Independent Natural Extension for Infinite Spaces

Williams-coherence to the Rescue!

ECSQARU 2017
ISIPTA '17

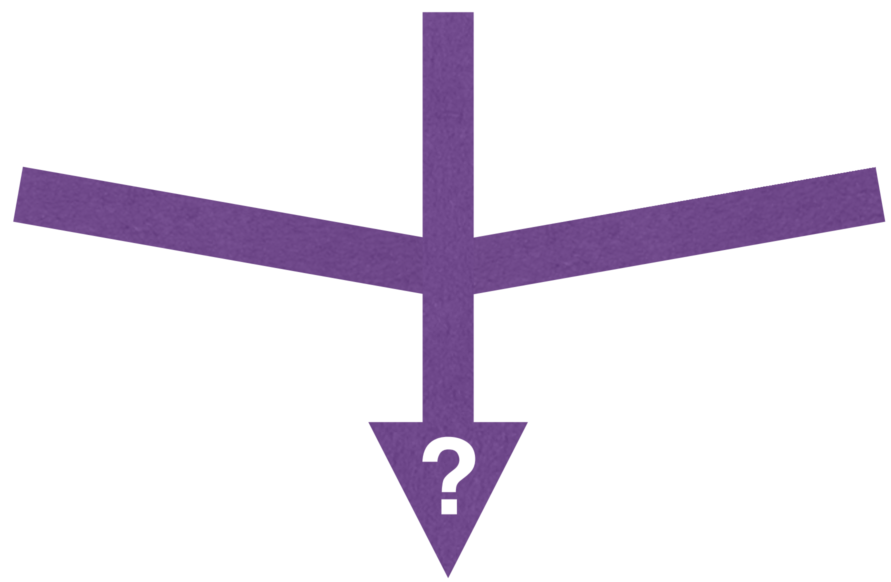
Jasper De Bock

Ghent University
Belgium

X_1 **independent** X_2

local uncertainty model

local uncertainty model



joint uncertainty model

$$P(X_1|X_2) = P(X_1)$$
$$P(X_2|X_1) = P(X_2)$$

$$X_1$$

local uncertainty model

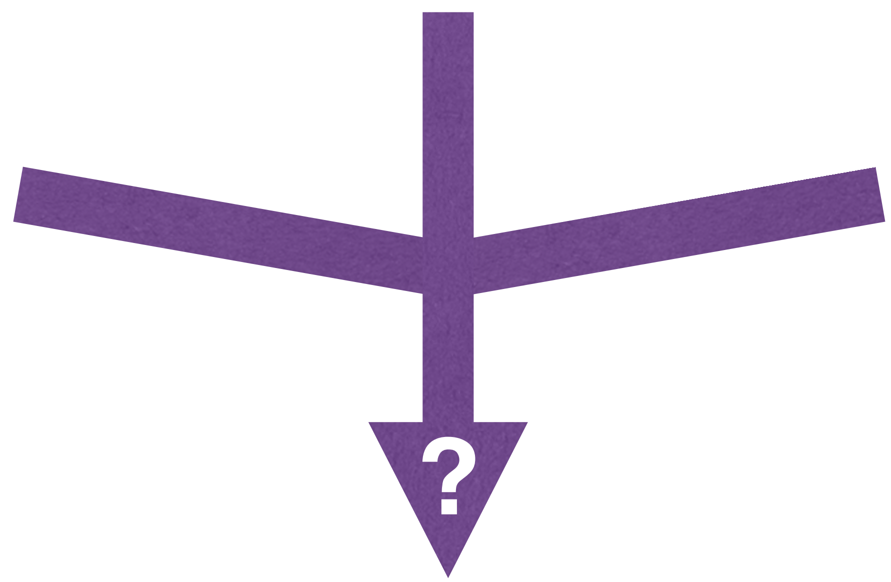
$$P(X_1)$$

independent

$$X_2$$

local uncertainty model

$$P(X_2)$$



joint uncertainty model

$$P(X_1, X_2)$$

$$P(X_1|X_2) = P(X_1)$$
$$P(X_2|X_1) = P(X_2)$$

X_1

independent

X_2

local uncertainty model

$P(X_1)$

local uncertainty model

$P(X_2)$

joint uncertainty model

$$P(X_1, X_2) = P(X_1)P(X_2)$$

$$X_1$$

local uncertainty model

$$P(X_1),\ \underline{P}(X_1),\ \mathcal{D}_1,\ P(f(X_1)),\ \underline{P}(f(X_1)),\ \ldots$$

$$E(f(X_1)) \quad \underline{E}(f(X_1))$$

$$X_1$$

local uncertainty model

$$P(X_1),\ \underline{P}(X_1),\ \mathcal{D}_1\ ,\ P(f(X_1))\ ,\ \underline{P}(f(X_1)),\ \ \ldots$$

$$E(f(X_1))\quad \underline{E}(f(X_1))$$

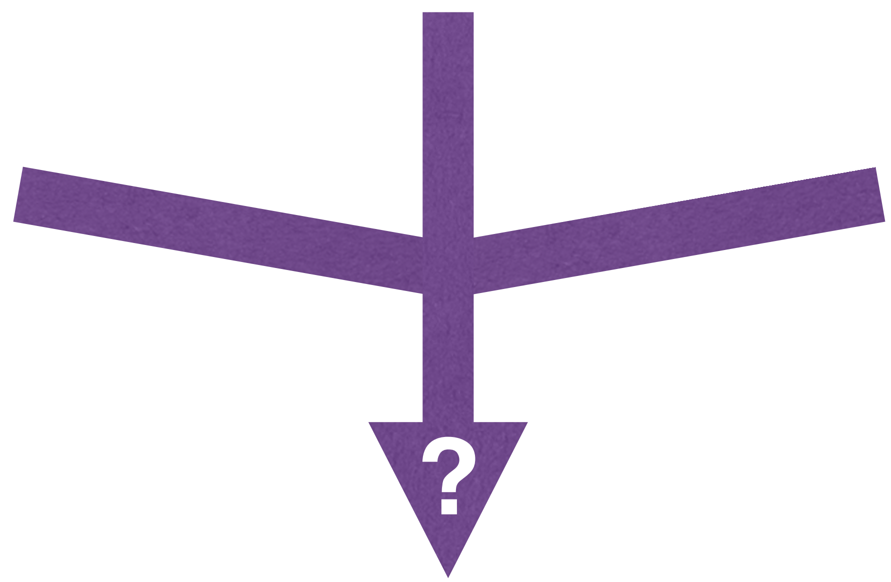
$$X_1 \quad \textbf{independent} \quad X_2$$

local uncertainty model

$\underline{P}(f(X_1))$

local uncertainty model

$\underline{P}(f(X_2))$



joint uncertainty model

$\underline{P}(f(X_1, X_2))$

$$X_1$$

local uncertainty model

$$\underline{P}(f(X_1))$$

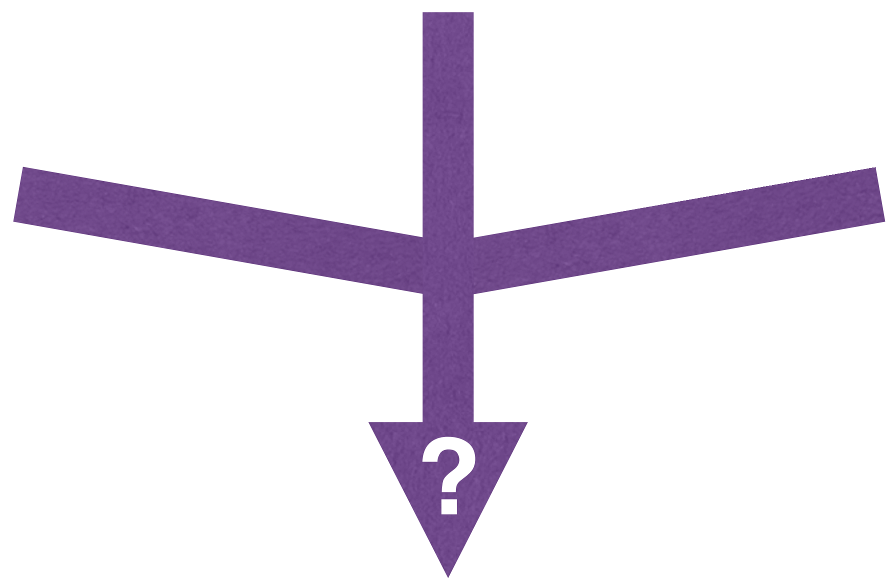
?

independent

$$X_2$$

local uncertainty model

$$\underline{P}(f(X_2))$$



joint uncertainty model

$$\underline{P}(f(X_1, X_2))$$

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

X_1

independent

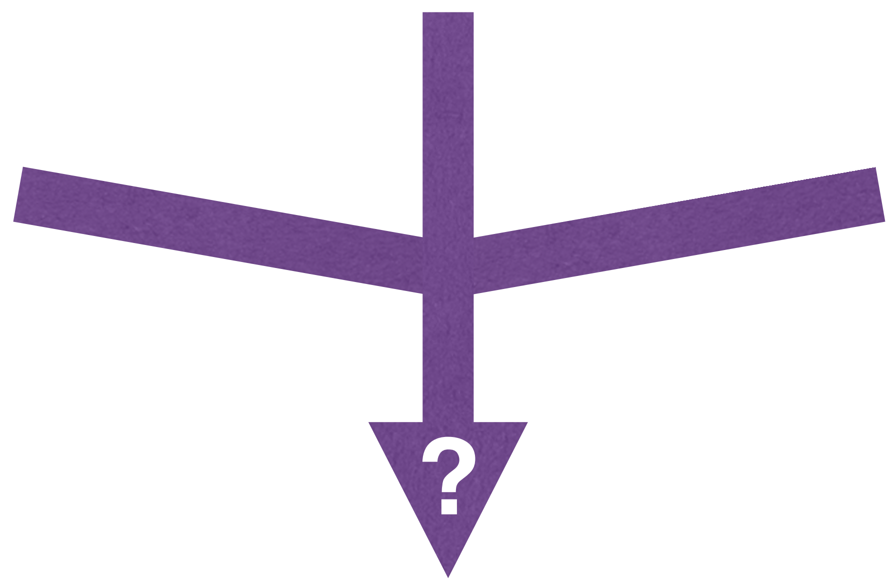
X_2

local uncertainty model

$\underline{P}(f(X_1))$

local uncertainty model

$\underline{P}(f(X_2))$



joint uncertainty model

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$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

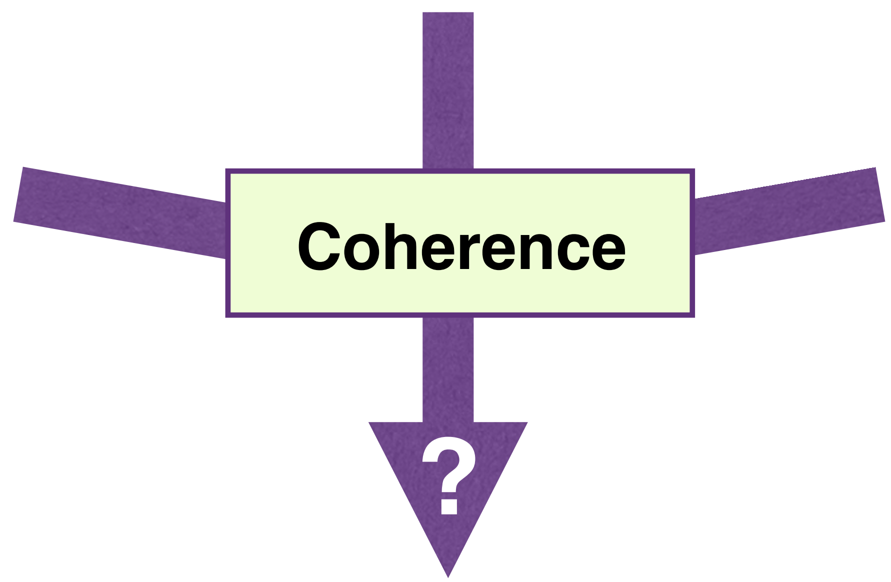
X_1

independent

X_2

local uncertainty model

$\underline{P}(f(X_1))$



local uncertainty model

$\underline{P}(f(X_2))$

joint uncertainty model

$$\underline{P}(f(X_1, X_2))$$

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

X_1

independent

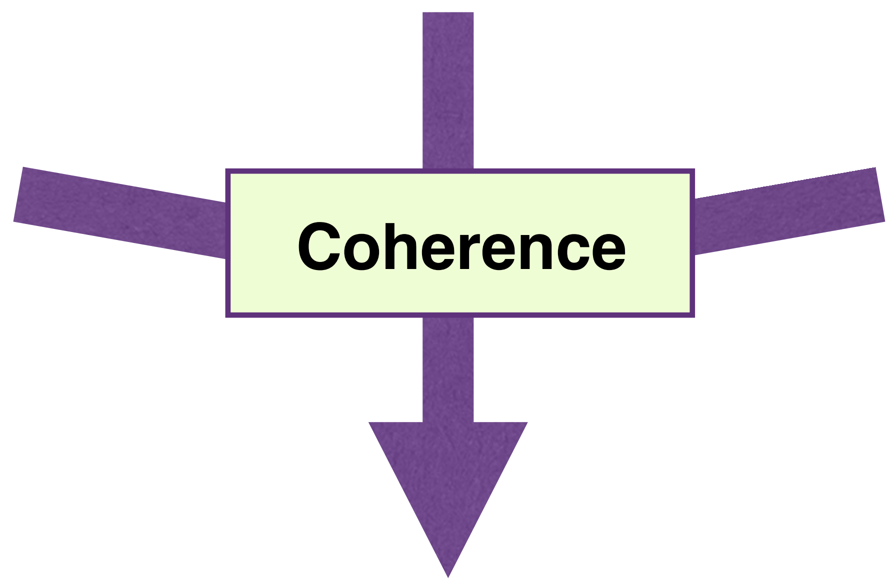
X_2

local uncertainty model

$$\underline{P}(f(X_1)) \\ || \\ \underline{P}_1(f)$$

local uncertainty model

$$\underline{P}(f(X_2)) \\ || \\ \underline{P}_2(f)$$



joint uncertainty model

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$

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Two very useful properties

External additivity

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

Factorisation

$$(\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2))$$
$$= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \overline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases}$$

$$\text{if } g \geq 0$$

DISCLAIMER!

All of this is well known, and has been for several years now…

de Cooman & Miranda 2012

de Cooman, Miranda & Zaffalon 2011

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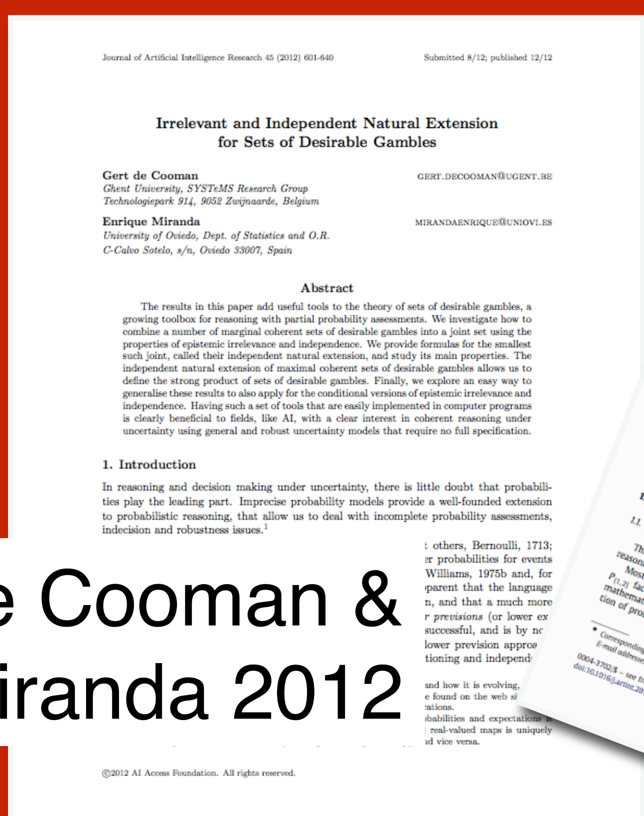


de Cooman & Miranda 2012

de Cooman, Miranda & Zaffalon 2011

…but only for finite spaces!

Independent Natural Extension
for Infinite Spaces

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

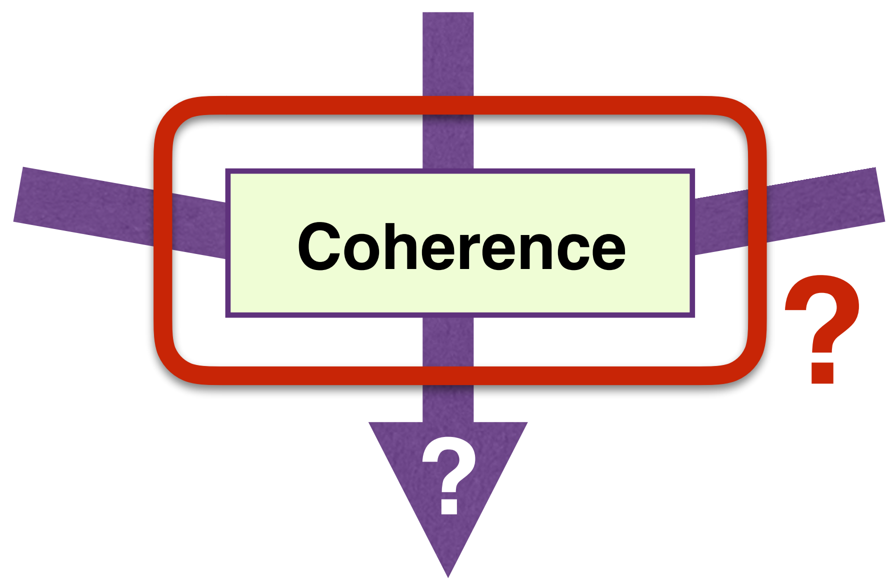
X_1

independent

X_2

local uncertainty model

$\underline{P}(f(X_1))$



local uncertainty model

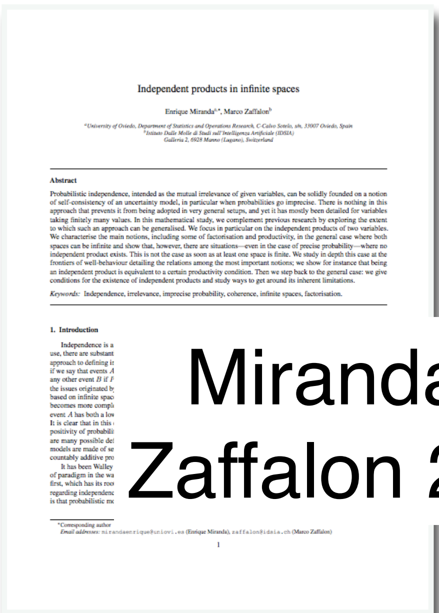
$\underline{P}(f(X_2))$

joint uncertainty model

$\boxed{\text{Coherence}}$?

Walley ⟷ **Williams**

Independent natural extension may not exist!



Independent products in infinite spaces

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Abstract

Probabilistic independence, intended as the mutual irrelevance of given variables, can be solidly founded on a notion of self-consistency of an uncertainty model, in particular when probabilities go imprecise. There is nothing in this approach that prevents it from being adopted in very general setups, and yet it has mostly been detailed for variables taking finitely many values. In this mathematical study, we complement previous research by exploring the extent to which such an approach can be generalised. We focus in particular on the independent products of two variables. We characterise the main notions, including some of factorisation and productivity, in the general case where both spaces can be infinite and show that, however, there are situations—even in the case of precise probability—where no independent product exists. This is not the case as soon as at least one space is finite. We study in depth this case at the frontiers of well-behaviour detailing the relations among the most important notions; we show for instance that being an independent product is equivalent to a certain productivity condition. Then we step back to the general case: we give conditions for the existence of independent products and study ways to get around its inherent limitations.

Keywords: Independence, irrelevance, imprecise probability, coherence, infinite spaces, factorisation.

1. Introduction

Miranda & Zaffalon 2015

Coherence

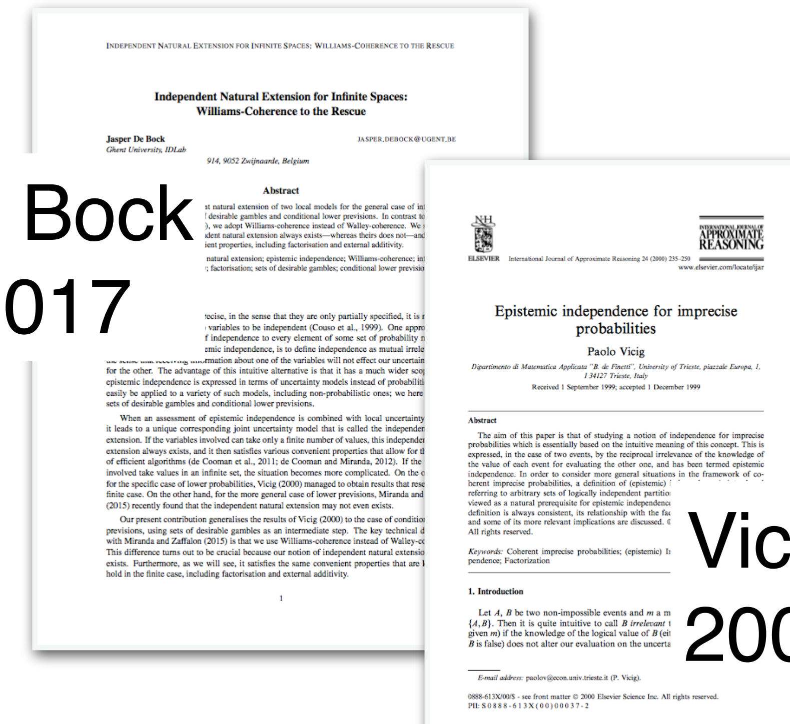
?

~~Walley~~ ⟷ Williams

Independent natural extension may not exist!

Miranda & Zaffalon 2015

Independent natural extension always exists!



De Bock 2017

Vicig 2000

Coherence

~~Walley~~ ⟷ **Williams** ✓

Independent Natural Extension for Infinite Spaces

Williams-coherence to the Rescue!

Two very useful properties

External additivity ?

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

Factorisation ?

$$(\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2))$$

$$= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \overline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases}$$

$$\text{if } g \geq 0$$

Two very useful properties

External additivity ✓

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

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$$\text{if } g \geq 0$$

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$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

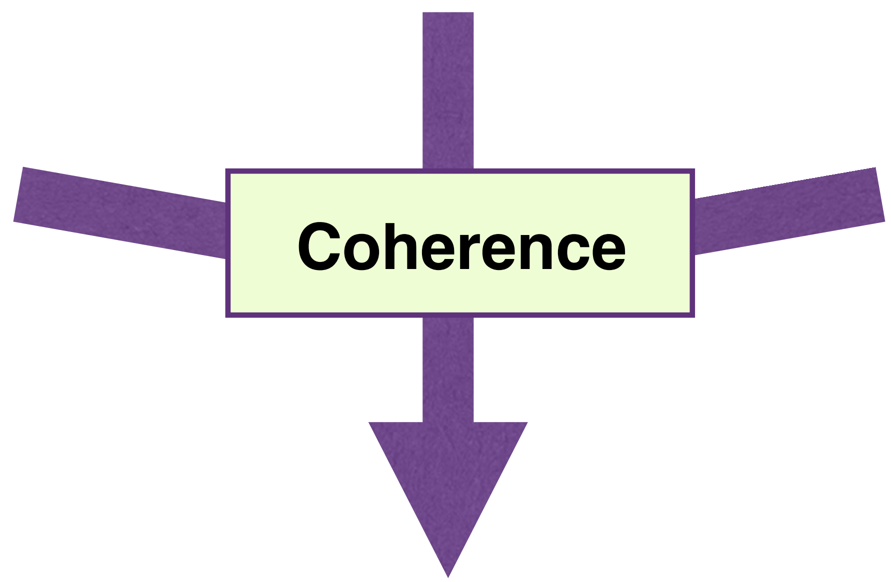
$$X_1$$

independent

$$X_2$$

local uncertainty model

$$\underline{P}(f(X_1)) \;||\; \underline{P}_1(f)$$



local uncertainty model

$$\underline{P}(f(X_2)) \;||\; \underline{P}_2(f)$$

joint uncertainty model

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$

$$\cancel{\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))}$$
$$\cancel{\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))}$$

independent

$$\Updownarrow$$

$$\underline{P}(f(X_1)|B_2) = \underline{P}(f(X_1)) \quad \forall B_2 \in \mathcal{B}_2$$
$$\underline{P}(f(X_2)|B_1) = \underline{P}(f(X_2)) \quad \forall B_1 \in \mathcal{B}_1$$

$$\cancel{\begin{aligned}\underline{P}(f(X_1)|X_2) &= \underline{P}(f(X_1))\\ \underline{P}(f(X_2)|X_1) &= \underline{P}(f(X_2))\end{aligned}}$$

independent

$\Updownarrow$

$$\underline{P}(f(X_1)|B_2) = \underline{P}(f(X_1)) \quad \forall B_2 \in \mathcal{B}_2$$
$$\underline{P}(f(X_2)|B_1) = \underline{P}(f(X_2)) \quad \forall B_1 \in \mathcal{B}_1$$

value-independence: $\mathcal{B}_i = \{\{x_i\} \colon x_i \in \mathcal{X}_i\}$

subset-independence: $\mathcal{B}_i = \mathcal{P}(\mathcal{X}_i) \setminus \{\emptyset\}$

Two very useful properties

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if $g \geq 0$

Two very useful properties

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$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

Factorisation ✓

$$(\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2))$$
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if $g \geq 0$ is $\mathcal{B}_1$-measurable

Independent natural extension may not exist!

~~Walley~~

~~value-independence~~

Factorisation may not hold!

Independent natural extension always exists!

Williams

subset-independence

Factorisation always holds!

See you at the poster?