

# Independent Natural Extension for Infinite Spaces

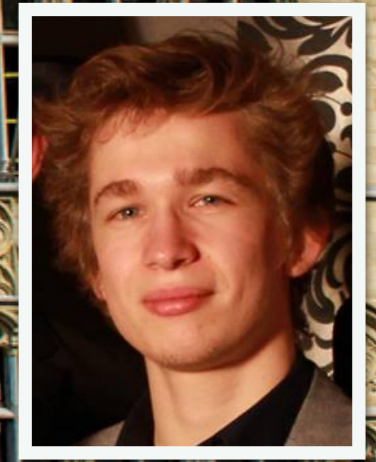
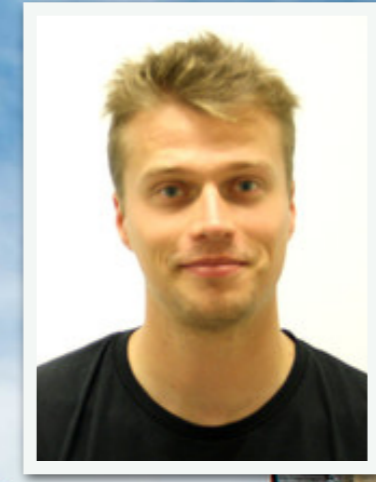
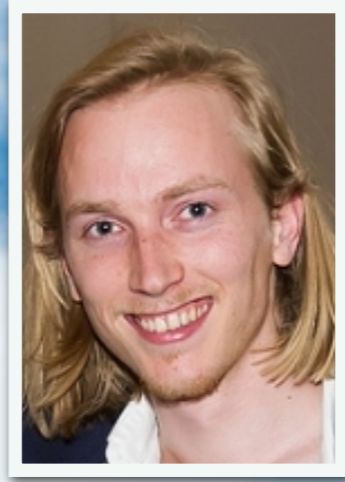
Williams-coherence  
to the Rescue!



**Jasper De Bock**

Ghent University  
Belgium











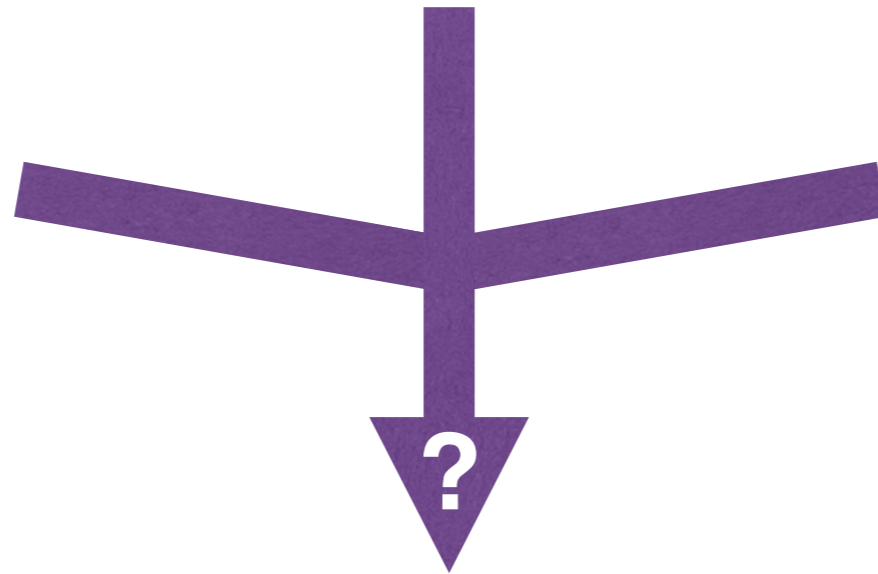
$X_1$

local  
uncertainty  
model

independent

$X_2$

local  
uncertainty  
model



joint uncertainty model

$$P(X_1|X_2) = P(X_1)$$

$$P(X_2|X_1) = P(X_2)$$

$X_1$

$X_2$

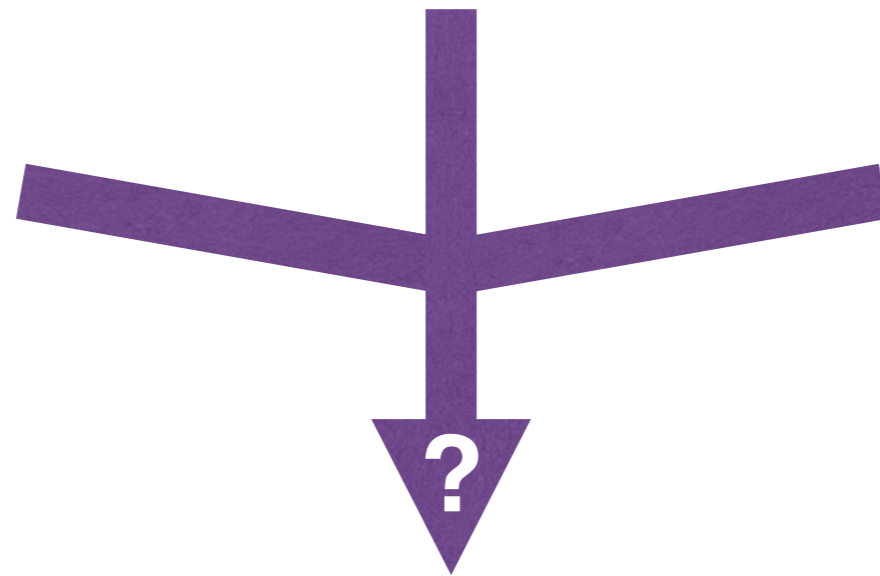
**independent**

**local  
uncertainty  
model**

**local  
uncertainty  
model**

$$P(X_1)$$

$$P(X_2)$$



**joint uncertainty model**

$$P(X_1, X_2)$$



$$P(X_1|X_2) = P(X_1)$$

$$P(X_2|X_1) = P(X_2)$$

$X_1$

**independent**

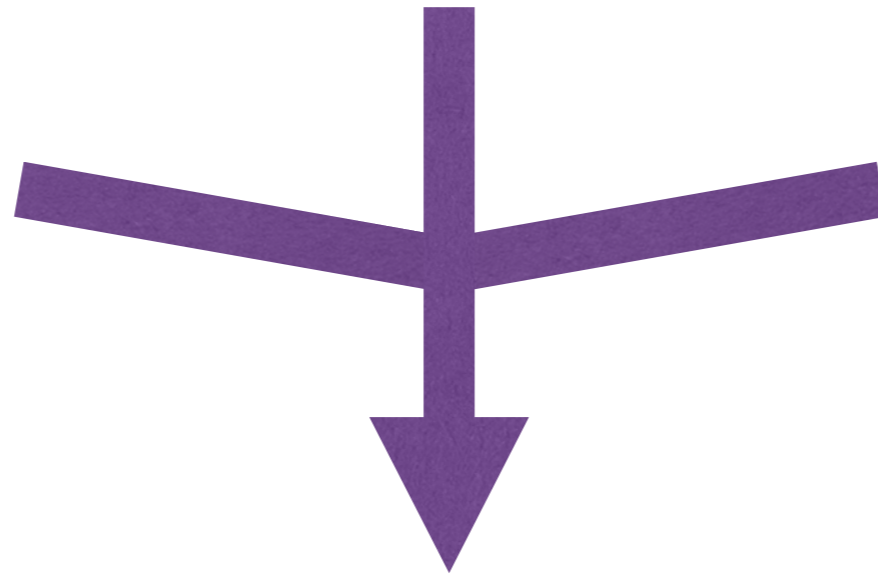
$X_2$

**local  
uncertainty  
model**

$$P(X_1)$$

**local  
uncertainty  
model**

$$P(X_2)$$



**joint uncertainty model**

$$P(X_1, X_2) = P(X_1)P(X_2)$$

# $X_1$

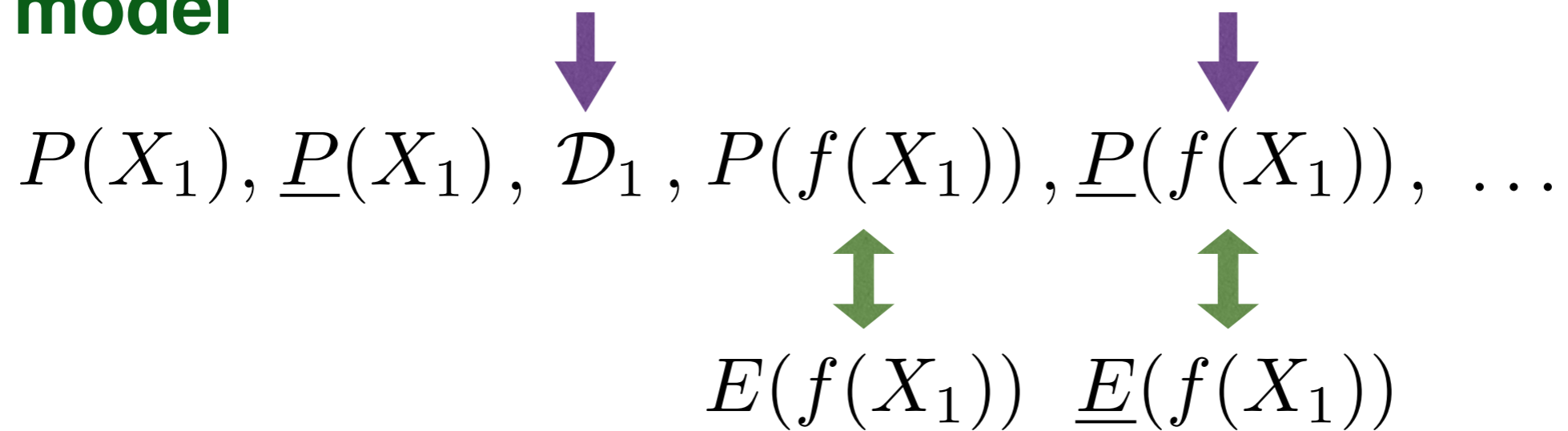
**local  
uncertainty  
model**

$P(X_1), \underline{P}(X_1), \mathcal{D}_1, P(f(X_1)), \underline{P}(f(X_1)), \dots$

$\updownarrow$                        $\updownarrow$   
 $E(f(X_1))$      $\underline{E}(f(X_1))$

# $X_1$

**local  
uncertainty  
model**



$X_1$

local  
uncertainty  
model

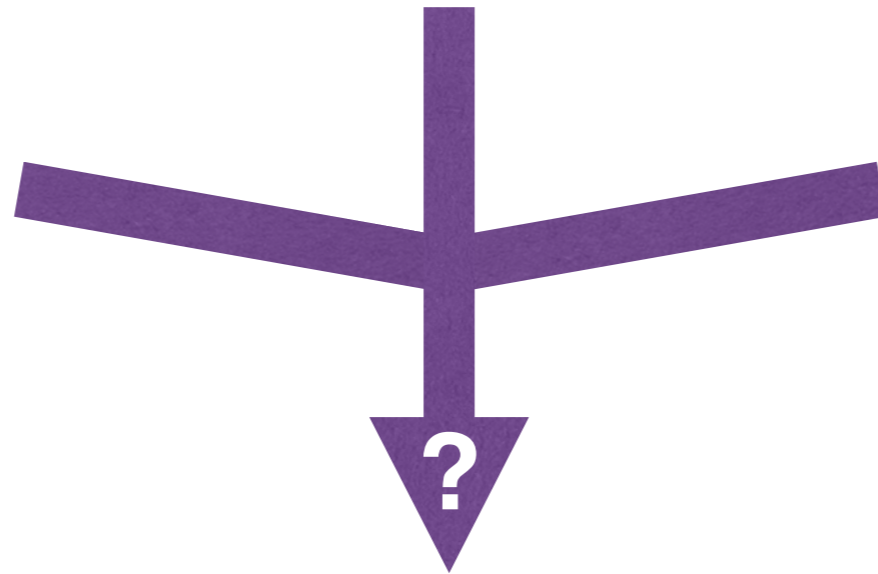
$$\underline{P}(f(X_1))$$

independent

$X_2$

local  
uncertainty  
model

$$\underline{P}(f(X_2))$$



joint uncertainty model

$$\underline{P}(f(X_1, X_2))$$

?

$X_1$

independent

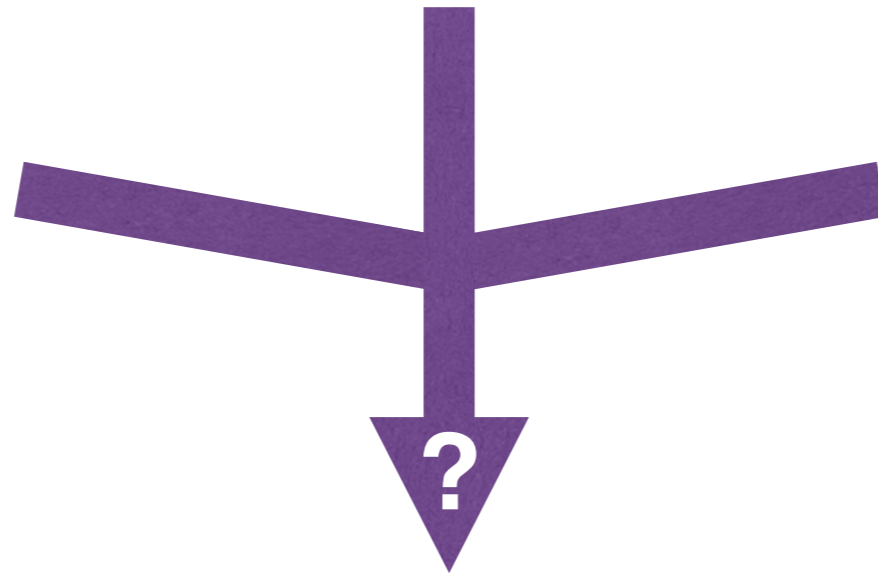
$X_2$

local  
uncertainty  
model

local  
uncertainty  
model

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



joint uncertainty model

$$\underline{P}(f(X_1, X_2))$$

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

$X_1$

$X_2$

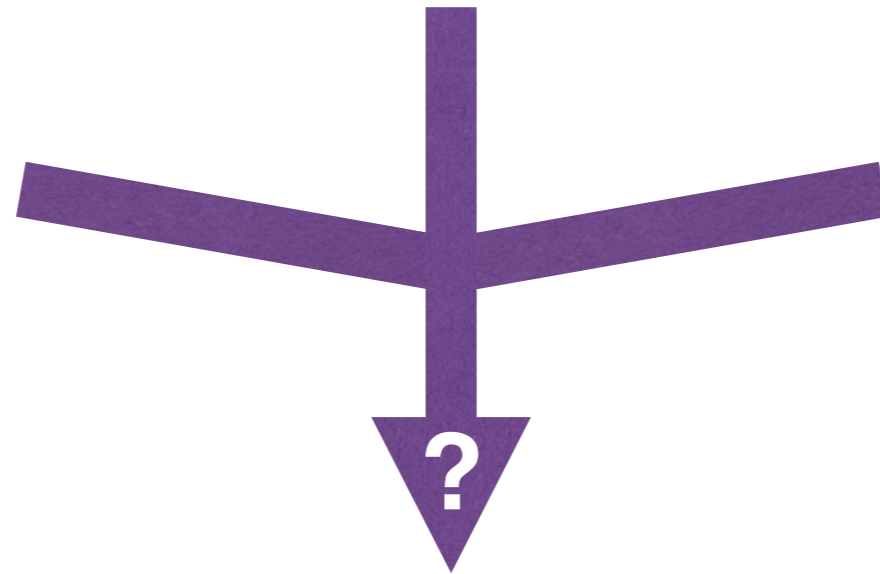
**independent**

**local  
uncertainty  
model**

**local  
uncertainty  
model**

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



**joint uncertainty model**

$$\underline{P}(f(X_1, X_2))$$

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

$X_1$

**independent**

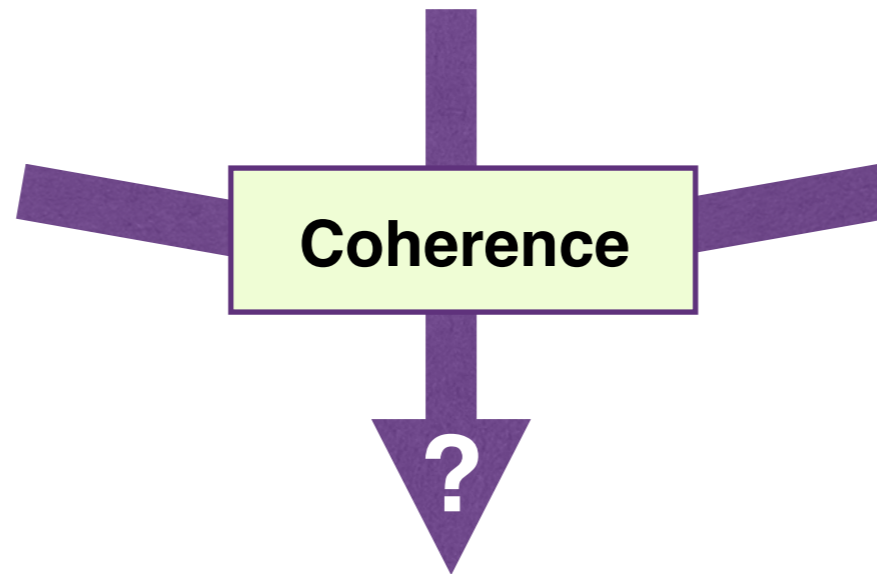
$X_2$

**local  
uncertainty  
model**

**local  
uncertainty  
model**

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



**joint uncertainty model**

$$\underline{P}(f(X_1, X_2))$$

$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

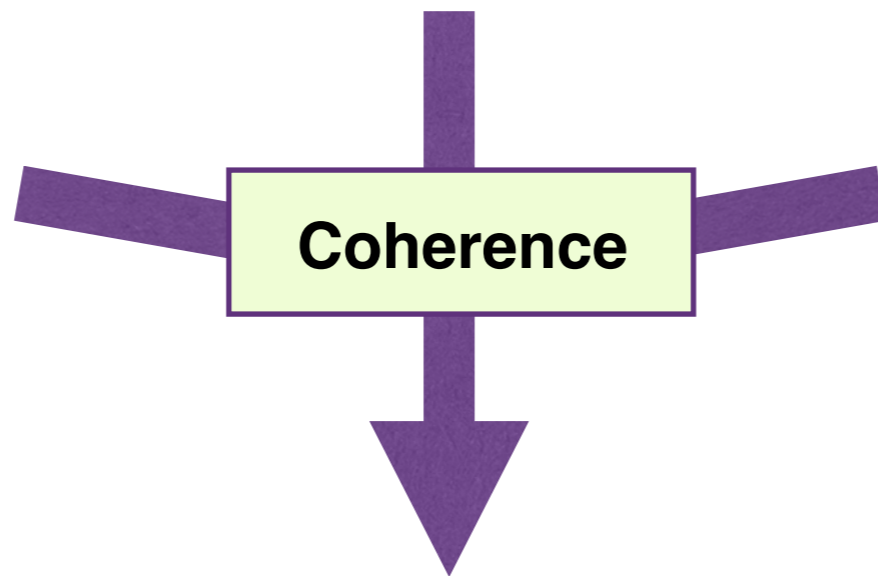
$X_1$

**independent**

$X_2$

**local  
uncertainty  
model**

**local  
uncertainty  
model**



$$\underline{P}(f(X_1))$$

||

$$\underline{P}_1(f)$$

$$\underline{P}(f(X_2))$$

||

$$\underline{P}_2(f)$$

**joint uncertainty model**

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$



# Independent Natural Extension for Infinite Spaces

Williams-coherence  
to the Rescue!



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# Two very useful properties

## External additivity

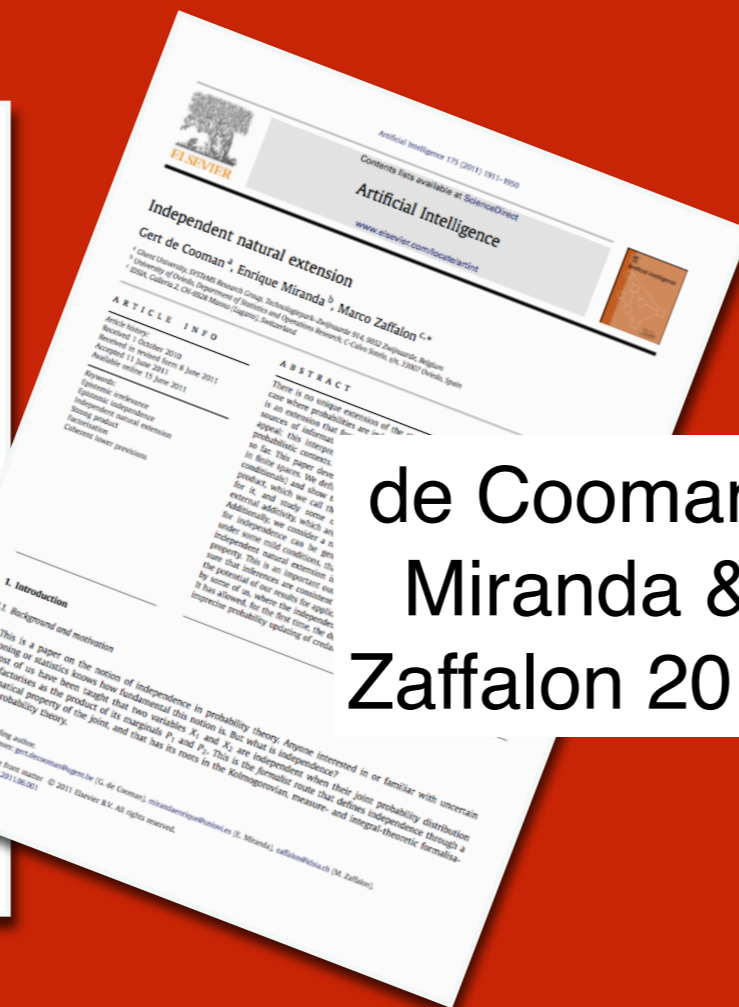
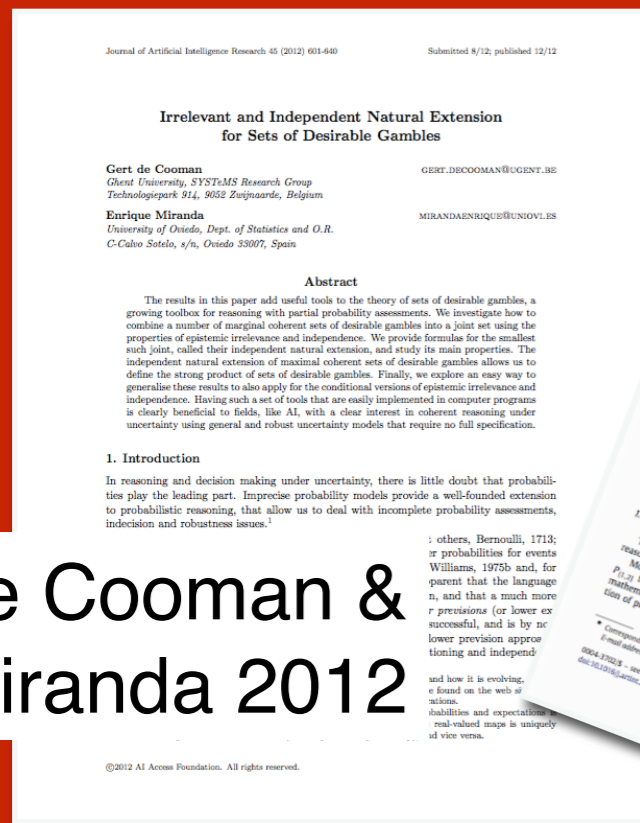
$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

## Factorisation

$$\begin{aligned} (\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2)) &= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \bar{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases} \\ &\qquad \qquad \qquad \text{if } g \geq 0 \end{aligned}$$

# DISCLAIMER!

All of this is well known, and has been for several years now...

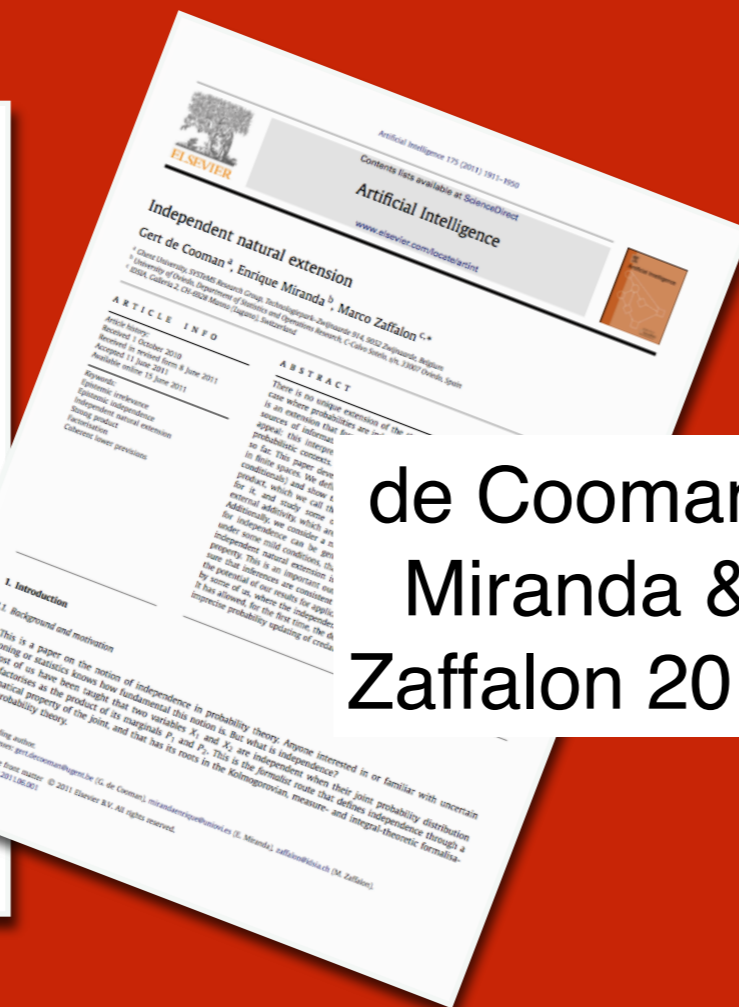
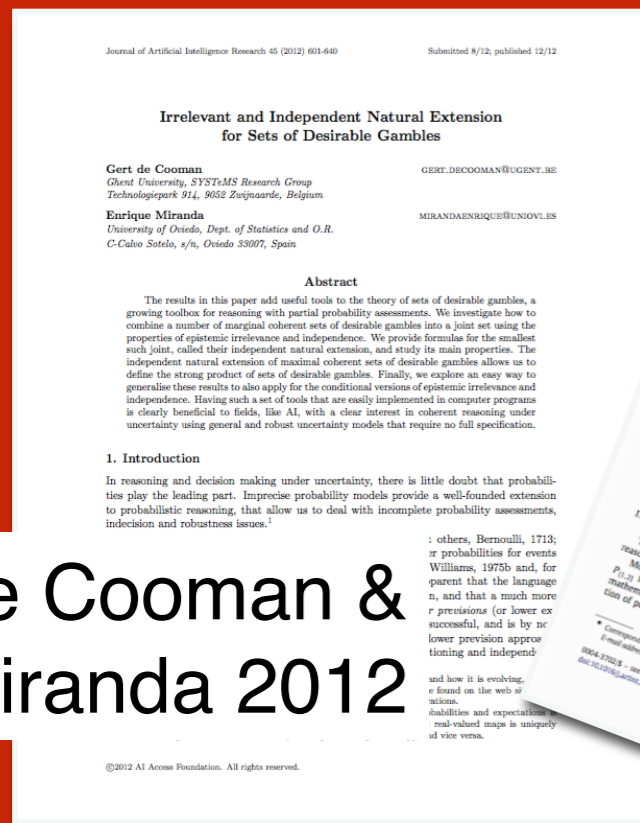


de Cooman & Miranda 2012

de Cooman, Miranda & Zaffalon 2011

# DISCLAIMER!

All of this is well known, and has been for several years now...



de Cooman & Miranda 2012

de Cooman, Miranda & Zaffalon 2011

...but only for finite spaces!

# Independent Natural Extension for **Infinite** Spaces



?



$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$
$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

$X_1$

**independent**

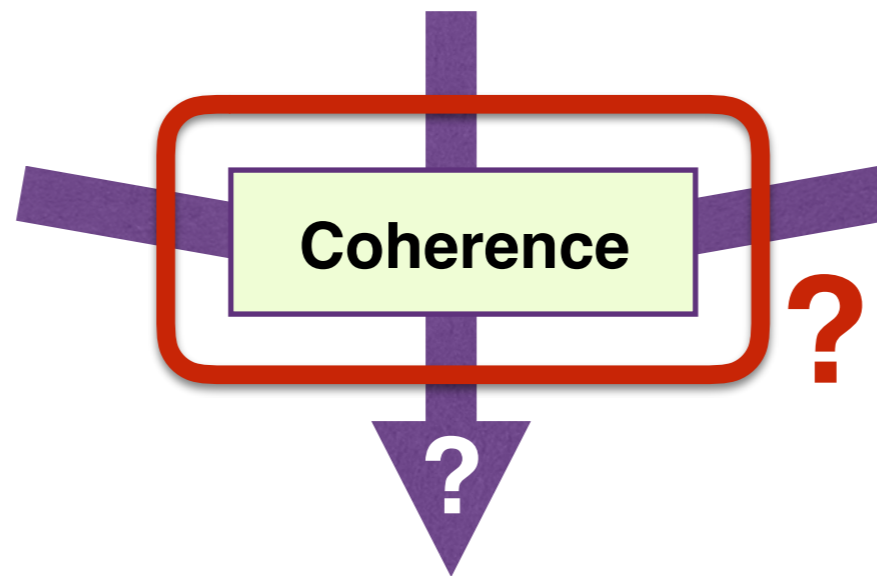
$X_2$

**local  
uncertainty  
model**

**local  
uncertainty  
model**

$$\underline{P}(f(X_1))$$

$$\underline{P}(f(X_2))$$



**joint uncertainty model**



Coherence

?

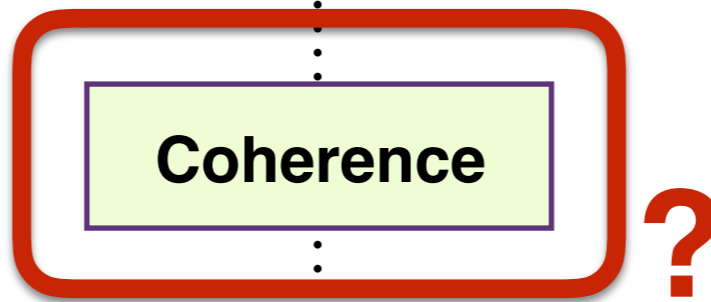
Walley ↔ Williams



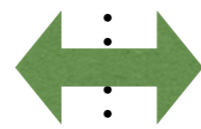
# Independent natural extension may not exist!



Miranda & Zaffalon 2015



~~Walley~~



Williams



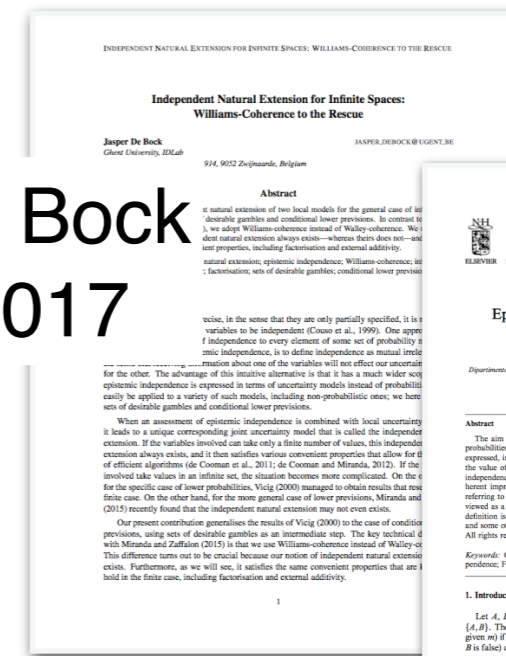


# Independent natural extension may not exist!

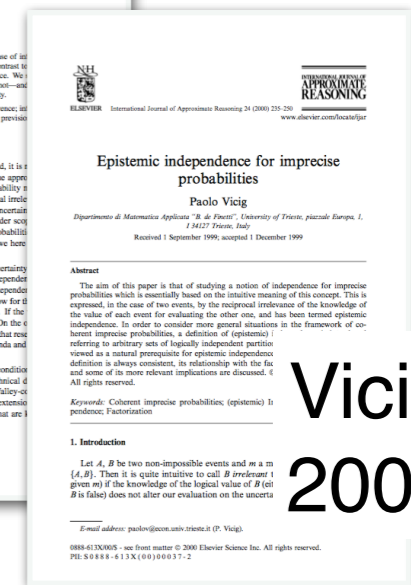


Miranda & Zaffalon 2015

# Independent natural extension always exists!



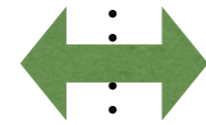
De Bock 2017



Vicig 2000

Coherence

~~Walley~~



Williams ✓



# Independent Natural Extension for Infinite Spaces

**Williams-coherence  
to the Rescue!**



# Two very useful properties

## External additivity ?

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

## Factorisation ?

$$\begin{aligned} &(\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2)) \\ &= \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \overline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases} \\ & \hspace{20em} \text{if } g \geq 0 \end{aligned}$$

# Two very useful properties

## External additivity ✓

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$$\underline{P}(f(X_1)|X_2) = \underline{P}(f(X_1))$$

$$\underline{P}(f(X_2)|X_1) = \underline{P}(f(X_2))$$

$X_1$

$X_2$

**independent**

**local  
uncertainty  
model**

**local  
uncertainty  
model**

$$\underline{P}(f(X_1))$$

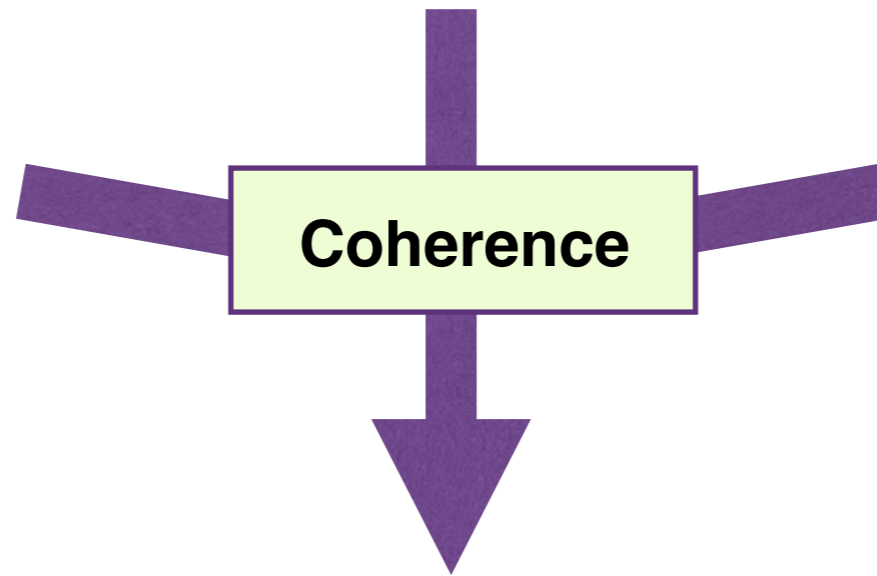
$$\parallel$$

$$\underline{P}_1(f)$$

$$\underline{P}(f(X_2))$$

$$\parallel$$

$$\underline{P}_2(f)$$

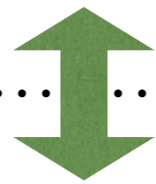


**joint uncertainty model**

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1, X_2))$$

$$\begin{aligned} \underline{P}(f(X_1)|X_2) &= \underline{P}(f(X_1)) \\ \underline{P}(f(X_2)|X_1) &= \underline{P}(f(X_2)) \end{aligned}$$

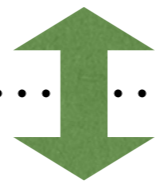
**independent**



$$\begin{aligned} \underline{P}(f(X_1)|B_2) &= \underline{P}(f(X_1)) \quad \forall B_2 \in \mathcal{B}_2 \\ \underline{P}(f(X_2)|B_1) &= \underline{P}(f(X_2)) \quad \forall B_1 \in \mathcal{B}_1 \end{aligned}$$

$$\begin{aligned} \underline{P}(f(X_1)|X_2) &= \underline{P}(f(X_1)) \\ \underline{P}(f(X_2)|X_1) &= \underline{P}(f(X_2)) \end{aligned}$$

**independent**



$$\begin{aligned} \underline{P}(f(X_1)|B_2) &= \underline{P}(f(X_1)) \quad \forall B_2 \in \mathcal{B}_2 \\ \underline{P}(f(X_2)|B_1) &= \underline{P}(f(X_2)) \quad \forall B_1 \in \mathcal{B}_1 \end{aligned}$$

**value-independence:**  $\mathcal{B}_i = \{\{x_i\} : x_i \in \mathcal{X}_i\}$

**subset-independence:**  $\mathcal{B}_i = \mathcal{P}(\mathcal{X}_i) \setminus \{\emptyset\}$

# Two very useful properties

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# Two very useful properties

## External additivity ✓

$$(\underline{P}_1 \otimes \underline{P}_2)(f(X_1) + h(X_2)) = \underline{P}_1(f(X_1)) + \underline{P}_2(h(X_2))$$

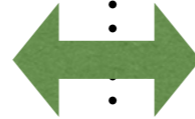
## Factorisation ✓

$$\begin{aligned} (\underline{P}_1 \otimes \underline{P}_2)(g(X_1)h(X_2)) \\ = \begin{cases} \underline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \geq 0 \\ \overline{P}_1(g(X_1))\underline{P}_2(h(X_2)) & \text{if } \underline{P}(h(X_2)) \leq 0 \end{cases} \end{aligned}$$

if  $g \geq 0$  is  $\mathcal{B}_1$ -measurable

**Independent  
natural extension  
may not exist!**

~~Walley~~



**Williams**



**Independent  
natural extension  
always exists!**

~~value-  
independence~~



**subset-  
independence**



**Factorisation  
may not hold!**


**Factorisation  
always holds!**

See you at the poster?



# Independent Natural Extension for Infinite Spaces

## Williams-Coherence to the Rescue!



**Jasper De Bock**  
jasper.debock@ugent.be  
Ghent University, Belgium

If you are not familiar with sets of desirable gambles, lower previsions, Williams-coherence, epistemic independence or independent natural extension, this poster may make little sense at first. I will do my very best to compensate with enthusiasm! If I fail, we can also simply go for a beer. In any case, the thought bubbles below may serve as a nice discussion starter.

### Modelling Uncertainty

A subject's uncertainty about a variable  $X$  that takes values  $x$  in a—possibly infinite—set  $\mathcal{X}$  can be modelled in various ways. We consider two very general and closely connected frameworks, the latter of which includes probabilities as a special case.

**Sets of desirable gambles.** The basic idea here is to consider the subject's attitude towards gambles on  $\mathcal{X}$ , which are bounded real-valued functions  $f$  on  $\mathcal{X}$  whose value  $f(x)$  represents the—possibly negative—payoff for the outcome  $x$ . In particular, we consider the gambles that she finds desirable, in the sense that she prefers them over not betting at all. We gather all these gambles in a so-called set of desirable gambles  $\mathcal{D}$ , which is a subset of the set  $\mathcal{G}(\mathcal{X})$  of all gambles.

**Conditional lower previsions.** Here too, the idea is to model a subject's uncertainty about  $X$  by considering her attitude towards gambles on  $\mathcal{X}$ . However, in this case, instead of considering sets of gambles, we consider the prices at which a subject is willing to buy these gambles. Let  $\mathcal{V}(\mathcal{X})$  be the set of all pairs  $(f, B)$ , where  $f$  is a gamble on  $\mathcal{X}$  and  $B$  is a non-empty subset of  $\mathcal{X}$ —an event. A conditional lower prevision  $P$  on a domain  $\mathcal{C} \subseteq \mathcal{V}(\mathcal{X})$  is then a map  $P: \mathcal{C} \rightarrow \bar{\mathbb{R}}: (f, B) \mapsto P(f|B)$ , where  $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$ . For any  $(f, B)$  in  $\mathcal{C}$ , the lower prevision  $P(f|B)$  of  $f$  conditional on  $B$  is interpreted as the subject's supremum price  $\mu$  for buying  $f$ , provided that the transaction is cancelled if  $B$  does not happen. In other words,  $P(f|B)$  is the supremum value of  $\mu$  for which she is eager to engage in a transaction where she receives  $f(x) - \mu$  if  $x \in B$  and zero otherwise. If  $B = \mathcal{X}$ , we write  $P(f) := P(f|\mathcal{X})$  and call  $P(f)$  the lower prevision of  $f$ .

**Their connection.** These two uncertainty frameworks are closely connected. In particular, because of their interpretation in terms of buying prices for gambles, a set of desirable gambles  $\mathcal{D}$  can be easily derived from a conditional lower prevision  $P_{\mathcal{D}}$  on  $\mathcal{V}(\mathcal{X})$ , the corresponding conditional lower prevision  $P_{\mathcal{D}}$  is defined by  $P_{\mathcal{D}}(f|B) := \sup\{\mu \in \mathbb{R} : [f - \mu]_B \in \mathcal{D}\}$  for every  $(f, B) \in \mathcal{V}(\mathcal{X})$ .

**Coherence.** For an uncertainty model to represent a rational subject's beliefs, it needs to satisfy a set of rationality criteria; if it does, it is called coherent. For a set of desirable gambles  $\mathcal{D}$ , coherence means that for any gambles  $f, g \in \mathcal{G}(\mathcal{X})$  and any real number  $\lambda > 0$ :

- D1. If  $f \geq 0$  and  $f \neq 0$ , then  $f \in \mathcal{D}$ .
- D2. If  $f \in \mathcal{D}$  then  $\lambda f \in \mathcal{D}$ .
- D3. If  $f, g \in \mathcal{D}$ , then  $f + g \in \mathcal{D}$ .
- D4. If  $f \leq 0$ , then  $f \notin \mathcal{D}$ .

A conditional lower prevision  $P$  on a domain  $\mathcal{C} \subseteq \mathcal{V}(\mathcal{X})$  is then said to be coherent if there is a coherent set of desirable gambles  $\mathcal{D}$  on  $\mathcal{X}$  such that  $P$  coincides with  $P_{\mathcal{D}}$  on  $\mathcal{C}$ . Equivalently,  $P$  is coherent if it satisfies the structure-free notion of Williams-coherence that was developed by Pelessoni and Vicig (2009).

### Modelling Independence

We say that  $X_1$  and  $X_2$  are independent if our uncertainty model for  $X_1$  is not affected by conditioning on information about  $X_2$ , and vice versa. This information can easily be applied to a probability measure, and then yields the usual notion of independence. More generally, it can just as easily be applied to lower previsions, sets of desirable gambles, or any other type of uncertainty model, and is then referred to as epistemic independence.

We consider a very general definition of epistemic independence. In particular, for every  $i \in \{1, 2\}$ , we consider any set of conditioning events  $\mathcal{A}_i$  for the variable  $X_i$ , that is, any subset of the set  $\mathcal{P}_k(\mathcal{X}_i)$  of all non-empty subsets of  $\mathcal{X}_i$ .

A coherent conditional lower prevision  $P$  on  $\mathcal{V}(\mathcal{X}_1 \times \mathcal{X}_2)$  is then called epistemically independent if for any  $i$  and  $j$  such that  $\{i, j\} = \{1, 2\}$ :

$$P(f_j|B_i, f_i|B_j) = P(f_j|B_j)$$

for all  $(f_i, B_i) \in \mathcal{V}(\mathcal{X}_i)$  and  $B_j \in \mathcal{A}_j$ .

Similarly, a coherent set of desirable gambles  $\mathcal{D}$  on  $\mathcal{X}_1 \times \mathcal{X}_2$  is epistemically independent if for any  $i$  and  $j$  such that  $\{i, j\} = \{1, 2\}$  and for any  $B_j \in \mathcal{A}_j$  that  $\text{marg}(\mathcal{D}|B_j) = \text{marg}(\mathcal{D})$  in the sense that for all  $f \in \mathcal{G}(\mathcal{X}_i)$ :

$$f(X_i) \mathbb{1}_{B_j}(X_j) \in \mathcal{D} \Leftrightarrow f(X_i) \in \mathcal{D}$$

where  $\mathbb{1}_{B_j}(X_j)$  is the indicator of  $B_j$ , defined otherwise.

Two special cases are particularly important. If  $\mathcal{A}_1 = \mathcal{P}_k(\mathcal{X}_1)$  and  $\mathcal{A}_2 = \mathcal{P}_k(\mathcal{X}_2)$ , we obtain the special case of epistemic value-independence, which is the most commonly called epistemic independence. If obtain what we call epistemic subset-independence. As we will see, the latter has superior properties.

### Independent Natural Extension

For all  $i \in \{1, 2\}$ , let  $\mathcal{D}_i$  be a local coherent set of desirable gambles on  $\mathcal{X}_i$ . The independent natural extension of  $\mathcal{D}_1$  and  $\mathcal{D}_2$  is then the smallest—most conservative—epistemically independent coherent set of desirable gambles on  $\mathcal{X}_1 \times \mathcal{X}_2$  that extends them, meaning that  $(\forall i \in \{1, 2\}) \mathcal{D}_i \subseteq \text{marg}(\mathcal{D}) := \{f \in \mathcal{G}(\mathcal{X}_i) : f(X_i) \in \mathcal{D}_i\}$ .

For lower previsions, the local models  $P_1$  and  $P_2$  are coherent conditional lower previsions on  $\mathcal{V}_i \subseteq \mathcal{V}(\mathcal{X}_i)$ , respectively. The independent natural extension of  $P_1$  and  $P_2$  is then the smallest—most conservative—epistemically independent coherent lower prevision on  $\mathcal{V}(\mathcal{X}_1 \times \mathcal{X}_2)$  that extends them, meaning that  $(\forall i \in \{1, 2\}) P_i(f_i|B_i) = P(f_i|B_i)$  for all  $(f_i, B_i) \in \mathcal{V}_i$ .

**Existence.** In both of our two frameworks, the independent natural extension always exists; we denote it by  $\mathcal{D}_1 \otimes \mathcal{D}_2$  and  $P_1 \otimes P_2$ , respectively. For lower previsions, this result crucially depends on our use of Williams-coherence; for Walley-coherence, as shown by Mirandis and Zaffalon (2015) for epistemic value-independence, this may no longer hold.

**Properties.** Let  $\{i, j\} = \{1, 2\}$  and consider any  $h \in \mathcal{G}(\mathcal{X}_j)$  and  $f, g \in \mathcal{G}(\mathcal{X}_i)$  such that  $f \geq 0$  is  $\mathcal{D}_i$ -measurable—a technical condition that coincides with the usual notion when  $\mathcal{A}_i = \mathcal{P}_k(\mathcal{X}_i)$  is a  $\sigma$ -field. Then all the terms are well-defined—if  $\mathcal{V}_i$  and  $\mathcal{V}_j$  are large enough—we have that

$$(P_1 \otimes P_2)(f + gh) = P_1(f + gP_2(h))$$

As a direct consequence, we find that

$$(P_1 \otimes P_2)(f + h) = P_1(f) + P_2(h)$$

and—with  $P_i(h) = -P_i(-h)$ —that

$$(P_1 \otimes P_2)(gh) = P_1(gP_2(h)) = \begin{cases} P_1(g)P_2(h) & \text{if } P_2(h) \geq 0 \\ P_1(g)P_2(h) & \text{if } P_2(h) \leq 0 \end{cases}$$

known as external additivity and factorisation, respectively. Crucially, for epistemic subset-independence,  $\mathcal{D}_i$ -measurability is trivially satisfied, and factorisation then always holds.

All of this seems very abstract. Does it have any practical use?

That's weird! Shouldn't the right-hand side be unconditional?

You said that probabilities are a special case. Yeah right... how does that work?

So why is there no  $B_i$  here?

What happens if there are more than two variables?