Imprecise Markov chains
From basic theory to applications II
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Imprecise continuous-time Markov chains
We will next construct the convex combination that satisfies Equation (112). So, consider any $t > 0$, $v \in \mathcal{U}_{<t}$ and $x_v \in \mathcal{X}_v$. We again distinguish between two cases: $t \leq \max u$ and $t > \max u$. If $t \leq \max u$, then for all $\Delta \in (0, t - \max v)$ and $x, y \in \mathcal{X}$, we see that $(X_t = y, (X_{t-\Delta} = x, X_v = x_v)) \in \mathcal{C}_0$, and therefore, since $P$ is an extension of $\tilde{P}$, it follows from Equation (96) that

$$P(X_t = y|X_{t-\Delta} = x, X_v = x_v) = P_0(X_t = y|X_{t-\Delta} = x, X_v = x_v).$$

Hence, if we let $\mathcal{I} := \{i\}$, $v^* := v$, $\lambda_i := 1$, $iP := P_0$ and $i_{x_v^*} := x_v$, Equation (112) is satisfied by choosing $\delta := t - \max v$. If $t > \max u$, then for all $\Delta \in (0, t - \max(v \cup u))$, it follows from Equation (110) (with $s := t$ and $w := v \cup t - \Delta$) that

$$P(X_t = y|X_{t-\Delta} = x, X_v = x_v) = \sum_{x_u \in \mathcal{X}_u \setminus v} P_{x_u}(X_t = y|X_{t-\Delta} = x, X_u \cup (v \setminus [0, \max u]) = x_u \cup (v \setminus [0, \max u]))$$

$$P^*(X_u \setminus v = x_u \setminus v|X_{t-\Delta} = x, X_v = x_v).$$

Therefore, if we let $\mathcal{I} := \mathcal{X}_u \setminus v$, $v^* := u \cup (v \setminus [0, \max u])$ and, for all $x_u \setminus v \in \mathcal{I}$,

$$\lambda_{x_u \setminus v} := P^*(X_u \setminus v = x_u \setminus v|X_{t-\Delta} = x, X_v = x_v),$$

$x_u \setminus v P = P_{x_u}$ and $x_u \setminus v x_v^* := x_u \cup (v \setminus [0, \max u])$, Equation (112) is satisfied by choosing $\delta := t - \max(v \cup u)$. Hence, Equation (112) can be satisfied both when $t \leq \max u$ and when $t > \max u$. 
Imprecise continuous-time Markov chains
Continuous-time Markov chains

\[ X_0 \quad X_{t_1} \quad X_{t_n} \quad X_t \quad X_{t+\Delta} \]

\[ 0 \quad t_1 \quad t_n \quad t \quad t + \Delta \]

\[ P(X_0 = x) \]

\[ P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \]
Continuous-time Markov chains

\[ P(X_0 = x) \]

\[ P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) = P(X_{t+\Delta} = y | X_t = x) \]

Markov assumption
Continuous-time Markov chains...

\[ P(X_0 = x) \]
\[ P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \]
\[ = P(X_{t+\Delta} = y | X_t = x) \]
\[ \approx I(x, y) + \Delta Q_t(x, y) \]
Continuous-time Markov chains

\[ P(X_0 = x) \]

\[ P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) = P(X_{t+\Delta} = y | X_t = x) \]

\[ \approx I(x, y) + \Delta Q_t(x, y) \]
Continuous-time Markov chains...

Let's assume that this does not depend on time!
Continuous-time Markov chains...

...that are homogeneous

$P(X_0 = x)$

$Q(x, y)$
Continuous-time Markov chains

\[ X_0, X_{t_1}, X_{t_n}, X_t, X_{t+\Delta} \]

\[ P(X_0 = x) \]

\[ Q(x, y) \]
Continuous-time Markov chains

\[ X_0 \quad X_{t_1} \quad X_{t_n} \quad X_t \quad X_{t+\Delta} \]

0 \quad t_1 \quad t_n \quad t \quad t + \Delta

\[ \mathbb{P}(X_0 = x) \]

that's just a probability mass function \( \pi_0(x) \)

initial distribution

transition rate matrix

\[
\sum_y Q(x, y) = 0
\]

\[
(\forall y \neq x) Q(x, y) \geq 0
\]

\[
(\forall x) Q(x, x) \leq 0
\]
<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-4</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>4</td>
<td>-6</td>
<td>2</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3</td>
<td>-6</td>
</tr>
<tr>
<td>D</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

$Q = \begin{pmatrix} -4 & 3 & 1 & 0 \\ 4 & -6 & 2 & 0 \\ 2 & 3 & -6 & 1 \\ 0 & 0 & 2 & -2 \end{pmatrix}$

- Amorous
- Bickering
- Confusion
- Depression
What is $P(X_t = y | X_0 = x)$?
What is \( P(X_t = y | X_0 = x) \)?

**transition matrix**

\[
T_t(x, y) := P(X_t = y | X_0 = x)
\]

**backward Kolmogorov differential equation**

\[
\frac{d}{dt} T_t = QT_t, \text{ with } T_0 = I
\]

\[
\Rightarrow T_t = e^{Qt} = \lim_{n \to +\infty} (I + \frac{t}{n}Q)^n
\]
What is $P(X_t = y | X_0 = x)$? \[ e^{Q_t}(x, y) \]

**transition matrix**

$T_t(x, y) := P(X_t = y | X_0 = x)$

**backward Kolmogorov differential equation**

\[
\frac{d}{dt} T_t = QT_t, \text{ with } T_0 = I
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\[ \Rightarrow T_t = e^{Q_t} = \lim_{n \to +\infty} \left( I + \frac{t}{n} Q \right)^n \]
What is \( P(X_t = y | X_0 = x) \)?

\[ e^{Qt}(x, y) \]

What is \( E(f(X_t) | X_0 = x) \)?

\[ e^{Qt} f(x) \]

What is \( P(X_t = y) \)?

\[ \pi_0 e^{Qt}(y) \]

What is \( E(f(X_t)) \)?

\[ \pi_0 e^{Qt} f \]

\[ \vdots \]
What is $P(X_t = y|X_0 = x)$?

The following limit always exists!

$$\lim_{t \to +\infty} P(X_t = y|X_0 = x) = \lim_{t \to +\infty} e^{Qt(x, y)}$$

And often does not depend on $x$!

$$\pi_\infty(y) = \lim_{t \to +\infty} P(X_t = y) = \lim_{t \to +\infty} \pi_0 e^{Qt(y)}$$
That’s all fine and well, but what can you use it for?
Reliability engineering (failure probabilities, …)

Queuing theory (waiting in line …)
- optimising supermarket waiting times
- dimensioning of call centers
- airport security lines
- router queues on the internet

Cell division in biology (how long does it take?)
Message passing in optical links

\[ F \quad n_2 F \quad S = m_1 F \]

\( m_1 \) channels

**type I** messages require 1 channel

**type II** messages require \( n_2 \) channels

We want to **minimise** the blocking probability of messages by finding an **optimal** policy
Message passing in optical links

\[ F \quad \frac{n_2 F}{S = m_1 F} \]

\[ m_1 \text{ channels} \quad m_2 = \frac{m_1}{n_2} \text{ superchannels} \]

type I messages require 1 channel

type II messages require \( n_2 \) channels

We want to minimise the blocking probability of messages by finding an optimal policy
$$\mathcal{X}_{\text{det}} := \left\{ (i_0, \ldots, i_{n_2}) \in \mathbb{N}^{(n_2+1)} : \sum_{k=0}^{n_2} i_k \leq m_2 \right\}$$

\[ I := \sum_{k=0}^{n_2} i_k \quad R := \sum_{k=0}^{n_2-1} i_k (n_2 - k) \]
So how about imprecision?
Imprecise continuous-time Markov chains
Imprecise continuous-time Markov chains

What if we don’t know these (exactly)

\[ P(X_0 = x) \]

\[ Q(x, y) \]
Imprecise continuous-time Markov chains

What if we don’t know these (exactly)
What is $P(X_t = y|X_0 = x)$?

What is $E(f(X_t)|X_0 = x)$?

What is $P(X_t = y)$?

What is $E(f(X_t))$?

Optimising with respect to $\pi_0 \in \mathcal{P}$ and $Q \in \mathcal{Q}$ yields lower and upper bounds.

$e^{Q_t}(x, y)$

$e^{Q_t}f(x)$

$\pi_0 e^{Q_t}(y)$

$\pi_0 e^{Q_t}f$
Imprecise continuous-time Markov chains

Let's assume that this does not depend on time!

\[ Q_t(x, y) \]
Imprecise continuous-time Markov chains

\[ Q_t(x, y) \]

Let’s assume that this does not depend on time!
Imprecise continuous-time Markov chains

\[ X_0, X_{t_1}, X_{t_n}, X_t, X_{t+\Delta} \]

0 \quad t_1 \quad t_n \quad t \quad t + \Delta

In that case, all we know is that

\[
P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\
= P(X_{t+\Delta} = y | X_t = x) \\
\approx I(x, y) + \Delta \sum_{Q_t(x, y) \cap Q} Q
\]
What is $P(X_t = y | X_0 = x)$ ?

What is $E(f(X_t) | X_0 = x)$ ?

What is $P(X_t = y)$ ?

What is $E(f(X_t))$ ?

Optimising with respect to $\pi_0 \in \mathcal{P}$ and $Q_t \in \mathcal{Q}$ yields lower and upper bounds.
Optimising with respect to $\pi_0 \in \mathcal{P}$ and $Q_t \in \mathcal{Q}$ yields lower and upper bounds.

(in many cases)

this turns out to be surprisingly simple
What is $E(f(X_t)|X_0 = x)$?

Lower transition rate operator

$T_t f(x) = E(f(X_t)|X_0 = x) = \min_{Q \in \mathcal{Q}} E(f(X_t)|X_0 = x)$

backward Kolmogorov differential equation

$\frac{d}{dt} T_t = QT_t$, with $T_0 = I$

$\Rightarrow T_t = e^{Qt} = \lim_{n \to +\infty} (I + \frac{t}{n}Q)^n$

Lower transition operator

$Qf(x) = \min_{Q \in \mathcal{Q}} Qf(x)$
What is \( E(f(X_t)|X_0 = x) \)?

\[ \geq e^{Qt} f(x) \]

**Lower transition operator**

\[
T_t f(x) = E(f(X_t)|X_0 = x) = \min_{Q \in \mathcal{Q}} E(f(X_t)|X_0 = x)
\]

**backward Kolmogorov differential equation**

\[
\frac{d}{dt} T_t = QT_t, \text{ with } T_0 = I
\]

\[
\Rightarrow T_t = e^{Qt} = \lim_{n \to +\infty} (I + \frac{t}{n}Q)^n
\]

**Lower transition rate operator**

\[
Qf(x) = \min_{Q \in \mathcal{Q}} Qf(x)
\]
What is $E(f(X_t)|X_0 = x)$?

\[ \geq e^{Q_t} f(x) \]
\[ \leq -(e^{Q_t}(-f))(x) \]

What is $P(X_t = y|X_0 = x)$?

\[ \geq e^{Q_t}I_y(x) \]
\[ \leq -(e^{Q_t}(-I_y))(x) \]

What is $E(f(X_t))$?

::
What is $E(f(X_t) | X_0 = x)$?

The following limit always exists!

$$
\lim_{t \to +\infty} E(f(X_t) | X_0 = x) = \lim_{t \to +\infty} e^{Qt} f(x)
$$

And often does not depend on $x$!

$$
E_{\infty} f = \lim_{t \to +\infty} E(f(X_t))
$$

with $E(f(X_t)) = \min_{\pi_0 \in \mathcal{P}} \min_{Q \in \mathcal{Q}} E(f(X_t)) = \min_{\pi_0 \in \mathcal{P}} \pi_0 e^{Qt} f$
Imprecise continuous-time Markov chains

\[ \begin{align*}
X_0 & \quad X_{t_1} & \quad X_{t_n} & \quad X_t & \quad X_{t+\Delta} \\
0 & \quad t_1 & \quad t_n & \quad t & \quad t + \Delta
\end{align*} \]

\[
P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) = P(X_{t+\Delta} = y | X_t = x) 
\approx I(x, y) + \Delta Q_t(x, y)
\]

Markov assumption
Imprecise continuous-time Markov chains

\[ P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \]

\[ \approx I(x, y) + \Delta Q_{t, x_1, \ldots, x_n}(x, y) \]

Markov assumption
Imprecise continuous-time Markov chains

In that case, all we know is that

\[
P(X_{t+\Delta} = y | X_{t_1} = x_1, \ldots, X_{t_n} = x_n, X_t = x) \\ \approx I(x, y) + \Delta Q_{t, x_1, \ldots, x_n} (x, y) \\
\]
What is \( P(X_t = y | X_0 = x) \)?

\[ e^{Q_t}(x, y) \]

What is \( E(f(X_t) | X_0 = x) \)?

\[ e^{Q_t}f(x) \]

What is \( P(X_t = y) \)?

\[ \pi_0 e^{Q^*_t}(y) \]

What is \( E(f(X_t)) \)?

\[ \pi_0 e^{Q^*_t}f \]

Optimising with respect to \( \pi_0 \in \mathcal{P} \) and \( Q_{t,x_1,...,x_n} \in \mathcal{Q} \) yields lower and upper bounds.
Optimising with respect to \( \pi_0 \in \mathcal{P} \) and \( Q_{t,x_1,...,x_n} \in \mathcal{Q} \) yields lower and upper bounds.

(in many cases)

this turns out to (still) be surprisingly simple
What is $E(f(X_t)|X_0 = x)$ ?

$$\geq e^{Qt} f(x)$$
$$\leq -(e^{Qt}(-f))(x)$$

What is $P(X_t = y|X_0 = x)$ ?

$$\geq e^{Qt} I_y(x)$$
$$\leq -(e^{Qt}(-I_y))(x)$$

What is $E(f(X_t))$ ?

$$\vdots$$
That’s enough! Too confusing! And time is running out...
Advantages of imprecise (continuous-time) Markov chains over their precise counterpart

- Partially specified $\pi_0$ and $Q$ are allowed
- Time homogeneity can be dropped
- The Markov assumption can be dropped
- Efficient computations remain possible
- ...
\[ \mathcal{X}_{\text{det}} := \left\{ (i_0, \ldots, i_{n_2}) \in \mathbb{N}^{(n_2+1)} : \sum_{k=0}^{n_2} i_k \leq m_2 \right\} \]

\[ I := \sum_{k=0}^{n_2} i_k \quad R := \sum_{k=0}^{n_2-1} i_k (n_2 - k) \]
\[ Q = \begin{bmatrix}
  A & B & C & D \\
  A & -4 & 3 & 1 & 0 \\
  B & 4 & -6 & 2 & 0 \\
  C & 2 & 3 & -6 & 1 \\
  D & 0 & 0 & 2 & -2 \\
\end{bmatrix} \]
\[ \mathcal{K}_{\text{red}} := \{(i, j, e) \in \mathbb{N}^3 : m_2 \leq i + j + e, i + (j + e)n_2 \leq m_1\} \]

\[ R := m_1 - i - jn_2 \]
References


References (2)

