Supplementary Material
Global Sensitivity Analysis
for MAP Inference in Graphical Models

Jasper De Bock
Ghent University, SYSTeMS
Ghent (Belgium)
jasper.debock@ugent.be

Cassio P. de Campos
Queen’s University
Belfast (UK)
c.decampos@qub.ac.uk

Alessandro Antonucci
IDSIA
Lugano (Switzerland)
alessandro@idsia.ch

1 Proof of Theorem 1

Theorem 1. Let \(X\) be a variable taking values in a finite set \(\text{Val}(X)\) and let \(\mathcal{P}\) be a set of candidate mass functions over \(X\). Let \(\hat{x}\) be a MAP instantiation for a mass function \(P \in \mathcal{P}\). Then \(\hat{x}\) is the unique MAP instantiation for every \(P' \in \mathcal{P}\) (equivalently \(\text{Val}^*(X)\) has cardinality one) if and only if

\[
\min_{P' \in \mathcal{P}} P'(\hat{x}) > 0 \quad \text{and} \quad \max_{x \in \text{Val}(X) \setminus \{\hat{x}\}} \max_{P \in \mathcal{P}} \frac{P'(x)}{P'(\hat{x})} < 1,
\]

(1)

where the first inequality should be checked first because if it fails, then the left-hand side of the second inequality is ill-defined.

Proof. We start by noticing that \(\hat{x}\) is the unique MAP instantiation for every \(P' \in \mathcal{P}\) if and only if

\[
\forall P' \in \mathcal{P}, \forall x \in \text{Val}(X) \setminus \{\hat{x}\} : P'(\hat{x}) > P'(x).
\]

(2)

In order for this condition to be satisfied, it is clearly necessary that \(P'(\hat{x})\) be strictly positive for each \(P' \in \mathcal{P}\) or, equivalently, by the compactness of \(\mathcal{P}\), that the leftmost part of Eq. (1) be satisfied. Under this condition, Eq. (2) can be rewritten as

\[
\forall P' \in \mathcal{P}, \forall x \in \text{Val}(X) \setminus \{\hat{x}\} : \frac{P'(x)}{P'(\hat{x})} < 1 \iff \max_{x \in \text{Val}(X) \setminus \{\hat{x}\}} \max_{P' \in \mathcal{P}} \frac{P'(x)}{P'(\hat{x})} < 1,
\]

(3)

where the compactness of \(\mathcal{P}\) implies the existence of the final maximum. \(\Box\)

2 Proof of Theorem 2

Theorem 2. Let \(X = (X_1, \ldots, X_n)\) be a vector of variables taking values in their respective finite domains \(\text{Val}(X_1), \ldots, \text{Val}(X_n)\), let \(I_1, \ldots, I_m\) be a collection of index sets such that \(I_1 \cup \cdots \cup I_m = [n]\) and, for every \(k \in [m]\), let \(\psi_k\) be a compact set of nonnegative factors over \(X_{I_k}\) such that \(\Psi = \times_{k=1}^m \psi_k\) is a family of PGMs.

Consider now a PGM \(\Phi \in \Psi\) and a MAP instantiation \(\hat{x}\) for \(P_\Phi\) and define, for every \(k \in [m]\) and every \(x_{I_k} \in \text{Val}(X_{I_k})\):

\[
\alpha_k := \min_{\phi_k' \in \psi_k} \phi_k'(\hat{x}_{I_k}) \quad \text{and} \quad \beta_k(x_{I_k}) := \max_{\phi_k' \in \psi_k} \phi_k'(x_{I_k}).
\]

(4)

Then \(\hat{x}\) is the unique MAP instantiation for every \(P' \in \mathcal{P}_\Phi\) if and only if

\[(\forall k \in [m]) \quad \alpha_k > 0 \quad \text{and} \quad \prod_{k=1}^m \beta_k(x_{I_k}^{(2)}) < 1, \quad (\text{RMAP})\]
where $\mathbf{x}^{(2)}$ is an arbitrary second best MAP instantiation for the distribution $P_k$, that corresponds to the PGM $\Phi := \{\beta_1, \ldots, \beta_m\}$. The first criterion in (RMAP) should be checked first because $\beta_k(\mathbf{x}^{(2)}_k)$ is ill-defined if $\alpha_k = 0$.

**Proof.** Since every set of factors $\psi_k$ is compact, $\mathcal{P}_\Psi$ is compact as well. Therefore, by Th. 1, $\tilde{x}$ is the unique MAP instantiation for every $P' \in \mathcal{P}_\Psi$ if and only if

$$\min_{P' \in \mathcal{P}_\Psi} P'(\tilde{x}) > 0 \quad \text{and} \quad \max_{\mathbf{x} \in \text{Val}(\mathbf{X}) \setminus \{\tilde{x}\}} \max_{P' \in \mathcal{P}_\Psi} \frac{P'(\mathbf{x})}{P'(\tilde{x})} < 1. \tag{5}$$

Hence, we are left to prove that Eq. (5) is equivalent to (RMAP). By the compactness of $\mathcal{P}_\Psi$:

$$\min_{P' \in \mathcal{P}_\Psi} P'(\tilde{x}) > 0 \iff (\forall P' \in \mathcal{P}_\Psi) P'(\tilde{x}) > 0 \iff (\forall \Phi' \in \Psi) P_{\Phi'}(\tilde{x}) > 0$$

$$\iff (\forall \Phi' \in \Psi) \frac{1}{Z_{\Phi'}} \prod_{k=1}^m \phi'_k(\tilde{x}_k) > 0 \iff (\forall \Phi' \in \Psi) \prod_{k=1}^m \phi'_k(\tilde{x}_k) > 0$$

$$\iff (\forall \Phi' \in \Psi)(\forall k \in [m]) \phi'_k(\tilde{x}_k) > 0 \iff (\forall k \in [m])(\forall \phi'_k \in \psi_k) \phi_k(\tilde{x}_k) > 0.$$

Thus, given the compactness of the sets $\psi_k$, the first inequality in Eq. (5) is equivalent to the first criterion in (RMAP).

If this first criterion holds, again using the compactness of the sets $\psi_k$, we find that all the $\beta_k(\mathbf{x}_k)$ are well-defined and nonnegative. Also, if the first criterion holds, then for all $\mathbf{x} \in \text{Val}(\mathbf{X})$:

$$f(\mathbf{x}) := \max_{P' \in \mathcal{P}_\Psi} \frac{P'(\mathbf{x})}{P'(\tilde{x})} = \max_{\Phi' \in \Psi} \frac{P_{\Phi'}(\mathbf{x})}{P_{\Phi'}(\tilde{x})} = \max_{\Phi' \in \Psi} \prod_{k=1}^m \phi'_k(\mathbf{x}_k) = \prod_{k=1}^m \phi'_k(\mathbf{x}_k) = \prod_{k=1}^m \beta_k(\mathbf{x}_k).$$

Thus, since $f(\tilde{x}) = 1$, $\Phi = \{\beta_1, \ldots, \beta_m\}$ is indeed a PGM. To conclude the proof, we show that the second inequality in Eq. (5), which can now be reformulated as

$$c := \max_{\mathbf{x} \in \text{Val}(\mathbf{X}) \setminus \{\tilde{x}\}} f(\mathbf{x}) < 1,$$

is equivalent to $f(\mathbf{x}^{(2)}) < 1$. Let $\mathbf{x}^{(1)}$ be (one of) the MAP instantiation(s) for $P_\Phi$ that enable(s) $\mathbf{x}^{(2)}$ to satisfy Eq. (5, main paper). First, assume that $f(\mathbf{x}^{(2)}) < 1$. Then by Eq. (5, main paper) and because $f(\tilde{x}) = 1$, we see that $\mathbf{x}^{(1)} = \tilde{x}$ and therefore that $c = f(\mathbf{x}^{(2)}) < 1$. Next, assume that $c < 1$. Then by Eq. (5, main paper) and because $f(\tilde{x}) = 1$, we find that $\mathbf{x}^{(1)} = \tilde{x}$ and $f(\mathbf{x}^{(2)}) < 1$. \(\square\)