Credal networks
an overview of different approaches

Jasper De Bock

11 June 2013
Bayesian networks: basic setup

- Graphical structure: DAG

\[ \forall s \in G: P(s), D(s), N(s) \]
Bayesian networks: basic setup

- Graphical structure: DAG
  \[ \forall s \in G: P(s), D(s), N(s) \]
- Variables $X_s$ take values $x_s$ in a finite non-empty set $X_s$
Bayesian networks: basic setup

- Graphical structure: DAG
  \[ \forall s \in G: \text{P}(s), \text{D}(s), \text{N}(s) \]
- Variables \( X_s \) take values \( x_s \) in a finite non-empty set \( X_s \)
- Local uncertainty models: mass functions \( q(X_s | x_{P(s)}) \)
  
  Example: \( q(X_4 | x_{\{2,3\}}) \)
Bayesian networks: basic setup

- Graphical structure: DAG
  \[ \forall s \in G: P(s), D(s), N(s) \]
- Variables \( X_s \) take values \( x_s \) in a finite non-empty set \( \mathcal{X}_s \)
- Local uncertainty models: mass functions \( q(X_s | x_{P(s)}) \)
- Independence assumptions:
  \[ \forall s \in G: I(N(s), s | P(s)) \]
Bayesian networks: joint model $p(X_G)$

- **Graphical structure:** DAG
  \[\forall s \in G: P(s), D(s), N(s)\]

- **Variables $X_s$** take values $x_s$ in a finite non-empty set $\mathcal{X}_s$

- **Local uncertainty models:**
  mass functions $q(X_s | x_{P(s)})$

- **Independence assumptions:**
  \[\forall s \in G: I(N(s), s | P(s))\]
Bayesian networks: joint model $p(X_G)$?

\[ p(x_s | x_{P(s)}, x_{N(s)}) = p(x_s | x_{P(s)}) \]

EQUIVALENT

\[ p(x_{N(s)} | x_{P(s)}, x_s) = p(x_{N(s)} | x_{P(s)}) \]

- Independence assumptions:
  \[ \forall s \in G: I(N(s), s | P(s)) \]
Bayesian networks: joint model \( p(X_G) \)?

- Local uncertainty models: mass functions \( q(X_s | x_{P(s)}) \)
- Independence assumptions:
  \( \forall s \in G: I(N(s), s | P(s)) \)
Bayesian networks: joint model $p(X_G)$?

\[
p(x_s | x_{p(s)}, x_{N(s)}) = p(x_s | x_{p(s)}) \quad \text{s.t.} \quad q(x_s | x_{p(s)})
\]

- Local uncertainty models: mass functions $q(X_s | x_{p(s)})$
- Independence assumptions:
  \[\forall s \in G: I(N(s), s | P(s))\]
Bayesian networks: joint model $p(X_G)$

\[
p(x_s | x_{P(s)}, x_{N(s)}) = \frac{p(x_s | x_{P(s)})}{q(x_s | x_{P(s)})} = p(x_G) = \prod_{s \in G} q(x_s | x_{P(s)})
\]
Credal networks: basic setup

- Graphical structure: DAG
  \[ \forall s \in G: P(s), D(s), N(s) \]
- Variables \( X_s \) take values \( x_s \) in a finite non-empty set \( X_s \)
- Local uncertainty models: credal sets \( \mathcal{M}(X_s | x_{P(s)}) \)
  - Closed and convex set of mass functions \( q(X_s | x_{P(s)}) \)
- Independence assumptions
  \[ \forall s \in G: \text{?I?}(N(s), s | P(s)) \]
Credal networks: joint model $\mathcal{M}(X_G)$?

- Graphical structure: DAG
  \[ \forall s \in G: P(s), D(s), N(s) \]
- Variables $X_s$ take values $x_s$ in a finite non-empty set $X_s$
- Local uncertainty models: credal sets $\mathcal{M}(X_s | x_{P(s)})$?
  \[ \text{Closed and convex set of mass functions } q(X_s | x_{P(s)}) \]
- Independence assumptions?
  \[ \forall s \in G: \forall ?I?(N(s), s | P(s)) \]
Credal networks: joint model $\mathcal{M}(X_G)$?

**Conditioning:**

$\mathcal{M}(X_G)$

$\mathcal{M}(X_s | x_{P(s)})$

$= \{p(X_s | x_{P(s)}): p(X_G) \in \mathcal{M}(X_G)\}$
Credal networks: joint model $\mathcal{M}(X_G)$?

Conditioning:

$\mathcal{M}(X_G) \iff \mathcal{M}(X_s | x_{P(s)}) = \{ p(X_s | x_{P(s)}): p(X_G) \in \mathcal{M}(X_G) \}$

Given local credal set
Credal networks: joint model $\mathcal{M}(X_G)$?

? Independence assumptions ?

- Strong independence
- Epistemic irrelevance
- Epistemic independence
Credal networks: joint model $\mathcal{M}(X_G)$?

? Independence assumptions?
- Strong independence
- Epistemic irrelevance
- Epistemic independence
Strong independence

∀ s ∈ G: SI(N(s), s | P(s))

\( \mathcal{M}(X_G) \) is the convex hull of a set of mass functions \( p(X_G) \) that satisfy the usual independence assumption:

∀ s ∈ G: I(N(s), s | P(s))
Credal networks: joint model $\mathcal{M}(X_G)$?

**Strong independence**

\[ \forall s \in G: SI(N(s), s|P(s)) \]

+ Local models $\mathcal{M}(X_s|X_{P(s)})$

\[ \mathcal{M}(X_G) \]

not unique!
Credal networks: joint model $\mathcal{M}^{\text{str}}(X_G)$?

Strong independence

$\forall s \in G: \text{SI}(N(s), s|P(s))$

+ Local models $\mathcal{M}(X_s| x_{P(s)})$

The strong extension $\mathcal{M}^{\text{str}}(X_G)$?
Credal networks: joint model \( \mathcal{M}^{\text{str}}(X_G) \)

**Strong independence**

\[ \forall s \in G: \text{SI}(N(s), s | P(s)) \]

+ Local models \( \mathcal{M}(X_s | x_{P(s)}) \)

\[ \downarrow \text{largest solution!} \]

The strong extension \( \mathcal{M}^{\text{str}}(X_G) \) is the convex hull of those \( p(X_G) \) for which

\[ p(x_G) = \prod_{s \in G} q(x_s | x_{P(s)}) \cap M(X_s | x_{P(s)}) \]
Credal networks: joint model $\mathcal{M}(X_G)$?

? Independence assumptions?

- Strong independence
- Epistemic irrelevance
- Epistemic independence
Credal networks: joint model $\mathcal{M}(X_G)$?

Epistemic irrelevance

$\forall s \in G: \text{IR}(N(s), s \mid P(s))$

$\mathcal{M}(X_s \mid x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s \mid x_{P(s)})$
Credal networks: joint model $\mathcal{M}(X_G)$?

Epistemic irrelevance

$$\forall s \in G: IR(N(s), s|P(s))$$

+ local credal set

$$\mathcal{M}(X_s|x_{P(s)},x_{N(s)}) = \mathcal{M}(X_s|x_{P(s)})$$

not unique!

$$\mathcal{M}(X_G)$$
Credal networks: joint model $\mathcal{M}^{\text{irr}}(X_G)$?

Epistemic irrelevance

$\forall s \in G: \text{IR}(N(s), s \mid P(s))$

$\mathcal{M}(X_s \mid x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s \mid x_{P(s)})$

local credal set

largest solution!

The irrelevant (natural) extension $\mathcal{M}^{\text{irr}}(X_G)$?
Credal networks: joint model $\mathcal{M}^{\text{irr}}(X_G)$

Epistemic irrelevance

$\forall s \in G: \text{IR}(N(s), s | P(s))$

$\mathcal{M}(X_s | x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s | x_{P(s)})$

The irrelevant (natural) extension $\mathcal{M}^{\text{irr}}(X_G)$ is the set of $p(X_G)$ for which

$P(X_s | x_{P(s)}, x_{N(s)}) \in \mathcal{M}(X_s | x_{P(s)})$
Credal networks: joint model $\mathcal{M}(X_G)$?

? Independence assumptions?

- Strong independence
- Epistemic irrelevance
- Epistemic independence
Credal networks: joint model $\mathcal{M}(X_G)$?

Epistemic independence

$\forall s \in G: \text{EI}(N(s), s|P(s))$

$\mathcal{M}(X_s|x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s|x_{P(s)})$

and $\mathcal{M}(X_{N(s)}|x_{P(s)}, x_s) = \mathcal{M}(X_{N(s)}|x_{P(s)})$
Credal networks: joint model $\mathcal{M}(X_G)$?

Epistemic independence

$\forall s \in G: EI(N(s), s | P(s))$

+ local credal set

$\mathcal{M}(X_s | x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s | x_{P(s)})$

and $\mathcal{M}(X_{N(s)} | x_{P(s)}, x_s) = \mathcal{M}(X_{N(s)} | x_{P(s)})$
Credal networks: joint model $\mathcal{M}(X_G)$

**Epistemic independence**

$\forall s \in G: \text{EI}(N(s), s | P(s))$

\[ \mathcal{M}(X_s | x_{P(s)} , x_{N(s)}) = \mathcal{M}(X_s | x_{P(s)}) \]

and

\[ \mathcal{M}(X_{N(s)} | x_{P(s)}, x_s) = \mathcal{M}(X_{N(s)} | x_{P(s)}) \]

not unique!

\[ \mathcal{M}(X_G) \]
Credal networks: joint model $\mathcal{M}^{\text{ind}}(X_G)$?

Epistemic independence

∀ s ∈ G: $\text{EI}(N(s), s \mid P(s))$

+ local credal set

$\mathcal{M}(X_s \mid x_{P(s)}, x_{N(s)}) = \mathcal{M}(X_s \mid x_{P(s)})$

and $\mathcal{M}(X_{N(s)} \mid x_{P(s)}, x_s) = \mathcal{M}(X_{N(s)} \mid x_{P(s)})$

The independent (natural) extension $\mathcal{M}^{\text{ind}}(X_G)$?
Epistemic independence
\[ \forall s \in G: \text{EI}(N(s), s \mid P(s)) \]

\[ M(X_N \mid x_P, x_N) = M(X_N \mid x_P) \]

And
\[ M(X_N \mid x_P, x) = M(X_N \mid x_P) \]

The independent (natural) extension
\[ M^{\text{ind}}(X_G) \]

is the set of \( p(X_G) \) for which
\[ p(X_s \mid x_P, x_N) \in M(X_s \mid x_P) \]

and ... (no easy description!)
Credal networks

Questions?