Imprecise Markov chains

From basic theory to applications II

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UTOPIÆ Uncertainty Treatment and Optimisation in Aerospace Engineering
Handling the unknown at the edge of tomorrow
MARIE CURIE ACTIONS


Puma

Imprecise continuous-time Markov chains

We will next construct the convex combination that satisfies Equation (112). So, consider any $t > 0$, $v \in \mathscr{U}_{<t}$ and $x_v \in \mathscr{X}_v$. We again distinguish between two cases: $t \leq \max u$ and $t > \max u$. If $t \leq \max u$, then for all $\Delta \in (0, t - \max v)$ and $x, y \in \mathscr{X}$, we see that $(X_t = y, (X_{t-\Delta} = x, X_v = x_v)) \in \mathscr{C}_\emptyset$, and therefore, since P is an extension of $\tilde{P}$, it follows from Equation (96) that

$$P(X_t = y | X_{t-\Delta} = x, X_v = x_v) = P_\emptyset(X_t = y | X_{t-\Delta} = x, X_v = x_v).$$

Hence, if we let $\mathscr{I} \coloneqq \{i\}$, $v^* \coloneqq v$, $\lambda_i \coloneqq 1$, ${}^iP \coloneqq P_\emptyset$ and ${}^ix_{v^*} \coloneqq x_v$, Equation (112) is satisfied by choosing $\delta \coloneqq t - \max v$. If $t > \max u$, then for all $\Delta \in (0, t - \max(v \cup u))$, it follows from Equation (110) (with $s \coloneqq t$ and $w \coloneqq v \cup t - \Delta$) that

$$\begin{aligned} &P(X_t = y | X_{t-\Delta} = x, X_v = x_v) \\ &\quad = \sum_{x_{u \setminus v} \in \mathscr{X}_{u \setminus v}} P_{x_u}(X_t = y | X_{t-\Delta} = x, X_{u \cup (v \setminus [0, \max u])} = x_{u \cup (v \setminus [0, \max u])}) \\ &\qquad\qquad\qquad\qquad\qquad\qquad P^*(X_{u \setminus v} = x_{u \setminus v} | X_{t-\Delta} = x, X_v = x_v). \end{aligned}$$

Therefore, if we let $\mathscr{I} \coloneqq \mathscr{X}_{u \setminus v}$, $v^* \coloneqq u \cup (v \setminus [0, \max u])$ and, for all $x_{u \setminus v} \in \mathscr{I}$,

$$\lambda_{x_{u \setminus v}} \coloneqq P^*(X_{u \setminus v} = x_{u \setminus v} | X_{t-\Delta} = x, X_v = x_v),$$

${}^{x_{u \setminus v}}P = P_{x_u}$ and ${}^{x_{u \setminus v}}x_{v^*} \coloneqq x_{u \cup (v \setminus [0, \max u])}$, Equation (112) is satisfied by choosing $\delta \coloneqq t - \max(v \cup u)$. Hence, Equation (112) can be satisfied both when $t \leq \max u$ and when $t > \max u$.

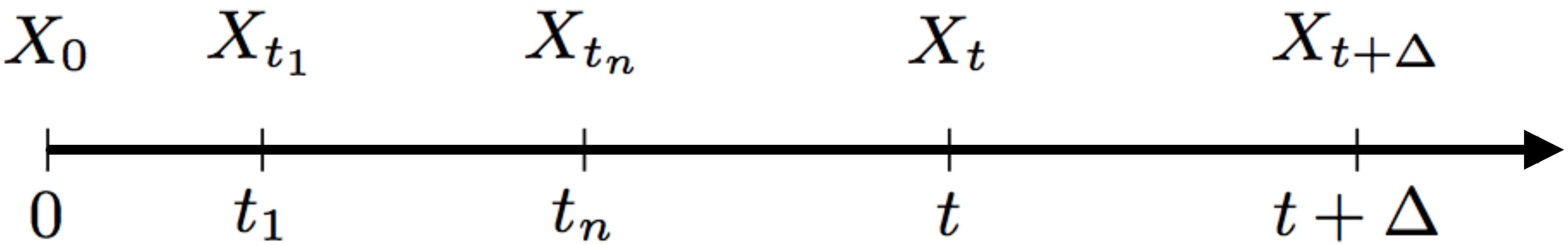
Imprecise continuous-time Markov chains

Continuous-time Markov chains

$$\mathrm{P}(X_0 = x)$$

$$\mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$

Continuous-time Markov chains



$$P(X_0 = x)$$

$$P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$
$$= P(X_{t+\Delta} = y | X_t = x)$$

Markov assumption

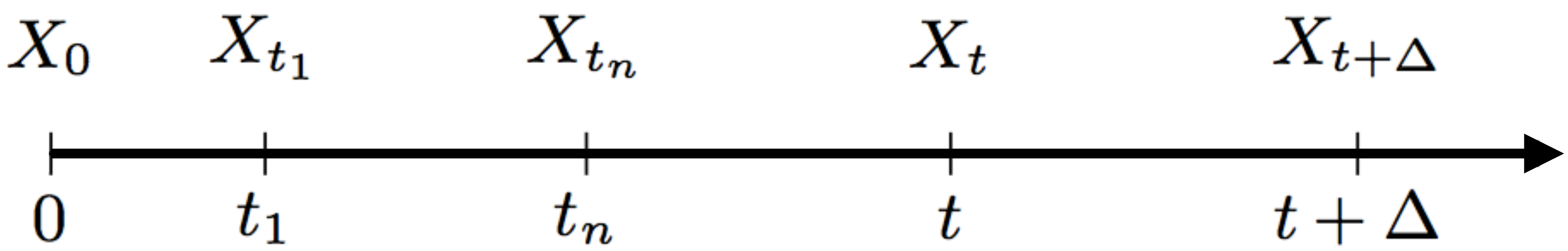
Continuous-time Markov chains…

$$\mathrm{P}(X_0 = x)$$

$$\begin{aligned}\mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)& \\ = \mathrm{P}(X_{t+\Delta} = y | X_t = x)& \\ \approx I(x,y) + \Delta\ Q_t(x,y)&\end{aligned}$$

…that are
nice enough

Continuous-time Markov chains



$$\boxed{\mathrm{P}(X_0 = x)}$$

$$\begin{aligned} \mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\ = \mathrm{P}(X_{t+\Delta} = y | X_t = x) \\ \approx I(x,y) + \Delta \boxed{Q_t(x,y)} \end{aligned}$$

Continuous-time Markov chains…

$$\boxed{\mathrm{P}(X_0 = x)}$$

Let's assume that this does not depend on time!

$$\boxed{Q_t(x, y)}$$

Continuous-time Markov chains…

$\boxed{\mathrm{P}(X_0 = x)}$

…that are homogeneous

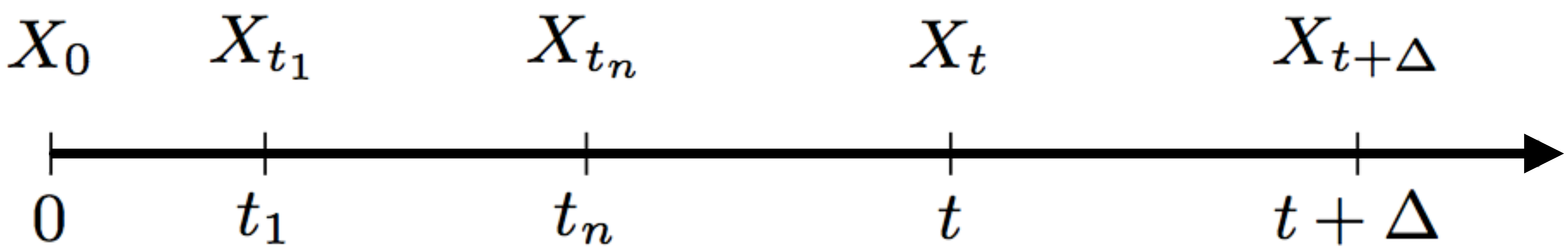
$\boxed{Q(x, y)}$

Continuous-time Markov chains

$$\boxed{\mathrm{P}(X_0 = x)}$$

$$\boxed{Q(x, y)}$$

Continuous-time Markov chains



$\boxed{\mathrm{P}(X_0 = x)}$ that's just a probability mass function $\pi_0(x)$

initial distribution

transition rate matrix

$$\sum_y Q(x, y) = 0$$
$$(\forall y \neq x)\, Q(x, y) \geq 0$$
$$(\forall x)\, Q(x, x) \leq 0$$

$\boxed{Q(x, y)}$

Amorous

Bickering

Confusion

Depression

$$Q = \begin{array}{c} \\ \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{array} \\ \begin{bmatrix} -4 & 3 & 1 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix} \end{array}$$

What is $P(X_t = y | X_0 = x)$?

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transition matrix

$$T_t(x,y) := P(X_t = y | X_0 = x)$$

backward Kolmogorov differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} T_t = Q T_t \,, \text{with } T_0 = I$$

$$\Longrightarrow T_t = e^{Qt} = \lim_{n \to +\infty} (I + \frac{t}{n} Q)^n$$

What is $P(X_t = y | X_0 = x)$ **?** $\rightarrow$ $\boxed{e^{Qt}(x, y)}$

transition matrix

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What is $P(X_t = y | X_0 = x)$ **?** $\Rightarrow$ $\boxed{e^{Qt}(x,y)}$

What is $E(f(X_t)|X_0 = x)$ **?** $\Rightarrow$ $\boxed{e^{Qt}f(x)}$

What is $P(X_t = y)$ **?** $\Rightarrow$ $\boxed{\pi_0 e^{Qt}(y)}$

What is $E(f(X_t))$ **?** $\Rightarrow$ $\boxed{\pi_0 e^{Qt} f}$

$\vdots$ $\qquad\qquad$ $\vdots$

What is $P(X_t = y | X_0 = x)$ **?** $\longrightarrow$ $\boxed{e^{Qt}(x, y)}$

The following limit always exists!

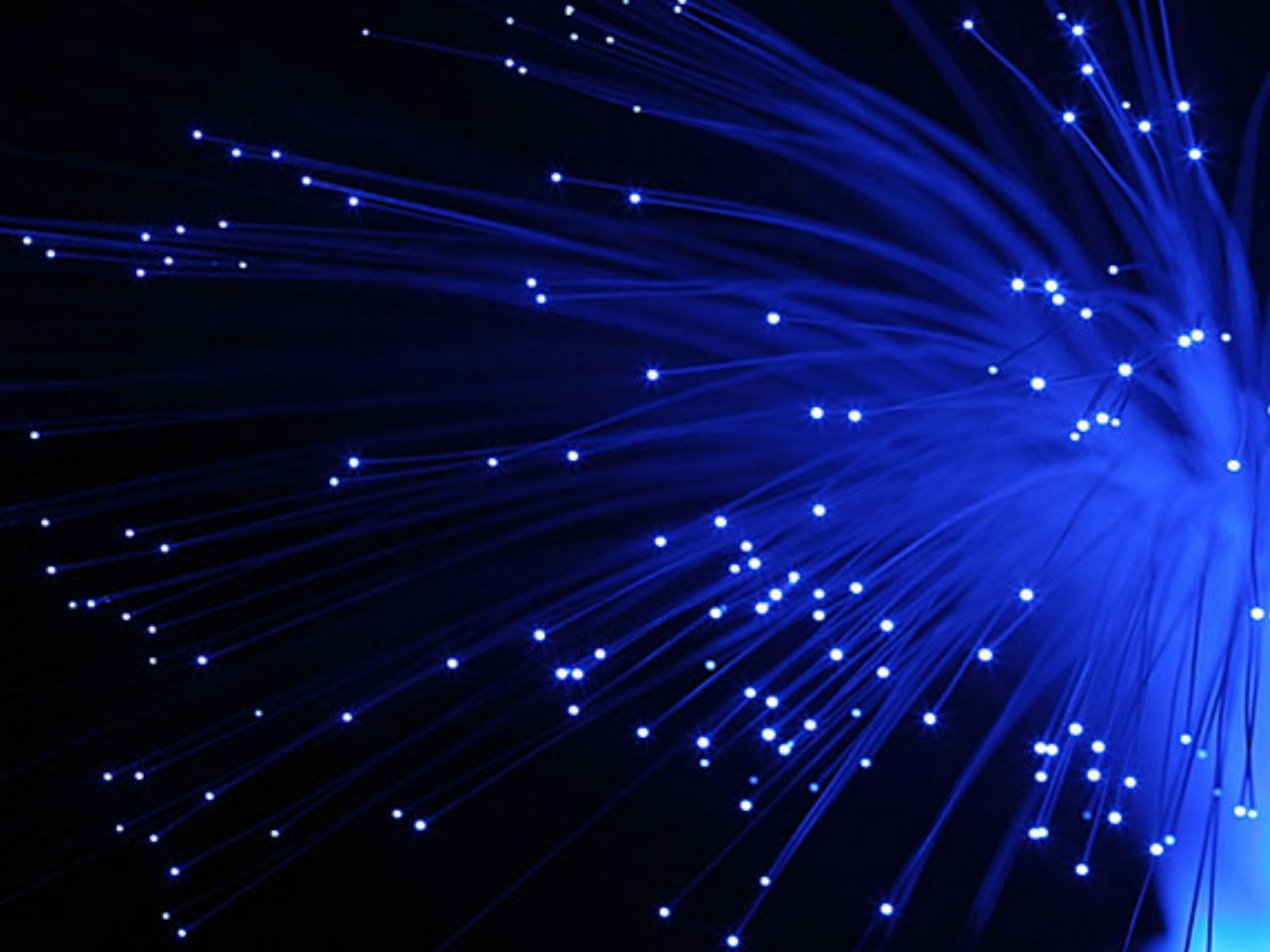
$$\lim_{t \to +\infty} P(X_t = y | X_0 = x) = \lim_{t \to +\infty} e^{Q}t(x, y)$$

And often does not depend on x **!**

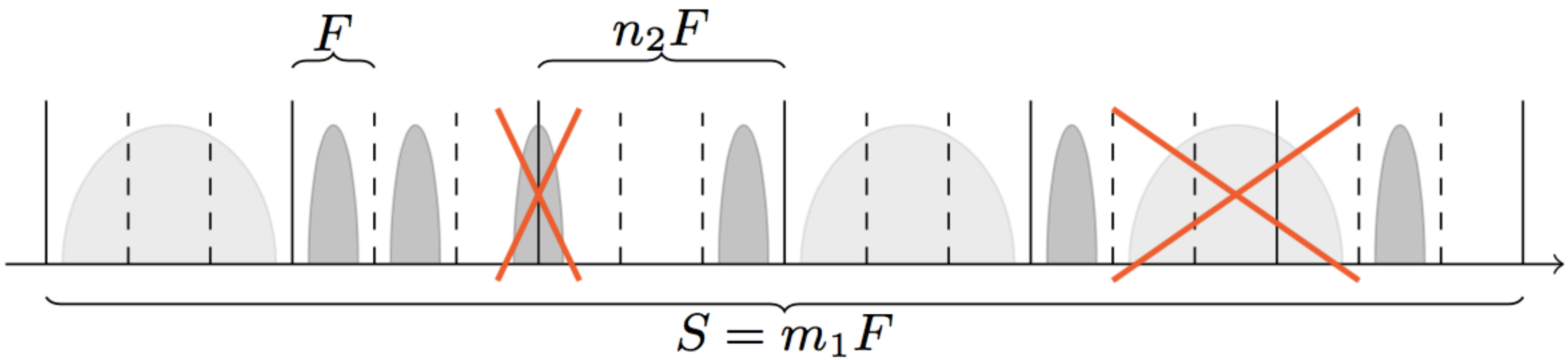
$$\boxed{\pi_\infty(y)} = \lim_{t \to +\infty} P(X_t = y) = \lim_{t \to +\infty} \pi_0 e^{Q}t(y)$$


That’s all fine and well, but what can you use it for?

- **Reliability engineering** (failure probabilities, …)
- **Queuing theory** (waiting in line …)
  - optimising supermarket waiting times
  - dimensioning of call centers
  - airport security lines
  - router queues on the internet
- **Cell division in biology** (how long does it take?)
- **…**



Message passing in optical links



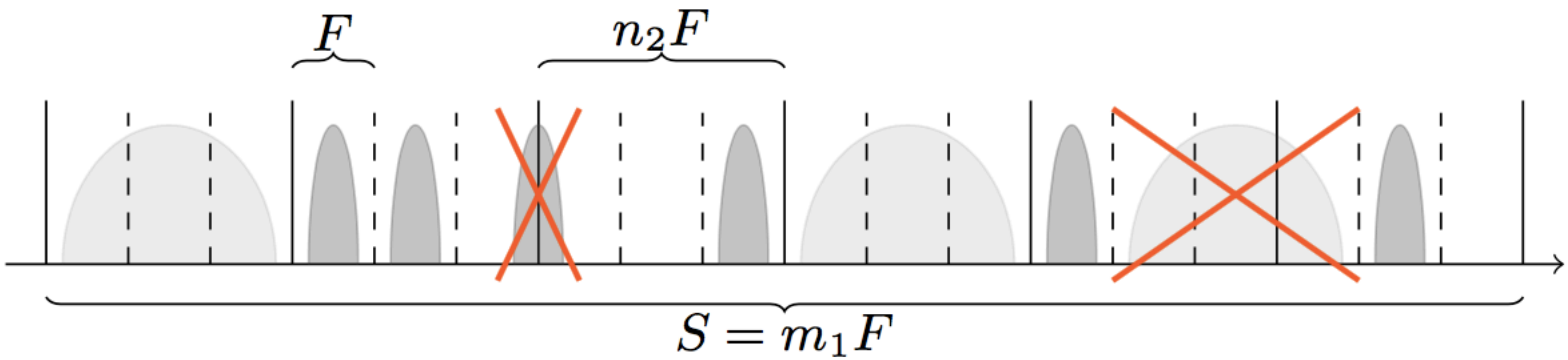
m_1 **channels**

type I messages require 1 channel

type II messages require n_2 channels

We want to minimise the blocking probability of messages by finding an optimal policy

Message passing in optical links



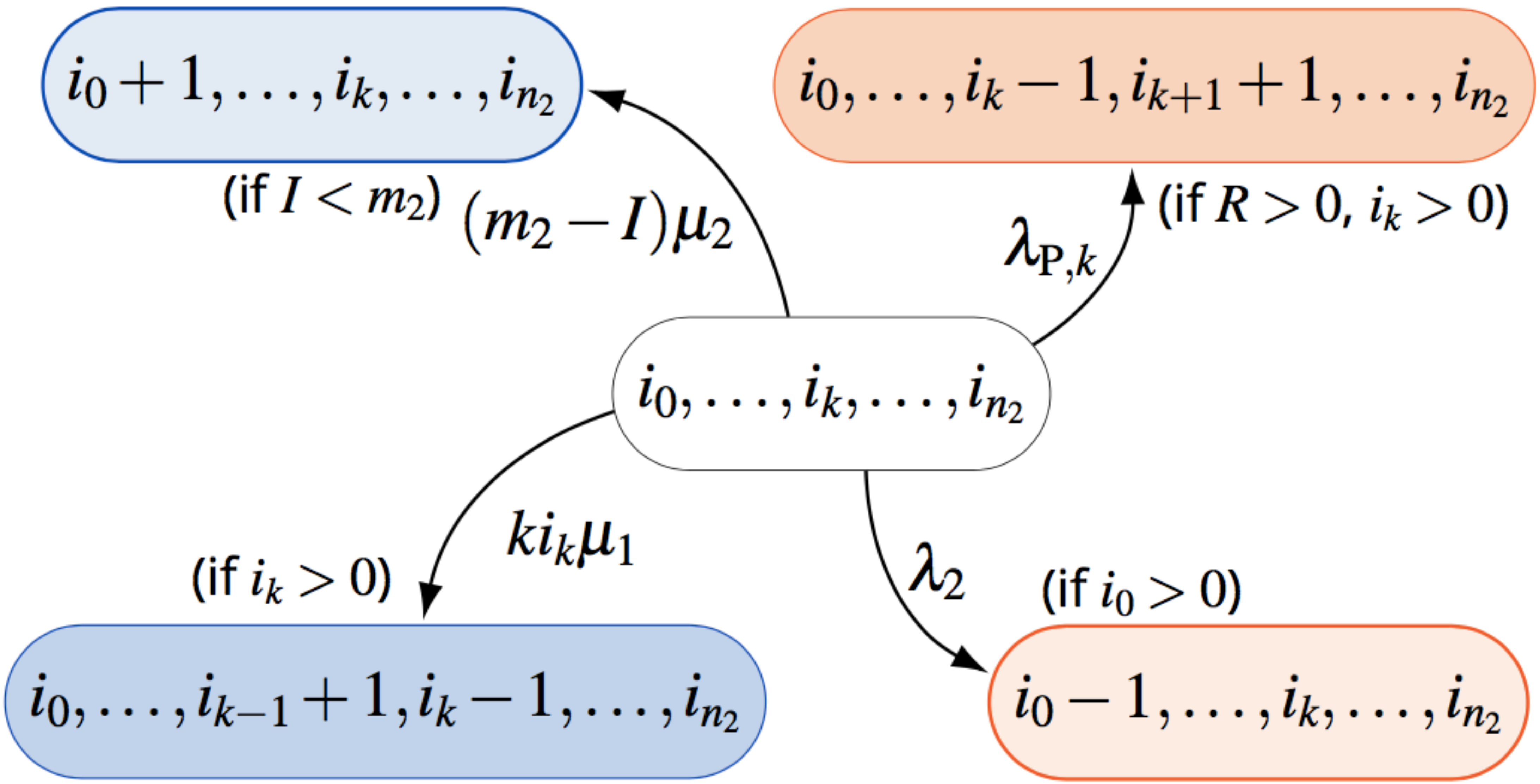
m_1 **channels** $m_2 = \dfrac{m_1}{n_2}$ **superchannels**

type I messages require 1 channel

type II messages require n_2 channels

We want to minimise the blocking probability of messages by finding an optimal policy

$$\mathscr{X}_{\det} := \left\{ (i_0, \ldots, i_{n_2}) \in \mathbb{N}^{(n_2+1)} : \sum_{k=0}^{n_2} i_k \leq m_2 \right\}$$



$I := \sum_{k=0}^{n_2} i_k \qquad R := \sum_{k=0}^{n_2-1} i_k (n_2 - k)$

So how about imprecision?

Imprecise continuous-time Markov chains

Imprecise continuous-time Markov chains

$\boxed{\mathrm{P}(X_0 = x)}$

What if we don't know these (exactly)

?

$\boxed{Q(x, y)}$

Imprecise continuous-time Markov chains

$\boxed{\mathrm{P}(X_0 = x)}$

$\cap$

$\mathcal{P}$

What if we don't
know these
(exactly)

?

$\mathcal{Q}$

$\cup$

$\boxed{Q(x, y)}$

What is $P(X_t = y | X_0 = x)$ **?** $\rightarrow$ $\boxed{e^{Qt}(x,y)}$

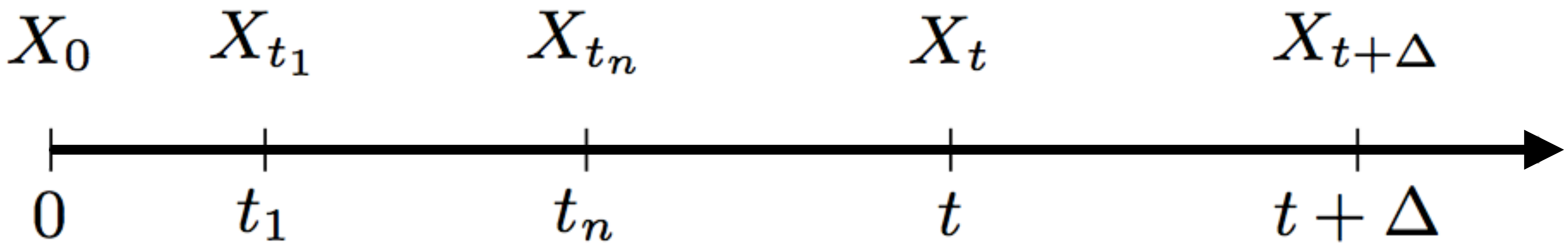
What is $E(f(X_t) | X_0 = x)$ **?** $\rightarrow$ $\boxed{e^{Qt} f(x)}$

What is $P(X_t = y)$ **?** $\rightarrow$ $\boxed{\pi_0 e^{Qt}(y)}$

What is $E(f(X_t))$ **?** $\rightarrow$ $\boxed{\pi_0 e^{Qt} f}$

Optimising with respect to $\pi_0 \in \mathcal{P}$ **and** $Q \in \mathcal{Q}$ **yields lower and upper bounds**

Imprecise continuous-time Markov chains



Let's assume that this does not depend on time!

$\boxed{Q_t(x,y)}$

Imprecise continuous-time Markov chains

~~Let's assume that this does not depend on time!~~

$\boxed{Q_t(x,y)}$

Imprecise continuous-time Markov chains

In that case, all we know is that

$$\begin{aligned}
\mathrm{P}(X_{t+\Delta} = y \mid X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\
&= \mathrm{P}(X_{t+\Delta} = y \mid X_t = x) \\
&\approx I(x,y) + \Delta\, \underset{\mathcal{Q}}{\underset{\cap}{Q_t}}(x,y)
\end{aligned}$$

What is $P(X_t = y | X_0 = x)$ **?** $\rightarrow$ $\cancel{\boxed{e^{Qt}(x,y)}}$

What is $E(f(X_t) | X_0 = x)$ **?** $\rightarrow$ $\cancel{\boxed{e^{Qt}f(x)}}$

What is $P(X_t = y)$ **?** $\rightarrow$ $\cancel{\boxed{\pi_0 e^{Qt}(y)}}$

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Optimising with respect to $\pi_0 \in \mathcal{P}$ **and** $Q_t \in \mathcal{Q}$ **yields lower and upper bounds**

(in many cases)

! this turns out to be surprisingly simple !

Optimising with respect to $\pi_0 \in \mathcal{P}$ and $Q_t \in \mathcal{Q}$ yields lower and upper bounds

What is $E(f(X_t)|X_0 = x)$ **?** $\Rightarrow$ $\boxed{\cancel{e^{Qt}f(x)}}$

Lower transition operator

$$\underline{T}_t f(x) = \underline{E}(f(X_t)|X_0 = x) = \min_{Q \in \mathcal{Q}} E(f(X_t)|X_0 = x)$$

backward Kolmogorov differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\underline{T}_t = \underline{Q}\,\underline{T}_t, \text{ with } \underline{T}_0 = I$$

$$\Rightarrow \underline{T}_t = e^{\underline{Q}t} = \lim_{n \to +\infty}(I + \frac{t}{n}\underline{Q})^n$$

Lower transition rate operator

$$\underline{Q}f(x) = \min_{Q \in \mathcal{Q}} Qf(x)$$

What is $E(f(X_t)|X_0 = x)$ **?** $\Rightarrow$ $\boxed{\geq e^{\underline{Q}t} f(x)}$

Lower transition operator

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Lower transition rate operator

$$\underline{Q}f(x) = \min_{Q \in \mathcal{Q}} Qf(x)$$

What is $E(f(X_t)|X_0 = x)$ **?**

$$\geq e^{\underline{Q}t} f(x)$$

$$\leq -(e^{\underline{Q}t}(-f))(x)$$

What is $P(X_t = y|X_0 = x)$ **?**

$$\geq e^{\underline{Q}t} I_y(x)$$

$$\leq -(e^{\underline{Q}t}(-I_y))(x)$$

What is $E(f(X_t))$ **?**

$\vdots$ $\vdots$

What is $E(f(X_t)|X_0 = x)$ **?** $\rightarrow$ $\boxed{\geq e^{\underline{Q}t} f(x)}$

The following limit always exists!

$$\lim_{t\to+\infty} \underline{E}(f(X_t)|X_0 = x) = \lim_{t\to+\infty} e^{\underline{Q}t} f(x)$$

And often does not depend on x **!**

$$\boxed{\underline{E}_\infty f} = \lim_{t\to+\infty} \underline{E}(f(X_t))$$

with $\underline{E}(f(X_t)) = \min_{\pi_0 \in \mathcal{P}} \min_{Q \in \mathcal{Q}} E(f(X_t) = \min_{\pi_0 \in \mathcal{P}} \pi_0 e^{\underline{Q}t} f$

Imprecise continuous-time Markov chains

$$\mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$
$$= \mathrm{P}(X_{t+\Delta} = y | X_t = x)$$
$$\approx I(x,y) + \Delta\ Q_t(x,y)$$

Markov assumption

Imprecise continuous-time Markov chains

$$\mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$
$$\approx I(x, y) + \Delta\, Q_{t,x_1,\ldots,x_n}(x, y)$$

~~Markov assumption~~

Imprecise continuous-time Markov chains

In that case, all we know is that

$$\mathrm{P}(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x)$$

$$\approx I(x,y) + \Delta\, \underset{\substack{\pitchfork \\ \mathcal{Q}}}{Q_{t,x_1,\ldots,x_n}}(x,y)$$

What is $P(X_t = y | X_0 = x)$ **?** $\rightarrow$ $\cancel{\boxed{e^{Qt}(x,y)}}$

What is $E(f(X_t)|X_0 = x)$ **?** $\rightarrow$ $\cancel{\boxed{e^{Qt}f(x)}}$

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Optimising with respect to $\pi_0 \in \mathcal{P}$ **and** $Q_{t,x_1,\ldots,x_n} \in \mathcal{Q}$ **yields lower and upper bounds**

(in many cases)

! this turns out to (still) be surprisingly simple !

Optimising with respect to $\pi_0 \in \mathcal{P}$ and $Q_{t,x_1,\ldots,x_n} \in \mathcal{Q}$ yields lower and upper bounds

What is $E(f(X_t)|X_0 = x)$ **?**

$$\geq e^{\underline{Q}t} f(x)$$

$$\leq -(e^{\underline{Q}t}(-f))(x)$$

What is $P(X_t = y|X_0 = x)$ **?**

$$\geq e^{\underline{Q}t} I_y(x)$$

$$\leq -(e^{\underline{Q}t}(-I_y))(x)$$

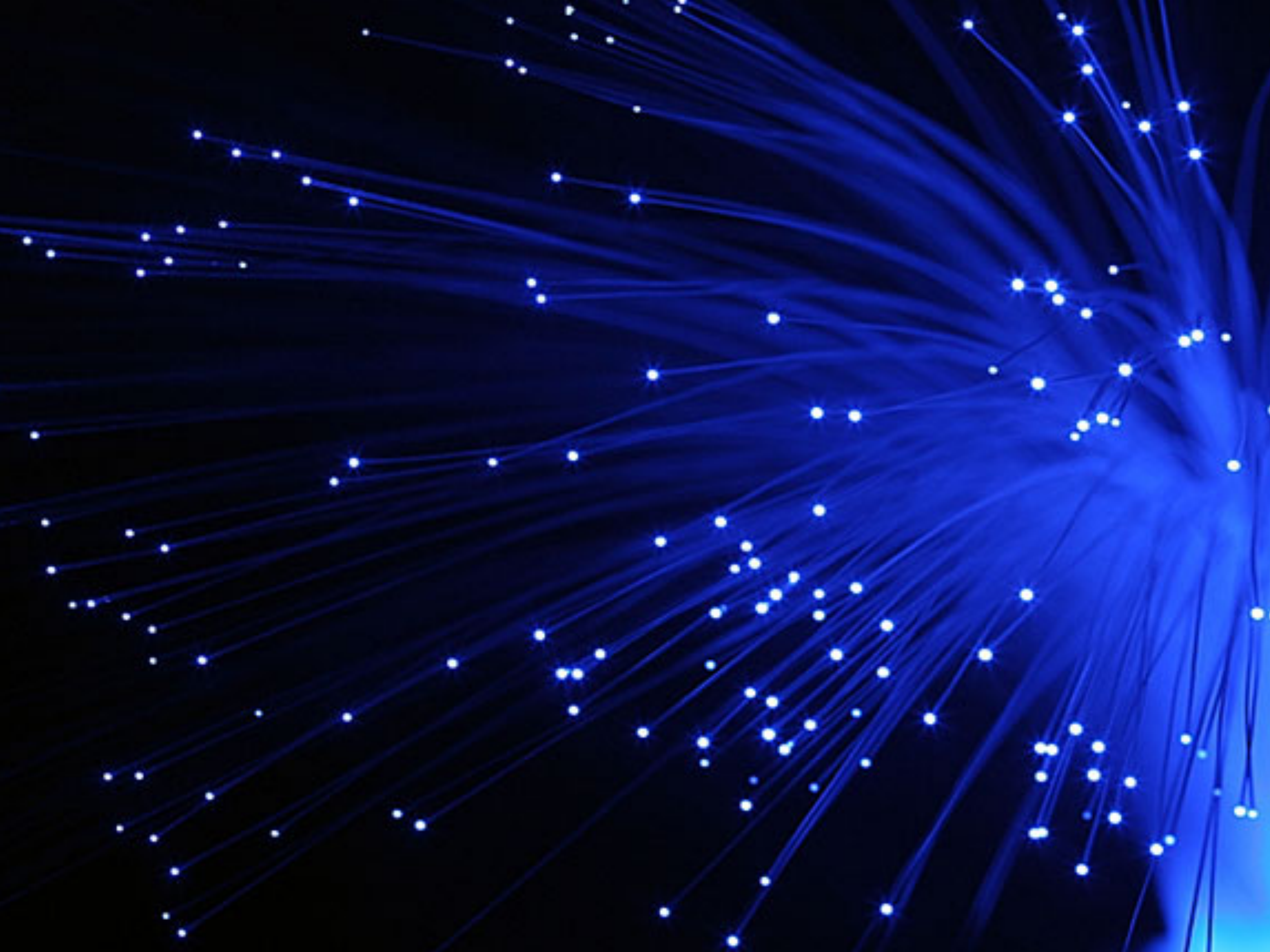
What is $E(f(X_t))$ **?**

$\vdots$ $\vdots$

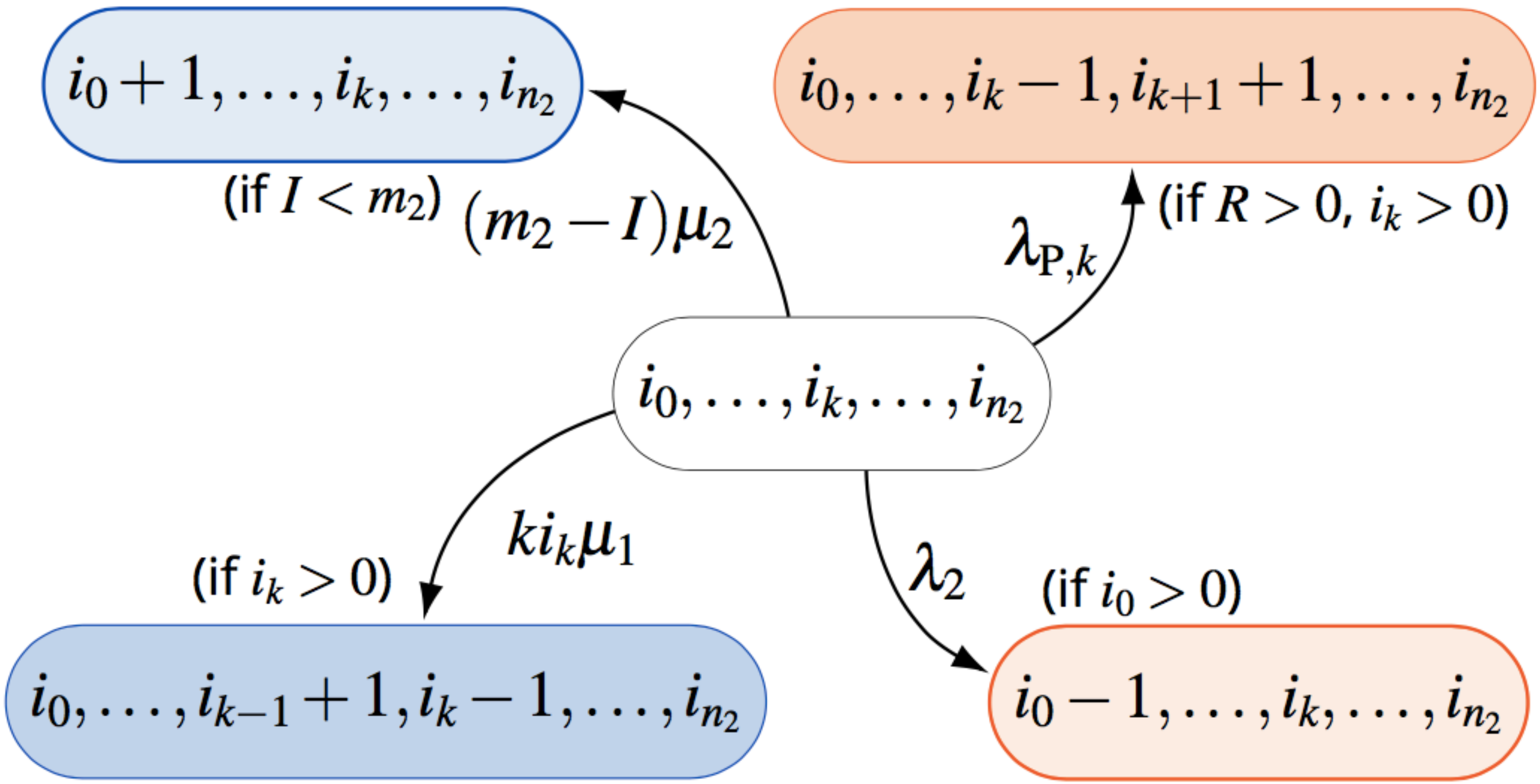
That’s enough! Too confusing! And time is running out…

Advantages of imprecise (continuous-time) Markov chains over their precise counterpart

- ✓ Partially specified π_0 and Q are allowed
- ✓ Time homogeneity can be dropped
- ✓ The Markov assumption can be dropped
- ✓ Efficient computations remain possible
- ✓ …



$$\mathscr{X}_{\mathrm{det}} := \left\{ (i_0, \ldots, i_{n_2}) \in \mathbb{N}^{(n_2+1)} : \sum_{k=0}^{n_2} i_k \leq m_2 \right\}$$



$$I := \sum_{k=0}^{n_2} i_k \qquad R := \sum_{k=0}^{n_2-1} i_k (n_2 - k)$$

Amorous GOOD

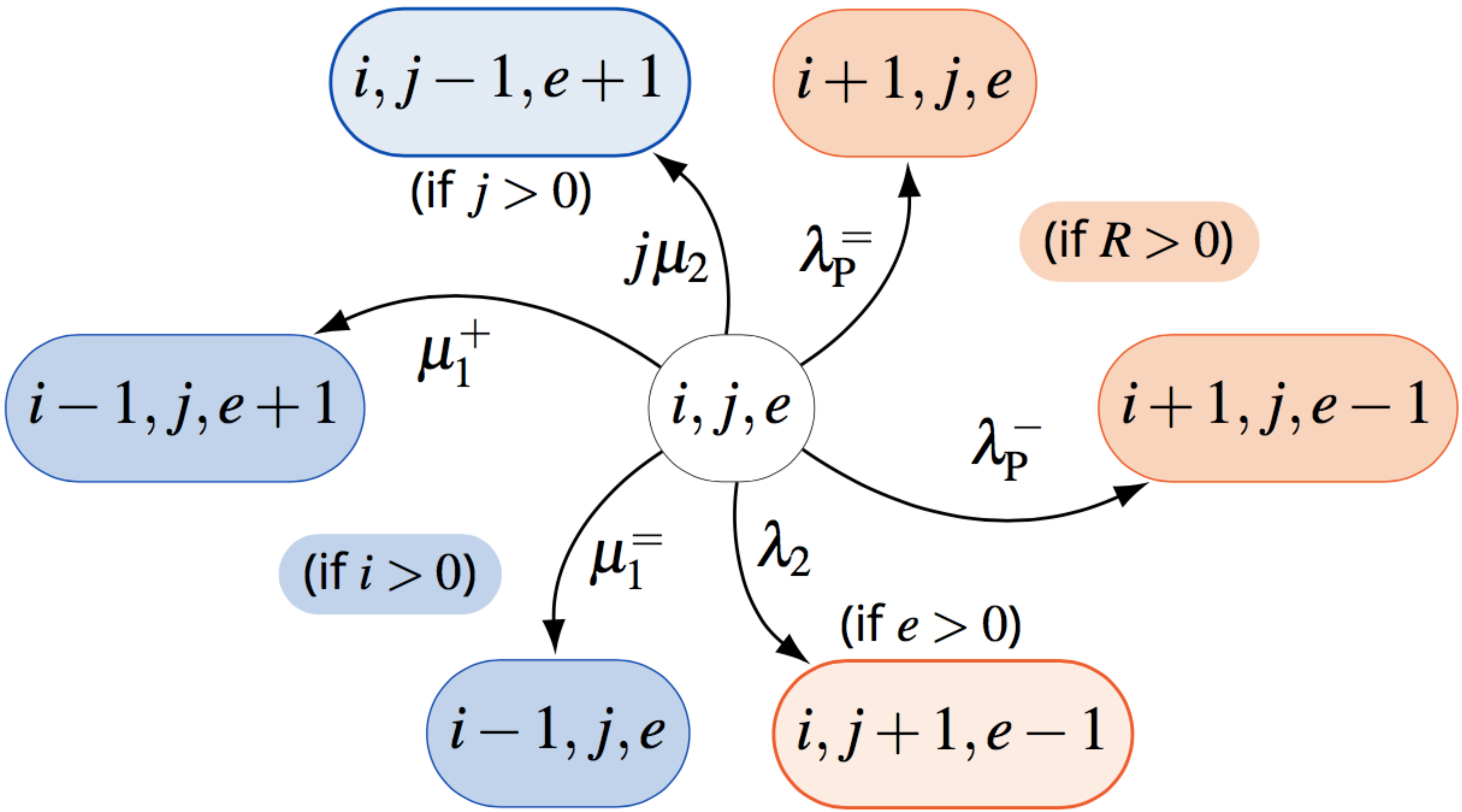
Bickering

Confusion NOT GOOD

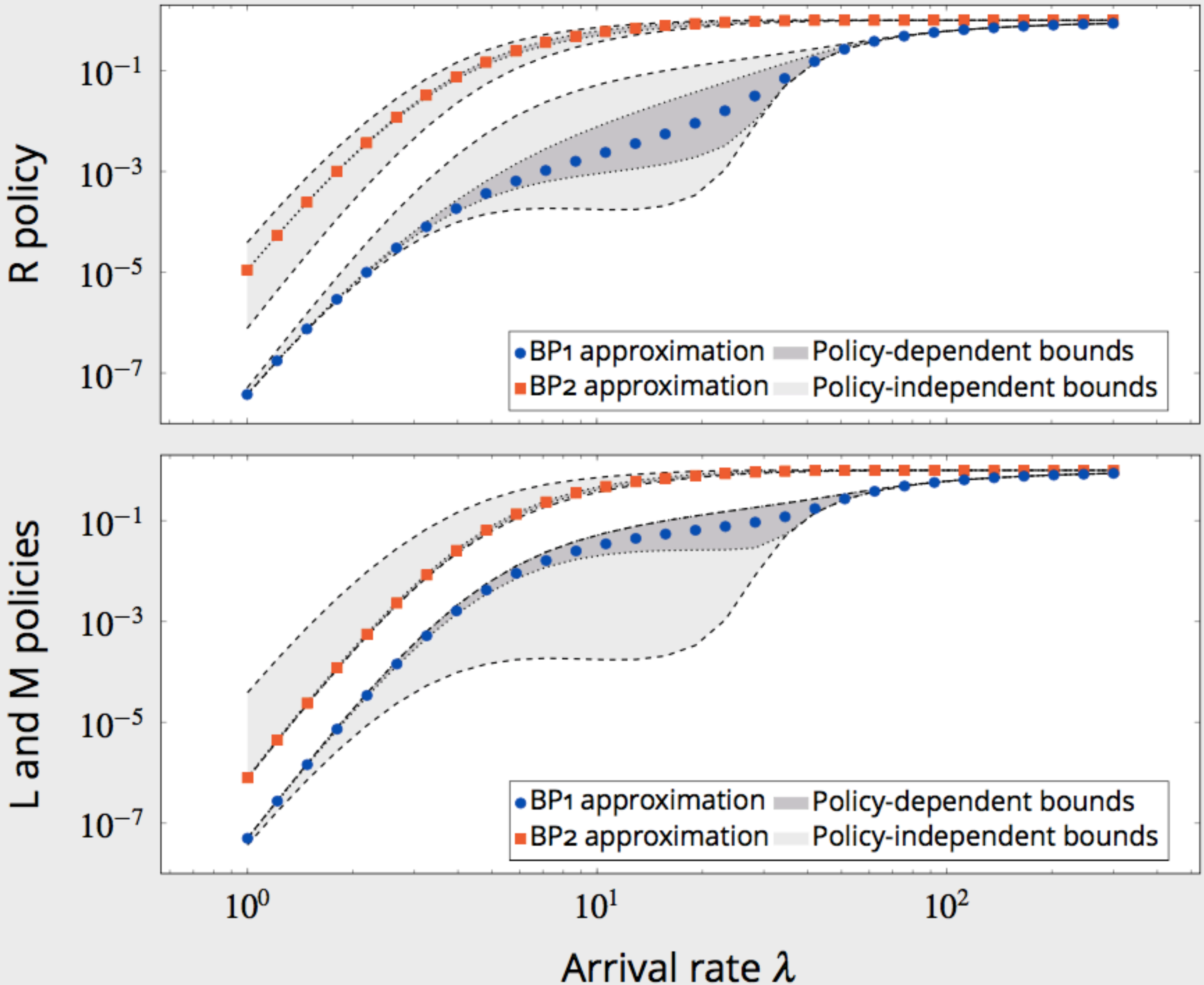
Depression

$$
Q = \begin{array}{c c}
 & \begin{array}{cccc} \mathbf{A} & \mathbf{B} & \mathbf{C} & \mathbf{D} \end{array} \\
\begin{array}{c} \mathbf{A} \\ \mathbf{B} \\ \mathbf{C} \\ \mathbf{D} \end{array} & \begin{bmatrix} -4 & 3 & 1 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}
\end{array}
$$

$$\mathscr{X}_{\text{red}} := \{(i,j,e) \in \mathbb{N}^3 : m_2 \leq i+j+e, i+(j+e)n_2 \leq m_1\}$$



$$R := m_1 - i - jn_2$$


R policy
L and M policies
10^{-1}
10^{-3}
10^{-5}
10^{-7}
BP1 approximation
BP2 approximation
Policy-dependent bounds
Policy-independent bounds
10^{0}
10^{1}
10^{2}
Arrival rate λ

References

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