RANDOMNESS AND IMPRECISION: FROM SUPERMARTINGALES TO RANDOMNESS TESTS

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ABSTRACT. We generalise the randomness test definitions in the literature for both the Martin-Löf and Schnorr randomness of a series of binary outcomes, in order to allow for interval-valued rather than merely precise forecasts for these outcomes, and prove that under some computability conditions on the forecasts, our definition of Martin-Löf test randomness is related to Levin's uniform randomness. We also show that these new randomness notions are, under some computability and non-degeneracy conditions on the forecasts, equivalent to the martingale-theoretic versions we introduced in earlier papers. In addition, we prove that our generalised notion of Martin-Löf randomness can be characterised by universal supermartingales and universal randomness tests.

1. INTRODUCTION

In a number of recent papers [4, 9, 10], two of us (De Cooman and De Bock) have shown how to associate various notions of algorithmic randomness with interval—rather than precise—forecasts for a sequence of binary outcomes, and argued why that is useful and interesting. Providing such interval forecasts for binary outcomes is a way to allow for *imprecision* in the resulting probability models. Still more recent papers [19–21] by the three of us explore these ideas further, and identify interesting relations between randomness associated with imprecise (interval) and precise forecasts.

All of this work follows the *martingale-theoretic* approach to randomness, where a sequence of outcomes is considered to be random if there's some specific type of supermartingale that becomes unbounded on that sequence in some specific way. How a supermartingale is defined in this context, is closely related to the interval forecasts involved, and how they can be interpreted.

There are, of course, other ways to approach and define algorithmic randomness, besides the martingale-theoretic one [2]: via randomness tests [14, 18, 24], via Kolmogorov complexity [14, 17, 18, 24, 25], via order-preserving transformations of the event tree associated with a sequence of outcomes [24], via specific limit laws (such as Lévy's zero-one law) [15, 44], and so on.

Here, we consider one of these alternatives, the randomness test approach, and we show how we can define specific tests involving interval forecasts that allow us to introduce two new flavours of *test(-theoretic) randomness* for imprecise forecasts: one reminiscent of the original Martin-Löf approach, and another of the original Schnorr approach. We then proceed to show that these test-theoretic notions of randomness are, under some computability and non-degeneracy conditions on the forecasts, equivalent to the martingale-theoretic notions introduced in our earlier papers [4, 9, 10]. We thus succeed in extending, to our more general imprecise probabilities context, earlier results by Schnorr [24] and Levin [16] showing that the test and martingale-theoretic randomness notions are essentially equivalent for precise forecasts.[1]

Given the state of the art in algorithmic randomness, it may seem unsurprising that there should be a connection between martingale-theoretic and randomness test approaches to

Key words and phrases. Martin-Löf randomness; Schnorr randomness; game-theoretic probability; interval forecast; supermartingale; randomness test.

[1]Schnorr proves this for fair-coin forecasts only.

randomness for imprecise (interval-valued) forecasting systems, as they are known to be there for their precise (point-valued) special cases; indeed, our suspicion that there might be such a connection in more general contexts is what made us look for it, initially. That is not to say, however, that proving that there is such a link is a straightforward matter, especially since a number of the techniques used for additive probabilities and linear expectations become unworkable, or need a fundamentally different approach, when dealing with imprecise or game-theoretic probabilities and expectations, which are typically non-additive and non-linear. The fact that we can identify *new* ways of establishing the connection between martingale-theoretic and randomness test approaches in a more general and arguably more abstract setting would argue in favour of our method of approach.

How have we structured our argumentation? When we work with precise forecasts, there are suitable notions of corresponding supermartingales and of corresponding measures on the set of all outcome sequences. These allow us to formulate randomness definitions that follow, respectively, a martingale-theoretic and a randomness test approach. Unsurprisingly therefore, we'll need to suitably extend such notions of supermartingale and measure to allow for interval forecasts, in order to help us broaden the existing randomness definitions on both approaches. In Section 2 we present an overview of the mathematical tools required to achieve this generalisation: we deal with generalised supermartingales in Section 2.3, and we extend the notion of a measure to that of an upper expectation in Section 2.4. All of these results are by now well established in the field of imprecise probabilities [1, 5, 37, 41] and game-theoretic probability [26, 27], so this section is intended as a basic overview of relevant results in that literature.

The basic ideas and results from computability theory that we'll need to rely on, are summarised briefly in Section 3.

In Section 4, we summarise the main ideas in our earlier paper [4], which allowed us to extend the existing martingale-theoretic versions of Martin-Löf randomness and Schnorr randomness to deal with interval forecasts. Extending, on the other hand, the existing randomness test definitions of Martin-Löf randomness and Schnorr randomness to deal with interval forecasts is the subject of Section 5.

In Section 6 we provide sufficient conditions for the martingale-theoretic and randomness test approaches to Martin-Löf randomness to be equivalent, and we do the same for Schnorr randomness in Section 8.

In Section 7 we prove that our notion of Martin-Löf test randomness for a(n interval-valued) forecasting system can be reinterpreted as an application of Levin's [3, 16] notion of Martin-Löf test randomness—also known as *uniform randomness*—to *effectively compact classes of measures*.

As a bonus, we use our argumentation in the earlier sections to prove in Section 9 that there are *universal* test supermartingales and *universal* randomness tests for our generalisations of Martin-Löf randomness.

This paper unites results from two distinct areas of research, imprecise and game-theoretic probabilities on the one hand and algorithmic randomness on the other. We realise that the intersection of both research communities is fairly small, and we've therefore tried to make the introductory discussion in Sections 2 and 3 as self-contained as possible, by including relevant results and even proofs from both research fields, in order to make it serve as a footbridge between them.

In order not to interrupt the flow of the discussion too much, we've moved all proofs to appendices: the proofs for the introductory discussion, which are based on material in the literature, to Appendix A, and the proofs of new results to Appendix B.

2. FORECASTING SYSTEMS, SUPERMARTINGALES AND UPPER EXPECTATIONS

2.1. **Forecast for a single outcome.** Let's begin by describing a single forecast as a game played by two players, a *Forecaster* and a *Sceptic*.[2]

We consider a *variable* X that may assume any of the two values in the doubleton $\{0,1\}$, and whose actual value is initially unknown.

A Forecaster specifies an interval bound $I = [\underline{p}, \overline{p}] \subseteq [0,1]$ for the expectation of X—or equivalently, for the probability that $X = 1$. This *interval forecast* I is interpreted as a commitment for Forecaster to adopt $\underline{p}$ as his *maximum acceptable buying price* and $\overline{p}$ as his *minimum acceptable selling price* for the uncertain reward (also called *gamble*) X—expressed in units of some linear utility scale, called *utiles*.[3]

We take this to imply that *Forecaster* commits to offering to some *Sceptic* (any combination of) the following gambles, whose uncertain pay-offs are also expressed in utiles:

(i) for all real $q \leq \underline{p}$ and all real $\alpha \geq 0$, Forecaster offers the gamble $\alpha[q - X]$ to Sceptic;
(ii) for all real $r \geq \overline{p}$ and all real $\beta \geq 0$, Forecaster offers the gamble $\beta[X - r]$ to Sceptic.

Sceptic can then pick any combination of the gambles offered to him by Forecaster, or in other words, she accepts the gamble (with reward function)

$$\alpha[q - X] + \beta[X - r] \text{ for some choice of } q \leq \underline{p},\ r \geq \overline{p} \text{ and } \alpha, \beta \geq 0.$$

Then finally, when the actual value x of the variable X in $\{0,1\}$ becomes known to both Forecaster and Sceptic, the corresponding reward $\alpha[q - x] + \beta[x - r]$ is paid by Forecaster to Sceptic.

This game already allows us to introduce some of the terminology, definitions and notation that we'll use further on. We call elements x of $\{0,1\}$ *outcomes*, and elements p of the real unit interval $[0,1]$ serve as *precise forecasts*. We denote by $\mathscr{I}$ the set of all *imprecise*, or *interval*, *forecasts* I: non-empty and closed subintervals of the real unit interval $[0,1]$. Any interval forecast I has a smallest element $\min I$ and a greatest element $\max I$, so $I = [\min I, \max I]$. We'll also use the generic notations $\underline{p} := \min I$ and $\overline{p} := \max I$ for its lower and upper bound, respectively. Clearly, an interval forecast $I = [\underline{p}, \overline{p}]$ is precise when $\underline{p} = \overline{p} =: p$, and we then make *no distinction* between a singleton interval forecast $\{p\} \in \mathscr{I}$ and the corresponding precise forecast $p \in [0,1]$; this also means we'll consider the set of precise forecasts $[0,1]$ to be a subset of the set of imprecise forecasts $\mathscr{I}$.

Any *gamble* on the variable X is completely determined by its reward (in utiles) when $X = 1$ and when $X = 0$. It can therefore be represented as a map $f\colon \{0,1\} \to \mathbb{R}$, or equivalently, as a point $(f(1), f(0))$ in the two-dimensional linear space $\mathbb{R}^2$. We denote the set of all such maps $f\colon \{0,1\} \to \mathbb{R}$ by $\mathscr{G}(\{0,1\})$. The gamble $f(X)$ is then the corresponding (possibly negative) increase in Sceptic's capital, as a function of the variable X. As we indicated above, the gambles $f(X)$ that Forecaster actually offers to Sceptic as a result of his interval forecast I constitute a closed convex cone $\mathscr{A}_I$ in $\mathbb{R}^2$:

$$\mathscr{A}_I := \left\{ \alpha[q - X] + \beta[X - r] \colon q \leq \underline{p},\ r \geq \overline{p} \text{ and } \alpha, \beta \in \mathbb{R}_{\geq 0} \right\}, \tag{1}$$

where we use $\mathbb{R}_{\geq 0}$ to denote the set of non-negative real numbers.

It turns out that this cone is quite easily characterised by an upper expectation functional, as we'll now explain. It won't surprise the reader if we associate with any precise forecast $p \in [0,1]$ the *expectation* (functional) E_p, defined by

$$E_p(f) := pf(1) + (1-p)f(0) \text{ for any gamble } f\colon \{0,1\} \to \mathbb{R}.$$

But it so happens that we can just as well associate (lower and upper) expectation functionals with an interval forecast $I \in \mathscr{I}$. The *lower expectation* (functional) $\underline{E}_I$ associated

[2]The names *Sceptic* and *Forecaster* are borrowed from Shafer and Vovk's work [26, 27].

[3]Our exposition here uses *maximum* rather than the more common [41] *supremum* acceptable buying prices, and *minimum* rather the more common *infimum* acceptable selling prices. We show in Ref. [4, App. A] that the difference is of no consequence.

with I is defined by

$$\underline{E}_I(f) := \min_{p\in I} E_p(f) = \min_{p\in I}\big[pf(1)+(1-p)f(0)\big] = \begin{cases} E_{\underline{p}}(f) & \text{if } f(1)\geq f(0) \\ E_{\overline{p}}(f) & \text{if } f(1)\leq f(0) \end{cases}$$
$$\text{for any gamble } f\in\mathscr{G}(\{0,1\}),$$

and similarly, the *upper expectation* (functional) $\overline{E}_I$ is defined by

$$\overline{E}_I(f) := \max_{p\in I} E_p(f) = \begin{cases} E_{\overline{p}}(f) & \text{if } f(1)\geq f(0) \\ E_{\underline{p}}(f) & \text{if } f(1)\leq f(0) \end{cases} = -\underline{E}_I(-f)$$
$$\text{for any gamble } f\in\mathscr{G}(\{0,1\}), \quad (2)$$

where the last equality identifies the *conjugacy* relationship between the lower and upper expectations $\underline{E}_I$ and $\overline{E}_I$. If we now combine the characterisation (1) of the gambles available to Sceptic with the properties of the upper expectation $\overline{E}_I$, listed in Proposition 1 below, then it is easy to see that[4]

$$f(X)\in\mathscr{A}_I \Leftrightarrow \overline{E}_I(f)\leq 0, \text{ for all } f\in\mathscr{G}(\{0,1\}).$$

In fact, the condition $\overline{E}_I(f)\leq 0$ is equivalent to $(\forall p\in I)E_p(f)\leq 0$, so the convex cone of all available gambles is the intersection of all half-planes determined by $E_p(f)\leq 0$ for all $p\in I$.

The functionals $\underline{E}_I$ and $\overline{E}_I$ have the following fairly immediate *coherence properties*, typical for the more general lower and upper expectation functionals defined on arbitrary gamble spaces [37, 41]; see also Proposition 2 further on.

Proposition 1. *Consider any interval forecast $I\in\mathscr{I}$. Then for all gambles $f,g\in\mathscr{G}(\{0,1\})$, all $\mu\in\mathbb{R}$ and all $\lambda\in\mathbb{R}_{\geq 0}$:*

C1. $\min f\leq \underline{E}_I(f)\leq \overline{E}_I(f)\leq \max f$; [bounds]

C2. $\overline{E}_I(\lambda f)=\lambda\overline{E}_I(f)$ and $\underline{E}_I(\lambda f)=\lambda\underline{E}_I(f)$; [non-negative homogeneity]

C3. $\overline{E}_I(f+g)\leq\overline{E}_I(f)+\overline{E}_I(g)$ and $\underline{E}_I(f+g)\geq\underline{E}_I(f)+\underline{E}_I(g)$; [sub/super-additivity]

C4. $\overline{E}_I(f+\mu)=\overline{E}_I(f)+\mu$ and $\underline{E}_I(f+\mu)=\underline{E}_I(f)+\mu$; [constant additivity]

C5. if $f\leq g$ then $\overline{E}_I(f)\leq\overline{E}_I(g)$ and $\underline{E}_I(f)\leq\underline{E}_I(g)$; [monotonicity]

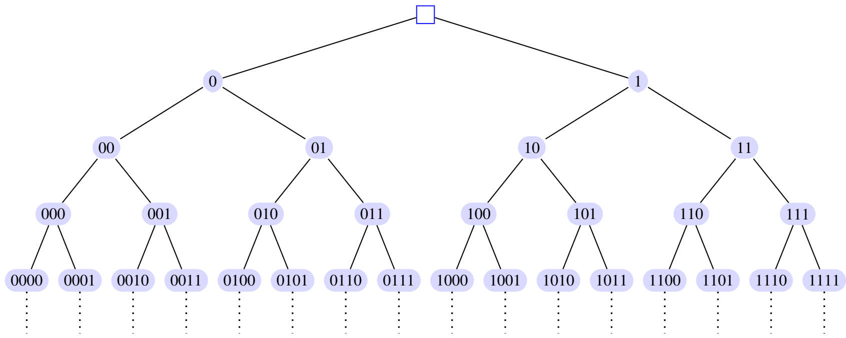
C6. if the sequence f_n of gambles in $\mathscr{G}(\{0,1\})$ converges uniformly to the gamble f, then $\overline{E}_I(f_n)\to\overline{E}_I(f)$ and $\underline{E}_I(f_n)\to\underline{E}_I(f)$. [uniform continuity]

2.2. **Forecasting for a sequence of outcomes: event trees and forecasting systems.** In a next step, we extend this set-up by considering a sequence of repeated versions of the forecasting game in the previous section. The ideas behind this extension are rather straightforward and can be sketched as follows. At each successive stage $k\in\mathbb{N}$, Forecaster presents an interval forecast $I_k=[\underline{p}_k,\overline{p}_k]$ for the unknown variable X_k. This effectively allows Sceptic to choose any gamble $f_k(X_k)$ such that $\overline{E}_{I_k}(f_k)\leq 0$. When the value x_k for X_k becomes known, this results in a gain in capital $f_k(x_k)$ for Sceptic at stage k. This gain $f_k(x_k)$ can, of course, be negative, resulting in an actual decrease in Sceptic's capital. Here and in what follows, $\mathbb{N}$ denotes the set of all natural numbers, without zero. We'll also use the notation $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ for the set of all non-negative integers.

Let's now describe the formal framework that will allow us to better investigate several interesting aspects of this extended forecasting set-up.

We call $(x_1,x_2,\dots,x_n,\dots)$ an *outcome sequence*, and collect all such outcome sequences in the set $\Omega:=\{0,1\}^{\mathbb{N}}$. Finite outcome sequences $x_{1:n}:=(x_1,\dots,x_n)$ are collected in the set $\mathbb{S}:=\{0,1\}^*=\bigcup_{n\in\mathbb{N}_0}\{0,1\}^n$. Such finite outcome sequences s in $\mathbb{S}$ and infinite outcome sequences ω in Ω constitute the nodes—called *situations*—and *paths* in an event tree with unbounded horizon, partially depicted below.

[4] The proof is straightforward; see also Refs. [9] and [21, Prop. 2].



The empty sequence $x_{1:0} =: \square$ is also called the *initial* situation. Any path $\omega \in \Omega$ is an infinite outcome sequence, and can therefore be identified with (the binary expansion of) a real number in the unit interval $[0,1]$.

For any path $\omega \in \Omega$, the initial sequence that consists of its first n elements is a situation in $\{0,1\}^n$ that is denoted by $\omega_{1:n}$. Its n-th element belongs to $\{0,1\}$ and is denoted by ω_n. As a convention, we let its 0-th element be the *initial* situation $\omega_{1:0} = \omega_0 := \square$.

For any situation $s \in \mathbb{S}$ and any path $\omega \in \Omega$, we say that ω *goes through* s if there's some $n \in \mathbb{N}_0$ such that $\omega_{1:n} = s$. We denote by $[\![s]\!]$ the so-called *cylinder set* of all paths $\omega \in \Omega$ that go through s. More generally, if $S \subseteq \mathbb{S}$ is some set of situations, then we denote by $[\![S]\!] := \bigcup_{s \in S} [\![s]\!]$ the set of all paths that go through some situation in S.

We write that $s \sqsubseteq t$, and say that the situation s *precedes* the situation t, when every path that goes through t also goes through s—so s is a precursor of t. An equivalent condition is of course that $[\![t]\!] \subseteq [\![s]\!]$. We may then also write $t \sqsupseteq s$ and say that t *follows* s.

We say that the situation s *strictly precedes* the situation t, and write $s \sqsubset t$, when $s \sqsubseteq t$ and $s \neq t$, or equivalently, when $[\![t]\!] \subset [\![s]\!]$.

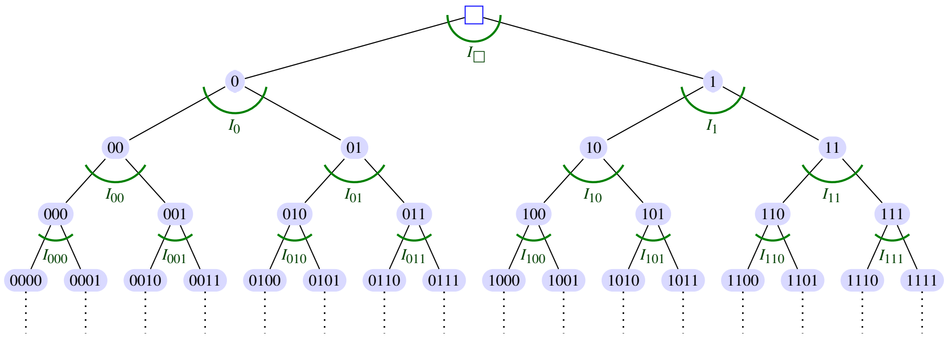
Finally, we say that two situations s and t are *incomparable*, and write $s \parallel t$, when neither $s \sqsubseteq t$ nor $t \sqsubseteq s$, or equivalently, when $[\![s]\!] \cap [\![t]\!] = \emptyset$, so there's no path that goes through both s and t.

For any situation $s = (x_1, \dots, x_n) \in \mathbb{S}$, we call $n = |s|$ its depth in the tree, so $|s| \geq |\square| = 0$. We use a similar notational convention for situations as for paths: we let $s_k := x_k$ and $s_{1:k} := (x_1, \dots, x_k)$ for all $k \in \{1, \dots, n\}$, and $s_{1:0} = s_0 := \square$. Also, for any $x \in \{0,1\}$, we denote by sx the situation $(x_1, \dots, x_n, x)$.

A subset K of $\mathbb{S}$ is called a *partial cut*—the term '*prefix free set*' is also commonly used in the algorithmic randomness literature—if its elements are mutually incomparable, or in other words constitute an anti-chain for the partial order $\sqsubseteq$, meaning that $s \parallel t$, or equivalently, $[\![s]\!] \cap [\![t]\!] = \emptyset$, for all $s, t \in K$ with $s \neq t$. With such a partial cut K, there corresponds a set $[\![K]\!] := \bigcup_{s \in K} [\![s]\!]$, which contains all paths that go through (some situation in) K, and the corresponding collection of cylinder sets $\{[\![s]\!] \colon s \in K\}$ constitutes a partition of $[\![K]\!]$.

For any situation s and any partial cut K, there are a number of possibilities. We say that s *precedes* K, and write $s \sqsubseteq K$, if s precedes some situation in K: $(\exists t \in K) s \sqsubseteq t$. Similarly, we say that s *strictly precedes* K, and write $s \sqsubset K$, if s strictly precedes some situation in K: $(\exists t \in K) s \sqsubset t$. We say that s *follows* K, and write $s \sqsupseteq K$, if s follows some—then necessarily unique—situation in K: $(\exists t \in K) s \sqsupseteq t$. Similarly for s *strictly follows* K, written as $s \sqsupset K$. Of course, the situations in K are the only ones that both precede and follow K. And, finally, we say that s is *incomparable* with K, and write $s \parallel K$, if s neither follows nor precedes (any situation in) K: $(\forall t \in K) s \parallel t$.

In the set-up described above, Forecaster only provides interval forecasts I_k after observing an actual sequence $(x_1, \dots, x_{k-1})$ of outcomes, and a corresponding sequence of available gambles $(f_1, \dots, f_{k-1})$ that Sceptic has chosen. This is the essence of *prequential* *forecasting* [6–8]. In our present discussion, it will be advantageous to consider an alternative setting where, before the start of the game, Forecaster specifies a forecast I_s in each of the possible situations s in the event tree $\mathbb{S}$; see the figure below.



This leads us to the notion of a forecasting system.

Definition 1 (Forecasting system)**.** A *forecasting system* is a map $\varphi\colon \mathbb{S}\to\mathscr{I}$ that associates an interval forecast $\varphi(s)\in\mathscr{I}$ with every situation s in the event tree $\mathbb{S}$. With any forecasting system φ we can associate two maps $\underline{\varphi},\overline{\varphi}\colon \mathbb{S}\to[0,1]$, defined by $\underline{\varphi}(s):=\min\varphi(s)$ and $\overline{\varphi}(s):=\max\varphi(s)$ for all $s\in\mathbb{S}$. A forecasting system φ is called *precise* if $\underline{\varphi}=\overline{\varphi}$. We denote by Φ the set $\mathscr{I}^{\mathbb{S}}$ of all forecasting systems and by Φ_{pr} its subset $[0,1]^{\mathbb{S}}$ of all precise forecasting systems.

We use the notation $\varphi\subseteq\varphi^*$ to express that the forecasting system φ^* is *at least as conservative* as φ, meaning that $\varphi(s)\subseteq\varphi^*(s)$ for all $s\in\mathbb{S}$.

In each situation $s\in\mathbb{S}$, the interval forecast $\varphi(s)$ corresponds to a so-called *local* upper expectation $\overline{E}_{\varphi(s)}$. These forecasts and their associated upper expectations allow us to turn the event tree into a so-called *imprecise probability tree*, with associated supermartingales and *global* upper (and lower) expectations. In the next two sections, we give a brief outline of how to do this. For more details, we refer to earlier papers [5, 12, 13], inspired by Shafer and Vovk's work [26–28, 39].

As an example, consider the following forecasting system borrowed from the imprecise probabilities literature, based on the *imprecise Dirichlet model* [11, 42, 43]:

$$\varphi(s):=\Big[\frac{n_1(s)}{n_0(s)+n_1(s)+\tau},\frac{n_1(s)+\tau}{n_0(s)+n_1(s)+\tau}\Big]\text{ for all }s\in\mathbb{S},$$

where τ is some strictly positive constant, $n_0(s)$ is the number of observations of outcome 0 in situation s, and $n_1(s)$ is the number of observations of outcome 1 in s. It starts with the vacuous forecast $\varphi(\square)=[0,1]$, becomes more precise as more observations are made, and converges to the observed relative frequency of ones in the long run.

2.3. **Supermartingales.** Recall that we use a forecasting system φ to identify Forecaster's forecasts $\varphi(s)$ in each of the possible situations $s\in\mathbb{S}$. In a similar way, we can introduce a strategy as a way to identify Sceptic's choice of gamble in each of the situations.

Definition 2 (Strategy)**.** A *strategy* is a map $\sigma\colon\mathbb{S}\to\mathscr{G}(\{0,1\})$. It allows us to associate a gamble $\sigma(s)\in\mathscr{G}(\{0,1\})$ with each situation s in the event tree $\mathbb{S}$. We call a strategy σ *compatible with a forecasting system* φ if it only selects gambles that are offered by the (Forecaster with) forecasting system φ in the sense that $\overline{E}_{\varphi(s)}(\sigma(s))\leq 0$ for all $s\in\mathbb{S}$.

We infer from the example of strategies and forecasting systems above that it can be useful to associate objects with situations, or in other words, to consider maps on $\mathbb{S}$. We'll call any map F defined on $\mathbb{S}$ a *process*. We now discuss other useful special cases besides forecasting systems and strategies.

A *real process* is a real-valued process: it associates a real number $F(s)\in\mathbb{R}$ with every situation $s\in\mathbb{S}$. Similarly, a *rational process* is a process that assumes values in the set $\mathbb{Q}$

of all rational numbers, and is therefore a special real process. A real process is called *non-negative* if it is non-negative in all situations, and a *positive* real process is (strictly) positive in all situations.

With any real process F, we can associate a process ΔF, called its *process difference*, defined as follows: for every situation $s \in \mathbb{S}$, $\Delta F(s)$ is the gamble on $\{0,1\}$ defined by

$$\Delta F(s)(x) := F(sx) - F(s) \text{ for all } x \in \{0,1\},$$

or in shorthand, with obvious notations, $\Delta F(s) = F(s\cdot) - F(s)$, where the '$\cdot$' in '$s\cdot$' is a placeholder for any element x of $\{0,1\}$. The *initial value* of a process F is its value $F(\square)$ in the initial situation $\square$. Clearly, a real process is completely determined by its initial value and its process difference, since

$$F(x_1,\dots,x_n) = F(\square) + \sum_{k=0}^{n-1} \Delta F(x_1,\dots,x_k)(x_{k+1}) \text{ for all } (x_1,\dots,x_n) \in \mathbb{S}.$$

Now, if we consider any strategy σ for Sceptic, then for any $s \in \mathbb{S}$,

$$F_\sigma(s) := F_\sigma(\square) + \sum_{k=0}^{|s|-1} \sigma(s_{1:k})(s_{k+1})$$

is the capital she has accumulated in situation s by starting in the initial situation $\square$ with initial capital $F_\sigma(\square)$ and selecting the gamble $\sigma(s_{1:k})$ in each of the situations $s_{1:k}$ strictly preceding s. This tells us that, as soon as we fix the initial values $F(\square)$, there's a one-to-one correspondence between real processes F and strategies σ by letting $\sigma \mapsto F_\sigma$ and, conversely, $F \mapsto \sigma_F := \Delta F$. Any real process F can therefore be seen as a capital process for Sceptic, generated by a suitably chosen strategy $\sigma := \Delta F$ and initial capital $F(\square)$.

We now turn to the special case of the capital processes for those strategies that are compatible with a given forecasting system φ. A *supermartingale* M for φ is a real process such that

$$\overline{E}_{\varphi(s)}(\Delta M(s)) \leq 0, \text{ or equivalently, } \overline{E}_{\varphi(s)}(M(s\cdot)) \leq M(s), \text{ for all } s \in \mathbb{S}, \tag{3}$$

or in other words, such that the corresponding strategy $\sigma_M := \Delta M$ is compatible with φ. Supermartingale differences have non-positive upper expectation, so roughly speaking supermartingales are real processes that Forecaster expects to decrease.

A real process M is a *submartingale* for φ if $-M$ is a supermartingale, meaning that $\underline{E}_{\varphi(s)}(\Delta M(s)) \geq 0$ for all $s \in \mathbb{S}$. Submartingale differences have non-negative lower expectation, so roughly speaking submartingales are real processes that Forecaster expects to increase. We denote the set of all supermartingales for a given forecasting system φ by $\overline{\mathbb{M}}^\varphi$, and $\underline{\mathbb{M}}^\varphi := -\overline{\mathbb{M}}^\varphi$ denotes the set of all submartingales for φ.

We call *test supermartingale* for φ any non-negative supermartingale M for φ with initial value $M(\square) = 1$. These test supermartingales will play a crucial part further on in this paper. They correspond to the capital processes that Sceptic can build by starting with unit capital and selecting, in each situation, a gamble that is offered there as a result of Forecaster's specification of the forecasting system φ, and that make sure that she never needs to resort to borrowing.

2.4. **Upper expectations.** A *gamble* on Ω, also called a *global gamble*, is a bounded real-valued map defined on the so-called *sample space*, which is the set Ω of all paths. We denote the set of all global gambles by $\mathscr{G}(\Omega)$. A *global event* G is a subset of Ω, and its *indicator* $\mathbb{I}_G$ is the gamble on Ω that assumes the value 1 on G and 0 elsewhere.

The super(- and sub)martingales for a forecasting system φ can be used to associate so-called *global* conditional upper and lower expectation functionals—defined on global gambles—with the forecasting system φ:[5]

$$\overline{E}^{\varphi}(g|s) := \inf\big\{M(s)\colon M \in \overline{\mathbb{M}}^{\varphi} \text{ and } \liminf M \geq_s g\big\} \tag{4}$$

$$\underline{E}^{\varphi}(g|s) := \sup\big\{M(s)\colon M \in \underline{\mathbb{M}}^{\varphi} \text{ and } \limsup M \leq_s g\big\} \tag{5}$$

for all gambles g on Ω and all situations $s \in \mathbb{S}$. In these expressions, we use the notations

$$\liminf M(\omega) := \liminf_{n\to\infty} M(\omega_{1:n}) \text{ and } \limsup M(\omega) := \limsup_{n\to\infty} M(\omega_{1:n}) \text{ for all } \omega \in \Omega,$$

and take $g \geq_s h$ to mean that the global gamble g dominates the global gamble h on the cylinder set $[\![s]\!]$—in all paths through s—or in other words that $(\forall \omega \in [\![s]\!])g(\omega) \geq h(\omega)$. Similarly, $g \leq_s h$ means that $(\forall \omega \in [\![s]\!])g(\omega) \leq h(\omega)$. Thus, for instance, $\overline{E}^{\varphi}(g|s)$ is the infimum capital that Sceptic needs to start with in situation s in order to be able to hedge the gamble g on all paths that go through s.

In the particular case that $s = \square$, we find the (so-called *unconditional*) upper and lower expectations $\overline{E}^{\varphi} := \overline{E}^{\varphi}(\cdot|\square)$ and $\underline{E}^{\varphi} := \underline{E}^{\varphi}(\cdot|\square)$.

Upper and lower expectations are clearly related to each other through *conjugacy*:

$$\underline{E}^{\varphi}(g|s) = -\overline{E}^{\varphi}(-g|s) \text{ for all gambles } g \text{ on } \Omega \text{ and all situations } s \in \mathbb{S}. \tag{6}$$

These upper and lower expectations satisfy a number of very useful properties, which we list below. We'll make repeated use of them in what follows, and we provide most of their proofs in the Appendix A for the sake of completeness and easy reference, even if proofs for similar results can also be found elsewhere [5, 26, 27, 31, 34, 36].

For any global gamble g and any situation $s \in \mathbb{S}$, we'll use the notations $\inf(g|s) := \inf\{g(\omega)\colon \omega \in [\![s]\!]\}$ and $\sup(g|s) := \sup\{g(\omega)\colon \omega \in [\![s]\!]\}$. Observe that then $\inf(g|\square) = \inf g$ and $\sup(g|\square) = \sup g$. Also, with any so-called *local gamble* f on $\{0,1\}$ and any situation $s \in \mathbb{S}$, we associate the global gamble f_s, defined by

$$f_s(\omega) := \begin{cases} f(x) & \text{if } \omega \in [\![sx]\!] \text{ with } x \in \{0,1\} \\ 0 & \text{otherwise, so if } \omega \notin [\![s]\!] \end{cases} \text{ for all } \omega \in \Omega.$$

Proposition 2 (Properties of upper/lower expectations)**.** *Consider any forecasting system $\varphi \in \Phi$. Then for all gambles g, g_n, h on Ω, with $n \in \mathbb{N}_0$, for all gambles f on $\{0,1\}$, all $\lambda \in \mathbb{R}_{\geq 0}$, and all situations $s \in \mathbb{S}$:*

E1. $\inf(g|s) \leq \underline{E}^{\varphi}(g|s) \leq \overline{E}^{\varphi}(g|s) \leq \sup(g|s)$; [bounds]

E2. $\overline{E}^{\varphi}(\lambda g|s) = \lambda \overline{E}^{\varphi}(g|s))$ *and* $\underline{E}^{\varphi}(\lambda g|s) = \lambda \underline{E}^{\varphi}(g|s)$; [non-negative homogeneity]

E3. $\underline{E}^{\varphi}(g|s) + \underline{E}^{\varphi}(h|s) \leq \underline{E}^{\varphi}(g+h|s) \leq \underline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(h|s) \leq \overline{E}^{\varphi}(g+h|s) \leq \overline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(h|s)$; [mixed sub/super-additivity]

E4. $\overline{E}^{\varphi}(g+h|s) = \overline{E}^{\varphi}(g|s) + h_s$ *and* $\underline{E}^{\varphi}(g+h|s) = \underline{E}^{\varphi}(g|s) + h_s$ *if h assumes the constant value h_s on $[\![s]\!]$*; [constant additivity]

E5. $\overline{E}^{\varphi}(g|s) = \overline{E}^{\varphi}(g\mathbb{I}_{[\![s]\!]}|s)$ *and* $\underline{E}^{\varphi}(g|s) = \underline{E}^{\varphi}(g\mathbb{I}_{[\![s]\!]}|s)$; [restriction]

E6. *if* $g \leq_s h$ *then* $\overline{E}^{\varphi}(g|s) \leq \overline{E}^{\varphi}(h|s)$ *and* $\underline{E}^{\varphi}(g|s) \leq \underline{E}^{\varphi}(h|s)$; [monotonicity]

E7. $\overline{E}^{\varphi}(f_s|s) = \overline{E}_{\varphi(s)}(f)$ *and* $\underline{E}^{\varphi}(f_s|s) = \underline{E}_{\varphi(s)}(f)$; [locality]

E8. $\overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot)) = \overline{E}^{\varphi}(g|s)$ *and* $\underline{E}_{\varphi(s)}(\underline{E}^{\varphi}(g|s\,\cdot)) = \underline{E}^{\varphi}(g|s)$; [sub/super-martingale]

E9. *if $g_n \nearrow g$ point-wise on $[\![s]\!]$, then* $\overline{E}^{\varphi}(g|s) = \sup_{n\in\mathbb{N}_0} \overline{E}^{\varphi}(g_n|s)$. [convergence]

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[5]Several related expressions appear in the literature, the domain of which typically also includes unbounded and even extended real-valued functions on Ω; see for example Refs. [36, Def. 3] and [35, p. 12]. These expressions are similar, but require supermartingales to be bounded below and submartingales to be bounded above, and often allow both to be extended real-valued. When applied to gambles, however, all of these expressions are equivalent; see Refs. [5, Prop. 10] and [36, Prop. 36]. This allows us to apply properties that were proved for these alternative expressions in our context as well. In particular, we'll make use of Ref. [36, Thm. 23] in our proof of Proposition 2, Ref. [35, Prop. 10 and Thm. 6] in our proof of Proposition 7 and Ref. [32, Thm. 13] in our proof of Theorem 19.

Property E7 essentially shows that the global models are extensions of the local ones. Property E8 shows in particular that for any global gamble g, the real process $\overline{E}^{\varphi}(g|\bullet)$ is a supermartingale for φ.

Extensive discussion in related contexts about why expressions such as (4) and (5) are interesting as well as useful, can be found in Refs. [5, 13, 26, 27, 31–36].[6] We mention explicitly that for precise forecasting systems, they result in models that coincide with the ones found in measure-theoretic probability theory; see Refs. [31, 35], and the brief discussion in Section 2.5 further on. Related results can also be found in Refs. [26, Ch. 8] and [27, Ch. 9]. In particular, for the precise *fair-coin forecasting system* $\varphi_{1/2}$, where all local forecasts equal $1/2$, these models coincide on all measurable global gambles with the usual uniform (Lebesgue) expectations. More generally, for an interval-valued forecasting system φ, the upper and lower expectation $\overline{E}^{\varphi}$ and $\underline{E}^{\varphi}$ provide tight upper and lower bounds on the measure-theoretic expectation of measurable global gambles for every precise forecasting system φ_{pr} that is compatible with φ, in the sense that $\varphi_{\mathrm{pr}} \subseteq \varphi$; see Refs. [31, 32] for related discussion and, in particular, Theorem 13 in Ref. [32] and Theorem 5.5.10 in Ref. [31].

For any global event $G \subseteq \Omega$ and any situation $s \in \mathbb{S}$, the corresponding (conditional) upper and lower *probabilities* are defined by $\overline{P}^{\varphi}(G|s) := \overline{E}^{\varphi}(\mathbb{I}_G|s)$ and $\underline{P}^{\varphi}(G|s) := \underline{E}^{\varphi}(\mathbb{I}_G|s)$. The following *conjugacy relationship for global events* follows at once from E4:

$$\underline{P}^{\varphi}(G|s) = 1 - \overline{P}^{\varphi}(G^c|s) \text{ for all } G \subseteq \Omega \text{ and } s \in \mathbb{S},$$

where $G^c := \Omega \setminus G$ is the complement of the global event G.

We'll have occasion to use the following direct corollary a number of times.

Corollary 3. *Consider any forecasting system φ, any partial cut $K \subseteq \mathbb{S}$, and any $s \in \mathbb{S}$. Then the following statements hold for the real process $\overline{P}^{\varphi}([\![K]\!]|\bullet)$:*
- (i) $\overline{P}^{\varphi}([\![K]\!]|s) = \overline{E}_{\varphi(s)}(\overline{P}^{\varphi}([\![K]\!]|s\,\cdot))$*;*
- (ii) $\overline{P}^{\varphi}([\![K]\!]|\bullet)$ *is a supermartingale for φ;*
- (iii) $0 \leq \overline{P}^{\varphi}([\![K]\!]|s) \leq 1$*;*
- (iv) $s \sqsupseteq K \Rightarrow \overline{P}^{\varphi}([\![K]\!]|s) = 1$ *and* $s \parallel K \Rightarrow \overline{P}^{\varphi}([\![K]\!]|s) = 0$*;*
- (v) $\liminf \overline{P}^{\varphi}([\![K]\!]|\bullet) \geq \mathbb{I}_{[\![K]\!]}$.

It will also prove useful to have expressions for the upper and lower probabilities of the cylinder sets. Unlike those for more general global events, they turn out to be particularly simple and elegant.

Proposition 4. *Consider any forecasting system $\varphi \in \Phi$ and any situation $s \in \mathbb{S}$, then*

$$\overline{P}^{\varphi}([\![s]\!]) = \prod_{k=0}^{|s|-1} \overline{\varphi}(s_{1:k})^{s_{k+1}} [1 - \underline{\varphi}(s_{1:k})]^{1-s_{k+1}}$$

$$\underline{P}^{\varphi}([\![s]\!]) = \prod_{k=0}^{|s|-1} \underline{\varphi}(s_{1:k})^{s_{k+1}} [1 - \overline{\varphi}(s_{1:k})]^{1-s_{k+1}}.$$

The idea for the following elegant and powerful inequality, in its simplest form, is due to Ville [38].

Proposition 5 (Ville's inequality [26, 27])**.** *Consider any forecasting system φ, any non-negative supermartingale T for φ, and any $C > 0$, then*

$$\overline{P}^{\varphi}\left(\left\{\omega \in \Omega \colon \sup_{n \in \mathbb{N}_0} T(\omega_{1:n}) \geq C\right\}\right) \leq \frac{1}{C} T(\Box).$$

Finally, we can see that more conservative forecasting systems lead to more conservative (larger) upper expectations.

[6] See footnote 5 for more details.

Proposition 6. *Consider any two forecasting systems $\varphi,\psi\in\Phi$ such that $\varphi\subseteq\psi$. Then*
(i) *any supermartingale for ψ is also a supermartingale for φ, so $\overline{\mathbb{M}}^{\psi}\subseteq\overline{\mathbb{M}}^{\varphi}$;*
(ii) *$\overline{E}^{\varphi}(f|s)\leq\overline{E}^{\psi}(f|s)$ for all global gambles $g\in\mathscr{G}(\Omega)$ and all situations $s\in\mathbb{S}$.*

2.5. **Precise forecasting systems and probability measures.** Let's consider any forecasting system φ and any situation $s\in\mathbb{S}$. It's an immediate consequence of E1–E3 that the set

$$\mathscr{G}_{\varphi,s}(\Omega):=\{g\in\mathscr{G}(\Omega)\colon \underline{E}^{\varphi}(g|s)=\overline{E}^{\varphi}(g|s)\}$$

of all global gambles g whose (conditional) lower expectation $\underline{E}^{\varphi}(g|s)$ and (conditional) upper expectation $\overline{E}^{\varphi}(g|s)$ in s coincide, is a real linear space; we refer to Ref. [37, Chs. 8 and 9] for a closer study of such linear spaces.

For any such gamble $g\in\mathscr{G}_{\varphi,s}(\Omega)$, we call the common value

$$E^{\varphi}(g|s):=\underline{E}^{\varphi}(g|s)=\overline{E}^{\varphi}(g|s)$$

the (precise conditional) *expectation* of the global gamble g in the situation s. Similarly, for any global event $G\subseteq\Omega$ such that $\mathbb{I}_G\in\mathscr{G}_{\varphi,s}(\Omega)$, we call the common value

$$P^{\varphi}(G|s):=\underline{P}^{\varphi}(G|s)=\overline{P}^{\varphi}(G|s)$$

the (precise conditional) *probability* of the global event G in the situation s.

It's then again an immediate consequence of E1–E3 that

$$E^{\varphi}(\lambda f+\mu g|s)=\lambda E^{\varphi}(f|s)+\mu E^{\varphi}(g|s)\text{ for all }f,g\in\mathscr{G}_{\varphi,s}(\Omega)\text{ and }\lambda,\mu\in\mathbb{R},$$

so the expectation $E^{\varphi}(\bullet|s)$ is a real linear functional on the linear space $\mathscr{G}_{\varphi,s}(\Omega)$, which, by the way, contains all constant global gambles, by E1. That $E^{\varphi}(\bullet|s)$ is bounded in the sense of E1, normalised as a consequence of E1, and monotone in the sense of E6, also justifies our calling it an 'expectation'.

It's not hard to see that $\varphi\subseteq\psi$ implies that $\mathscr{G}_{\varphi,s}(\Omega)\supseteq\mathscr{G}_{\psi,s}(\Omega)$, so we gather that the more precise φ, the larger $\mathscr{G}_{\varphi,s}(\Omega)$. The linear space $\mathscr{G}_{\varphi,s}(\Omega)$ will be maximally large when φ is precise, or in other words when $\underline{\varphi}=\overline{\varphi}$.

Let's now assume that the forecasting system $\varphi=\varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}}$ is indeed *precise*, and take a better look at the linear space $\mathscr{G}_{\varphi_{\mathrm{pr}},\square}(\Omega)$ of those global gambles g that have a precise (unconditional) expectation $E^{\varphi_{\mathrm{pr}}}(g)=E^{\varphi_{\mathrm{pr}}}(g|\square)$. As is quite often done, we provide the set of all paths Ω with the Cantor topology, whose base is the collection of all cylinder sets $\{[\![s]\!]\colon s\in\mathbb{S}\}$; see for instance Ref. [14, Sec. 1.2]. All these cylinder sets $[\![s]\!]$ are clopen in this topology. The corresponding Borel algebra $\mathscr{B}(\Omega)$ is the σ-algebra generated by this Cantor topology.

We'll need the following proposition further on in Section 5.2 (and in particular Proposition 13) to show that our newly proposed notion of a Schnorr test for a forecasting system properly generalises Schnorr's original notion of a totally recursive sequential test for the fair-coin forecasting system $\varphi_{1/2}$, and in Section 7 to relate our version of Martin-Löf test randomness to uniform randomness.

Proposition 7. *Assume that the forecasting system $\varphi=\varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}}$ is precise. Then $\mathscr{G}_{\varphi_{\mathrm{pr}},\square}(\Omega)$ includes the linear space of all Borel measurable global gambles, and $E^{\varphi_{\mathrm{pr}}}$ corresponds on that space with the usual expectation of the countably additive probability measure given by Ionescu Tulcea's extension theorem [30, Thm. II.9.2]. In particular, for any partial cut $K\subseteq\mathbb{S}$, we have that $\mathbb{I}_{[\![K]\!]}\in\mathscr{G}_{\varphi_{\mathrm{pr}},\square}(\Omega)$ and*

$$P^{\varphi_{\mathrm{pr}}}([\![K]\!])=\sum_{s\in K}\prod_{k=0}^{|s|-1}\varphi_{\mathrm{pr}}(s_{1:k})^{s_{k+1}}[1-\varphi_{\mathrm{pr}}(s_{1:k})]^{1-s_{k+1}}.$$

As a direct consequence, we can associate with any precise forecasting system $\varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}}$ a probability measure $\mu^{\varphi_{\mathrm{pr}}}$ on the measurable space $(\Omega,\mathscr{B}(\Omega))$ defined by restricting the probability $P^{\varphi_{\mathrm{pr}}}$ to the Borel measurable events:

$$\mu^{\varphi_{\mathrm{pr}}}(G):=P^{\varphi_{\mathrm{pr}}}(G)=E^{\varphi_{\mathrm{pr}}}(\mathbb{I}_G)\text{ for all }G\in\mathscr{B}(\Omega).\tag{7}$$

If we consider any Borel measurable gamble g, then the result above tells us that

$$E^{\varphi_{\mathrm{pr}}}(g) = \int_\Omega g(\omega)\mathrm{d}\mu^{\varphi_{\mathrm{pr}}}(\omega).$$

3. Notions of computability

Computability theory studies what it means for a mathematical object to be implementable, or in other words, achievable by some computation on a machine. It considers as basic building blocks *partial recursive* natural maps $\phi\colon \mathbb{N}_0 \to \mathbb{N}_0$, which are maps that can be computed by a Turing machine. This means that there's some Turing machine that halts on the input $n \in \mathbb{N}_0$ and outputs the natural number $\phi(n) \in \mathbb{N}_0$ if $\phi(n)$ is defined, and doesn't halt otherwise. By the Church–Turing (hypo)thesis, this is equivalent to the existence of a finite algorithm that, given any input $n \in \mathbb{N}_0$, outputs the non-negative integer $\phi(n) \in \mathbb{N}_0$ if $\phi(n)$ is defined, and never finishes otherwise; in what follows, we'll often use this equivalence without mentioning it explicitly. If the Turing machine halts for all inputs $n \in \mathbb{N}_0$, that is, if the Turing machine computes the non-negative integer $\phi(n)$ in a finite number of steps for every $n \in \mathbb{N}_0$, then the map ϕ is defined for all arguments and we call it *total recursive*, or simply *recursive* [14, Ch. 2].

Instead of $\mathbb{N}_0$, we'll also consider functions with domain or codomain $\{0,1\}$, $\mathbb{N}$, $\mathbb{S}$, $\mathbb{S} \times \mathbb{N}_0$, $\mathbb{Q}$ or any other countable set $\mathscr{D}$ whose elements can be encoded by the natural numbers; the choice of encoding isn't important, provided we can algorithmically decide whether a natural number is an encoding of an object and, if this is the case, we can find an encoding of the same object with respect to the other encoding [29, p. xvi]. A function $\phi\colon \mathscr{D} \to \mathscr{D}'$ is then called *partial recursive* if there's a Turing machine that, when given the natural-valued encoding of any $d \in \mathscr{D}$, outputs the natural-valued encoding of $\phi(d) \in \mathscr{D}'$ if $\phi(d)$ is defined, and never halts otherwise. By the Church–Turing thesis, this is equivalent to the existence of a finite algorithm that, when given the input $d \in \mathscr{D}$, outputs the object $\phi(d) \in \mathscr{D}'$ if $\phi(d)$ is defined, and never finishes otherwise. If the Turing machine halts on all natural numbers that encode some element $d \in \mathscr{D}$, or equivalently, if the finite algorithm outputs an element $\phi(d) \in \mathscr{D}'$ for every $d \in \mathscr{D}$, then we call ϕ *total recursive*, or simply *recursive*. When $\mathscr{D}' = \mathbb{Q}$, then for any rational number $\alpha \in \mathbb{Q}$ and any two recursive rational maps $q_1, q_2\colon \mathscr{D} \to \mathbb{Q}$, the following rational maps are clearly recursive as well: $q_1 + q_2$, $q_1 \cdot q_2$, q_1/q_2 with $q_2(d) \neq 0$ for all $d \in \mathscr{D}$, $\max\{q_1, q_2\}$, αq_1 and $\lceil q_1 \rceil$. Since a finite number of finite algorithms can always be combined into one, it follows from the foregoing that the rational maps $\min\{q_1, q_2\}$ and $\lfloor q_1 \rfloor$ are also recursive.

We'll also consider notions of implementability for sets of objects. For any countable set $\mathscr{D}$ whose elements can be encoded by the natural numbers, a subset $D \subseteq \mathscr{D}$ is called *recursively enumerable* if there's a Turing machine that halts on every natural number that encodes an element $d \in D$, but never halts on any natural number that encodes an element $d \in \mathscr{D} \setminus D$ [14, Def. 2.2.1]. If both the set D and its complement $\mathscr{D} \setminus D$ are recursively enumerable, then we call D *recursive*. This is equivalent to the existence of a recursive indicator $\mathbb{I}_D\colon \mathscr{D} \to \{0,1\}$ that outputs 1 for all $d \in D$, and outputs 0 otherwise [14, p. 11]. For any indexed family $(D_{d'})_{d' \in \mathscr{D}'}$, with $D_{d'} \subseteq \mathscr{D}$ for all $d' \in \mathscr{D}'$ and $\mathscr{D}'$ a countable set whose elements can be encoded by the natural numbers, we say that $D_{d'}$ is *recursive(ly enumerable) effectively in $d' \in \mathscr{D}'$* if there's a recursive(ly enumerable) set $\mathfrak{D} \subseteq \mathscr{D}' \times \mathscr{D}$ such that $D_{d'} = \{d \in \mathscr{D}\colon (d', d) \in \mathfrak{D}\}$ for all $d' \in \mathscr{D}'$.

Countably infinite sets can also be used to come up with a notion of implementability for uncountably infinite sets of objects. Consider, as an example, a set of paths $G \subseteq \Omega$. It's called *effectively open* if there's some recursively enumerable set $A \subseteq \mathbb{S}$ such that $G = [\![A]\!]$. For any indexed family $(G_d)_{d \in \mathscr{D}}$, with $G_d \subseteq \Omega$ for all $d \in \mathscr{D}$, we say that G_d is *effectively open, effectively in $d \in \mathscr{D}$* if there's some recursively enumerable set $\mathfrak{D} \subseteq \mathscr{D} \times \mathbb{S}$ such that $G_d = \bigcup\{[\![s]\!] \subseteq \Omega\colon (d, s) \in \mathfrak{D}\}$ for all $d \in \mathscr{D}$.

Recursive functions and recursively enumerable sets can also be used to define notions of implementability for maps whose codomain is uncountably infinite, such as real-valued maps. For any countable set $\mathscr{D}$ whose elements can be encoded by the natural numbers, a real map $r\colon \mathscr{D} \to \mathbb{R}$ is called *lower semicomputable* if there's some recursive rational map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $q(d,n+1) \geq q(d,n)$ and $r(d) = \lim_{m\to\infty} q(d,m)$ for all $d \in \mathscr{D}$ and $n \in \mathbb{N}_0$. Equivalently, a real map $r\colon \mathscr{D} \to \mathbb{R}$ is lower semicomputable if and only if the set $\{(d,q) \in \mathscr{D} \times \mathbb{Q}\colon r(d) > q\}$ is recursively enumerable [14, Sec. 5.2]; in this case, we also say that the set $\{(d,x) \in \mathscr{D} \times \mathbb{R}\colon r(d) > x\}$ is *effectively open*.[7] A real map $r\colon \mathscr{D} \to \mathbb{R}$ is called *upper semicomputable* if $-r$ is lower semicomputable. If a real map $r\colon \mathscr{D} \to \mathbb{R}$ is both lower and upper semicomputable, then we call it *computable*; we then also say that $r(d)$ is a computable real *effectively* in $d \in \mathscr{D}$. This is equivalent to the existence of a recursive rational map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|r(d) - q(d,N)| \leq 2^{-N}$ for all $d \in \mathscr{D}$ and $N \in \mathbb{N}_0$ [4, Props. 3 and 4]. It is also equivalent to the existence of two recursive maps $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{Q}$ and $e\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{N}_0$ such that $|r(d) - q(d,\ell)| \leq 2^{-N}$ for all $d \in \mathscr{D}$, $N \in \mathbb{N}_0$ and $\ell \geq e(d,N)$ [4, Prop. 3]. A real number $\alpha \in \mathbb{R}$ is then called computable if it is computable as a real map on a singleton. For any computable real number $\alpha \in \mathbb{R}$ and any two computable real maps $r_1, r_2\colon \mathscr{D} \to \mathbb{R}$, the following real maps are computable as well: $r_1 + r_2$, $r_1 \cdot r_2$, r_1/r_2 with $r_2(d) \neq 0$ for all $d \in \mathscr{D}$, $\max\{r_1, r_2\}$, αr_1, $\exp(r_1)$ and $\log_2(r_1)$ with $r_1(d) > 0$ for all $d \in \mathscr{D}$ [23, Sec. 0.2]. Moreover, a forecasting system $\varphi \in \Phi$ is called *computable* if the two real processes $\underline{\varphi}, \overline{\varphi}$ are computable.

Computable real maps can also be used to show that another real map $r\colon \mathscr{D} \to \mathbb{R}$ is computable or lower semicomputable. If there's some computable real map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{R}$ such that $|r(d) - q(d,N)| \leq 2^{-N}$ for all $d \in \mathscr{D}$ and $N \in \mathbb{N}_0$, then the real map r is computable and we say that q *converges effectively* to r [23, Sec 0.2]. Equivalently, the real map r is computable if and only if there's some computable real map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{R}$ and some recursive map $e\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{N}_0$ such that $|r(d) - q(d,\ell)| \leq 2^{-N}$ for all $d \in \mathscr{D}$, $N \in \mathbb{N}_0$ and $\ell \geq e(d,N)$, and we then also say that q converges effectively to r [23, Sec 0.2]. Finally, if there's some computable real map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{R}$ such that $q(d,n+1) \geq q(d,n)$ and $r(d) = \lim_{m\to\infty} q(d,m)$ for all $n \in \mathbb{N}_0$, then the real map r is lower semicomputable; since we haven't found an explicit proof for this last property in the relevant literature, we provide one in Appendix A.

Proposition 8. *Consider any countable set $\mathscr{D}$ whose elements can be encoded by the natural numbers. Then a real map $r\colon \mathscr{D} \to \mathbb{R}$ is lower semicomputable if there's a computable real map $q\colon \mathscr{D} \times \mathbb{N}_0 \to \mathbb{R}$ such that $q(d,n+1) \geq q(d,n)$ and $r(d) = \lim_{m\to\infty} q(d,m)$ for all $d \in \mathscr{D}$ and $n \in \mathbb{N}_0$.*

4. RANDOMNESS VIA SUPERMARTINGALES

We now turn to the martingale-theoretic notions of Martin-Löf and Schnorr randomness associated with an interval-valued forecasting system φ, which we borrow from our earlier paper on randomness and imprecision [4]. We limit ourselves here to a discussion of the definitions of these randomness notions, and refer to that earlier work for an extensive account of their properties, relevance and usefulness.

Definition 3 (Martin-Löf randomness [4])**.** Consider any forecasting system $\varphi\colon \mathbb{S} \to \mathscr{I}$ and any path $\omega \in \Omega$. We call ω *Martin-Löf random for* φ if all lower semicomputable test supermartingales T for φ remain bounded above on ω, meaning that there's some $B_T \in \mathbb{R}$ such that $T(\omega_{1:n}) \leq B_T$ for all $n \in \mathbb{N}_0$, or equivalently, that $\sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) < \infty$. We then also say that the forecasting system φ *makes* ω *Martin-Löf random.*

[7] This is the second time we encounter the term 'effectively open' in this section. Both definitions, the one for effectively open subsets of Ω and the one for effectively open subsets of $\mathscr{D} \times \mathbb{R}$, are instances of a general definition of effective openness; see for instance the appendix on Effective Topology in Ref. [40].

In other words, Martin-Löf randomness of a path means that there's no strategy leading to a lower semicomputable capital process that starts with unit capital and avoids borrowing, and that allows Sceptic to increase her capital without bounds by exploiting the bets on the outcomes along the path that are made available to her by Forecaster's specification of the forecasting system φ.

When the forecasting system φ is non-degenerate,[8] precise and computable, our definition reduces to that of *Martin-Löf randomness* on the Schnorr–Levin martingale-theoretic account.[9] We propose to continue speaking of Martin-Löf randomness also when φ is no longer precise, computable, or non-degenerate. We'll adopt the same proposal for the other randomness notions in this paper, and choose to do so because some of our results continue to hold under these weakened conditions; see in particular Propositions 14 and 20 further on.

We provide a clear motivation for allowing for non-computable forecasting systems $\varphi \in \Phi$ in this way—that is, without providing them as oracles—in Ref. [22], where we show that a path $\omega \in \Omega$ is Martin-Löf random for a *stationary* forecasting system $\varphi \in \Phi$ if and only if it is Martin-Löf random for at least one (possibly non-computable) compatible precise forecasting system $\varphi_{\mathrm{pr}} \subseteq \varphi$; a forecasting system $\varphi \in \Phi$ is called *stationary* if there's some interval forecast $I \in \mathscr{I}$ such that $\varphi(s) = I$ for all $s \in \mathbb{S}$, and then we also denote it by φ_I. That φ's non-computability is an essential ingredient for this result is made obvious by our Theorem 37 in Ref. [4], as that implies that for any stationary forecasting system φ_I with $\min I < \max I$, there is at least one path $\omega \in \Omega$ that is Martin-Löf random for φ, but not for any *computable* compatible precise forecasting system $\varphi_{\mathrm{pr}} \subseteq \varphi$.

We can also use the ideas in our earlier paper on randomness and imprecision [4] to extend Schnorr's original randomness definition [24, Ch. 9] for the afore-mentioned fair-coin forecasting system $\varphi_{1/2}$ to more general—not necessarily precise nor necessarily computable—forecasting systems. We begin with a definition borrowed from Schnorr's seminal work; see Refs. [24, Ch. 9] and [25].

Definition 4 (Growth function)**.** We call a map $\rho\colon \mathbb{N}_0 \to \mathbb{N}_0$ a *growth function* if
- (i) it is recursive;
- (ii) it is non-decreasing: $(\forall n_1, n_2 \in \mathbb{N}_0)(n_1 \leq n_2 \Rightarrow \rho(n_1) \leq \rho(n_2))$;
- (iii) it is unbounded.

We say that a real-valued map $\mu\colon \mathbb{N}_0 \to \mathbb{R}$ is *computably unbounded* if there's some growth function ρ such that $\limsup_{n\to\infty}[\mu(n) - \rho(n)] > 0$.

Clearly, if a real-valued map $\mu\colon \mathbb{N}_0 \to \mathbb{R}$ is computably unbounded, it is also unbounded above [4, Prop. 13]. Similarly to before, we choose to continue speaking of Schnorr randomness also when φ is no longer the precise, computable, and non-degenerate fair-coin forecasting system $\varphi_{1/2}$.

Definition 5 (Schnorr randomness [4])**.** Consider any forecasting system $\varphi\colon \mathbb{S} \to \mathscr{I}$ and any path $\omega \in \Omega$. We call ω *Schnorr random for* φ if no computable test supermartingale T for φ is computably unbounded on ω, or in other words, if $\limsup_{n\to\infty}[T(\omega_{1:n}) - \rho(n)] \leq 0$ for all computable test supermartingales T for φ and all growth functions ρ. We then also say that the forecasting system φ *makes* ω *Schnorr random*.

Clearly, Schnorr randomness is implied by Martin-Löf randomness. Furthermore, without any loss of generality, we can focus on recursive *positive* and *rational* test supermartingales in the definition above.

[8] Further on, we will define *non-degenerate* as never assuming the precise 'degenerate' values $\{0\}$ or $\{1\}$.

[9] For an historical overview with many relevant references, see Ref. [2]. Schnorr's martingale-theoretic definition focuses on the fair-coin forecasting system $\varphi_{1/2}$; see Ref. [24, Ch. 5]. Levin's approach [16, 45] works for computable probability measures (equivalent with computable precise forecasting systems), and uses semimeasures (equivalent with supermartingales). In these discussions, supermartingales may be infinite-valued, whereas we only allow for real-valued supermartingales, but this difference in approach has no consequences as long as the forecasting systems involved are non-degenerate; see also the discussion in Ref. [4, Sec. 5.3].

Proposition 9. *Consider any forecasting system φ and any path $\omega \in \Omega$. Then ω is Schnorr random for φ if and only if no recursive positive rational test supermartingale for φ is computably unbounded on ω.*

5. Randomness via randomness tests

Next, we turn to a 'measure-theoretic', or *randomness test*, approach to defining Martin-Löf and Schnorr randomness for (interval-valued) forecasting systems, which will be inspired by the existing corresponding notions for fair-coin, or more generally, computable precise forecasts [14, 16, 18, 24, 45].

To this end, we consider a forecasting system φ and the upper and lower expectations for global gambles associated with the corresponding imprecise probability tree, given by Equations (4) and (5).

5.1. **Martin-Löf tests.**

Let's begin our discussion of Martin-Löf tests with a few notational conventions that will prove useful for the remainder of this paper. With any subset A of $\mathbb{N}_0 \times \mathbb{S}$, we can associate a sequence A_n of subsets of $\mathbb{S}$, defined by

$$A_n := \{s \in \mathbb{S} \colon (n,s) \in A\} \text{ for all } n \in \mathbb{N}_0.$$

With each such A_n, we can associate the set of paths $[\![A_n]\!]$. If the set A is recursively enumerable, then we'll say that the $[\![A_n]\!]$ constitute a *computable sequence of effectively open sets*.[10]

The following definition trivially generalises the idea of a randomness test, as introduced by Martin-Löf [18], from the fair-coin forecasting system—and more generally from a computable precise forecasting system—to our present context. It will lead in Section 5.3 further on to a suitable generalisation of Martin-Löf's randomness definition that allows for *interval-valued* forecasting systems. Here too, we'll continue to speak of Martin-Löf tests also when φ is no longer precise, computable, or non-degenerate.

Definition 6 (Martin-Löf test)**.** We call a sequence of global events $G_n \subseteq \Omega$ a *Martin-Löf test* for a forecasting system φ if there's some recursively enumerable subset A of $\mathbb{N}_0 \times \mathbb{S}$ such that for the associated computable sequence of effectively open sets $[\![A_n]\!]$, we have that $G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$.

We may always—and often will—assume without loss of generality that the subsets A_n of the event tree $\mathbb{S}$ that constitute the Martin-Löf test are *partial cuts*. Moreover, we can even assume the set A to be *recursive* rather than merely recursively enumerable, because there's actually a single algorithm that turns any recursively enumerable set $B \subseteq \mathbb{S}$ into a recursive partial cut $B' \subseteq \mathbb{S}$ such that $[\![B]\!] = [\![B']\!]$. We refer to Ref. [14, Sec. 2.19] for discussion and proofs; see also the related discussions in Refs. [24, Korollar 4.10, p. 37] and [29, Lemma 2, Section 5.6].

Corollary 10. *A sequence of global events G_n is a Martin-Löf test for a forecasting system φ if and only if there's some* recursive *subset A of $\mathbb{N}_0 \times \mathbb{S}$ such that A_n is a* partial cut*, $G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$.*

In what follows, we'll also use the term *Martin-Löf test* to refer to a subset A of $\mathbb{N}_0 \times \mathbb{S}$ that *represents* the Martin-Löf test G_n in the specific sense that $G_n = [\![A_n]\!]$ for all $n \in \mathbb{N}_0$. Due to Corollary 10, we can always assume such subsets A of $\mathbb{N}_0 \times \mathbb{S}$ to be recursive, and the corresponding A_n to be partial cuts.

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[10] We've borrowed this terminology from Ref. [40]. For a justification of the term 'computable', we also refer to the discussion in Ref. [14, Sec. 2.19].

5.2. **Schnorr tests.** In order to propose a suitable generalisation of Schnorr's definition of a totally recursive sequential test [24, Def. (8.1), p. 63] for the fair-coin forecasting system $\varphi_{1/2}$, we need a few more notations. Starting from any subset A of $\mathbb{N}_0 \times \mathbb{S}$, we let

$$\left.\begin{aligned} A_n^{<\ell} &:= A_n \cap \{t \in \mathbb{S}\colon |t| < \ell\} \\ A_n^{\geq\ell} &:= A_n \cap \{t \in \mathbb{S}\colon |t| \geq \ell\} \end{aligned}\right\} \text{ for all } n, \ell \in \mathbb{N}_0. \tag{8}$$

In the important special case that A_n is a partial cut, the global event $[\![A_n]\!]$ is the disjoint union of the global events $[\![A_n^{<\ell}]\!]$ and $[\![A_n^{\geq\ell}]\!]$, implying that $\mathbb{I}_{[\![A_n]\!]} = \mathbb{I}_{[\![A_n^{<\ell}]\!]} + \mathbb{I}_{[\![A_n^{\geq\ell}]\!]}$.

Here as well, we'll continue to speak of Schnorr tests also when φ is no longer the precise, computable and non-degenerate $\varphi_{1/2}$.

Definition 7 (Schnorr test)**.** We call a sequence of global events $G_n \subseteq \Omega$ a *Schnorr test* for a forecasting system φ if there's some *recursive* subset A of $\mathbb{N}_0 \times \mathbb{S}$—called its *representation*—such that $G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and *additionally*, if there's some recursive map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$—called its *tail bound*—such that

$$\overline{P}^{\varphi}\big([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]\big) \leq 2^{-N} \text{ for all } (N,n) \in \mathbb{N}_0^2 \text{ and all } \ell \geq e(N,n). \tag{9}$$

As for the case of Martin-Löf tests, we can assume without loss of generality that the representation A is such that the A_n are partial cuts, at which point $[\![A_n]\!] \setminus [\![A_n^{<\ell}]\!] = [\![A_n^{\geq\ell}]\!]$ in Equation (9). Moreover, we can assume without loss of generality that there's no dependence of the tail bound e on the index n of the $[\![A_n^{\geq\ell}]\!]$. The proposition below also shows that these simplifications can be implemented independently.

Proposition 11. *Consider any Schnorr test G_n for a forecasting system φ with representation $C \subseteq \mathbb{N}_0 \times \mathbb{S}$. Then*

(i) *it also has a representation A such that A_n is a partial cut for all $n \in \mathbb{N}_0$;*
(ii) *it has a tail bound e that does not depend on the index n of the $[\![C_n]\!] \setminus [\![C_n^{<\ell}]\!]$, meaning that $e(N,n) = e(N,n') =: e(N)$ for all $N, n, n' \in \mathbb{N}_0$, and that moreover is a growth function.*

We'll also use the term *Schnorr test* to refer its representation A. So, a Schnorr test is a Martin-Löf test with the additional property that it is always assumed to be recursive rather than merely recursively enumerable, and that the upper probabilities of its 'tail global events' converge to zero effectively. As indicated above, we can, and often will, assume that the sets A_n are partial cuts and that the tail bound is a univariate growth function. But we'll never assume that these simplifications are in place without explicitly saying so.

Let's now investigate our notion of a Schnorr test in some more detail. First of all, we study how it relates to Schnorr's definition of a totally recursive sequential test [24, Def. (8.1), p. 63] for the (precise) fair-coin forecasting system $\varphi_{1/2}$ that associates a constant precise forecast $\varphi_{1/2}(s) := 1/2$ with each situation $s \in \mathbb{S}$.

Recall that Schnorr calls a recursive subset A of $\mathbb{N}_0 \times \mathbb{S}$ a *totally recursive sequential test* provided that $P^{\varphi_{1/2}}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and *additionally*, that the sequence of real numbers $P^{\varphi_{1/2}}([\![A_n]\!])$ is computable. Our additional condition (9) in Definition 7 above therefore seems somewhat more involved than Schnorr's additional computability requirement for the sequence $P^{\varphi_{1/2}}([\![A_n]\!])$.

Let's now show, by means of Propositions 12 and 13 below, that that is only an illusion. Indeed, in Proposition 12 we show that our additional condition (9) implies the Schnorr-like additional computability requirement, even in the case of more general computable *interval-valued* forecasting systems. And in Proposition 13, we prove that for general computable but *precise* forecasting systems the Schnorr-like additional requirement implies our additional effective convergence condition.

Proposition 12. *If $A \subseteq \mathbb{N}_0 \times \mathbb{S}$ is a Schnorr test for a computable forecasting system φ, then the $\overline{P}^{\varphi}([\![A_n]\!])$ constitute a computable sequence of real numbers.*

The next proposition is concerned with the special case of precise forecasting systems φ_{pr}. We recall from Proposition 7 [with $s = \square$] that the martingale-theoretic approach of defining global upper and lower expectations through Eqs. (4) and (5) then recovers the standard probability measure $P^{\varphi_{\mathrm{pr}}}$ associated with the local mass functions implicit in φ_{pr}, and that for each partial cut K, the corresponding set of paths $[\![K]\!]$ is Borel measurable, so $\underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = P^{\varphi_{\mathrm{pr}}}([\![K]\!])$. We'll use this fact implicitly and freely in the formulation and proof (in Appendix B) of the result below.

Proposition 13. *Consider a Martin-Löf test G_n for a computable* precise *forecasting system φ_{pr}. If the $P^{\varphi_{\mathrm{pr}}}(G_n)$ constitute a computable sequence of real numbers, then G_n is a Schnorr test.*

5.3. **Defining Martin-Löf and Schnorr test randomness.** With the definitions of Martin-Löf and Schnorr tests for a forecasting system at hand, we are now in a position to generalise both Martin-Löf's and Schnorr's definition for randomness using randomness tests, from fair-coin to interval-valued forecasting systems.

Definition 8 (Test randomness)**.** Consider a forecasting system φ. Then we call a sequence $\omega \in \Omega$

(i) *Martin-Löf test random* for φ if $\omega \notin \bigcap_{m \in \mathbb{N}_0} [\![A_m]\!]$, for all Martin-Löf tests A for φ;
(ii) *Schnorr test random* for φ if $\omega \notin \bigcap_{m \in \mathbb{N}_0} [\![A_m]\!]$, for all Schnorr tests A for φ.

We want to show in the next two sections that for forecasting systems that are *computable* and satisfy a simple additional non-degeneracy condition, our 'test' and 'martingale-theoretic' notions of both Martin-Löf and Schnorr randomness are equivalent.

6. Equivalence of Martin-Löf and Martin-Löf test randomness

Let's start by considering Martin-Löf randomness. Our claim, in Theorem 16 further on, that the 'test' and 'martingale-theoretic' versions for this type of randomness are equivalent, follows the spirit of a reasonably similar proof in a paper on precise prequential Martin-Löf randomness by Vovk and Shen [40, Proof of Theorem 1]. It allows us to extend Schnorr's line of reasoning for this equivalence [24, Secs. 5–9] from fair-coin to computable interval-valued forecasting systems.

We begin with the more easily proved side of the equivalence, the actual proof of which in Appendix B relies rather heavily on Ville's inequality.

Proposition 14. *Consider any path ω in Ω and any forecasting system φ. If ω is Martin-Löf test random for φ then it is also Martin-Löf random for φ.*

For the converse result, whose proof in Appendix B is definitely more involved, the following definition introduces a useful additional condition.

Definition 9 (Non-degeneracy)**.** We call a forecasting system φ *non-degenerate* when $\underline{\varphi}(s) < 1$ and $\overline{\varphi}(s) > 0$ for all $s \in \mathbb{S}$, and *degenerate* otherwise.

So, a forecasting system φ is degenerate as soon as there's some situation s for which either $\underline{\varphi}(s) = \overline{\varphi}(s) = 0$, or $\underline{\varphi}(s) = \overline{\varphi}(s) = 1$, meaning that according to Forecaster, after observing s, the next outcome will be almost surely 1, or almost surely 0.

With this definition, we are now ready to state a converse to Proposition 14.

Proposition 15. *Consider any path ω in Ω and any non-degenerate computable forecasting system φ. If ω is Martin-Löf random for φ then it is also Martin-Löf test random for φ.*

Compared to the classical (precise) setting, non-degeneracy is required in the above proposition, as the following counterexample reveals. This is, essentially, a consequence of our preferring not to allow for extended real-valued test supermartingales; see also the discussion in Section 5.3 of our Ref. [4].

Counterexample. Consider any non-degenerate computable forecasting system $\varphi \in \Phi$ and any path $\omega \in \Omega$ that is Martin-Löf random for φ; that there always is such a path follows from our Corollary 20 in Ref. [4]. Let the degenerate forecasting system $\varphi_o \in \Phi$ be defined by letting $\varphi_o(\square) := 1 - \omega_1$ and $\varphi_o(s) := \varphi(s)$ for all $s \in \mathbb{S} \setminus \{\square\}$. We'll show that ω is Martin-Löf random but not Martin-Löf test random for φ_o.

To show that ω isn't Martin-Löf test random for φ_o, consider the recursive set $A := \bigcup_{n\in\mathbb{N}_0}\{(n,\omega_1)\} \subseteq \mathbb{N}_0 \times \mathbb{S}$, for which $A_n = \{\omega_1\}$ for all $n \in \mathbb{N}_0$, and therefore, obviously, $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. A is moreover a Martin-Löf test for φ_o, because, by Proposition 4, $\overline{P}^{\varphi_o}([\![A_n]\!]) = \overline{P}^{\varphi_o}([\![\omega_1]\!]) = (1-\omega_1)^{\omega_1}\,\omega_1{}^{1-\omega_1} = 0$ for all $n \in \mathbb{N}_0$. Hence, ω can't be Martin-Löf test random for φ_o.

To show that ω is Martin-Löf random for φ_o, assume towards contradiction that there's some lower semicomputable test supermartingale T_o for φ_o such that $\limsup_{n\to\infty} T_o(\omega_{1:n}) = \infty$. Fix any $M \in \mathbb{N}$ for which $\max\{T_o(1), T_o(0)\} < M$, and define the real process $T\colon \mathbb{S} \to \mathbb{R}$ by letting $T(\square) := 1$ and $T(s) := M^{-1}T_o(s)$ for all $s \in \mathbb{S} \setminus \{\square\}$; it is easy to check that T is a lower semicomputable test supermartingale for φ. Clearly, $\limsup_{n\to\infty} T(\omega_{1:n}) = \limsup_{n\to\infty} M^{-1}T_o(\omega_{1:n}) = \infty$, which is the desired contradiction.

When reading the proof of Proposition 15, you'll see that it is one by contradiction: we fix a path $\omega \in \Omega$, assume that it fails a Martin-Löf test A, and then show the existence of a lower semicomputable test supermartingale W that becomes unbounded on ω. We want to draw attention to the interesting fact that the test supermartingale W constructed in this proof not only becomes unbounded but actually *converges to* ∞ on every path in the global event $\bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$ associated with the Martin-Löf test A. We'll come back to this in Section 9, where we'll show that Martin-Löf randomness for a non-degenerate computable forecasting system can be checked using a single (universal) lower semicomputable supermartingale, or equivalently, using a single (universal) Martin-Löf test; see in particular Corollary 24.

If we now combine Propositions 14 and 15, we find the desired equivalence result.

Theorem 16. *Consider any path ω in Ω and any non-degenerate computable forecasting system φ. Then ω is Martin-Löf random for φ if and only if it is Martin-Löf test random for φ.*

7. The relation between uniform and Martin-Löf test randomness

Alexander Shen has recently pointed out to us that the idea of testing randomness for a set of measures has been explored before. In 1973, Levin [3, 16] defined a *uniform randomness test* that depends uniformly on all probability measures, leading to a test-theoretic randomness notion nowadays known as *uniform randomness*. This randomness notion turns out to nicely allow for testing randomness for even more general sets of measures than ours—*effectively compact classes of measures*: a path is random for an effectively compact class of measures if and only if it is uniformly random for at least one probability measure in the class.

Below, we give a brief account of how uniform randomness allows for testing a path's randomness w.r.t. effectively compact sets of probability measures, and explain how our notion of Martin-Löf test randomness, when restricted to *computable* forecasting systems, fits into that framework. To introduce this specific application of uniform randomness, we need to define a notion of effective compactness for sets of probability measures.

7.1. Effectively compact classes of probability measures.

We denote by $\mathscr{M}(\Omega)$ the set of all probability measures over the measurable space $(\Omega, \mathscr{B}(\Omega))$, and recall from the discussion in Section 2.5 that every precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}$ leads to a probability measure $\mu^{\varphi_{\mathrm{pr}}} \in \mathscr{M}(\Omega)$. Conversely, for any measure $\mu \in \mathscr{M}(\Omega)$, there's at least one precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}$ such that $\mu = \mu^{\varphi_{\mathrm{pr}}}$, for instance the one defined by

$$\varphi_{\mathrm{pr}}(s) := \begin{cases} \frac{\mu(\llbracket s1 \rrbracket)}{\mu(\llbracket s \rrbracket)} & \text{if } \mu(\llbracket s \rrbracket) > 0 \\ 1/2 & \text{if } \mu(\llbracket s \rrbracket) = 0 \end{cases} \text{ for all } s \in \mathbb{S}.$$

This tells us that we can essentially identify probability measures and precise forecasting systems (although forecasting systems are slightly more informative, as they provide full conditional information):

$$\mathscr{M}(\Omega) = \{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}\}. \tag{10}$$

With any $b \Subset \mathbb{Q} \times \mathbb{S} \times \mathbb{Q}$, where '$\Subset$' is taken to mean 'is a finite subset of', we associate a so-called *basic open set* in the set of probability measures $\mathscr{M}(\Omega)$, denoted by $b(\Omega)$, and given by

$$b(\Omega) := \big\{\mu \in \mathscr{M}(\Omega) \colon u < \mu(\llbracket s \rrbracket) < v \text{ for all } (u,s,v) \in b\big\};$$

we collect all generators b of basic open sets $b(\Omega)$ in the set $\mathscr{P}_{\mathrm{fin}}(\mathbb{Q} \times \mathbb{S} \times \mathbb{Q})$. The basic open set $b(\Omega)$ consists of all probability measures that satisfy the finite collection of conditions characterised by b. A subset $\mathscr{C} \subseteq \mathscr{M}(\Omega)$ is then called *effectively open* if there is a recursively enumerable set $B \subseteq \mathscr{P}_{\mathrm{fin}}(\mathbb{Q} \times \mathbb{S} \times \mathbb{Q})$ such that $\bigcup_{b \in B} b(\Omega) = \mathscr{C}$.[11] A subset $\mathscr{C} \subseteq \mathscr{M}(\Omega)$ is called *effectively closed* if $\mathscr{M}(\Omega) \setminus \mathscr{C}$ is effectively open. A subset $\mathscr{C} \subseteq \mathscr{M}(\Omega)$ is called *effectively compact* if it is compact and if the set

$$\left\{B \colon B \Subset \mathscr{P}_{\mathrm{fin}}(\mathbb{Q} \times \mathbb{S} \times \mathbb{Q}) \text{ and } \bigcup_{b \in B} b(\Omega) \supseteq \mathscr{C}\right\}$$

is recursively enumerable.

With any forecasting system φ, we can associate a collection of *compatible* precise forecasting systems $\{\varphi_{\mathrm{pr}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi\}$, and therefore also, falling back on Equation (7), a collection of probability measures $\{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi\}$. We begin by uncovering a sufficient condition on φ for the corresponding collection of probability measures to be effectively compact.[12]

Proposition 17. *Consider a computable forecasting system φ for which $\underline{\varphi}$ is lower semicomputable and $\overline{\varphi}$ is upper semicomputable. Then the collection of probability measures $\{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}$ and $\varphi_{\mathrm{pr}} \subseteq \varphi\}$ is effectively compact.*

On the other hand, not every effectively compact set of probability measures is a collection that corresponds to a (computable) forecasting system. Consider, as a counterexample, the set $Ber := \{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } (\exists p \in [0,1])(\forall s \in \mathbb{S})\varphi_{\mathrm{pr}}(s) = p\}$ that consists of all Bernoulli (iid) probability measures. As is mentioned by Bienvenu et al. [3, Sec. 5.3], this set *Ber* is an example of an effectively compact set of measures.

But, there is no forecasting system φ for which $Ber = \{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi\}$. Indeed, consider any forecasting system φ for which $Ber \subseteq \{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi\}$, then necessarily $p \in \varphi(s)$ for all $p \in (0,1)$ and all $s \in \mathbb{S}$, which implies that $\varphi(s) = [0,1]$ for all $s \in \mathbb{S}$. This means that φ can only be the so-called *vacuous forecasting system* $\varphi_{[0,1]}$, for which, by Equation (10), $\{\mu^{\varphi_{\mathrm{pr}}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi_{[0,1]}\} = \mathscr{M}(\Omega) \neq Ber$.

We conclude in particular that *the collections of probability measures that correspond to computable forecasting systems constitute only a strict subset of the effectively compact sets of probability measures*.

[11] This is a third instance in this paper of the general definition of effective openness; see for instance the appendix on Effective Topology in Ref. [40].

[12] This serves as a warning that our argument to show that Martin-Löf test randomness associated with forecasting systems φ fits into the uniform randomness framework needn't work when these forecasting systems φ aren't in some sense effectively describable.

7.2. **The connection with uniform randomness.** Using this notion of effective compactness, we now work towards a randomness notion that allows for testing randomness w.r.t. a class of probability measures, by associating tests with effectively compact classes of probability measures.

Definition 10. We call a map $\tau\colon \Omega \to [0,+\infty]$ a *$\mathscr{C}$-test* for an effectively compact class of probability measures $\mathscr{C} \subseteq \mathscr{M}(\Omega)$ if the set $\{\omega \in \Omega\colon \tau(\omega) > r\}$ is effectively open, effectively in $r \in \mathbb{Q}$ and if $\int \tau(\omega)\,\mathrm{d}\mu(\omega) \leq 1$ for all $\mu \in \mathscr{C}$.

A few clarifications are in order here. The conditions for a $\mathscr{C}$-test τ require in particular that $\{\omega \in \Omega\colon \tau(\omega) > r\}$ should be open, and therefore belong to $\mathscr{B}(\Omega)$, for all rational r, implying that the map τ is Borel measurable. This implies that the integral $\int \tau(\omega)\,\mathrm{d}\mu(\omega)$, which we'll also denote by $\mu(\tau)$, exists.

Going from tests to the corresponding randomness notion is now but a small step.

Definition 11. Consider an effectively compact class of probability measures $\mathscr{C} \subseteq \mathscr{M}(\Omega)$. Then a path $\omega \in \Omega$ is *$\mathscr{C}$-random* if $\tau(\omega) < \infty$ for every $\mathscr{C}$-test τ.

This is considered an interesting randomness notion, because it allows for a sensitivity analysis interpretation through the well-known notion of uniform randomness.

Proposition 18 ([3, Defs. 5.2&5.22, Thm. 5.23])**.** *Consider an effectively compact class of probability measures $\mathscr{C} \subseteq \mathscr{M}(\Omega)$. Then a path $\omega \in \Omega$ is $\mathscr{C}$-random if and only if it is uniformly random for a probability measure $\mu \in \mathscr{C}$.*

With this interpretation in terms of uniform randomness in place, we now show that our definition of Martin-Löf test randomness for a forecasting system φ is a special case of $\mathscr{C}$-randomness, where the effectively compact class $\mathscr{C}$ takes the specific form $\mathscr{C}^{\varphi} := \{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}} \subseteq \varphi\}$. Observe, by the way, that Proposition 7 and the properties of integrals guarantee that

$$\mu^{\varphi_{\mathrm{pr}}}(\tau) = \sup_{n\in\mathbb{N}} \mu^{\varphi_{\mathrm{pr}}}(\min\{\tau,n\}) = \sup_{n\in\mathbb{N}} E^{\varphi_{\mathrm{pr}}}(\min\{\tau,n\}) \tag{11}$$

for every $\mathscr{C}^{\varphi}$-test τ and all precise forecasting systems $\varphi_{\mathrm{pr}} \subseteq \varphi$ compatible with φ.

Theorem 19. *Consider any computable forecasting system φ. Then a path $\omega \in \Omega$ is Martin-Löf test random for φ if and only if it is $\mathscr{C}^{\varphi}$-random for the effectively compact class of probability measures $\mathscr{C}^{\varphi}$.*

8. Equivalence of Schnorr and Schnorr test randomness

Next, we turn to Schnorr randomness. Our argumentation that the 'test' and 'martingale-theoretic' versions for this type of randomness are equivalent, in Theorem 22 below, adapts and simplifies a line of reasoning in Downey and Hirschfeldt's book [14, Thm. 7.1.7], in order to still make it work in our more general context. Here too, it allows us to extend Schnorr's argumentation [24, Secs. 5–9] for this equivalence from fair-coin to computable and non-degenerate interval forecasts.

As was the case for Martin-Löf randomness, we begin with the implication that is easier to prove.

Proposition 20. *Consider any path ω in Ω and any forecasting system φ. If ω is Schnorr test random for φ then it is Schnorr random for φ.*

Non-degeneracy and computability of the forecasting system are enough to guarantee that the converse implication also holds. That non-degeneracy is a necessary condition can be shown by essentially the same simple counter-example as in the case of Martin-Löf randomness.

Proposition 21. *Consider any path ω in Ω and any non-degenerate computable forecasting system φ. If ω is Schnorr random for φ then it is Schnorr test random for φ.*

If we now combine Propositions 20 and 21, we find the desired result.

Theorem 22. *Consider any path ω in Ω and any non-degenerate computable forecasting system φ. Then ω is Schnorr random for φ if and only if it is Schnorr test random for φ.*

9. UNIVERSAL MARTIN-LÖF TESTS AND UNIVERSAL LOWER SEMICOMPUTABLE TEST SUPERMARTINGALES

In our definition of Martin-Löf randomness of a path ω, *all* lower semicomputable test supermartingales T must remain bounded on ω. Similarly, for ω to be Martin-Löf test random, we require that $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$ for *all* Martin-Löf tests A.

In his seminal paper [18], Martin-Löf proved that test randomness of a path can also be checked using a single, *universal*, Martin-Löf test. A few years later, Schnorr proved in his doctoral thesis on algorithmic randomness for fair-coin forecasts that Martin-Löf randomness can also be checked using a single, *universal*, lower semicomputable test supermartingale.

Let's now prove that something similar is still possible in our more general context. We begin by proving the existence of a universal Martin-Löf test.

Proposition 23. *Consider any computable forecasting system φ. Then there's a* universal *Martin-Löf test U for φ such that a path $\omega \in \Omega$ is Martin-Löf test random for φ if and only if $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$.*

We continue by proving the existence of a universal lower semicomputable supermartingale that, as mentioned in the discussion in Section 6, tends to infinity on every non-Martin-Löf random path $\omega \in \Omega$, instead of merely being unbounded.

Corollary 24. *Consider any non-degenerate computable forecasting system φ. Then there's a* universal *lower semicomputable test supermartingale T for φ such that any path $\omega \in \Omega$ is not Martin-Löf (test) random for φ if and only if $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$.*

10. CONCLUSION AND FUTURE WORK

The conclusion to be drawn from our argumentation is straightforward: Martin-Löf and Schnorr randomness for binary sequences can also be associated with interval, or imprecise, forecasts, and they can furthermore—like their precise forecast counterparts—be defined using a martingale-theoretic and a randomness test approach; both turn out to lead to the same randomness notions, at least under computability and non-degeneracy conditions on the forecasts. In addition, our Martin-Löf randomness notion for computable interval-valued forecasting systems can be characterised by a single universal lower semicomputable test supermartingale, or equivalently, for forecasting systems that are moreover non-degenerate, by a single universal Martin-Löf-test, as is the case for precise forecasts.

Why do we believe our results to merit interest?

Our study of randomness notions for imprecise forecasts aims at generalising martingale-theoretic and measure-theoretic randomness notions, by going from global probability measures to the global upper expectations—or equivalently, *sets of global probability measures*—that can be associated with interval-valued forecasting systems; see Section 2.4. We have already argued extensively elsewhere [4] why we believe this generalisation to be useful and important, so let's focus here on other arguments, specifically related to the results in this paper.

We have shown in Section 7 that computable imprecise forecasting systems correspond to at least a subset of effectively compact classes of measures, and that for this subset, the Martin-Löf test randomness we have introduced above coincides with an existing measure-theoretic randomness notion: a path is Martin-Löf (test) random for a computable imprecise forecasting system $\varphi \in \Phi$ if and only if it is $\mathscr{C}^{\varphi}$-random for the effectively compact class of probability measures $\mathscr{C}^{\varphi}$. In this sense, our results here provide this measure-theoretic notion with a martingale-theoretic account, at least for this subset of effectively compact classes of measures.

In addition to the fact that our martingale-theoretic and this measure-theoretic notion of randomness coincide for computable non-degenerate forecasting systems, they also carry similar interpretations. Indeed, one of our earlier results, Corollary 11 in Ref. [22], indicates that a path is martingale-theoretically random for a stationary forecasting system if and only if it is martingale-theoretically random for some probability measure compatible with it. We are furthermore convinced that this result can be extended to non-stationary forecasting systems as well. On the other hand, as we've seen in Proposition 18, $\mathscr{C}$-randomness for an effectively closed class of probability measures $\mathscr{C}$ tests whether a path is uniformly random with respect to some probability measure compatible with it. So, not only do both notions define randomness with respect to a set of probability measures, they also test whether a path is random with respect to some compatible measure.

In our work so far, we have focused on extending martingale-theoretic and randomness test definitions of randomness to deal with interval forecasts. As a final remark, we'd like to point out that in the precise-probabilistic setting, there are also other approaches to defining the classical notions of Martin-Löf and Schnorr randomness, besides the randomness test and martingale-theoretic ones: via Kolmogorov complexity [14, 17, 18, 24, 25], order-preserving transformations of the event tree associated with a sequence of outcomes [24], or specific limit laws (such as Lévy's zero-one law) [15, 44]. It remains to be investigated whether our interval forecast extensions can also be arrived at via such alternative routes.

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COMPETING INTERESTS

We declare that we, the authors, have no competing interests as defined by Springer, or other interests that might be perceived to influence the results and/or discussion reported in this paper.

DATA AVAILABILITY

We declare that we, the authors, don't have any research data outside the submitted manuscript file.

References

[1] Thomas Augustin et al., eds. *Introduction to Imprecise Probabilities*. John Wiley & Sons, 2014.

[2] Laurent Bienvenu, Glenn Shafer and Alexander Shen. 'On the history of martingales in the study of randomness'. In: *Electronic Journal for History of Probability and Statistics* 5 (2009).

[3] Laurent Bienvenu et al. 'Algorithmic tests and randomness with respect to a class of measures'. In: *Computing Research Repository - CORR* 274 (2011).

[4] Gert de Cooman and Jasper De Bock. 'Randomness is inherently imprecise'. In: *International Journal of Approximate Reasoning* 141 (2021), pp. 28–68.

[5] Gert de Cooman, Jasper De Bock and Stavros Lopatatzidis. 'Imprecise stochastic processes in discrete time: global models, imprecise Markov chains, and ergodic theorems'. In: *International Journal of Approximate Reasoning* 76 (2016), pp. 18–46.

[6] A. Philip Dawid. 'The Well-Calibrated Bayesian'. In: *Journal of The American Statistical Association* 77 (1982), pp. 605–610.

[7] A. Philip Dawid. 'Statistical theory: The prequential approach'. In: *Journal of the Royal Statistical Society, Series A* 147 (1984), pp. 278–292.

[8] A. Philip Dawid and Vladimir G. Vovk. 'Prequential probability: Principles and properties'. In: *Bernoulli* 5 (1999), pp. 125–162.

[9] Gert de Cooman and Jasper De Bock. 'Computable randomness is inherently imprecise'. In: *arXiv e-prints* (2017). arXiv: `1703.00931 [math.PR]`.

[10] Gert de Cooman and Jasper De Bock. 'Computable randomness is inherently imprecise'. In: *Proceedings of Machine Learning Research* 62 (2017). Ed. by Alessandro Antonucci et al. ISIPTA 2017 — Proceedings of the Tenth International Symposium on Imprecise Probability: Theories and Applications, pp. 133–144.

[11] Gert de Cooman, Jasper De Bock and Márcio Alves Diniz. 'Coherent predictive inference under exchangeability with imprecise probabilities'. In: *Artificial Intelligence Journal* 52 (2015), pp. 1–95.

[12] Gert de Cooman, Jasper De Bock and Stavros Lopatatzidis. 'A pointwise ergodic theorem for imprecise Markov chains'. In: *ISIPTA 2015 – Proceedings of the Ninth International Symposium on Imprecise Probability: Theories and Applications*. Ed. by Thomas Augustin et al. Aracne Editrice, 2015, pp. 107–116.

[13] Gert de Cooman and Filip Hermans. 'Imprecise probability trees: Bridging two theories of imprecise probability'. In: *Artificial Intelligence* 172 (2008), pp. 1400–1427.

[14] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. New York: Springer, 2010.

[15] Simon Huttegger, Sean Walsh and Francesca Zaffora Blando. 'Algorithmic randomness and Lévy's Upward Theorem'. In preparation, based on Chapter 3 of [44].

[16] Leonid A. Levin. 'On the notion of a random sequence'. In: *Soviet Math. Doklady* 14 (1973), pp. 1413–1416.

[17] Ming Li and Paul M. B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Fourth. Cham: Springer, 2019.

[18] Per Martin-Löf. 'The definition of random sequences'. In: *Information and Control* 9 (1966), pp. 602–619.

[19] Floris Persiau, Jasper De Bock and Gert de Cooman. 'Computable randomness is about more than probabilities'. In: *arXiv e-prints* (2020). Expanded version, with proofs, of [20]. arXiv: `2005.00471 [math.PR]`.

[20] Floris Persiau, Jasper De Bock and Gert de Cooman. 'Computable randomness is about more than probabilities'. In: *Scalable Uncertainty Management, 14th International Conference, SUM 2020*. Ed. by J. Davis and K. Tabia. Vol. 12322. Singapore: Springer, 2020, pp. 172–186.

[21] Floris Persiau, Jasper De Bock and Gert de Cooman. 'A remarkable equivalence between non-stationary precise and stationary imprecise uncertainty models in computable randomness'. In: *Proceedings of Machine Learning Research* 147 (2021). ISIPTA 2021 — Proceedings of the Twelfth International Symposium on Imprecise Probability: Theories and Applications, pp. 244–253.

[22] Floris Persiau, Jasper De Bock and Gert de Cooman. 'On the (dis)similarities between stationary imprecise and non-stationary precise uncertainty models in algorithmic randomness'. In: *International Journal of Approximate Reasoning* 151 (2022). Expanded version, with proofs, of [21], 272–291.

[23] Marian Boykan Pour-El and Jonathan Ian Richards. *Computability in analysis and physics*. Cambridge University Press, 2016.

[24] Claus Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit: Eine algorithmische Begründung der Wahrscheinlichkeitstheorie*. Springer, 1971.

[25] Claus Peter Schnorr. 'Process complexity and effective random tests'. In: *Journal of Computer and System Sciences* 7 (1973), pp. 376–388.

[26] Glenn Shafer and Vladimir Vovk. *Probability and Finance: It's Only a Game!* New York: Wiley, 2001.

[27] Glenn Shafer and Vladimir Vovk. *Game-Theoretic Foundations for Probability and Finance*. Hoboken, NJ: Wiley, 2019.

[28] Glenn Shafer, Vladimir Vovk and Akimichi Takemura. 'Lévy's zero–one law in game-theoretic probability'. In: *Journal of Theoretical Probability* 25 (2012), pp. 1–24.

[29] Alexander Shen, Vladimir A. Uspensky and Nikolai Vereshchagin. *Kolmogorov Complexity and Algorithmic Randomness*. Vol. 220. Mathematical surveys and monographs. American Mathematical Society, 2017.

[30] Albert N. Shiryaev. *Probability-1*. 3rd ed. Graduate Texts in Mathematics. New York: Springer, 2016.

[31] Natan T'Joens. 'Upper expectations for discrete-time imprecise stochastic processes: In practice, they are all the same!' PhD thesis. Faculty of Engineering and Architecture, Ghent University, Ghent, Belgium, 2022.

[32] Natan T'Joens and Jasper De Bock. 'Global upper expectations for discrete-time stochastic processes: In practice, they are all the same!' In: *Proceedings of Machine Learning Research* 147 (2021). ISIPTA 2021 — Proceedings of the Twelfth International Symposium on Imprecise Probability: Theories and Applications, pp. 310–319.

[33] Natan T'Joens, Jasper De Bock and Gert de Cooman. 'Continuity properties of game-theoretic upper expectations'. In: *arXiv e-prints* (2019). arXiv: `1902.09406 [math.PR]`.

[34] Natan T'Joens, Jasper De Bock and Gert de Cooman. 'In search of a global belief model for discrete-time uncertain processes'. In: *Proceedings of Machine Learning Research* 103 (2019). ISIPTA 2019 — Proceedings of the Eleventh International Symposium on Imprecise Probability: Theories and Applications, pp. 377–385.

[35] Natan T'Joens, Jasper De Bock and Gert de Cooman. 'A particular upper expectation as global belief model for discrete-time finite-state uncertain processes'. In: *International Journal of Approximate Reasoning* 131 (2021), pp. 30–55.

[36] Natan T'Joens, Jasper De Bock and Gert de Cooman. 'Game-theoretic upper expectations for discrete-time finite-state uncertain processes'. In: *Journal of Mathematical Analysis and Applications* 504 (2021).

[37] Matthias C. M. Troffaes and Gert de Cooman. *Lower Previsions*. Wiley, 2014.

[38] Jean Ville. *Étude critique de la notion de collectif*. Paris: Gauthier-Villars, 1939.

[39] Vladimir Vovk and Glenn Shafer. 'Game-theoretic probability'. In: *Introduction to Imprecise Probabilities*. Ed. by Thomas Augustin et al. John Wiley & Sons, 2014.

[40] Vladimir G. Vovk and Alexander Shen. 'Prequential randomness and probability'. In: *Theoretical Computer Science* 411 (2010), pp. 2632–2646.
[41] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. London: Chapman and Hall, 1991.
[42] Peter Walley. 'Inferences from multinomial data: learning about a bag of marbles'. In: *Journal of the Royal Statistical Society, Series B* 58 (1996). With discussion, pp. 3–57.
[43] Peter Walley and Jean-Marc Bernard. *Imprecise Probabilistic Prediction for Categorical Data*. Tech. rep. Laboratoire Cognition et Activitées Finalisées, Université de Paris 8, 1999.
[44] Francesca Zaffora Blando. 'Patterns and Probabilities: A Study in Algorithmic Randomness and Computable Learning'. PhD thesis. Department of Philosophy, Stanford University, Stanford CA, USA, 2020.
[45] Alexander K. Zvonkin and Leonid A. Levin. 'The complexity of finite objects and the development of the concepts of information and randomness by means of the theory of algorithms'. In: *Russian Mathematical Surveys* 25 (1970), pp. 83–124.

APPENDIX A. PROOFS OF RESULTS IN SECTIONS 2 AND 3

For some of the proofs below, we'll need the following version of a well-known basic result a number of times; see also Refs. [28, Lemma 2] and [5, Lemma 1] for related but slightly stronger statements.

Lemma 25. *Consider any supermartingale M for φ and any situation $s \in \mathbb{S}$, then there's some path $\omega \in [\![s]\!]$ such that $M(s) \geq \sup_{n\geq|s|} M(\omega_{1:n})$.*

Proof. Since M is a supermartingale, we know that $\overline{E}_{\varphi(s)}(M(s\,\cdot)) \leq M(s)$, and therefore, by C1, that $\min M(s\,\cdot) \leq \overline{E}_{\varphi(s)}(M(s\,\cdot)) \leq M(s)$, implying that there's some $x \in \{0,1\}$ such that $M(sx) \leq M(s)$. Repeating the same argument over and over again[13] leads us to conclude that there's some $\omega \in [\![s]\!]$ such that $M(\omega_{1:|s|+n}) \leq M(s)$ for all $n \in \mathbb{N}_0$, whence indeed $\sup_{n\geq|s|} M(\omega_{1:n}) \leq M(s)$. □

Proof of Proposition 2. We'll give proofs for E1–E8, in the interest of making this paper as self-contained as possible. The proof of E9 would take us too far afield, however; we refer the interested reader to Ref. [36, Thm. 23], which is applicable in our context as well.[14]

We begin by proving that $\inf(g|s) \leq \overline{E}^{\varphi}(g|s) \leq \sup(g|s)$. Conjugacy will then imply that also $\inf(g|s) \leq \underline{E}^{\varphi}(g|s) \leq \sup(g|s)$, and therefore that both $\underline{E}^{\varphi}(g|s)$ and $\overline{E}^{\varphi}(g|s)$ are real numbers. This important fact will be used a number of times in the remainder of this proof. The remaining inequality in E1 will be proved further on below. Since all constant real processes are supermartingales [by C1], we infer from Equation (4) that, almost trivially,

$$\overline{E}^{\varphi}(g|s) \leq \inf\{\alpha \in \mathbb{R} \colon \alpha \geq g(\omega) \text{ for all } \omega \in [\![s]\!]\} = \sup(g|s).$$

For the other inequality, consider any supermartingale $M \in \overline{\mathbb{M}}^{\varphi}$ such that $\liminf M \geq_s g$ [there clearly is such a supermartingale since g is bounded]. We derive from Lemma 25 that there's some path $\omega \in [\![s]\!]$ such that $M(s) \geq M(\omega_{1:|s|+n})$ for all $n \in \mathbb{N}_0$, and therefore also that $M(s) \geq \liminf M(\omega) \geq g(\omega) \geq \inf(g|s)$. Equation (4) then guarantees that, indeed,

$$\overline{E}^{\varphi}(g|s) = \inf\big\{M(s) \colon M \in \overline{\mathbb{M}}^{\varphi} \text{ and } \liminf M \geq_s g\big\} \geq \inf(g|s).$$

In particular, we then find for $g = 0$ that

$$\underline{E}^{\varphi}(0|s) = \overline{E}^{\varphi}(0|s) = 0. \tag{12}$$

E2. We prove the first equality; the second equality then follows from conjugacy. It follows from Equation (12) that we may assume without loss of generality that $\lambda > 0$. The

[13] This argument requires the axiom of dependent choice.

[14] See footnote 5 for an explanation and more details.

desired equality now follows at once from Equation (4) and the equivalences $M \in \overline{\mathbb{M}}^{\varphi} \Leftrightarrow \lambda^{-1}M \in \overline{\mathbb{M}}^{\varphi}$ and $\liminf M \geq_s \lambda g \Leftrightarrow \liminf \lambda^{-1}M \geq_s g$.

E3. We prove the third and fourth inequalities; the remaining inequalities will then follow from conjugacy. For the fourth inequality, we consider any real α and β such that $\alpha > \overline{E}^{\varphi}(g|s)$ and $\beta > \overline{E}^{\varphi}(h|s)$. Then it follows from Equation (4) that there are supermartingales $M_1, M_2 \in \overline{\mathbb{M}}^{\varphi}$ such that $\liminf M_1 \geq_s g$, $\liminf M_2 \geq_s h$, $\alpha > M_1(s)$ and $\beta > M_2(s)$. But then $M := M_1 + M_2$ is a supermartingale for φ [use C3] with

$$\liminf M = \liminf(M_1 + M_2) \geq_s \liminf M_1 + \liminf M_2 \geq_s g + h,$$

and we therefore infer from Equation (4) that

$$\overline{E}^{\varphi}(g+h|s) \leq M(s) = M_1(s) + M_2(s) < \alpha + \beta.$$

Since this inequality holds for all real $\alpha > \overline{E}^{\varphi}(g|s)$ and $\beta > \overline{E}^{\varphi}(h|s)$, and since we've proved above that (conditional) upper expectations of global gambles are real-valued, we find that, indeed, $\overline{E}^{\varphi}(g+h|s) \leq \overline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(h|s)$.

For the third inequality, observe that $h = (g+h) - g$, so we infer from the inequality we've just proved that

$$\overline{E}^{\varphi}(h|s) = \overline{E}^{\varphi}((g+h) - g|s) \leq \overline{E}^{\varphi}(g+h|s) + \overline{E}^{\varphi}(-g|s) = \overline{E}^{\varphi}(g+h|s) - \underline{E}^{\varphi}(g|s),$$

whence, indeed, $\overline{E}^{\varphi}(g+h|s) \geq \underline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(h|s)$, since we've already proved above that (conditional) upper and lower expectations of global gambles are real-valued.

E1. It's only left to prove that $\underline{E}^{\varphi}(g|s) \leq \overline{E}^{\varphi}(g|s)$. Since $g - g = 0$, we infer from E3 and Equation (12) that $0 = \overline{E}^{\varphi}(g-g|s) \leq \overline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(-g|s) = \overline{E}^{\varphi}(g|s) - \underline{E}^{\varphi}(g|s)$. The desired inequality now follows from the fact that (conditional) upper and lower expectations of global gambles are real-valued, as proved above.

E4. We prove the first equality; the second will then follow from conjugacy. Infer from E1 that $\underline{E}^{\varphi}(h|s) = \overline{E}^{\varphi}(h|s) = h_s$, and then E3 indeed leads to

$$\overline{E}^{\varphi}(g|s) + h_s = \overline{E}^{\varphi}(g|s) + \overline{E}^{\varphi}(h|s) \geq \overline{E}^{\varphi}(g+h|s) \geq \overline{E}^{\varphi}(g|s) + \underline{E}^{\varphi}(h|s) = \overline{E}^{\varphi}(g|s) + h_s.$$

E5. We prove the first equality; the second will then follow from conjugacy. Since the global gambles g and $g\mathbb{I}_{[\![s]\!]}$ coincide on the global event $[\![s]\!]$, we see that $\liminf M \geq_s g$ is equivalent to $\liminf M \geq_s g\mathbb{I}_{[\![s]\!]}$ for all supermartingales M for φ, and therefore the desired equality follows readily from Equation (4).

E6. We prove the first implication; the second will then follow from conjugacy. Assume that $g \leq_s h$, then $\sup(g-h|s) \leq 0$, so we infer from E1 and E3 that,

$$0 \geq \sup(g-h|s) \geq \overline{E}^{\varphi}(g-h|s) \geq \overline{E}^{\varphi}(g|s) + \underline{E}^{\varphi}(-h|s) = \overline{E}^{\varphi}(g|s) - \overline{E}^{\varphi}(h|s).$$

The desired inequality now follows from the fact that (conditional) upper and lower expectations of global gambles are real-valued, as proved above.

E7. We prove the first equality; the second will then follow from conjugacy.

First of all, it follows from C4 and E4 that we may assume without loss of generality that $f \geq 0$, and therefore also $f_s \geq 0$. Now consider any supermartingale M for φ such that $\liminf M \geq_s f_s$. An argument similar to the one involving Lemma 25 near the beginning of this proof allows us to conclude that there's some $\omega \in [\![s1]\!]$ such that $M(s1) \geq \liminf M(\omega)$, and similarly, that there's some $\varpi \in [\![s0]\!]$ such that $M(s0) \geq \liminf M(\varpi)$. Since $\liminf M \geq_s f_s$, this implies that $M(s1) \geq f(1)$ and $M(s0) \geq f(0)$, and therefore $M(s\cdot) \geq f$. But then we find that $\Delta M(s) \geq f - M(s)$, and therefore also, using C5 and C4, that

$$0 \geq \overline{E}_{\varphi(s)}(\Delta M(s)) \geq \overline{E}_{\varphi(s)}(f - M(s)) = \overline{E}_{\varphi(s)}(f) - M(s),$$

whence $M(s) \geq \overline{E}_{\varphi(s)}(f)$. Equation (4) then leads to $\overline{E}^{\varphi}(f_s|s) \geq \overline{E}_{\varphi(s)}(f)$.

To prove the converse inequality, consider the real process M_o defined by

$$M_o(t) := \begin{cases} f(1) & \text{if } s1 \sqsubseteq t \\ f(0) & \text{if } s0 \sqsubseteq t \\ \overline{E}_{\varphi(s)}(f) & \text{otherwise.} \end{cases}$$

It's clear that $\Delta M_o(t) = 0$ for all $t \neq s$. To check that M_o is a supermartingale for φ, it is therefore enough to observe that, using C4,

$$\begin{aligned}\overline{E}_{\varphi(s)}(\Delta M_o(s)) &= \overline{E}_{\varphi(s)}(M_o(s\,\cdot) - M_o(s)) \\ &= \overline{E}_{\varphi(s)}(M_o(s\,\cdot)) - M_o(s) = \overline{E}_{\varphi(s)}(f) - \overline{E}_{\varphi(s)}(f) = 0.\end{aligned}$$

Since clearly also $\liminf M_o \geq_s f_s$, Equation (4) allows us to conclude that, indeed, also $\overline{E}^{\varphi}(f_s|s) \leq M_o(s) = \overline{E}_{\varphi(s)}(f)$.

E8. We prove the first equality; the second equality will then follow from conjugacy.

Consider any supermartingale M for φ such that $\liminf M \geq_s g$. Then also $\liminf M \geq_{sx} g$, and therefore, using Equation (4), $M(sx) \geq \overline{E}^{\varphi}(g|sx)$, for all $x \in \{0,1\}$. Hence, $M(s\,\cdot) \geq \overline{E}^{\varphi}(g|s\,\cdot)$ and therefore,

$$M(s) \geq \overline{E}_{\varphi(s)}(M(s\,\cdot)) \geq \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot)),$$

where the first inequality follows from the supermartingale condition, and the second one from C5. Equation (4) then guarantees that $\overline{E}^{\varphi}(g|s) \geq \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot))$.

For the converse inequality, fix any real $\varepsilon > 0$. For any $x \in \{0,1\}$, we infer from E1 that $\overline{E}^{\varphi}(g|sx)$ is real, and therefore Equation (4) tells us that there's some $M_x \in \overline{\mathbb{M}}^{\varphi}$ such that $\liminf M_x \geq_{sx} g$ and $M_x(sx) \leq \overline{E}^{\varphi}(g|sx) + \varepsilon$. We now define the real process M by letting

$$M(t) := \begin{cases} M_x(t) & \text{if } t \sqsupseteq sx \text{ with } x \in \{0,1\} \\ \overline{E}_{\varphi(s)}(M(s\,\cdot)) & \text{otherwise} \end{cases} \quad \text{for all } t \in \mathbb{S}.$$

On the one hand, observe that, in particular, by construction,

$$M(sx) = M_x(sx) \leq \overline{E}^{\varphi}(g|sx) + \varepsilon \text{ for all } x \in \{0,1\},$$

and therefore also

$$M(s) = \overline{E}_{\varphi(s)}(M(s\,\cdot)) \leq \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot) + \varepsilon) = \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot)) + \varepsilon, \tag{13}$$

where the inequality follows from C5 and the second equality from C4. On the other hand, a straightforward verification shows that M is a supermartingale for φ. Moreover, again by construction,

$$\liminf M(\omega) = \liminf M_x(\omega) \geq g(\omega) \text{ for all } \omega \in [\![sx]\!] \text{ with } x \in \{0,1\},$$

and therefore $\liminf M \geq_s g$, so we infer from Equation (4) that $M(s) \geq \overline{E}^{\varphi}(g|s)$. Combined with the inequality in Equation (13), this leads to $\overline{E}^{\varphi}(g|s) \leq \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot)) + \varepsilon$. Since this holds for all $\varepsilon > 0$, we find that, indeed, also, $\overline{E}^{\varphi}(g|s) \leq \overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(g|s\,\cdot))$. ☐

Proof of Corollary 3. Statements (i) and (ii) follow at once from E8.

Statement (iii) follows from E1, taking into account that $0 \leq \mathbb{I}_{[\![K]\!]} \leq 1$.

For (iv), observe on the one hand that $s \sqsupseteq K$ implies that the global gamble $\mathbb{I}_{[\![K]\!]}$ assumes the constant value 1 on $[\![s]\!]$, and use E1. If, on the other hand, $s \parallel K$, then $\mathbb{I}_{[\![K]\!]}$ assumes the constant value 0 on $[\![s]\!]$, and the desired result again follows from E1.

For (v), observe that it follows from E1 that $\overline{P}^{\varphi}([\![K]\!]|\bullet) \geq 0$. It therefore suffices to consider any $\omega \in [\![K]\!]$ and to prove that then $\liminf \overline{P}^{\varphi}([\![K]\!]|\omega) = 1$. But if $\omega \in [\![K]\!]$, then there must be some $s \in K$ such that $\omega \in [\![s]\!]$. Hence, $\omega_{1:n} \sqsupseteq K$ and therefore, by (iv), also $\overline{P}^{\varphi}([\![K]\!]|\omega_{1:n}) = 1$ for all $n \geq |s|$. ☐

Proof of Proposition 4. We give the proof for the upper probability. The proof for the lower probability is completely similar.

First of all, fix any $\ell \in \{0,1,\dots,|s|-1\}$. For any $x \in \{0,1\}$,

$$\begin{aligned}\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell}x) &= \overline{E}^{\varphi}(\mathbb{I}_{\llbracket s\rrbracket}|s_{1:\ell}x) = \overline{E}^{\varphi}(\mathbb{I}_{\llbracket s\rrbracket}\mathbb{I}_{\llbracket s_{1:\ell}x\rrbracket}|s_{1:\ell}x) = \overline{E}^{\varphi}(\mathbb{I}_{\llbracket s\rrbracket}\mathbb{I}_{\{s_{\ell+1}\}}(x)|s_{1:\ell}x)\\ &= \overline{E}^{\varphi}(\mathbb{I}_{\llbracket s\rrbracket}|s_{1:\ell}x)\mathbb{I}_{\{s_{\ell+1}\}}(x) = \overline{E}^{\varphi}(\mathbb{I}_{\llbracket s\rrbracket}|s_{1:\ell+1})\mathbb{I}_{\{s_{\ell+1}\}}(x)\\ &= \overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1})\mathbb{I}_{\{s_{\ell+1}\}}(x),\end{aligned}$$

where $\mathbb{I}_{\{s_{\ell+1}\}}$ is the indicator (gamble) on $\{0,1\}$ of the singleton $\{s_{\ell+1}\}$, and where the second equality follows from E5 and the fourth equality from E2. Hence,

$$\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell}\,\cdot) = \overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1})\mathbb{I}_{\{s_{\ell+1}\}}, \tag{14}$$

so we can infer from the recursion equation in Corollary 3(i) that

$$\begin{aligned}\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell}) &= \overline{E}_{\varphi(s_{1:\ell})}\big(\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell}\,\cdot)\big) = \overline{E}_{\varphi(s_{1:\ell})}\big(\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1})\mathbb{I}_{\{s_{\ell+1}\}}\big)\\ &= \overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1})\overline{E}_{\varphi(s_{1:\ell})}(\mathbb{I}_{\{s_{\ell+1}\}}),\end{aligned}$$

where the second equality follows from Equation (14) and the third equality from C2 and the fact that $\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1}) \geq 0$ [use E1]. Since Equation (2) now tells us that

$$\overline{E}_{\varphi(s_{1:\ell})}(\mathbb{I}_{\{s_{\ell+1}\}}) = \begin{cases}\overline{\varphi}(s_{1:\ell}) & \text{if } s_{\ell+1}=1\\ 1-\underline{\varphi}(s_{1:\ell}) & \text{if } s_{\ell+1}=0\end{cases} = \overline{\varphi}(s_{1:\ell})^{s_{\ell+1}}[1-\underline{\varphi}(s_{1:\ell})]^{1-s_{\ell+1}},$$

this leads to

$$\overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell}) = \overline{P}^{\varphi}(\llbracket s\rrbracket|s_{1:\ell+1}))\overline{\varphi}(s_{1:\ell})^{s_{\ell+1}}[1-\underline{\varphi}(s_{1:\ell})]^{1-s_{\ell+1}}.$$

A simple iteration on ℓ now shows that, indeed,

$$\begin{aligned}\overline{P}^{\varphi}(\llbracket s\rrbracket) = \overline{P}^{\varphi}(\llbracket s\rrbracket|\Box) &= \overline{P}^{\varphi}(\llbracket s\rrbracket|s)\prod_{k=0}^{|s|-1}\overline{\varphi}(s_{1:k})^{s_{k+1}}[1-\underline{\varphi}(s_{1:k})]^{1-s_{k+1}}\\ &= \prod_{k=0}^{|s|-1}\overline{\varphi}(s_{1:k})^{s_{k+1}}[1-\underline{\varphi}(s_{1:k})]^{1-s_{k+1}},\end{aligned}$$

where the last equality follows from $\overline{P}^{\varphi}(\llbracket s\rrbracket|s)=1$, as is guaranteed by E1, or alternatively, by Corollary 3(iv). □

The proof of Proposition 5 is based on Shafer and Vovk's work on game-theoretic probabilities [26, 27].

Proof of Proposition 5. Let $G_C \coloneqq \{\omega \in \Omega\colon \sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) \geq C\}$. Consider any $0 < \varepsilon < C$, and let T_ε be the real process given for all $s \in \mathbb{S}$ by

$$T_\varepsilon(s) \coloneqq \begin{cases}T(t) & \text{if there's some first } t \sqsubseteq s \text{ such that } T(t) \geq C-\varepsilon\\ T(s) & \text{if } T(t) < C-\varepsilon \text{ for all } t \sqsubseteq s,\end{cases}$$

so T_ε is the version of T that mimics the behaviour of T but is stopped—kept constant—as soon as it reaches a value of at least $C-\varepsilon$. Observe that $T_\varepsilon(\Box) = T(\Box)$, and that $\frac{1}{C-\varepsilon}T_\varepsilon$ is still a non-negative supermartingale for φ. For any $\omega \in G_C$, we have that $\sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) \geq C > C-\varepsilon$, so there's some $n \in \mathbb{N}_0$ such that $T(\omega_{1:n}) > C-\varepsilon$, implying that $T_\varepsilon(\omega_{1:m}) = T_\varepsilon(\omega_{1:n}) > C-\varepsilon$ for all $m \geq n$, and therefore $\liminf_{n\to\infty}\frac{1}{C-\varepsilon}T_\varepsilon(\omega_{1:n}) \geq 1$. Hence

$$\liminf_{n\to\infty}\frac{1}{C-\varepsilon}T_\varepsilon(\omega_{1:n}) \geq \mathbb{I}_{G_C}(\omega) \text{ for all } \omega \in \Omega,$$

and therefore Equation (4) tells us that $\overline{P}^{\varphi}(G_C) \leq \frac{1}{C-\varepsilon}T_\varepsilon(\Box) = \frac{1}{C-\varepsilon}T(\Box)$. Since this holds for all $0 < \varepsilon < C$, we are done. □

Proof of Proposition 6. For (i), consider any supermartingale M for ψ, which means that $\overline{E}_{\psi(s)}(M(s\,\cdot)) \leq M(s)$ for all $s \in \mathbb{S}$. Now simply observe that also

$$\overline{E}_{\varphi(s)}(M(s\,\cdot)) = \sup_{p\in\varphi(s)} E_p(M(s\,\cdot)) \leq \sup_{p\in\psi(s)} E_p(M(s\,\cdot)) = \overline{E}_{\psi(s)}(M(s\,\cdot)) \leq M(s),$$

where the first inequality holds because $\varphi(s) \subseteq \psi(s)$.

For (ii), we use Equation (4):

$$\begin{aligned}\overline{E}^{\varphi}(g|s) &= \inf\big\{M(s)\colon M \in \overline{\mathbb{M}}^{\varphi} \text{ and } \liminf M \geq_s g\big\}\\ &\leq \inf\big\{M(s)\colon M \in \overline{\mathbb{M}}^{\psi} \text{ and } \liminf M \geq_s g\big\} = \overline{E}^{\psi}(g|s),\end{aligned}$$

where the inequality holds because we have just shown that $\overline{\mathbb{M}}^{\psi} \subseteq \overline{\mathbb{M}}^{\varphi}$. □

Proof of Proposition 7. The first statement follows from combining Proposition 10 and Theorem 6 in Ref. [35], which are applicable in the present context as well.[15] The second statement involving the partial cuts then follows from the first, as any cut K is necessarily countable, as a subset of the countable set $\mathbb{S}$. This implies that $[\![K]\!]$ is a countable union of clopen sets, and therefore belongs to the Borel algebra.

To make this paper more self-contained, we nevertheless provide an alternative and more direct proof for the last statement involving partial cuts K, which is all we'll really need for the purposes of this paper. Let, for ease of notation,

$$p(s) \coloneqq \prod_{k=0}^{|s|-1} \varphi_{\mathrm{pr}}(s_{1:k})^{s_{k+1}}[1-\varphi_{\mathrm{pr}}(s_{1:k})]^{1-s_{k+1}}, \text{ for all } s \in \mathbb{S}.$$

First of all, let's assume that K is finite, then it follows from E3 and Proposition 4 that

$$\overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) \leq \sum_{s\in K} p(s) \leq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]),$$

and then E1 guarantees that

$$P^{\varphi_{\mathrm{pr}}}([\![K]\!]) \coloneqq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \sum_{s\in K} p(s). \tag{15}$$

Next, let's consider the more involved (and only remaining) case that K is countably infinite. Let $K^{\leq n} \coloneqq \{s \in K\colon |s| \leq n\}$, for all $n \in \mathbb{N}$, then $K^{\leq n}$ is an increasing nested sequence of finite partial cuts, with $K = \bigcup_{n\in\mathbb{N}} K^{\leq n}$, and similarly $[\![K]\!] = \bigcup_{n\in\mathbb{N}}[\![K^{\leq n}]\!]$. It now follows from E9 and the non-negativity of the $p(s)$ that

$$\overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \sup_{n\in\mathbb{N}} \overline{P}^{\varphi_{\mathrm{pr}}}([\![K^{\leq n}]\!]) = \sup_{n\in\mathbb{N}} P^{\varphi_{\mathrm{pr}}}([\![K^{\leq n}]\!]) = \sup_{n\in\mathbb{N}} \sum_{s\in K^{\leq n}} p(s) = \sum_{s\in K} p(s). \tag{16}$$

On the other hand, it follows from E1, E3 and Equation (15) that, for all $n \in \mathbb{N}$,

$$\begin{aligned}0 \leq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!] \setminus [\![K^{\leq n}]\!]) &= \underline{E}^{\varphi_{\mathrm{pr}}}\big(\mathbb{I}_{[\![K]\!]} - \mathbb{I}_{[\![K^{\leq n}]\!]}\big) \leq \underline{E}^{\varphi_{\mathrm{pr}}}\big(\mathbb{I}_{[\![K]\!]}\big) + \overline{E}^{\varphi_{\mathrm{pr}}}\big(-\mathbb{I}_{[\![K^{\leq n}]\!]}\big)\\ &= \underline{E}^{\varphi_{\mathrm{pr}}}\big(\mathbb{I}_{[\![K]\!]}\big) - \underline{E}^{\varphi_{\mathrm{pr}}}\big(\mathbb{I}_{[\![K^{\leq n}]\!]}\big) = \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) - P^{\varphi_{\mathrm{pr}}}([\![K^{\leq n}]\!])\\ &= \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) - \sum_{s\in K^{\leq n}} p(s),\end{aligned}$$

and therefore $\sum_{s\in K^{\leq n}} p(s) \leq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!])$. Taking the supremum over $n \in \mathbb{N}$ on both sides of this inequality leads to

$$\sum_{s\in K} p(s) = \sup_{n\in\mathbb{N}} \sum_{s\in K^{\leq n}} p(s) \leq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]),$$

which, together with Equation (16) and E1, leads to

$$P^{\varphi_{\mathrm{pr}}}([\![K]\!]) \coloneqq \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \sum_{s\in K} p(s).$$ □

[15] See footnote 5 for an explanation and more details.

Proof of Proposition 8. Suppose there's a computable real map $q\colon \mathscr{D}\times\mathbb{N}_0\to\mathbb{R}$ such that $q(d,n+1)\geq q(d,n)$ and $r(d)=\lim_{m\to\infty}q(d,m)$ for all $d\in\mathscr{D}$ and $n\in\mathbb{N}_0$. Since q is computable, there's some recursive rational map $p\colon \mathscr{D}\times\mathbb{N}_0^2\to\mathbb{Q}$ such that

$$|q(d,m)-p(d,m,n)|\leq 2^{-n} \text{ for all } d\in\mathscr{D} \text{ and } m,n\in\mathbb{N}_0. \tag{17}$$

Let $q'\colon \mathscr{D}\times\mathbb{N}_0\to\mathbb{Q}$ be defined as $q'(d,n):=\max_{k=0}^{n}[p(d,k,k)-2^{-k}]$ for all $d\in\mathscr{D}$ and $n\in\mathbb{N}_0$. This map is clearly rational and recursive. Furthermore,

$$q'(d,n+1)=\max_{k=0}^{n+1}[p(d,k,k)-2^{-k}]\geq\max_{k=0}^{n}[p(d,k,k)-2^{-k}]=q'(d,n)$$

and

$$q'(d,n)=\max_{k=0}^{n}[p(d,k,k)-2^{-k}]\leq\sup_{k\in\mathbb{N}_0}[p(d,k,k)-2^{-k}]\leq\sup_{k\in\mathbb{N}_0}q(d,k)=r(d)$$

for all $d\in\mathscr{D}$ and $n\in\mathbb{N}_0$, where the last inequality holds by Equation (17). We end this proof by showing that $\lim_{n\to\infty}q'(d,n)=r(d)$. To this end, assume towards contradiction that there's some $N\in\mathbb{N}_0$ such that $\lim_{n\to\infty}q'(d,n)+2^{-N}<r(d)$. Since $r(d)=\lim_{m\to\infty}q(d,m)$, there's some natural $M>N+1$ such that $q(d,M)>r(d)-2^{-(N+1)}$. As a consequence, we have that, also taking into account Equation (17),

$$\begin{aligned}q'(d,M)<r(d)-2^{-N}&<q(d,M)-2^{-N}+2^{-(N+1)}=q(d,M)-2^{-(N+1)}\\&\leq p(d,M,M)+2^{-M}-2^{-(N+1)}\leq p(d,M,M)-2^{-M}\leq q'(d,M),\end{aligned}$$

which is clearly a contradiction. □

Appendix B. Proofs of new results

B.1. **Proofs of results in Section 4.**

Proof of Proposition 9. Since any recursive positive supermartingale is also computable and non-negative, it clearly suffices to prove the 'if' part. So suppose that no recursive positive and rational test supermartingale for φ is computably unbounded on ω. To prove that ω is Schnorr random, consider any computable test supermartingale T for φ, and assume towards contradiction that T is computably unbounded on ω, so there's some growth function ρ such that $\limsup_{n\to\infty}[T(\omega_{1:n})-\rho(n)]>0$. If we consider the map $\rho'\colon\mathbb{N}_0\to\mathbb{N}_0$ defined by $\rho'(n):=\lfloor\frac{1}{4}\rho(n)\rfloor$ for all $n\in\mathbb{N}_0$, then it is clear that ρ' is a growth function too, and that $\rho\geq4\rho'$. Now observe that, for all $n\in\mathbb{N}_0$,

$$T(\omega_{1:n})-\rho(n)=T(\omega_{1:n})-4R(\omega_{1:n})+4R(\omega_{1:n})-\rho(n)\leq 4\cdot2^{-n}+4[R(\omega_{1:n})-\rho'(n)],$$

where R is the recursive positive rational test supermartingale R for φ constructed in Lemma 26. Hence also $\limsup_{n\to\infty}[R(\omega_{1:n})-\rho'(n)]>0$, so R is computably unbounded on ω, a contradiction. □

The proof above makes use of the following lemma, which is a simplified—to fit our present purpose—version of one we proved earlier in Ref. [19, Lemma 24]. We include it and its proof in the interest of making this paper as self-contained as possible.

Lemma 26. *For any computable test supermartingale T for φ, there's a recursive positive rational test supermartingale R for φ such that $|4R(s)-T(s)|\leq4\cdot2^{-|s|}$ for all $s\in\mathbb{S}$.*

Proof. Consider any computable test supermartingale T. Since T is computable, there's some recursive rational map $q\colon\mathbb{S}\times\mathbb{N}_0\to\mathbb{Q}$ such that

$$|T(s)-q(s,N)|\leq2^{-N} \text{ for all } s\in\mathbb{S} \text{ and } N\in\mathbb{N}_0. \tag{18}$$

Observe that, since $T(\square)=1$, we can assume without loss of generality that $q(\square,0)=1$. Define the rational process R by letting

$$R(s):=\frac{q(s,|s|)+3\cdot2^{-|s|}}{4} \text{ for all } s\in\mathbb{S}.$$

Since the maps $|\bullet|$ and q are recursive, so is the rational process R. Furthermore, it follows from Equation (18) that

$$\left.\begin{aligned} q(sx,|sx|) &\leq T(sx)+\frac{1}{2}\cdot 2^{-|s|} \\ T(s) &\leq q(s,|s|)+2^{-|s|} \end{aligned}\right\} \text{ for all } s\in\mathbb{S} \text{ and } x\in\{0,1\}. \tag{19}$$

Moreover, $R(\square)=\frac{q(\square,0)+3}{4}=1$, and the bottom inequality in Equation (19) also guarantees that R is positive:

$$R(s)=\frac{q(s,|s|)+3\cdot 2^{-|s|}}{4}\geq\frac{T(s)+2\cdot 2^{-|s|}}{4}\geq\frac{2\cdot 2^{-|s|}}{4}>0 \text{ for all } s\in\mathbb{S}.$$

Next, we show that R is a supermartingale. By combining the inequalities in Equation (19), we find that for all $s\in\mathbb{S}$,

$$q(s\cdot,|s\cdot|)-q(s,|s|)\leq T(s\cdot)-T(s)+\frac{3}{2}\cdot 2^{-|s|},$$

and therefore also, again using the inequalities in Equation (19),

$$\begin{aligned} \Delta R(s)=R(s\cdot)-R(s) &= \frac{q(s\cdot,|s\cdot|)+3\cdot 2^{-|s\cdot|}}{4}-\frac{q(s,|s|)+3\cdot 2^{-|s|}}{4} \\ &= \frac{q(s\cdot,|s\cdot|)-q(s,|s|)-\frac{3}{2}\cdot 2^{-|s|}}{4} \\ &\leq \frac{T(s\cdot)-T(s)+\frac{3}{2}\cdot 2^{-|s|}-\frac{3}{2}\cdot 2^{-|s|}}{4}=\frac{\Delta T(s)}{4}. \end{aligned}$$

This implies that, indeed,

$$\overline{E}_{\varphi(s)}(\Delta R(s))\leq\overline{E}_{\varphi(s)}\left(\frac{\Delta T(s)}{4}\right)=\frac{1}{4}\overline{E}_{\varphi(s)}(\Delta T(s))\leq 0 \text{ for all } s\in\mathbb{S},$$

where the first inequality follows from C5, the equality follows from C2, and the last inequality follows from the supermartingale inequality $\overline{E}_{\varphi(s)}(\Delta T(s))\leq 0$.

This shows that R is a recursive positive rational test supermartingale for φ. For the rest of the proof, consider that, by Equation (18), indeed

$$\begin{aligned} |4R(s)-T(s)|=\left|q(s,|s|)+3\cdot 2^{-|s|}-T(s)\right| &\leq 3\cdot 2^{-|s|}+\left|q(s,|s|)-T(s)\right| \\ &\leq 3\cdot 2^{-|s|}+2^{-|s|}=4\cdot 2^{-|s|} \text{ for all } s\in\mathbb{S}. \square \end{aligned}$$

B.2. **Proofs of results in Section 5.** We begin by proving the following general and powerful lemma, various instantiations of which will help us through many a complicated argument further on.

Lemma 27 (Workhorse Lemma)**.** *Consider any computable forecasting system φ, any countable set $\mathscr{D}$ whose elements can be encoded by the natural numbers, and any recursive set $C\subseteq\mathscr{D}\times\mathbb{N}_0\times\mathbb{S}$ such that $|s|\leq p$ for all $(d,p,s)\in C$. Then $\overline{P}^{\varphi}([\![C_d^p]\!]|s)$ is a computable real effectively in d, p and s, with $C_d^p:=\{s\in\mathbb{S}\colon (d,p,s)\in C\}$ for all $p\in\mathbb{N}_0$ and $d\in\mathscr{D}$.*

Proof. We start by observing that C_d^p is a finite recursive set of situations, effectively in d and p. Similarly,

$$C_d^{p\prime}:=\{t\in\mathbb{S}\colon |t|=p \text{ and } C_d^p\sqsubseteq t\}$$

is clearly also a finite recursive set of situations, effectively in d and p. Moreover, it is a partial cut.

Another important observation is that there are, in principle, three mutually exclusive possibilities for any of the sets C_d^p and any $t\in\mathbb{S}$. The first possibility is that $C_d^p\sqsubseteq t$, which can be checked recursively. In that case, we know from Corollary 3(iv) that $\overline{P}^{\varphi}([\![C_d^p]\!]|t)=1$. The second possibility is that $t\parallel C_d^p$, which can be checked recursively as well. In that case, we know from Corollary 3(iv) that $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | t) = 0$. The third, final, and most involved possibility is that $t \sqsubset C_d^p$, which can also be checked recursively.

It's clear from this discussion that the computability of $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | s)$ is trivial when $C_d^p \sqsubseteq s$ or $s \parallel C_d^p$, so we'll from now on only pay attention to the case that $s \sqsubset C_d^p$. Since, obviously, $\llbracket C_d^{p\prime} \rrbracket = \llbracket C_d^p \rrbracket$ and in this case also $s \sqsubset C_d^{p\prime}$, we'll focus on the computability of $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | s)$.

For any $t \sqsupseteq s$ with $|t| = p$, we infer from the discussion above that $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) = 1$ if $t \in C_d^{p\prime}$ and $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) = 0$ otherwise. Clearly then, $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t)$ is a computable real effectively in d, p and t with $|t| = p$.

In a next step, we find by applying Corollary 3(i) that, for any $t \sqsupseteq s$ with $|t| = p - 1$,

$$\begin{aligned}\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) &= \overline{E}_{\varphi(t)}\big(\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t\,\cdot)\big)\\ &= \max\Big\{\underline{\varphi}(t)\overline{P}^{\varphi}\big(\llbracket C_d^{p\prime} \rrbracket \big| t1\big) + [1-\underline{\varphi}(t)]\overline{P}^{\varphi}\big(\llbracket C_d^{p\prime} \rrbracket \big| t0\big),\\ &\qquad\qquad \overline{\varphi}(t)\overline{P}^{\varphi}\big(\llbracket C_d^{p\prime} \rrbracket \big| t1\big) + [1-\overline{\varphi}(t)]\overline{P}^{\varphi}\big(\llbracket C_d^{p\prime} \rrbracket \big| t0\big)\Big\},\end{aligned}$$

which is clearly a computable real effectively in d, p and t with $|t| = p - 1$, simply because φ is computable.

By applying Corollary 3(i) to situations $t \sqsupseteq s$ with successively smaller $|t|$, we eventually end up in the situation s after a finite number of steps, which implies that $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | s)$ is a computable real, effectively in d, p and s. □

Proof of Proposition 11. By assumption, the representation C is a recursive subset of $\mathbb{N}_0 \times \mathbb{S}$ such that $G_n = \llbracket C_n \rrbracket$ and $\overline{P}^{\varphi}\big(\llbracket C_n \rrbracket\big) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and such that there's some recursive map $e' \colon \mathbb{N}_0^2 \to \mathbb{N}_0$ such that $\overline{P}^{\varphi}\big(\llbracket C_n \rrbracket \setminus \llbracket C_n^{<\ell} \rrbracket\big) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0^2$ and all $\ell \geq e'(N,n)$.

For the proof of the first statement, consider for any $n \in \mathbb{N}_0$, the set of situations

$$A_n \coloneqq \{s \in C_n \colon (\forall t \sqsubset s) t \notin C_n\} \subseteq C_n,$$

which is clearly a partial cut and recursive effectively in n. Of course, the corresponding $A \coloneqq \{(n,s) \colon n \in \mathbb{N}_0 \text{ and } s \in A_n\} \subseteq C$ is then recursive. It follows readily from our construction that $\llbracket A_n \rrbracket = \llbracket C_n \rrbracket$ and $\llbracket A_n^{<\ell} \rrbracket = \llbracket C_n^{<\ell} \rrbracket$ for all $n, \ell \in \mathbb{N}_0$.

For proof of the second statement, define $e \colon \mathbb{N}_0 \to \mathbb{N}_0$ by letting

$$e(N) \coloneqq N + \max_{m=0}^{N} \max_{n=0}^{N} e'(m,n) \text{ for all } N \in \mathbb{N}_0.$$

Clearly, the map e is recursive because e' is. It's non-decreasing because

$$e(N+1) = N + 1 + \max_{m=0}^{N+1} \max_{n=0}^{N+1} e'(m,n) \geq N + \max_{m=0}^{N} \max_{n=0}^{N} e'(m,n) = e(N) \text{ for all } N \in \mathbb{N}_0,$$

and it is unbounded because $e(N) \geq N$ for all $N \in \mathbb{N}_0$. We conclude that e is a growth function. Now, fix any $N \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$, then there are two possibilities. The first is that $n \leq N$, and then for all $\ell \geq e(N)$ also $\ell \geq e'(N,n)$, and therefore, as we know from the beginning of this proof,

$$\overline{P}^{\varphi}(\llbracket C_n \rrbracket \setminus \llbracket C_n^{<\ell} \rrbracket) \leq 2^{-N}.$$

The other possibility is that $n > N$, and then trivially for all $\ell \geq e(N)$

$$\overline{P}^{\varphi}(\llbracket C_n \rrbracket \setminus \llbracket C_n^{<\ell} \rrbracket) \leq \overline{P}^{\varphi}(\llbracket C_n \rrbracket) \leq 2^{-n} \leq 2^{-N}.$$

where the first inequality follows from E6, and the penultimate one, as explained at the beginning of this proof, follows from the assumption. □

Proof of Proposition 12. Given the assumptions, an appropriate instantiation of our Workhorse Lemma 27 [with $\mathscr{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n,\ell,s) \in \mathbb{N}_0^2 \times \mathbb{S} \colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$] guarantees that the real map $(n,\ell) \mapsto \overline{P}^{\varphi}(\llbracket A_n^{<\ell} \rrbracket)$ is computable. Moreover, the following line of reasoning tells us that for all $n, \ell \in \mathbb{N}_0$,

$$\big|\overline{P}^{\varphi}(\llbracket A_n \rrbracket) - \overline{P}^{\varphi}(\llbracket A_n^{<\ell} \rrbracket)\big| = \big|\overline{E}^{\varphi}\big(\mathbb{I}_{\llbracket A_n \rrbracket}\big) - \overline{E}^{\varphi}\big(\mathbb{I}_{\llbracket A_n^{<\ell} \rrbracket}\big)\big| = \overline{E}^{\varphi}\big(\mathbb{I}_{\llbracket A_n \rrbracket}\big) - \overline{E}^{\varphi}\big(\mathbb{I}_{\llbracket A_n^{<\ell} \rrbracket}\big)$$

$$\leq \overline{E}^{\varphi}\big(\mathbb{I}_{[\![A_n]\!]} - \mathbb{I}_{[\![A_n^{<\ell}]\!]}\big) = \overline{P}^{\varphi}([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]), \tag{20}$$

where the second equality follows from E6, and the inequality follows from E3. Since A is a Schnorr test, we know that it has a tail bound, so there's some recursive map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$ such that $\overline{P}^{\varphi}\big([\![A_n]\!] \setminus [\![A_n^{\geq \ell}]\!]\big) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0^2$ and all $\ell \geq e(N,n)$, and if we combine this with the inequality in Equation (20), this leads to

$$\big|\overline{P}^{\varphi}([\![A_n]\!]) - \overline{P}^{\varphi}([\![A_n^{<\ell}]\!])\big| \leq 2^{-N} \text{ for all } (N,n) \in \mathbb{N}_0^2 \text{ and all } \ell \geq e(N,n).$$

Since this tells us that the computable real map $(n,\ell) \mapsto \overline{P}^{\varphi}[\![A_n^{<\ell}]\!]$ converges effectively to the sequence of real numbers $\overline{P}^{\varphi}([\![A_n]\!])$, we conclude that $\overline{P}^{\varphi}([\![A_n]\!])$ is a computable sequence of real numbers. □

Proof of Proposition 13. By Corollary 10, we may assume without loss of generality that there's a recursive $A \subseteq \mathbb{N}_0 \times \mathbb{S}$ such that A_n is a partial cut, $G_n = [\![A_n]\!]$ and $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$. Assume that the $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$ constitute a computable sequence of real numbers. Then, by Definition 7, it suffices to prove that there's some recursive map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0^2$ and all $\ell \geq e(N,n)$.

To do so, we start by proving that the real map $(n,\ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is computable and that $\lim_{\ell\to\infty} P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) = 0$ for all $n \in \mathbb{N}_0$. First of all, observe that the computability of the forecasting system φ_{pr}, the recursive character of the finite partial cuts $A_n^{<\ell}$ and an appropriate instantiation of our Workhorse Lemma 27 [with $\mathscr{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n,\ell,s) \in \mathbb{N}_0^2 \times \mathbb{S}\colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$] allow us to infer that the real map $(n,\ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!])$ is computable. Since the forecasting system φ_{pr} is precise, and since $\mathbb{I}_{[\![A_n]\!]} = \mathbb{I}_{[\![A_n^{<\ell}]\!]} + \mathbb{I}_{[\![A_n^{\geq \ell}]\!]}$ for all $(n,\ell) \in \mathbb{N}_0^2$ due to A_n being a partial cut, we infer from Proposition 7 that

$$P^{\varphi_{\mathrm{pr}}}\big([\![A_n^{\geq \ell}]\!]\big) = P^{\varphi_{\mathrm{pr}}}\big([\![A_n]\!]\big) - P^{\varphi_{\mathrm{pr}}}\big([\![A_n^{<\ell}]\!]\big). \tag{21}$$

Since $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$ is a computable sequence of real numbers and $(n,\ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!])$ is a computable real map, it follows from Equation (21) that $(n,\ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is a computable real map. Furthermore, since $\mathbb{I}_{[\![A_n^{<\ell}]\!]} \nearrow \mathbb{I}_{[\![A_n]\!]}$ point-wise as $\ell \to \infty$, it follows from E9 that $\lim_{\ell\to\infty} P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!]) = P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$, and therefore also that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \searrow 0$ as $\ell \to \infty$, for all $n \in \mathbb{N}_0$.

We are now ready to prove that there's some recursive map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0^2$ and all $\ell \geq e(N,n)$. Since $(n,\ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is a computable real map, there's some recursive rational map $q\colon \mathbb{N}_0^3 \to \mathbb{Q}$ such that

$$\big|P^{\varphi_{\mathrm{pr}}}\big([\![A_n^{\geq \ell}]\!]\big) - q(n,\ell,N)\big| \leq 2^{-N} \text{ for all } (n,\ell,N) \in \mathbb{N}_0^3. \tag{22}$$

Define the map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$ by

$$e(N,n) := \min\big\{\ell \in \mathbb{N}_0\colon q(n,\ell,N+2) < 2^{-(N+1)}\big\} \text{ for all } (N,n) \in \mathbb{N}_0^2. \tag{23}$$

Clearly, if we can prove that the set of natural numbers in the definition above is always non-empty, then the map e will be well-defined and recursive. To do so, fix any $(N,n) \in \mathbb{N}_0^2$, and observe that since $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \searrow 0$ as $\ell \to \infty$, there always is some $\ell_o \in \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell_o}]\!]) < 2^{-(N+2)}$. For this same ℓ_o, it then indeed follows from Equation (22) that

$$q(n,\ell_o,N+2) \leq P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell_o}]\!]) + 2^{-(N+2)} < 2^{-(N+2)} + 2^{-(N+2)} = 2^{-(N+1)}.$$

To complete the proof, consider any $n,N \in \mathbb{N}_0$ and any $\ell \geq e(N,n)$. Then, indeed,

$$\begin{aligned} P^{\varphi_{\mathrm{pr}}}\big([\![A_n^{\geq \ell}]\!]\big) &\leq P^{\varphi_{\mathrm{pr}}}\big([\![A_n^{\geq e(N,n)}]\!]\big) \leq q(n,e(N,n),N+2) + 2^{-(N+2)} \\ &< 2^{-(N+1)} + 2^{-(N+2)} < 2^{-N}, \end{aligned}$$

where the first inequality follows from $\ell \geq e(N,n)$ and E6, the second inequality follows from Equation (22), and the third inequality follows from Equation (23). □

B.3. **Proofs of results in Section 6.**

Proof of Proposition 14. We give a proof by contraposition. Assume that ω isn't Martin-Löf random for φ, which implies that there's some lower semicomputable test supermartingale T for φ that becomes unbounded on ω, so $\sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) = \infty$. Now, let us consider the following set

$$A := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon T(s) > 2^n\} \subseteq \mathbb{N}_0 \times \mathbb{S}.$$

That T is a lower semicomputable test supermartingale implies, by Lemma 28(iii)&(i), that A is a Martin-Löf test for φ with $[\![A_m]\!] := \{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\}$ for all $m \in \mathbb{N}_0$. That $\sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) = \infty$ then implies that $\omega \in [\![A_m]\!]$ for all $m \in \mathbb{N}_0$, so ω isn't Martin-Löf test random for φ either. $\square$

Lemma 28. *Consider any lower semicomputable test supermartingale T for φ, and let $A := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon T(s) > 2^n\}$. Then*
(i) $[\![A_m]\!] = \{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\}$ *for all $m \in \mathbb{N}_0$;*
(ii) $\overline{P}^{\varphi}([\![A_m]\!]) \leq 2^{-m}$ *for all $m \in \mathbb{N}_0$;*
(iii) *A is a Martin-Löf test.*

Proof. We begin with the proof of (i). Since, by its definition, $[\![A_m]\!] = \bigcup\{[\![s]\!] \colon s \in A_m\}$, we have the following chain of equivalences for any $\varpi \in \Omega$:

$$\begin{aligned}\varpi \in [\![A_m]\!] &\Leftrightarrow (\exists s \in A_m)(\varpi \in [\![s]\!]) \Leftrightarrow (\exists s \in \mathbb{S})(\varpi \in [\![s]\!] \text{ and } (m,s) \in A)\\ &\Leftrightarrow (\exists s \in \mathbb{S})(\varpi \in [\![s]\!] \text{ and } T(s) > 2^m) \Leftrightarrow (\exists n \in \mathbb{N}_0) T(\varpi_{1:n}) > 2^m,\end{aligned}$$

proving (i).

Next, we turn to the proof of (ii). If we recall that T is a nonnegative supermartingale for φ with $T(\square) = 1$ and let $C := 2^m > 0$ in Ville's inequality [Proposition 5], then we find, also taking into account (i) and E6, that indeed,

$$\begin{aligned}\overline{P}^{\varphi}([\![A_m]\!]) &= \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\right\}\right)\\ &\leq \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) \geq 2^m\right\}\right) \leq \frac{1}{2^m} T(\square) = 2^{-m}.\end{aligned}$$

For (iii), it now only remains to prove that the set $A = \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon T(s) > 2^n\}$ is recursively enumerable. The argument is a standard one. That T is lower semicomputable means that there's some recursive map $q_T \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{Q}$ such that $q_T(m+1,s) \geq q_T(m,s)$ and $T(s) = \sup_{\ell\in\mathbb{N}_0} q_T(\ell,s)$ for all $m \in \mathbb{N}_0$ and all $s \in \mathbb{S}$. Now observe the following chain of equivalences, for any $(n,s) \in \mathbb{N}_0 \times \mathbb{S}$:

$$(n,s) \in A \Leftrightarrow T(s) > 2^n \Leftrightarrow \sup_{m\in\mathbb{N}_0} q_T(m,s) > 2^n \Leftrightarrow (\exists m \in \mathbb{N}_0) q_T(m,s) > 2^n.$$

Since q_T is rational-valued and recursive, the inequality $q_T(m,s) > 2^n$ is decidable, which makes it clear that A is indeed recursively enumerable. $\square$

With any non-degenerate forecasting system φ, we can associate the (clearly) positive real processes c_φ and C_φ, defined by

$$c_\varphi(s) := \min\{1 - \underline{\varphi}(s), \overline{\varphi}(s)\} \text{ and } C_\varphi(s) := \prod_{k=0}^{|s|-1} c_\varphi(s_{1:k})^{-1} \text{ for all } s \in \mathbb{S}.$$

Observe that $C_\varphi(\square) = 1$, and that $0 < c_\varphi(s) \leq 1$ and therefore also $C_\varphi(s) \geq 1$ for all $s \in \mathbb{S}$. Also, if φ is computable, then so are c_φ and C_φ.

Interestingly, the map C_φ can be used to bound non-negative supermartingales for non-degenerate forecasting systems.

Proposition 29. *Consider any non-degenerate forecasting system φ and any non-negative supermartingale M for φ. Then $0 \leq M(s) \leq M(\square) C_\varphi(s)$ for all $s \in \mathbb{S}$.*

Proof of Proposition 29. Fix any situation $s \in \mathbb{S}$ and simply observe that

$$\begin{aligned}
M(s) \geq \overline{E}_{\varphi(s)}(\Delta M(s\,\cdot)) &= \begin{cases} \underline{\varphi}(s)M(s1) + [1-\underline{\varphi}(s)]M(s0) & \text{if } M(s1) \leq M(s0) \\ \overline{\varphi}(s)M(s1) + [1-\overline{\varphi}(s)]M(s0) & \text{if } M(s1) > M(s0) \end{cases} \\
&\geq \begin{cases} [1-\underline{\varphi}(s)]M(s0) & \text{if } M(s1) \leq M(s0) \\ \overline{\varphi}(s)M(s1) & \text{if } M(s1) > M(s0) \end{cases} \\
&= \max M(s\,\cdot) \begin{cases} 1-\underline{\varphi}(s) & \text{if } M(s1) \leq M(s0) \\ \overline{\varphi}(s) & \text{if } M(s1) > M(s0) \end{cases} \\
&\geq \min\bigl\{1-\underline{\varphi}(s), \overline{\varphi}(s)\bigr\} \max M(s\,\cdot),
\end{aligned}$$

where the first inequality holds because M is a supermartingale for φ, and the other inequalities hold because M is non-negative. Hence, $\max M(s\,\cdot) \leq c_\varphi(s)^{-1}M(s)$. A simple induction argument now leads to the desired result. □

Proof of Proposition 15. Again, we give a proof by contraposition. Assume that ω isn't Martin-Löf test random for φ. This implies that there's some Martin-Löf test A such that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. The idea behind the proof is an altered, much simplified and stripped-down version of an argument borrowed in its essence from a different proof in a paper by Vovk and Shen about precise prequential Martin-Löf randomness [40, Proof of Theorem 1]. It's actually quite straightforward when we ignore its technical complexities: we'll use the Martin-Löf test A to construct a lower semicomputable test supermartingale W for φ that becomes unbounded on ω. Although it might not appear so at first sight from the way we go about it, this W is essentially obtained by summing the non-negative supermartingales $\overline{P}^\varphi([\![A_n]\!]|\bullet)$, each of which is 'fully turned on' as soon as the partial cut A_n is reached. The main technical difficulty will be to prove that this W is lower semicomputable, and we'll take care of this task in a roundabout way, in a number of lemmas [Lemmas 30–32 below].

Back to the proof now. Recall from Corollary 10 that we may assume without loss of generality that the set A is recursive and that the corresponding A_n are partial cuts. We also recall the definition of the partial cuts $A_n^{<\ell} := \{s \in \mathbb{S} \colon (n,s) \in A \text{ and } |s| < \ell\} \subseteq A_n$, for all $n, \ell \in \mathbb{N}_0$, with $[\![A_n]\!] = \bigcup_{\ell\in\mathbb{N}_0} [\![A_n^{<\ell}]\!]$. These same partial cuts also appear in Equation (8), where we prepared for the definition of a Schnorr test.

We begin by considering the real processes $W_n^\ell := \overline{P}^\varphi([\![A_n^{<\ell}]\!]|\bullet)$, where $n, \ell \in \mathbb{N}_0$. By Lemma 30, each W_n^ℓ is a non-negative computable supermartingale. We infer from E6 that $\overline{P}^\varphi([\![A_n]\!]) = \overline{E}^\varphi(\mathbb{I}_{[\![A_n]\!]}) \geq \overline{E}^\varphi(\mathbb{I}_{[\![A_n^{<\ell}]\!]}) = W_n^\ell(\square)$, and therefore, also invoking Lemma 30(ii) and the assumption that $\overline{P}^\varphi([\![A_n]\!]) \leq 2^{-n}$, we get that

$$0 \leq W_n^\ell(\square) \leq 2^{-n}. \tag{24}$$

Next, fix any $s \in \mathbb{S}$ and any $\ell \in \mathbb{N}_0$, and let $W^\ell(s) := \frac{1}{2}\sum_{n=0}^\infty W_n^\ell(s)$. Observe that, since all its terms $W_n^\ell(s)$ are non-negative by Lemma 30(ii), the series $W^\ell(s) = \frac{1}{2}\sum_{n=0}^\infty W_n^\ell(s)$ converges to some non-negative extended real number. We first check that it is real-valued, as in principle, the defining series might converge to ∞. Combine Equation (24) and Proposition 29 to find that:

$$0 \leq W_n^\ell(s) \leq W_n^\ell(\square)C_\varphi(s) \leq C_\varphi(s)2^{-n} \text{ for all } n \in \mathbb{N}_0, \tag{25}$$

whence also

$$0 \leq W^\ell(s) = \frac{1}{2}\sum_{n=0}^\infty W_n^\ell(s) \leq C_\varphi(s), \tag{26}$$

which shows that $W^\ell(s)$ is bounded above, and therefore indeed real. Moreover, it now follows from Lemma 30(ii) that $W^\ell(s) \leq W^{\ell+1}(s)$ for all $\ell \in \mathbb{N}_0$, which guarantees that the limit $W(s) := \lim_{\ell\to\infty} W^{\ell}(s) = \sup_{\ell\in\mathbb{N}_0} W^{\ell}(s)$ exists as an extended real number. It's moreover real-valued, because we infer from taking the limit in Equation (26) that also

$$0 \leq W(s) \leq C_{\varphi}(s). \tag{27}$$

We've thus defined a non-negative real process W, and we infer from Lemma 31 that W is a non-negative lower semicomputable supermartingale for φ. In addition, we infer from Equation (27) that $0 \leq W(\square) \leq 1$.

Moreover, since $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$, we see that W is unbounded on ω. Indeed, consider any $n \in \mathbb{N}_0$, then since $\omega \in [\![A_n]\!]$, there is some $m_n \in \mathbb{N}_0$ such that $W_n^{\ell}(\omega_{1:m}) = 1$ for all $m, \ell \geq m_n$ [To see this, observe that $\omega \in [\![A_n]\!]$ first of all implies that there is some (unique) $M_n \in \mathbb{N}_0$ for which $\omega_{1:M_n} \in A_n$, and secondly that then $\omega_{1:M_n} \in A_n^{<\ell} \Leftrightarrow \ell > M_n$; so if $\ell \geq M_n + 1$ then $\omega_{1:m} \sqsupseteq A_n^{<\ell}$ for all $m \geq M_n$; now use Lemma 30(iii) to find that then also $W_n^{\ell}(\omega_{1:m}) = 1$ for all $m \geq M_n$]. So, if we consider any $N \in \mathbb{N}_0$ and let $M_N := \max\{m_n \colon n \in \{0, 1, \dots, N\}\}$, then

$$W^{\ell}(\omega_{1:m}) \geq \frac{1}{2} \sum_{n=0}^{N} W_n^{\ell}(\omega_{1:m}) = \frac{1}{2}(N+1) \text{ for all } m, \ell \geq M_N,$$

and therefore also

$$W(\omega_{1:m}) \geq \frac{1}{2}(N+1) \text{ for all } m \geq M_N,$$

which shows that, in fact,

$$\lim_{m\to\infty} W(\omega_{1:m}) = \infty. \tag{28}$$

The relevant condition being $\overline{E}_{\varphi(\square)}(W(\square\,\cdot)) \leq W(\square)$, we see that replacing $W(\square) \leq 1$ by 1 does not change the supermartingale character of W, and doing so leads to a lower semicomputable test supermartingale for φ that is unbounded on ω. This tells us that, indeed, ω isn't Martin-Löf random for φ. $\square$

Lemma 30. *For any $n, \ell \in \mathbb{N}_0$, consider the real process W_n^{ℓ}, defined in the proof of Proposition 15 by $W_n^{\ell} := \overline{P}^{\varphi}([\![A_n^{<\ell}]\!]|\bullet)$. Then the following statements hold:*

(i) $W_n^{\ell}(s) = \overline{E}_{\varphi(s)}(W_n^{\ell}(s\,\cdot))$ *for all* $s \in \mathbb{S}$*;*
(ii) $0 \leq W_n^{\ell}(s) \leq W_n^{\ell+1}(s) \leq 1$ *for all* $s \in \mathbb{S}$*;*
(iii) $W_n^{\ell}(s) = 1$ *for all* $s \sqsupseteq A_n^{<\ell}$*;*
(iv) *the real map* $(n, \ell, s) \mapsto W_n^{\ell}(s)$ *is computable.*

In particular, for all $n, \ell \in \mathbb{N}_0$, $W_n^{\ell} := \overline{P}^{\varphi}([\![A_n^{<\ell}]\!]|\bullet)$ is a non-negative computable supermartingale for φ.

Proof. Statement (i) follows from Corollary 3(i), since $A_n^{<\ell}$ is a partial cut.

The first and third inequalities in (ii) follow from Corollary 3(iii). The second inequality is a consequence of $A_n^{<\ell} \subseteq A_n^{<\ell+1}$ and the monotone character of the conditional lower expectation $\overline{E}^{\varphi}(\bullet|s)$ [use E6].

Statement (iii) is an immediate consequence of Corollary 3(iv).

For the proof of (iv), consider that the partial cut A is recursive and that the forecasting system φ is computable, and apply an appropriate instantiation of our Workhorse Lemma 27 [with $\mathscr{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n, \ell, s) \in \mathbb{N}_0^2 \times \mathbb{S} \colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$].

The rest of the proof is now immediate. $\square$

Lemma 31. *The real process W, defined in the proof of Proposition 15, is a non-negative lower semicomputable supermartingale for φ.*

Proof. First of all, recall from Equation (27) in the proof of Proposition 15 that W is indeed non-negative.

Next, define, for any $m, \ell \in \mathbb{N}_0$, the real process V_m^{ℓ} by letting $V_m^{\ell}(s) := \frac{1}{2}\sum_{n=0}^{m} W_n^{\ell}(s)$ for all $s \in \mathbb{S}$. It follows from Lemma 30(ii) that V_m^{ℓ} is non-negative. By Lemma 30(iv), the real map $(n,\ell,s)\mapsto W_n^\ell(s)$ is computable, so we see that so is $(m,\ell,s)\mapsto V_m^\ell(s)$. Moreover, it is clear from the definition of the processes V_m^ℓ and W^ℓ that $V_m^\ell(s)\nearrow W^\ell(s)$ as $m\to\infty$, and that

$$\left|W^\ell(s)-V_m^\ell(s)\right|=\frac{1}{2}\sum_{n=m+1}^{\infty}W_n^\ell(s)\leq\frac{1}{2}C_\varphi(s)\sum_{n=m+1}^{\infty}2^{-n}=\frac{1}{2}C_\varphi(s)2^{-m}\leq 2^{-m+L_{C_\varphi}(s)-1}$$
$$\text{for all } \ell,m\in\mathbb{N}_0 \text{ and all } s\in\mathbb{S},$$

where the first inequality follows from Equation (25), and the second inequality is based on Lemma 32 and the notations introduced there. If we now consider the recursive map $e\colon\mathbb{N}_0\times\mathbb{S}\to\mathbb{N}_0$ defined by $e(N,s):=N+L_{C_\varphi}(s)-1$ [recall that L_{C_φ} is recursive by Lemma 32], then we find that $|W^\ell(s)-V_m^\ell(s)|\leq 2^{-N}$ for all $(N,s)\in\mathbb{N}_0\times\mathbb{S}$ and all $m\geq e(N,s)$, which guarantees that the real map $(\ell,s)\mapsto W^\ell(s)$ is computable.

Now, consider that for any $s\in\mathbb{S}$, $W^\ell(s)\nearrow W(s)$ as $\ell\to\infty$. Since we've just proved that $(\ell,s)\mapsto W^\ell(s)$ is a computable real map, we conclude that the process W is indeed lower semicomputable, as a point-wise limit of a non-decreasing sequence of computable processes [invoke Proposition 8].

To complete the proof, we show that W is a supermartingale. It follows from C2, C3 and the supermartingale character of the W_n^ℓ [Lemma 30] that

$$\overline{E}_{\varphi(s)}(\Delta V_m^\ell(s))=\overline{E}_{\varphi(s)}\left(\frac{1}{2}\sum_{n=0}^{m}\Delta W_n^\ell(s)\right)\leq\frac{1}{2}\sum_{n=0}^{m}\overline{E}_{\varphi(s)}(\Delta W_n^\ell(s))\leq 0 \text{ for all } s\in\mathbb{S},$$

so V_m^ℓ is also a supermartingale. Since $V_m^\ell(s)\to W^\ell(s)$, we also find that $\Delta V_m^\ell(s)\to\Delta W^\ell(s)$ for all $s\in\mathbb{S}$. Since the gambles $\Delta V_m^\ell(s)$ are defined on the finite domain $\{0,1\}$, this point-wise convergence also implies uniform convergence, so we can infer from C6 that

$$\overline{E}_{\varphi(s)}(\Delta W^\ell(s))=\overline{E}_{\varphi(s)}\left(\lim_{m\to\infty}\Delta V_m^\ell(s)\right)=\lim_{m\to\infty}\overline{E}_{\varphi(s)}(\Delta V_m^\ell(s))\leq 0 \text{ for all } s\in\mathbb{S}.$$

This shows that W^ℓ is also a supermartingale. And, since $W^\ell(s)\to W(s)$, we find that also $\Delta W^\ell(s)\to\Delta W(s)$ for all $s\in\mathbb{S}$. Since the gambles $\Delta W^\ell(s)$ are defined on the finite domain $\{0,1\}$, this point-wise convergence also implies uniform convergence, so we can again infer from C6 that

$$\overline{E}_{\varphi(s)}(\Delta W(s))=\overline{E}_{\varphi(s)}\left(\lim_{\ell\to\infty}\Delta W^\ell(s)\right)=\lim_{\ell\to\infty}\overline{E}_{\varphi(s)}(\Delta W^\ell(s))\leq 0 \text{ for all } s\in\mathbb{S}.$$

This shows that W is indeed a supermartingale. □

Lemma 32. *If the real process F is computable and $F\geq 1$, then there's some recursive map $L_F\colon\mathbb{S}\to\mathbb{N}$ such that $L_F\geq\log_2 F$, or equivalently, $F\leq 2^{L_F}$.*

Proof. That F is computable implies that the non-negative process $\log_2 F$ is computable as well. That the non-negative real process $\log_2 F$ is computable means that there's some recursive map $q_F\colon\mathbb{N}_0\times\mathbb{S}\to\mathbb{Q}$ such that $|\log_2 F(s)-q_F(n,s)|\leq 2^{-n}$ for all $(n,s)\in\mathbb{N}_0\times\mathbb{S}$, and therefore in particular that $|\log_2 F-q_F(0,\bullet)|\leq 1$. Hence, $0\leq\log_2 F\leq 1+q_F(0,\bullet)\leq 1+\lceil q_F(0,\bullet)\rceil$ and $L_F:=1+\lceil q_F(0,\bullet)\rceil$ is a recursive and $\mathbb{N}$-valued process. □

B.4. **Proofs of results in Section 7.**

Proof of Proposition 17. Proposition 5.5 in Ref. [3] tells us that every effectively closed subset of $\mathscr{M}(\Omega)$ is effectively compact, so it suffices to prove that $\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}$ is effectively closed, which we'll do by establishing the existence of a recursively enumerable set $B\subseteq\mathscr{P}_{\mathrm{fin}}(\mathbb{Q}\times\mathbb{S}\times\mathbb{Q})$ such that $\bigcup_{b\in B}b(\Omega)=\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}$.

Since $\underline{\varphi}$ is lower semicomputable and $\overline{\varphi}$ is upper semicomputable, there are two recursive rational maps $\underline{q},\overline{q}\colon\mathbb{S}\times\mathbb{N}_0\to\mathbb{Q}$ such that, for all $s\in\mathbb{S}$, $\underline{q}(s,n)\nearrow\underline{\varphi}(s)$ and $\overline{q}(s,n)\searrow\overline{\varphi}(s)$ as $n\to\infty$. Let

$$B := \bigcup_{r\in\mathbb{Q}\cap(0,2), s\in\mathbb{S}, n\in\mathbb{N}_0} \big\{\{(-1,s,r),(r\overline{q}(s,n),s1,2)\},\{(-1,s,r),(r(1-\underline{q}(s,n)),s0,2)\}\big\}.$$

This set is clearly recursively enumerable.

To show that $\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}\subseteq\bigcup_{b\in B}b(\Omega)$, we start by proving that for any measure $\mu\in\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}$ there must be some $t\in\mathbb{S}$ such that $\mu(\llbracket t\rrbracket)>0$ and $\mu(\llbracket t1\rrbracket)/\mu(\llbracket t\rrbracket)\notin\varphi(t)$. To this end, consider the precise (not necessarily computable) forecasting system φ'_{pr} defined by

$$\varphi'_{\mathrm{pr}}(s) := \begin{cases} \frac{\mu(\llbracket s1\rrbracket)}{\mu(\llbracket s\rrbracket)} & \text{if } \mu(\llbracket s\rrbracket)>0\\ \underline{\varphi}(s) & \text{if } \mu(\llbracket s\rrbracket)=0\end{cases} \text{ for all } s\in\mathbb{S}.$$

By construction, $\mu=\mu^{\varphi'_{\mathrm{pr}}}$. Since $\mu\in\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}$ by assumption, there is some $t\in\mathbb{S}$ such that $\varphi'_{\mathrm{pr}}(t)\notin\varphi(t)$. Since, for all $s\in\mathbb{S}$, $\varphi'_{\mathrm{pr}}(s)=\underline{\varphi}(s)\subseteq\varphi(s)$ if $\mu(\llbracket s\rrbracket)=0$, we infer that, indeed, $\mu(\llbracket t\rrbracket)>0$ and $\mu(\llbracket t1\rrbracket)/\mu(\llbracket t\rrbracket)\notin\varphi(t)$.

There are now two possible and mutually exclusive cases.

The first case is that $\mu(\llbracket t1\rrbracket)/\mu(\llbracket t\rrbracket)>\overline{\varphi}(t)$, and then there is some $\varepsilon\in(0,1)$ such that $\mu(\llbracket t1\rrbracket)>\overline{\varphi}(t)\mu(\llbracket t\rrbracket)+\varepsilon$. Then there are $r\in\mathbb{Q}\cap(0,2)$ and $n\in\mathbb{N}_0$ such that $\mu(\llbracket t\rrbracket)<r<\mu(\llbracket t\rrbracket)+\varepsilon/4$ and $\overline{\varphi}(t)\leq\overline{q}(t,n)<\overline{\varphi}(t)+\varepsilon/4$, and we then find that

$$\begin{aligned} 0\leq r\overline{q}(t,n) &< \Big(\mu(\llbracket t\rrbracket)+\frac{\varepsilon}{4}\Big)\Big(\overline{\varphi}(t)+\frac{\varepsilon}{4}\Big)=\mu(\llbracket t\rrbracket)\overline{\varphi}(t)+\mu(\llbracket t\rrbracket)\frac{\varepsilon}{4}+\overline{\varphi}(t)\frac{\varepsilon}{4}+\frac{\varepsilon^2}{16}\\ &\leq \mu(\llbracket t\rrbracket)\overline{\varphi}(t)+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon^2}{16}\\ &< \mu(\llbracket t\rrbracket)\overline{\varphi}(t)+\varepsilon<\mu(\llbracket t1\rrbracket)<2,\end{aligned}$$

implying that $\mu\in\bigcup_{b\in B}b(\Omega)$.

The second possible case is that $\mu(\llbracket t1\rrbracket)/\mu(\llbracket t\rrbracket)<\underline{\varphi}(t)$, and then there is some $\varepsilon\in(0,1)$ such that $\mu(\llbracket t1\rrbracket)<\underline{\varphi}(t)\mu(\llbracket t\rrbracket)-\varepsilon$, and for which then also $\mu(\llbracket t0\rrbracket)=\mu(\llbracket t\rrbracket)-\mu(\llbracket t1\rrbracket)>\mu(\llbracket t\rrbracket)(1-\underline{\varphi}(t))+\varepsilon$. Then there are $r\in\mathbb{Q}\cap(0,2)$ and $n\in\mathbb{N}_0$ such that $\mu(\llbracket t\rrbracket)<r<\mu(\llbracket t\rrbracket)+\varepsilon/4$ and $\underline{\varphi}(t)-\varepsilon/4<\underline{q}(t,n)\leq\underline{\varphi}(t)$, and then we find that

$$\begin{aligned} 0\leq r\big(1-\underline{q}(t,n)\big) &< \Big(\mu(\llbracket t\rrbracket)+\frac{\varepsilon}{4}\Big)\Big(1-\underline{\varphi}(t)+\frac{\varepsilon}{4}\Big)\\ &= \mu(\llbracket t\rrbracket)\big(1-\underline{\varphi}(t)\big)+\mu(\llbracket t\rrbracket)\frac{\varepsilon}{4}+\big(1-\underline{\varphi}(t)\big)\frac{\varepsilon}{4}+\frac{\varepsilon^2}{16}\\ &\leq \mu(\llbracket t\rrbracket)\big(1-\underline{\varphi}(t)\big)+\frac{\varepsilon}{4}+\frac{\varepsilon}{4}+\frac{\varepsilon^2}{16}\\ &< \mu(\llbracket t\rrbracket)\big(1-\underline{\varphi}(t)\big)+\varepsilon<\mu(\llbracket t0\rrbracket)<2,\end{aligned}$$

also implying that $\mu\in\bigcup_{b\in B}b(\Omega)$, so $\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}\subseteq\bigcup_{b\in B}b(\Omega)$.

To prove that $\bigcup_{b\in B}b(\Omega)\subseteq\mathscr{M}(\Omega)\setminus\{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}} \text{ and } \varphi_{\mathrm{pr}}\subseteq\varphi\}$, consider any $\varphi_{\mathrm{pr}}\subseteq\varphi$. For any $s\in\mathbb{S}$, $n\in\mathbb{N}_0$ and $r>\mu^{\varphi_{\mathrm{pr}}}(\llbracket s\rrbracket)$ it follows from Proposition 7 that

$$\mu^{\varphi_{\mathrm{pr}}}(\llbracket s1\rrbracket)=\mu^{\varphi_{\mathrm{pr}}}(\llbracket s\rrbracket)\varphi_{\mathrm{pr}}(s)\leq r\overline{\varphi}(s)\leq r\overline{q}(s,n)$$

and

$$\mu^{\varphi_{\mathrm{pr}}}(\llbracket s0\rrbracket)=\mu^{\varphi_{\mathrm{pr}}}(\llbracket s\rrbracket)\big(1-\varphi_{\mathrm{pr}}(s)\big)\leq r\big(1-\underline{\varphi}(s)\big)\leq r\big(1-\underline{q}(s,n)\big),$$

implying that $\mu^{\varphi_{\mathrm{pr}}}\notin\bigcup_{b\in B}b(\Omega)$. □

Proof of Theorem 19. For the 'only if'-direction, assume that there's some $\mathscr{C}^{\varphi}$-test τ such that $\tau(\omega)=\infty$. Then we must show that ω isn't Martin-Löf test random for φ. First of all, that τ is a $\mathscr{C}^{\varphi}$-test implies in particular that $\{\varpi\in\Omega\colon\tau(\varpi)>r\}$ is effectively open, effectively in $r\in\mathbb{Q}$, meaning that there's some recursively enumerable subset $B\subseteq\mathbb{Q}\times\mathbb{S}$ such that, with obvious notations, $\llbracket B_r\rrbracket=\{\varpi\in\Omega\colon\tau(\varpi)>r\}$ for all $r\in\mathbb{R}$. This in turn implies that $A:=\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon(2^n,s)\in B\}$ is a recursively enumerable subset of $\mathbb{N}_0\times\mathbb{S}$ such that $\llbracket A_n\rrbracket=\{\varpi\in\Omega\colon\tau(\varpi)>2^n\}$ for all $n\in\mathbb{N}_0$. If we fix any $n\in\mathbb{N}_0$, then by assumption $\tau(\omega) > 2^n$ and therefore $\omega \in [\![A_n]\!]$. Hence, $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$, so we are done if we can prove that A is a Martin-Löf test for φ. We already know that A is recursively enumerable. Suppose towards contradiction that there is some $m \in \mathbb{N}_0$ such that $\overline{P}^{\varphi}([\![A_m]\!]) > 2^{-m}$. By Theorem 13 in Ref. [32], and footnote 5 which explains why this theorem applies to our context, it holds that $\overline{P}^{\varphi}([\![A_m]\!]) = \sup_{\varphi_{\mathrm{pr}}\subseteq\varphi} P^{\varphi_{\mathrm{pr}}}([\![A_m]\!])$, and hence, there is some precise $\varphi_{\mathrm{pr}} \subseteq \varphi$ for which

$$1 < 2^m P^{\varphi_{\mathrm{pr}}}([\![A_m]\!]) = E^{\varphi_{\mathrm{pr}}}(2^m \mathbb{I}_{[\![A_m]\!]}) \overset{\text{Prop. 7}}{=} \mu^{\varphi_{\mathrm{pr}}}(2^m \mathbb{I}_{[\![A_m]\!]}) \leq \mu^{\varphi_{\mathrm{pr}}}(\tau),$$

a contradiction.

For the 'if'-direction, assume that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$ for some Martin-Löf test A for φ. If we let $C_n := \bigcup_{m>n} A_m$ for all $n \in \mathbb{N}_0$, then clearly the set

$$\begin{aligned} C :=& \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon s \in C_n\} \\ =& \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon (\exists m > n) s \in A_m\} = \bigcup_{(m,s)\in A, m>n\in\mathbb{N}_0} \{(n,s)\} \end{aligned}$$

is recursively enumerable because A is, and the $[\![C_n]\!]$ therefore constitute a computable sequence of effectively open sets. Moreover, clearly $[\![C_0]\!] \supseteq [\![C_1]\!] \supseteq \dots$, and

$$\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!] \subseteq \bigcap_{n\in\mathbb{N}} [\![A_n]\!] \subseteq \bigcap_{n\in\mathbb{N}_0} \bigcup_{m>n} [\![A_m]\!] = \bigcap_{n\in\mathbb{N}_0} [\![C_n]\!]. \tag{29}$$

Now define the map $\tau \colon \Omega \to [0, +\infty]$ as $\tau(\omega) := \sum_{n\in\mathbb{N}} \mathbb{I}_{[\![C_n]\!]}(\omega)$ for all $\omega \in \Omega$. It follows from Equation (29) that $\tau(\omega) = \infty$, so we're done if we can show that τ is a $\mathscr{C}^{\varphi}$-test.

It follows from the nestedness $[\![C_0]\!] \supseteq [\![C_1]\!] \supseteq \dots$ that $\{\omega \in \Omega \colon \tau(\omega) > n\} = [\![C_{n+1}]\!]$ for all $n \in \mathbb{N}_0$. Therefore, since the $[\![C_n]\!]$ constitute a computable sequence of effectively open sets, so do the $\{\omega \in \Omega \colon \tau(\omega) > n\}$. By observing that

$$\{\omega \in \Omega \colon \tau(\omega) > r\} = \begin{cases} \{\omega \in \Omega \colon \tau(\omega) > \lfloor r \rfloor\} = [\![C_{\lfloor r \rfloor + 1}]\!] & \text{if } r \geq 0 \\ \Omega = [\![\mathbb{S}]\!] & \text{if } r < 0 \end{cases} \text{ for all } r \in \mathbb{Q},$$

we infer that $\{\omega \in \Omega \colon \tau(\omega) > r\}$ is effectively open, effectively in $r \in \mathbb{Q}$. Furthermore, it holds for any $\varphi_{\mathrm{pr}} \subseteq \varphi$ that

$$\begin{aligned} \mu^{\varphi_{\mathrm{pr}}}(\tau) &= \mu^{\varphi_{\mathrm{pr}}}\left(\sum_{n\in\mathbb{N}} \mathbb{I}_{[\![C_n]\!]}\right) \leq \mu^{\varphi_{\mathrm{pr}}}\left(\sum_{n\in\mathbb{N}}\sum_{m>n} \mathbb{I}_{[\![A_m]\!]}\right) \leq \sum_{n\in\mathbb{N}}\sum_{m>n} \mu^{\varphi_{\mathrm{pr}}}\left(\mathbb{I}_{[\![A_m]\!]}\right) \\ &= \sum_{n\in\mathbb{N}}\sum_{m>n} E^{\varphi_{\mathrm{pr}}}\left(\mathbb{I}_{[\![A_m]\!]}\right) \leq \sum_{n\in\mathbb{N}}\sum_{m>n} \overline{E}^{\varphi}\left(\mathbb{I}_{[\![A_m]\!]}\right) \leq \sum_{n\in\mathbb{N}}\sum_{m>n} 2^{-m} = \sum_{n\in\mathbb{N}} 2^{-n} = 1, \end{aligned}$$

where the first two inequalities follow from the properties of integrals, the second equality follows from Proposition 7, and the third inequality follows from Proposition 6. □

B.5. **Proofs of results in Section 8.**

Proof of Proposition 20. We give a proof by contraposition. Assume that ω isn't Schnorr random for φ, which implies that there's some computable test supermartingale T that is computably unbounded on ω, meaning that there's some growth function ρ such that

$$\limsup_{n\to\infty} [T(\omega_{1:n}) - \rho(n)] > 0. \tag{30}$$

By Proposition 9, we may also assume without loss of generality that T is recursive and rational-valued. Drawing inspiration from Schnorr's proof [24, Satz (9.4), p. 73] and Downey and Hirschfeldt's simplified version [14, Thm. 7.1.7], we let

$$A := \{(n,t) \in \mathbb{N}_0 \times \mathbb{S} \colon T(t) \geq \rho(|t|) \geq 2^n\}. \tag{31}$$

Then A is a recursive subset of $\mathbb{N}_0 \times \mathbb{S}$ [because the inequalities in the expressions above are decidable, as all numbers involved are rational]. We also see that, for any $\varpi \in \Omega$,

$$\varpi \in [\![A_n]\!] \Leftrightarrow (\exists m \in \mathbb{N}_0) \varpi_{1:m} \in A_n \Leftrightarrow (\exists m \in \mathbb{N}_0) \big(T(\varpi_{1:m}) \geq \rho(m) \geq 2^n\big). \tag{32}$$

Hence, $[\![A_n]\!] \subseteq \{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^n\}$, so we infer from Ville's inequality [Proposition 5] and E6 that

$$\overline{P}^{\varphi}([\![A_n]\!]) \leq \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^n\right\}\right) \leq 2^{-n} \text{ for all } n \in \mathbb{N}_0.$$

This shows that A is a Martin-Löf test for φ. It also follows from Equations (30) and (32) that $\omega \in \bigcap_{m\in\mathbb{N}_0}[\![A_m]\!]$. So we'll find that ω isn't Schnorr test random for φ, provided we can prove that A is a Schnorr test.

To this end, we'll show that it has a tail bound. Define the map $e\colon \mathbb{N}_0^2 \to \mathbb{N}_0$ by letting $e(N,n) := \min\{k \in \mathbb{N}_0 \colon \rho(k) \geq 2^N\}$, for all $N,n \in \mathbb{N}_0$. Fix any $N,n \in \mathbb{N}_0$, then we infer from Equation (31) that

$$\varpi \in [\![A_n^{\geq \ell}]\!] \Leftrightarrow (\exists m \geq \ell) T(\varpi_{1:m}) \geq \rho(m) \geq 2^n), \text{ for all } \ell \in \mathbb{N}_0.$$

Hence, for all $\ell \geq e(N,n)$ and all $\varpi \in [\![A_n^{\geq \ell}]\!]$, there's some $m \geq \ell$ such that

$$T(\varpi_{1:m}) \geq \rho(m) \geq \rho(\ell) \geq \rho(e(N,n)) \geq 2^N,$$

which implies that $[\![A_n^{\geq \ell}]\!] \subseteq \{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^N\}$. Ville's inequality [Proposition 5] and E6 then guarantee that, for all $\ell \geq e(N,n)$, since $[\![A_n]\!] \setminus [\![A_n^{<\ell}]\!] \subseteq [\![A_n^{\geq \ell}]\!]$,

$$\overline{P}^{\varphi}\big([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]\big) \leq \overline{P}^{\varphi}\big([\![A_n^{\geq \ell}]\!]\big) \leq \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^N\right\}\right) \leq 2^{-N}. \quad \square$$

Proof of Proposition 21. For this converse result too, we give a proof by contraposition. Assume that ω isn't Schnorr test random for φ, which implies that there's some Schnorr test A for φ such that $\omega \in \bigcap_{n\in\mathbb{N}_0}[\![A_n]\!]$. It follows from Proposition 11 that we may assume without loss of generality that the sets of situations A_n are partial cuts for all $n \in \mathbb{N}_0$. We'll now use this A to construct a computable test supermartingale that is computably unbounded on ω.

We infer from Lemma 34 that there's some growth function ς such that

$$\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}\big([\![A_n^{\geq \varsigma(k)}]\!]\big) \leq 2^{-k} \text{ for all } k \in \mathbb{N}_0. \tag{33}$$

We use this growth function ς to define the following maps, all of which are non-negative supermartingales for φ, by Corollary 3 and C2, because the $A_n^{\geq \varsigma(k)}$ are partial cuts:

$$Z_{n,k}\colon \mathbb{S} \to \mathbb{R}\colon s \mapsto 2^k \overline{P}^{\varphi}\big([\![A_n^{\geq \varsigma(k)}]\!] \,\big|\, s\big), \text{ for all } n,k \in \mathbb{N}_0.$$

Since the forecasting system φ was assumed to be non-degenerate, Proposition 29 now implies that

$$0 \leq Z_{n,k}(s) \leq Z_{n,k}(\square) C_{\varphi}(s) = 2^k \overline{P}^{\varphi}\big([\![A_n^{\geq \varsigma(k)}]\!]\big) C_{\varphi}(s) \text{ for all } s \in \mathbb{S}. \tag{34}$$

If we also define the (possibly extended) real process $Z := \frac{1}{2}\sum_{n,k\in\mathbb{N}_0} Z_{n,k}$, then we infer from Equations (33) and (34) that

$$0 \leq Z(s) = \frac{1}{2}\sum_{n,k\in\mathbb{N}_0} Z_{n,k}(s) \leq \frac{1}{2} C_{\varphi}(s) \sum_{n,k\in\mathbb{N}_0} 2^k \overline{P}^{\varphi}\big([\![A_n^{\geq \varsigma(k)}]\!]\big) \leq C_{\varphi}(s) \frac{1}{2} \sum_{k\in\mathbb{N}_0} 2^{-k} = C_{\varphi}(s) \quad \text{for all } s \in \mathbb{S}. \tag{35}$$

This guarantees that Z is real-valued, and that, moreover, $Z(\square) \leq 1$.

Now, fix any $s \in \mathbb{S}$. Then we readily see that $\frac{1}{2}\sum_{n=0}^{N}\sum_{\ell=0}^{L} Z_{n,\ell}(s) \nearrow Z(s)$ and therefore also $\frac{1}{2}\sum_{n=0}^{N}\sum_{\ell=0}^{L} \Delta Z_{n,\ell}(s) \to \Delta Z(s)$ as $N,L \to \infty$. Since the gambles $\Delta Z_{n,\ell}(s)$ and $\Delta Z(s)$ are defined on the finite domain $\{0,1\}$, this point-wise convergence also implies uniform convergence, so we can infer from C6 and

$$\overline{E}_{\varphi(s)}\left(\frac{1}{2}\sum_{n=0}^{N}\sum_{\ell=0}^{L} \Delta Z_{n,\ell}(s)\right) \leq \frac{1}{2}\sum_{n=0}^{N}\sum_{\ell=0}^{L} \overline{E}_{\varphi(s)}(\Delta Z_{n,\ell}(s)) \leq 0,$$

which is implied by C2, C3 and the supermartingale character of the $Z_{n,\ell}$, that also

$$\overline{E}_{\varphi(s)}(\Delta Z(s)) = \lim_{N,L\to\infty} \overline{E}_{\varphi(s)}\bigg(\frac{1}{2}\sum_{n=0}^{N}\sum_{\ell=0}^{L}\Delta Z_{n,\ell}(s)\bigg) \leq 0. \tag{36}$$

This tells us that Z is a non-negative supermartingale for φ. It follows from Lemma 35 that Z is also computable.

The relevant condition being $\overline{E}_{\varphi(\square)}(Z(\square\cdot)) \leq Z(\square)$, we see that replacing $Z(\square) \leq 1$ by 1 does not change the supermartingale character of Z, and doing so leads to a computable test supermartingale Z' for φ.

To show that this Z' is computably unbounded on ω, we take two steps.

In a first step, we fix any $n \in \mathbb{N}_0$. Since $\omega \in \bigcap_{m\in\mathbb{N}_0} [\![A_m]\!]$, and since the A_m were assumed to be partial cuts, there's some (unique) $\ell_n \in \mathbb{N}_0$ such that $\omega_{1:\ell_n} \in A_n$. This tells us that if $\ell \leq \ell_n$, then also $\omega_{1:\ell_n} \in A_n^{\geq\ell}$, and therefore, by Corollary 3(iv), that $\overline{P}^{\varphi}([\![A_n^{\geq\ell}]\!]|\omega_{1:\ell_n}) = 1$ for all $\ell \leq \ell_n$. Hence,

$$\overline{P}^{\varphi}\big([\![A_n^{\geq\varsigma(k)}]\!]\,\big|\,\omega_{1:\ell_n}\big) = 1 \text{ for all } k \in \mathbb{N}_0 \text{ such that } \varsigma(k) \leq \ell_n.$$

Let's now define the map $\varsigma^\sharp\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\varsigma^\sharp(\ell) := \sup\{k \in \mathbb{N}_0\colon \varsigma(k) \leq \ell\}$ for all $\ell \in \mathbb{N}_0$, where we use the convention that $\sup\emptyset = 0$. It's clear that $\varsigma^\sharp$ is a growth function. Moreover, as soon as $\ell_n \geq \varsigma(0)$, we find that, in particular, $\varsigma(k) \leq \ell_n$ for $k = \varsigma^\sharp(\ell_n)$. Hence,

$$\overline{P}^{\varphi}\big([\![A_n^{\geq\varsigma(k)}]\!]\,\big|\,\omega_{1:\ell_n}\big) = 1 \text{ for } k = \varsigma^\sharp(\ell_n), \text{ if } \ell_n \geq \varsigma(0).$$

This leads us to the conclusion that for all $n \in \mathbb{N}_0$, there's some $\ell_n \in \mathbb{N}_0$ such that

$$Z'(\omega_{1:\ell_n}) \geq Z(\omega_{1:\ell_n}) \geq \frac{1}{2}Z_{n,\varsigma^\sharp(\ell_n)}(\omega_{1:\ell_n}) = 2^{\varsigma^\sharp(\ell_n)-1} \text{ if } \ell_n \geq \max\{\varsigma(0),1\}. \tag{37}$$

Since $\varsigma^\sharp$ is a growth function, so is the map $\rho\colon \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$\rho(m) := \max\big\{2^{\varsigma^\sharp(m)-1} - 1, 1\big\} \text{ for all } m \in \mathbb{N}_0.$$

We will therefore be done if we can now show that the sequence ℓ_n is unbounded as $n \to \infty$, because the inequality in Equation (37) will then guarantee that

$$\limsup_{m\to\infty}[Z'(\omega_{1:m}) - \rho(m)] > 0,$$

so the computable test supermartingale Z' is computably unbounded on ω.

Proving that ℓ_n is unbounded as $n \to \infty$ is therefore our second step. To accomplish this, we use the assumption that φ is non-degenerate. Assume, towards contradiction, that there's some natural number B such that $\ell_n \leq B$ for all $n \in \mathbb{N}_0$. The non-degenerate character of φ implies that $\min\{\overline{\varphi}(s), 1-\underline{\varphi}(s)\} > 0$ for all $s \in \mathbb{S}$, which implies in particular that there's some real $1 > \delta > 0$ such that $\min\{\overline{\varphi}(\omega_{1:k}), 1-\underline{\varphi}(\omega_{1:k})\} \geq \delta$ for all non-negative integers $k \leq B$, as they are finite in number. But this implies that, for any $n \in \mathbb{N}_0$,

$$2^{-n} \geq \overline{P}^{\varphi}([\![A_n]\!]) \geq \overline{P}^{\varphi}([\![\omega_{1:\ell_n}]\!]) = \prod_{k=0}^{\ell_n-1} \overline{\varphi}(\omega_{1:k})^{\omega_{k+1}}[1-\underline{\varphi}(\omega_{1:k})]^{1-\omega_{k+1}} \geq \delta^{\ell_n} \geq \delta^B,$$

where the first inequality follows from the properties of a Schnorr test, the second inequality from $[\![\omega_{1:\ell_n}]\!] \subseteq [\![A_n]\!]$ and E6, the equality from Proposition 4, and the fourth inequality from $1 > \delta > 0$ and $\ell_n \leq B$. However, since $1 > \delta > 0$ and $B \in \mathbb{N}$, there's always some $n \in \mathbb{N}_0$ such that $2^{-n} < \delta^B$, which is the desired contradiction. □

Lemma 33. *Consider any Schnorr test A for a non-degenerate computable forecasting system φ, such that the corresponding A_n are partial cuts for all $n \in \mathbb{N}_0$. Then there's some recursive map $\tilde{e}\colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ such that its partial maps $\tilde{e}(\bullet, s)$ are growth functions for all $s \in \mathbb{S}$, and such that*

$$\overline{P}^{\varphi}\big([\![A_n^{\geq\ell}]\!]\big|s\big) \leq 2^{-N} \textit{ for all } (N,n,s) \in \mathbb{N}_0^2 \times \mathbb{S} \textit{ and all } \ell \geq \tilde{e}(N,s).$$

Proof. Proposition 11(ii) guarantees that there's a growth function $e\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket\big) = \overline{P}^{\varphi}\big(\llbracket A_n \rrbracket \setminus \llbracket A_n^{<\ell} \rrbracket\big) \leq 2^{-M} \text{ for all } M,n \in \mathbb{N}_0 \text{ and all } \ell \geq e(M),$$

where the equality holds because the A_n are assumed to be partial cuts. Since the real process $\overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket \big| \bullet\big)$ is a non-negative supermartingale by Corollary 3, we infer from the non-degeneracy of φ, Proposition 29 and Lemma 32 [where we recall that $C_\varphi \geq 1$ is computable] that

$$0 \leq \overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket \big| s\big) \leq \overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket\big) C_\varphi(s) \leq 2^{-M} C_\varphi(s) \leq 2^{-M+L_{C_\varphi}(s)}$$
$$\text{for all } (M,n) \in \mathbb{N}_0^2 \text{ and all } \ell \geq e(M).$$

It's therefore clear that if we let

$$\tilde{e}(N,s) \coloneqq e\big(N + L_{C_\varphi}(s)\big) \text{ for all } (N,s) \in \mathbb{N}_0 \times \mathbb{S},$$

then

$$\overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket \big| s\big) \leq 2^{-N} \text{ for all } (N,n,s) \in \mathbb{N}_0^2 \times \mathbb{S} \text{ and all } \ell \geq \tilde{e}(N,s).$$

This $\tilde{e}$ is recursive because e and L_{C_φ} are [recall that L_{C_φ} is recursive by Lemma 32]. For any fixed s in $\mathbb{S}$, $\tilde{e}(\bullet, s)$ is clearly non-decreasing and unbounded, because e is. □

Lemma 34. *Consider any Schnorr test A for a computable forecasting system φ, such that the corresponding A_n are partial cuts for all $n \in \mathbb{N}_0$. Then there's some growth function $\varsigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that*

$$\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}\big(\big\llbracket A_n^{\geq \varsigma(k)} \big\rrbracket\big) \leq 2^{-k} \text{ for all } k \in \mathbb{N}_0.$$

Proof. Proposition 11(ii) guarantees that there's a growth function $e\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\overline{P}^{\varphi}\big(\llbracket A_n^{\geq \ell} \rrbracket\big) = \overline{P}^{\varphi}\big(\llbracket A_n \rrbracket \setminus \llbracket A_n^{<\ell} \rrbracket\big) \leq 2^{-N} \text{ for all } N,n \in \mathbb{N}_0 \text{ and all } \ell \geq e(N),$$

where the equality holds because the A_n are assumed to be partial cuts. Let $\varsigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varsigma(k) \coloneqq \max_{n=0}^{2k+1} e(2k+2+n)$ for all $k \in \mathbb{N}_0$. Clearly, ς is recursive because e is. It follows from the non-decreasingness and unboundedness of e that ς is non-decreasing, since

$$\varsigma(k+1) = \max_{n=0}^{2k+3} e(2k+4+n) \geq \max_{n=0}^{2k+1} e(2k+2+n) = \varsigma(k) \text{ for all } k \in \mathbb{N}_0,$$

and that ς is unbounded, since $\varsigma(k) \geq e(2k+2)$ for all $k \in \mathbb{N}_0$. So we conclude that ς is a growth function.

Now, for any $k \in \mathbb{N}_0$, we find that, indeed,

$$\begin{aligned}
\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}\big(\big\llbracket A_n^{\geq \varsigma(k)} \big\rrbracket\big) &= 2^k \sum_{n=0}^{2k+1} \overline{P}^{\varphi}\big(\big\llbracket A_n^{\geq \varsigma(k)} \big\rrbracket\big) + 2^k \sum_{n=2k+2}^{\infty} \overline{P}^{\varphi}\big(\big\llbracket A_n^{\geq \varsigma(k)} \big\rrbracket\big) \\
&\leq 2^k \sum_{n=0}^{2k+1} \overline{P}^{\varphi}\big(\big\llbracket A_n^{\geq e(2k+2+n)} \big\rrbracket\big) + 2^k \sum_{n=2k+2}^{\infty} \overline{P}^{\varphi}\big(\llbracket A_n \rrbracket\big) \\
&\leq 2^k \sum_{n=0}^{2k+1} 2^{-(2k+2+n)} + 2^k \sum_{n=2k+2}^{\infty} 2^{-n} = 2^{-(k+1)} \sum_{n=0}^{2k+1} 2^{-(n+1)} + 2^{-(k+1)} \\
&\leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k},
\end{aligned}$$

where the first inequality follows from E6. □

Lemma 35. *The non-negative supermartingale Z in the proof of Proposition 21 is computable.*

Proof. We use the notations in the proof of Proposition 21. We aim at obtaining a computable real map that converges effectively to Z. First of all, for any $p \in \mathbb{N}_0$,

$$Z(s) = \frac{1}{2}\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} Z_{n,k}(s) = \frac{1}{2}\sum_{k=0}^{p}\sum_{n=0}^{\infty} Z_{n,k}(s) + \underbrace{\frac{1}{2}\sum_{k=p+1}^{\infty}\sum_{n=0}^{\infty} Z_{n,k}(s)}_{R_1(p,s)},$$

where

$$\begin{aligned}
|R_1(p,s)| = R_1(p,s) &= \frac{1}{2}\sum_{k=p+1}^{\infty}\sum_{n=0}^{\infty} Z_{n,k}(s) \\
&\leq \frac{1}{2}\sum_{k=p+1}^{\infty}\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}\big(\llbracket A_n^{\geq \varsigma(k)} \rrbracket\big) C_\varphi(s) = \frac{1}{2} C_\varphi(s) \sum_{k=p+1}^{\infty}\left(\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}\big(\llbracket A_n^{\geq \varsigma(k)} \rrbracket\big)\right) \\
&\leq C_\varphi(s)\frac{1}{2}\sum_{k=p+1}^{\infty} 2^{-k} = C_\varphi(s) 2^{-(p+1)} \leq 2^{-(p+1-L_{C_\varphi}(s))}.
\end{aligned}$$

In this chain of (in)equalities, the first inequality follows from Equation (34) and the second inequality follows from Equation (33). The last inequality is based on Lemma 32 and the notations introduced there. If we therefore define the recursive map $e_1 \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ by $e_1(N,s) := N + L_{C_\varphi}(s)$ for all $(N,s) \in \mathbb{N}_0 \times \mathbb{S}$ [recall that L_{C_φ} is recursive by Lemma 32], then we find that

$$|R_1(p,s)| \leq 2^{-(N+1)} \text{ for all } (N,s) \in \mathbb{N}_0 \times \mathbb{S} \text{ and all } p \geq e_1(N,s).$$

Next, we consider any $p, q \in \mathbb{N}_0$ and look at

$$\frac{1}{2}\sum_{k=0}^{p}\sum_{n=0}^{\infty} Z_{n,k}(s) = \frac{1}{2}\sum_{k=0}^{p}\sum_{n=0}^{q} Z_{n,k}(s) + \underbrace{\frac{1}{2}\sum_{k=0}^{p}\sum_{n=q+1}^{\infty} Z_{n,k}(s)}_{R_2(p,q,s)},$$

where

$$\begin{aligned}
|R_2(p,q,s)| = R_2(p,q,s) &= \frac{1}{2}\sum_{k=0}^{p}\sum_{n=q+1}^{\infty} Z_{n,k}(s) \\
&\leq \frac{1}{2}\sum_{k=0}^{p}\sum_{n=q+1}^{\infty} 2^k \overline{P}^{\varphi}\big(\llbracket A_n^{\geq \varsigma(k)} \rrbracket\big) C_\varphi(s) \leq \frac{1}{2} C_\varphi(s) 2^p \sum_{k=0}^{p}\left(\sum_{n=q+1}^{\infty} \overline{P}^{\varphi}\big(\llbracket A_n \rrbracket\big)\right) \\
&\leq C_\varphi(s) 2^{p-1}(p+1)\sum_{n=q+1}^{\infty} 2^{-n} = C_\varphi(s) 2^{p-q-1}(p+1) \leq 2^{2p-q-1+L_{C_\varphi}(s)}.
\end{aligned}$$

In this chain of (in)equalities, the first inequality follows from Equation (34), the second inequality follows from E6 since $\llbracket A_n^{\geq \varsigma(k)} \rrbracket \subseteq \llbracket A_n \rrbracket$ for all $k, n \in \mathbb{N}_0$, and the third inequality follows from the assumption that A is a Schnorr test. The fourth inequality is based on Lemma 32 and the notations introduced there, and the fact that $p+1 \leq 2^p$ for all $p \in \mathbb{N}_0$. If we therefore define the recursive map $e_3 \colon \mathbb{N}_0^2 \times \mathbb{S} \to \mathbb{N}_0$ by $e_3(p,N,s) := N + 2p + L_{C_\varphi}(s)$ for all $(p,N,s) \in \mathbb{N}_0^2 \times \mathbb{S}$ [recall that L_{C_φ} is recursive by Lemma 32], then we find that

$$|R_2(p,q,s)| \leq 2^{-(N+1)} \text{ for all } (p,N,s) \in \mathbb{N}_0^2 \times \mathbb{S} \text{ and } q \geq e_3(p,N,s).$$

Now, consider the recursive map $e_2 \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ defined by $e_2(N,s) := e_3(e_1(N,s),N,s)$ for all $(N,s) \in \mathbb{N}_0 \times \mathbb{S}$, and let

$$V_N(s) := \frac{1}{2}\sum_{k=0}^{e_1(N,s)}\sum_{n=0}^{e_2(N,s)} Z_{n,k}(s) \text{ for all } N \in \mathbb{N}_0 \text{ and } s \in \mathbb{S}.$$

Since the real map $(n,k,s) \mapsto Z_{n,k}(s)$ is computable by Lemma 36, it follows that the real map $(N,s) \mapsto V_N(s)$ is computable as well, since by definition each $V_N(s)$ is a finite sum of a real numbers that are computable effectively in N and s, and all terms that are included in the sum are defined recursively as a function of N and s. From the argumentation above, we infer that

$$\begin{aligned}|Z(s)-V_N(s)| &= |R_1(e_1(N,s),s)+R_2(e_1(N,s),e_2(N,s),s)|\\ &\leq |R_1(e_1(N,s),s)|+|R_2(e_1(N,s),e_2(N,s),s)|\\ &\leq 2^{-(N+1)}+2^{-(N+1)}=2^{-N} \text{ for all } s\in\mathbb{S} \text{ and } N\in\mathbb{N}_0,\end{aligned}$$

proving that Z is indeed computable. □

Lemma 36. *For the non-negative supermartingales $Z_{n,k}$ defined in the proof of Proposition 21, the real map $(n,k,s)\mapsto Z_{n,k}(s)$ is computable.*

Proof. We use the notations and assumptions in the proof of Proposition 21. Clearly, it suffices to prove that the real map $(n,k,s)\mapsto Z_{n,k}(s)2^{-k}=\overline{P}^{\varphi}\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket\,\big|s\big)$ is computable. If we let

$$A_n^{k,\ell}:=A_n^{<\ell}\cap A_n^{\geq\varsigma(k)}=\{s\in A_n\colon \varsigma(k)\leq|s|<\ell\},$$

then $\llbracket A_n^{k,\ell}\rrbracket\subseteq\llbracket A_n^{\geq\varsigma(k)}\rrbracket$ and the global events $\llbracket A_n^{k,\ell}\rrbracket$ and $\llbracket A_n^{\geq\ell}\rrbracket$ are disjoint for all $\ell,n,k\in\mathbb{N}_0$, because the A_n have been assumed to be partial cuts. Moreover,

$$\llbracket A_n^{k,\ell}\rrbracket\subseteq\llbracket A_n^{\geq\varsigma(k)}\rrbracket\left\{\begin{array}{ll}=\llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket & \text{if } \ell>\varsigma(k)\\ \subseteq\llbracket A_n^{\geq\ell}\rrbracket=\llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket & \text{if } \ell\leq\varsigma(k)\end{array}\right\}\subseteq\llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket, \tag{38}$$

where the last equality holds because then $\llbracket A_n^{k,\ell}\rrbracket=\emptyset$. By Lemma 33, there's some recursive map $\tilde{e}\colon\mathbb{N}_0\times\mathbb{S}\to\mathbb{N}_0$ such that $\overline{P}^{\varphi}\big(\llbracket A_n^{\geq\ell}\rrbracket\big|s\big)\leq 2^{-N}$ for all $(N,n,s)\in\mathbb{N}_0^2\times\mathbb{S}$ and all $\ell\geq\tilde{e}(N,s)$. This allows us to infer that

$$\begin{aligned}\overline{P}^{\varphi}\big(\llbracket A_n^{k,\ell}\rrbracket\big|s\big) &\leq \overline{P}^{\varphi}\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket\,\big|s\big)\leq\overline{P}^{\varphi}\big(\llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket\big|s\big)\\ &\leq \overline{P}^{\varphi}\big(\llbracket A_n^{k,\ell}\rrbracket\big|s\big)+\overline{P}^{\varphi}\big(\llbracket A_n^{\geq\ell}\rrbracket\big|s\big)\\ &\leq \overline{P}^{\varphi}\big(\llbracket A_n^{k,\ell}\rrbracket\big|s\big)+2^{-N} \text{ for all } N,k,n\in\mathbb{N}_0 \text{ and } s\in\mathbb{S} \text{ and } \ell\geq\tilde{e}(N,s),\end{aligned}\tag{39}$$

where the first two inequalities follow from Equation (38) and E6, and the third inequality follows from E3, because $\llbracket A_n^{k,\ell}\rrbracket$ and $\llbracket A_n^{\geq\ell}\rrbracket$ are disjoint. Now, the sets $A_n^{k,\ell}$ are recursive effectively in n, k and ℓ, and it also holds that $|s|<\ell$ for all $s\in A_n^{k,\ell}$ and $n,k,\ell\in\mathbb{N}_0$. Hence, the real map $(n,k,\ell,s)\mapsto\overline{P}^{\varphi}(\llbracket A_n^{k,\ell}\rrbracket|s)$ is computable by an appropriate instantiation of our Workhorse Lemma 27 [with $\mathscr{D}\to\mathbb{N}_0^2$, $d\to(n,k)$, $p\to\ell$ and $C\mapsto\{(n,k,\ell,s)\in\mathbb{N}_0^3\times\mathbb{S}\colon s\in A_n^{k,\ell}\}$, and therefore $C_d^p\to A_n^{k,\ell}$], because the forecasting system φ is computable as well. The inequalities in Equation (39) tell us that this computable real map converges effectively to the real map $(n,k,s)\mapsto\overline{P}^{\varphi}(\llbracket A_n^{\geq\varsigma(k)}\rrbracket|s)$, which is therefore computable as well. □

B.6. **Proofs of results in Section 9.**

Proof of Proposition 23. It's a standard result in computability theory that the countable collection $\phi_i\colon\mathbb{N}_0\to\mathbb{N}_0$, with $i\in\mathbb{N}_0$, of all partial recursive maps is itself partial recursive, meaning that there's some partial recursive map $\phi\colon\mathbb{N}_0^2\to\mathbb{N}_0$ such that $\phi(i,n)=\phi_i(n)$ for all $i,n\in\mathbb{N}_0$; see for instance Ref. [14, Prop. 2.1.2]. Consequently, via encoding, we can infer that there's a recursively enumerable set $A\subseteq\mathbb{N}_0^2\times\mathbb{S}$ that contains all recursively enumerable sets $C\subseteq\mathbb{N}_0\times\mathbb{S}$, in the sense that for every recursively enumerable set $C\subseteq\mathbb{N}_0\times\mathbb{S}$ there's some $M\in\mathbb{N}_0$ such that $C={}^{M}A$, with ${}^{m}A:=\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon(m,n,s)\in A\}$ for all $m\in\mathbb{N}_0$. With every such ${}^{m}A$, we associate as usual the sets of situations ${}^{m}A_n$, defined for all $n\in\mathbb{N}_0$ by ${}^{m}A_n:=\{s\in\mathbb{S}\colon(n,s)\in{}^{m}A\}$. For reasons explained after Definition 6, we can and will assume, without changing the map of global events $(m,n)\mapsto\llbracket{}^{m}A_n\rrbracket$, that all these sets ${}^{m}A_n$ are partial cuts and recursive effectively in m and n; again, see Ref. [14, Sec. 2.19] for more discussion and proofs. For this A, we then have that for every recursively enumerable set $C\subseteq\mathbb{N}_0\times\mathbb{S}$ there's some $m_C\in\mathbb{N}_0$ such that $\llbracket C_n\rrbracket=\llbracket{}^{m_C}A_n\rrbracket$ for all $n\in\mathbb{N}_0$.

As a first step in the proof, we show that there's a single finite algorithm for turning, for any given $m \in \mathbb{N}_0$, the corresponding recursive set ${}^mA$ into a Martin-Löf test ${}^mB$ for φ. Let ${}^mA_n^{<\ell} := \{s \in \mathbb{S} \colon (m,n,s) \in A, |s| < \ell\}$ for all $m,n,\ell \in \mathbb{N}_0$. It's clear from the construction that the finite sets ${}^mA_n^{<\ell}$ are recursive effectively in m, n and ℓ. Observe that the computability of the forecasting system φ, the recursive character of the finite partial cuts ${}^mA_n^{<\ell}$ and an appropriate instantiation of our Workhorse Lemma 27 [with $\mathscr{D} \to \mathbb{N}_0^2$, $d \to (m,n)$, $p \to \ell$ and $C \to \{(m,n,\ell,s) \in \mathbb{N}_0^3 \times \mathbb{S} \colon s \in {}^mA_n^{<\ell}\}$, and therefore $C_d^p \to {}^mA_n^{<\ell}$] allow us to infer that the real map $(m,n,\ell) \mapsto \overline{P}^\varphi([\![{}^mA_n^{<\ell}]\!])$ is computable, meaning that there's some recursive rational map $q \colon \mathbb{N}_0^4 \to \mathbb{Q}$ such that

$$\left|\overline{P}^\varphi([\![{}^mA_n^{<\ell}]\!]) - q(m,n,\ell,N)\right| \leq 2^{-N} \text{ for all } m,n,\ell,N \in \mathbb{N}_0.$$

Observe that $q(m,n,\ell,n+2)$ is a rational approximation for $\overline{P}^\varphi([\![{}^mA_n^{<\ell}]\!])$ up to $2^{-(n+2)}$, since

$$\left|\overline{P}^\varphi([\![{}^mA_n^{<\ell}]\!]) - q(m,n,\ell,n+2)\right| \leq 2^{-(n+2)} \text{ for all } m,n,\ell \in \mathbb{N}_0. \tag{40}$$

Now consider the (obviously) recursive map $\lambda \colon \mathbb{N}_0^3 \to \mathbb{N}_0$, defined by

$$\lambda(m,n,\ell) := \max\Big\{p \in \{0,\dots,\ell\} \colon (\forall k \in \{0,\dots,p\})\, q(m,n,k,n+2) \leq 2^{-(n+1)} + 2^{-(n+2)}\Big\} \quad \text{for all } m,n,\ell \in \mathbb{N}_0. \tag{41}$$

Observe that $\lambda(m,n,0) = 0$, because

$$q(m,n,0,n+2) \leq \overline{P}^\varphi([\![{}^mA_n^{<0}]\!]) + 2^{-(n+2)} = \overline{P}^\varphi(\emptyset) + 2^{-(n+2)} = 2^{-(n+2)},$$

where the inequality follows from Equation (40), and the last equality from E1; this ensures that the map λ is indeed well-defined. Consequently, by construction,

$$q(m,n,\lambda(m,n,\ell),n+2) \leq 2^{-(n+1)} + 2^{-(n+2)} \text{ for all } m,n,\ell \in \mathbb{N}_0. \tag{42}$$

Also, the partial maps $\lambda(m,n,\bullet)$ are obviously non-decreasing.

Now let ${}^mB_n^\ell := {}^mA_n^{<\lambda(m,n,\ell)}$ for all $m,n,\ell \in \mathbb{N}_0$. It follows from Equations (40) and (42) that

$$\begin{aligned}\overline{P}^\varphi\big([\![{}^mB_n^\ell]\!]\big) = \overline{P}^\varphi\Big([\![{}^mA_n^{<\lambda(m,n,\ell)}]\!]\Big) &\leq q(m,n,\lambda(m,n,\ell),n+2) + 2^{-(n+2)} \\ &\leq \big(2^{-(n+1)} + 2^{-(n+2)}\big) + 2^{-(n+2)} = 2^{-n}.\end{aligned}$$

We now use the sets ${}^mB_n^\ell$ in the obvious manner to define

$${}^mB_n := \bigcup_{\ell \in \mathbb{N}_0} {}^mB_n^\ell \text{ and } {}^mB := \bigcup_{n \in \mathbb{N}_0} \{n\} \times {}^mB_n, \text{ for all } m,n \in \mathbb{N}_0,$$

so the set ${}^mB \subseteq \mathbb{N}_0 \times \mathbb{S}$ is recursively enumerable as a countable union of finite sets $\{n\} \times {}^mB_n^\ell$ that are recursive effectively in n and ℓ. Moreover, it follows from E9 and the non-decreasing character of the partial map $\lambda(m,n,\bullet)$ that

$$\overline{P}^\varphi\big([\![{}^mB_n]\!]\big) = \sup_{\ell \in \mathbb{N}_0} \overline{P}^\varphi\big([\![{}^mB_n^\ell]\!]\big) \leq 2^{-n}, \text{ for all } m,n \in \mathbb{N}_0,$$

and therefore each ${}^mB$ is a Martin-Löf test for φ.

As a second step in the proof, we now show that any path $\omega \in \Omega$ is Martin-Löf test random for φ if and only if $\omega \notin \bigcap_{n \in \mathbb{N}_0} [\![{}^mB_n]\!]$ for all $m \in \mathbb{N}_0$. Since each ${}^mB$ is a Martin-Löf test for φ, it suffices to show by Lemma 37 that for every recursively enumerable subset $C \subseteq \mathbb{N}_0 \times \mathbb{S}$ for which $\overline{P}^\varphi([\![C_n]\!]) \leq 2^{-(n+1)}$ for all $n \in \mathbb{N}_0$, there's some $m_C \in \mathbb{N}_0$ such that $[\![C_n]\!] = [\![{}^{m_C}B_n]\!]$ for all $n \in \mathbb{N}_0$; this is what we now set out to do.

Since we assumed that C is recursively enumerable, we know there's some $m_C \in \mathbb{N}_0$ such that $[\![C_n]\!] = [\![{}^{m_C}A_n]\!]$ for all $n \in \mathbb{N}_0$. This implies that $\overline{P}^\varphi([\![{}^{m_C}A_n]\!]) = \overline{P}^\varphi([\![C_n]\!]) \leq 2^{-(n+1)}$ for all $n \in \mathbb{N}_0$, so we see that for this m_C:

$$q(m_C,n,\ell,n+2) \leq \overline{P}^\varphi([\![{}^{m_C}A_n^{<\ell}]\!]) + 2^{-(n+2)} \leq \overline{P}^\varphi([\![{}^{m_C}A_n]\!]) + 2^{-(n+2)} \leq 2^{-(n+1)} + 2^{-(n+2)} \quad \text{for all } n,\ell \in \mathbb{N}_0,$$

where the first inequality follows from Equation (40), and the second inequality follows from E6. If we now look at the definition of the map λ in Equation (41), we see that $\lambda(m_C,n,\ell)=\ell$ for all $n,\ell\in\mathbb{N}_0$. Consequently,

$$^{m_C}A_n=\bigcup_{\ell\in\mathbb{N}_0}{}^{m_C}A_n^{<\ell}=\bigcup_{\ell\in\mathbb{N}_0}{}^{m_C}A_n^{<\lambda(m_C,n,\ell)}=\bigcup_{\ell\in\mathbb{N}_0}{}^{m_C}B_n^{\ell}={}^{m_C}B_n \text{ for all } n\in\mathbb{N}_0,$$

and therefore, indeed, $[\![C_n]\!]=[\![^{m_C}A_n]\!]=[\![^{m_C}B_n]\!]$ for all $n\in\mathbb{N}_0$.

As a third step in the proof, we show that we can combine the Martin-Löf tests ^{m}B for φ, with $m\in\mathbb{N}_0$, into a single Martin-Löf test U for φ. To this end, let $U_n:=\bigcup_{m\in\mathbb{N}_0}{}^{m}B_{n+m+1}$ for all $n\in\mathbb{N}_0$. Then $U:=\bigcup_{n\in\mathbb{N}_0}\{n\}\times U_n$ is clearly recursively enumerable as a countably infinite union of finite sets $\{n\}\times{}^{m}B^{l}_{n+m+1}$ that are recursive effectively in m, n and ℓ, given the construction in the first step of the proof. It is clear that

$$\begin{aligned}\overline{P}^{\varphi}([\![U_n]\!])&=\overline{P}^{\varphi}\Big(\bigcup_{m\in\mathbb{N}_0}[\![^{m}B_{n+m+1}]\!]\Big)=\sup_{k\in\mathbb{N}_0}\overline{P}^{\varphi}\Big(\bigcup_{m=0}^{k}[\![^{m}B_{n+m+1}]\!]\Big)\\&\leq\sup_{k\in\mathbb{N}_0}\sum_{m=0}^{k}\overline{P}^{\varphi}([\![^{m}B_{n+m+1}]\!])\leq\sup_{k\in\mathbb{N}_0}\sum_{m=0}^{k}2^{-(n+m+1)}=\sum_{m=0}^{\infty}2^{-(n+m+1)}=2^{-n},\end{aligned}$$

where the second equality follows from E9 and the first inequality from E6 and E3, given that $\mathbb{I}_{\bigcup_{m=0}^{k}[\![^{m}B_{n+m+1}]\!]}\leq\sum_{m=0}^{k}\mathbb{I}_{[\![^{m}B_{n+m+1}]\!]}$. We conclude that U is indeed a Martin-Löf test for φ.

We finish the argument, in a fourth and final step, by proving that any path $\omega\in\Omega$ is Martin-Löf test random for φ if and only if $\omega\notin\bigcap_{n\in\mathbb{N}_0}[\![U_n]\!]$. To this end, consider any path $\omega\in\Omega$. For necessity, assume that ω is Martin-Löf test random for φ. Then clearly also $\omega\notin\bigcap_{n\in\mathbb{N}_0}[\![U_n]\!]$ by Definition 8, since we've just proved that U is a Martin-Löf test for φ. For sufficiency, assume that $\omega\notin\bigcap_{n\in\mathbb{N}_0}[\![U_n]\!]$. To prove that ω is Martin-Löf test random, we must prove, as argued above, that $\omega\notin\bigcap_{n\in\mathbb{N}_0}[\![^{m}B_n]\!]$ for all $m\in\mathbb{N}_0$. Assume towards contradiction that there's some $m_o\in\mathbb{N}_0$ such that $\omega\in\bigcap_{n\in\mathbb{N}_0}[\![^{m_o}B_n]\!]$. By construction, clearly, $[\![^{m_o}B_{n+m_o+1}]\!]\subseteq[\![U_n]\!]$ for all $n\in\mathbb{N}_0$. This implies that $\omega\in[\![U_n]\!]$ for all $n\in\mathbb{N}_0$, a contradiction. □

In the above proof, we've used the following alternative characterisation of Martin-Löf test randomness.

Lemma 37. *A path $\omega\in\Omega$ is Martin-Löf test random for a forecasting system φ if and only if $\omega\notin\bigcap_{m\in\mathbb{N}_0}[\![C_m]\!]$ for all recursively enumerable subsets C of $\mathbb{N}_0\times\mathbb{S}$ such that $\overline{P}^{\varphi}([\![C_n]\!])\leq 2^{-(n+1)}$ for all $n\in\mathbb{N}_0$.*

Proof. It clearly suffices to prove the 'if' part. So assume towards contradiction that ω isn't Martin-Löf test random, meaning that there's some Martin-Löf test A for φ such that $\omega\in\bigcap_{m\in\mathbb{N}_0}[\![A_m]\!]$. Consider the recursively enumerable set $C\subseteq\mathbb{N}_0\times\mathbb{S}$ defined by

$$C:=\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon (n+1,s)\in A\},$$

then $C_n=A_{n+1}$, and therefore also $\overline{P}^{\varphi}([\![C_n]\!])=\overline{P}^{\varphi}([\![A_{n+1}]\!])\leq 2^{-(n+1)}$ for all $n\in\mathbb{N}_0$. Since $\bigcap_{n\in\mathbb{N}_0}[\![A_n]\!]\subseteq\bigcap_{n\in\mathbb{N}_0}[\![A_{n+1}]\!]=\bigcap_{n\in\mathbb{N}_0}[\![C_n]\!]$, we see that also $\omega\in\bigcap_{n\in\mathbb{N}_0}[\![C_n]\!]$, a contradiction. □

Proof of Corollary 24. Consider the universal Martin-Löf test U in Proposition 23, and the corresponding computable sequence of effectively open sets $[\![U_n]\!]$. The argumentation in the proof of Proposition 15 can now be used to construct the lower semicomputable test supermartingale T defined by $T(\square):=1$ and $T(x_{1:n}):=\frac{1}{2}\lim_{\ell\to\infty}\sum_{m=0}^{\infty}\overline{P}^{\varphi}([\![U_m^{<\ell}]\!]|x_{1:n})$ for all $x_{1:n}\in\mathbb{S}$ with $n\in\mathbb{N}$, which we claim does the job.

Indeed, consider any path $\omega\in\Omega$. Suppose that ω isn't Martin-Löf (test) random for φ, then we know from (Theorem 16 and) Proposition 23 that $\omega\in\bigcap_{n\in\mathbb{N}_0}[\![U_n]\!]$, and therefore the argumentation in the proof of Proposition 15 guarantees that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$. Conversely, suppose that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$. This tells us that ω isn't Martin-Löf random for φ, and therefore, by Proposition 14, not Martin-Löf test random for φ either. □

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