is strictly preferred to 0

0 is not desirable

$f > 0$ implies f desirable

f, g desirable implies

$\lambda f + \mu g$ desirable for $(\lambda, \mu) > 0$

f is (a) desirable (gamble)

$\Leftrightarrow$ f is strictly preferred to 0

$v \succ u$

$\Leftrightarrow$ v is strictly preferred to u

$\Leftrightarrow$ $v - u$ is desirable

1 2 3 4

u_1 u_2 u_3 u_4

$\exists\, v \in A : v - u$ is desirable

$\Updownarrow$

$u \in R(A) = \{$ 1 3 $\}$

u_1 u_3

$\exists$ desirable $f \in \{v - u : v \in A\}$

$\Updownarrow$

$\exists\, v \in A : v - u$ is desirable

$\Updownarrow$

$$u \in R(A) = \{ (1) \; (3) \}$$

u_1 u_3

$\exists$ desirable $f \in \{v - u : v \in A\}$

$\Updownarrow$

$\exists\, v \in A : v - u$ is desirable

$\Updownarrow$

$$u \in R(A) \Leftrightarrow u \notin C(A)$$

A is (a) desirable (gamble set)

$\Leftrightarrow \exists$ desirable $f \in A$

Let K be the set of all of them

possible assessments:

$$\{f\} \in K, \{f_1, f_2\} \in K, \ldots$$

$$\exists \text{ desirable } f \in \{v - u : v \in A\}$$

$$\Updownarrow$$

$$\{v - u : v \in A\} \in K$$

$$\Updownarrow$$

$$u \in R(A) \Leftrightarrow u \notin C(A)$$

A is (a) desirable (gamble set)

$\Leftrightarrow \exists$ desirable $f \in A$

Let K be the set of all of them

Which properties should we impose on it ?

A is (a) desirable (gamble set)

$\Leftrightarrow \exists$ desirable $f \in A$

Let K be the set of all of them

Definition 4 (Coherence). *A set of desirable gamble sets* $K \subseteq \mathcal{Q}$ *is called* coherent *if it satisfies the following axioms:*

K$_0$. $\emptyset \notin K$;

K$_1$. $A \in K \Rightarrow A \setminus \{0\} \in K$, *for all* $A \in \mathcal{Q}$;

K$_2$. $\{u\} \in K$, *for all* $u \in \mathcal{L}_{>0}$;

K$_3$. *if* $A_1, A_2 \in K$ *and if, for all* $u \in A_1$ *and* $v \in A_2$, $(\lambda_{u,v}, \mu_{u,v}) > 0$, *then*

$$\{\lambda_{u,v} u + \mu_{u,v} v \colon u \in A_1, v \in A_2\} \in K;$$

K$_4$. $A_1 \in K$ *and* $A_1 \subseteq A_2 \Rightarrow A_2 \in K$, *for all* $A_1, A_2 \in \mathcal{Q}$.

$$\emptyset \notin K$$

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$$u > 0 \Rightarrow \{u\} \in K$$

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$A_1, A_2 \in K$ implies

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if $A_1 \subseteq A_2$

then $A_1 \in K$ implies $A_2 \in K$

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K is coherent

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Theorem 7 (Representation). *Every coherent set of desirable gamble sets* $K \in \overline{\mathbf{K}}$ *is dominated by at least one binary set of desirable gamble sets:* $\overline{\mathbf{D}}(K) := \{D \in \overline{\mathbf{D}} \colon K \subseteq K_D\} \neq \emptyset$. *Moreover,* $K = \bigcap\{K_D : D \in \overline{\mathbf{D}}(K)\}$.

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K is coherent

K_D

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K is coherent

Fully determined by

- its singletons $\{f\}$
- basic desirability assessments
- **binary** preferences
- partial preference order

!

K_D

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K is coherent

$$\Updownarrow$$

$$K = \bigcap\{K_D \colon D \in \overline{\mathbf{D}}(K)\}$$

with $\overline{\mathbf{D}}(K) = \{D \in \mathbf{D} \colon K \subseteq K_D\}$

Fully determined by

- its singletons $\{f\}$
- basic desirability assessments
- **binary** preferences
- partial preference order

!

Theorem 8. *Let* $\{K_i\}_{i \in I}$ *be an arbitrary non-empty family of sets of desirable gamble sets, with intersection* $K := \bigcap_{i \in I} K_i$. *If* K_i *is coherent for all* $i \in I$, *then so is* K. *This implies that* $(\overline{\mathbf{K}}, \subseteq)$ *is a complete meet-semilattice.*

Theorem 10 (Natural extension). *Consider any assessment* $\mathcal{A} \subseteq \mathcal{Q}$. *Then* $\mathcal{A}$ *is consistent if and only if* $\emptyset \notin \mathcal{A}$ *and* $\{0\} \notin \mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A})$. *Moreover, if* $\mathcal{A}$ *is consistent, then* $\mathrm{Ex}(\mathcal{A}) = \mathrm{Rs}(\mathrm{Posi}(\mathcal{L}^{\mathrm{s}}_{>0} \cup \mathcal{A}))$.

existing decision models are special cases

similar results with (extra) convexity axiom