Proceedings of Machine Learning Research 215:153–164, 2023 ISIPTA 2023

Desirable Sets of Things and Their Logic

Gert de Cooman GERT.DECOOMAN@UGENT.BE
Foundations Lab for imprecise probabilities, Ghent University, Belgium

Arthur Van Camp ARTHUR.VANCAMP@BRISTOL.AC.UK
Department of Philosophy, University of Bristol, UK

Jasper De Bock JASPER.DEBOCK@UGENT.BE
Foundations Lab for imprecise probabilities, Ghent University, Belgium

Abstract

We identify the logic behind the recent theory of coherent sets of desirable (sets of) things, which generalise coherent sets of desirable (sets of) gambles and coherent choice functions, and show that this identification allows us to establish various representation results for such coherent models in terms of simpler ones.

Keywords: desirability, desirable sets of things, conservative inference, propositional logic, filter, prime filter, principal filter, representation

1. Introduction

The theory of *imprecise probabilities* [1, 15, 23, 25] allows for partial specification of probability models and, equally importantly, allows for *conservative inference*: if we specify bounds on the probabilities of a number of events, then the theory is concerned with, amongst other things, inferring the implied bounds on the probabilities of other events.

Conservative probabilistic inference can be represented intuitively and effectively by considering simple desirability statements [21, 26]: if some uncertain rewards—called *gambles*—are considered desirable to a subject, what does that imply about the desirability (or otherwise) of other gambles? That a subject considers a given gamble to be desirable is then a simple statement very much like asserting a proposition in a propositional logic context. Inferring from a collection of such desirability statements which other gambles are desirable, is then a matter of *deductive inference* based on a number of so-called *coherence* rules, very much like logical inference is based on the conjunction and modus ponens rules. This observation has led to a theory of *coherent sets of desirable gambles* [3, 5, 11, 12, 16, 17, 26]: sets of gambles that are deductively closed under the inference based on the coherence rules.

A desirability statement for a gamble is tantamount to a pairwise comparison—a *strict preference*—between this gamble and the zero gamble. Isaac Levi [15] recognised quite early on that certain aspects of conservative probabilistic inference demand looking *further than merely pairwise* preferences between gambles. This has led to the introduction of so-called *coherent choice functions* into the field of imprecise probabilities [14, 19, 22, 24]. Recently, we've shown [6, 8, 9, 10] that working with such choice functions is mathematically equivalent to doing inferences with *desirable sets of gambles*, rather than with desirable gambles, where a set of gambles is judged to be desirable as soon as at least one of its elements is: coherent choice functions can be seen as a special case of coherent—deductively closed—sets of desirable sets of gambles.

In very recent work [7], Jasper De Bock has taken this idea further: in his abstract generalisation, gambles are replaced by abstract objects—*things*—and it's assumed that some abstract property of things, called their *desirability*, can be inferred from the desirability of other things through inference rules that are summarised by the action of some closure operator. This leads to a theory of *coherent*—deductively closed—*sets of desirable sets of things*.

At about the same time, Catrin Campbell–Moore [2] showed that statements about, and the inference behind, the desirability of (some types of sets of) gambles can be represented by filters of sets of probability measures: the conservative inference mechanism behind such models can be interpreted as that of propositional logic involving statements about some 'ideal unknown' probability measure.[1]

Here, we draw inspiration from these recent developments, and show that the conservative inference mechanism behind coherent—deductively closed—sets of desirable sets of things is essentially that of propositional logic involving statements about some 'unknown' coherent set of desirable things. Our results allow us to prove powerful representation theorems for such coherent sets of desirable sets of things in terms of simpler, so-called *conjunctive*, models.

We'll freely use basic concepts and results from order theory [4], and assume familiarity with most of them.

[1] She has since (private communication) extended this idea to representations by filters of sets of coherent sets of desirable gambles, along the lines of, but independently from, what we'll achieve for the more general coherent sets of things in Section 6.

© 2023 G. de Cooman, A. Van Camp & J. De Bock.

Our argumentation is structured as follows. In Section 2, we summarise the basic ideas behind coherent sets of desirable sets of things, and identify the order-theoretic underpinnings of the inference mechanism behind them. We show that the coherent sets of desirable things can be embedded into the coherent sets of desirable sets of things, in the form of the *conjunctive* models. In Section 3, we explain how each desirability statement for a given set of things can be identified with a so-called *event*: the specific subset of the collection of all coherent sets of desirable things it's compatible with. In Section 4, we identify the order-theoretic nature of the collection of all such events as a bounded distributive lattice. Section 5 is a very short primer on order- and set-theoretic filters. In Sections 6–8, we relate by order isomorphisms the (various types of) coherent sets of desirable sets of things to (various types of) proper filters, and then explain how such order isomorphisms lead to representation of these (various types of) coherent sets of desirable sets of things in terms of conjunctive ones, providing simple alternative ways to derive similar results as in Ref. [7]. Due to a lack of space, we've had to exclude the proofs for our results; we refer the interested reader to a longer arXiv version [13] of this paper.

2. Sets of Desirable (Sets of) Things

We give a brief overview of (a version of) the theory of desirable (sets of) things [7] that'll be sufficient for our purposes here.

2.1. Desirable Things

Consider a set T of things t that may have a certain property; having this property makes a thing *desirable*.

You, our subject, may entertain ideas about which things are desirable, and You represent these ideas by providing a (not necessarily exhaustive) set of things that You find desirable. We'll call such a subset $S \subseteq T$ a *set of desirable things*, or SDT for short (plural: SDTs): a set with the property that You think *each of its elements* desirable. We denote by $\mathscr{P}(T)$ the set of all subsets S of T, or in other words, the collection of all candidate SDTs.

SDTs can be ordered by set inclusion. We interpret $S_1 \subseteq S_2$ to mean that S_1 is *less informative*, or *more conservative*, than S_2, simply because a subject with SDT S_1 finds fewer things desirable than a subject with SDT S_2.

Our basic assumption is that there are rules that underlie the notion of desirability for things, and that the net effect of these rules can be captured by a closure operator and a set of forbidden things.

We recall that a *closure operator* on a non-empty set G is a map $\mathrm{Cl}\colon \mathscr{P}(G) \to \mathscr{P}(G)$ satisfying:

C_1. $A \subseteq \mathrm{Cl}(A)$ for all $A \subseteq G$;

C_2. if $A \subseteq B$ then $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$ for all $A, B \subseteq G$;

C_3. $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$ for all $A \subseteq G$.

A closure operator Cl is called *finitary* if it's enough to know the closure of finite sets, in the following sense:

$$\mathrm{Cl}(A) = \bigcup\{\mathrm{Cl}(F)\colon F \in \mathscr{P}(G) \text{ and } F \Subset A\}, A \in \mathscr{P}(G),$$

where we use the notation '$\Subset$' to mean 'is a finite subset of', and agree to call the empty set $\emptyset$ finite.

First of all, as already suggested above, we assume that there's some inference mechanism that allows us to infer the desirability of a thing from the desirability of other things. This inference mechanism is represented by a closure operator $\mathrm{Cl}_{\mathbf{D}}\colon \mathscr{P}(T) \to \mathscr{P}(T)$, in the following sense:

D_1. if all things in S are desirable, then so are all things in $\mathrm{Cl}_{\mathbf{D}}(S)$.

The set $\mathbf{D} := \{S \in \mathscr{P}(T)\colon \mathrm{Cl}_{\mathbf{D}}(S) = S\}$ collects all *closed* sets of things.

Secondly, we assume that there's a set of so-called *forbidden* things T_-, which are *never desirable*:

D_2. no thing in T_- is desirable, so if all things in S are desirable, then $S \cap T_- = \emptyset$.

Of course, because we assume that all things in an SDT are desirable, it can never intersect the set T_-, so this leaves us with the collection

$$\overline{\mathbf{D}} := \{S \in \mathbf{D}\colon S \cap T_- = \emptyset\} \subseteq \mathbf{D}$$

of the closed SDTs that we'll call *coherent*. We'll use the generic notation D for such coherent SDTs. It's a standard result in order theory, and easy to check, that they constitute an *intersection structure*: the intersection of any non-empty family $I \neq \emptyset$ of them is still coherent:

$$\big((\forall i \in I) D_i \in \overline{\mathbf{D}}\big) \Rightarrow \bigcap_{i \in I} D_i \in \overline{\mathbf{D}}.$$

Clearly, an SDT $S \subseteq T$ can be extended to a coherent one iff $\mathrm{Cl}_{\mathbf{D}}(S) \in \overline{\mathbf{D}}$, or equivalently, if $\mathrm{Cl}_{\mathbf{D}}(S) \cap T_- = \emptyset$, and in that case we'll call this S *consistent*. For any consistent SDT S, it's easy to see that

$$\mathrm{Cl}_{\mathbf{D}}(S) = \bigcap\{D \in \overline{\mathbf{D}}\colon S \subseteq D\},$$

so $\mathrm{Cl}_{\mathbf{D}}(S)$ is the smallest, or most conservative, or least informative, coherent SDT that the consistent S can be extended to. In this sense, the closure operator $\mathrm{Cl}_{\mathbf{D}}$ represents *conservative inference*. The following result is then a standard conclusion in order theory [4, Chapter 7].

Proposition 1 *The partially ordered set $\langle \mathbf{D}, \subseteq \rangle$ is a complete lattice with bottom $0_{\mathbf{D}} = T_+$ and top $1_{\mathbf{D}} = T$. For any non-empty family S_i, $i \in I$ of elements of $\mathbf{D}$, we have for its infimum and its supremum that, respectively, $\inf_{i \in I} S_i = \bigcap_{i \in I} S_i$ and $\sup_{i \in I} S_i = \mathrm{Cl}_{\mathbf{D}}(\bigcup_{i \in I} S_i)$.*

The set $T_+ := \mathrm{Cl}_{\mathbf{D}}(\emptyset) = \bigcap \mathbf{D}$ is the smallest closed SDT. If T_+ is coherent, or in other words if the empty set $\emptyset$ is consistent, then T_+ is the smallest, or most conservative, coherent SDT. This will be the case iff

D_3. $T_+ \cap T_- = \emptyset$, or equivalently, $\overline{\mathbf{D}} \neq \emptyset$.

We'll from now on also always assume that this 'sanitary' condition's verified. All things in T_+ are then always implicitly desirable, regardless of any of the desirability statements You might make.

Running Example As a familiar example, consider a variable X whose value in a *finite* set $\mathscr{X}$ is unknown. Any map $h\colon \mathscr{X} \to \mathbb{R}$ then corresponds to a real-valued uncertain reward $h(X)$, and is called a *gamble* on X. $h(X)$ is typically expressed in units of some linear utility scale. The set $\mathscr{G}$ of all such gambles is a linear space.

In a typical decision problem, You are uncertain about the value of X, and are asked to express Your preferences between several possible decisions or acts, where each such act has an associated uncertain reward, or gamble.

We consider the strict vector ordering $>$, defined by $f > g \Leftrightarrow (\forall x \in \mathscr{X}) f(x) > g(x)$, as a *background ordering*, reflecting the minimal preferences You always have, regardless of Your beliefs about X. We denote by $\mathscr{G}_{>0}$ the set of all *positive gambles* $h > 0$, and let $\mathscr{G}_{\leq 0} := -\mathscr{G}_{>0} \cup \{0\}$.

Your *set of desirable gambles* $D \subseteq \mathscr{G}$ contains the gambles that are desirable to You in the sense that You strictly prefer them to 0. We'll call it *coherent* [10] (see also Refs. [3, 11, 18, 20, 26] for related definitions) when

GD_1. $0 \notin D$;

GD_2. $\mathscr{G}_{>0} \subseteq D$;

GD_3. if $f, g \in D$ and $(\lambda, \mu) > 0$,[2] then $\lambda f + \mu g \in D$, for all $f, g \in \mathscr{G}$ and $\lambda, \mu \in \mathbb{R}$.

We can identify gambles as special cases of the abstract things, let T correspond to the set of gambles $\mathscr{G}$, and let desirable gambles correspond to desirable things. The requirements GD_1–GD_3 correspond to the *finitary* closure operator $\mathrm{Cl}_{\mathbf{D}}$, determined by $\mathrm{Cl}_{\mathbf{D}}(S) = \mathrm{posi}(S \cup \mathscr{G}_{>0})$ for all consistent S, where $\mathrm{posi}(\bullet)$ is the set of all positive linear combinations of '$\bullet$', that the convex cone $\mathscr{G}_{\leq 0}$ plays the role of the set of forbidden things T_-, and that $\mathrm{posi}(\emptyset \cup \mathscr{G}_{>0}) = \mathrm{posi}(\mathscr{G}_{>0}) = \mathscr{G}_{>0}$ plays the role of the set T_+. With these identifications, the axioms D_1–D_3 are verified.

2.2. Desirable Sets of Things

Your claim that a set of things $S \subseteq T$ is a set of desirable things is tantamount to a *conjunctive* statement: You state that "all things in S are desirable". In the formalism described above, there's no way to deal with *disjunctive* statements of the type "at least one of the things in S is desirable". So let's look for a way to also allow for such disjunctive statements.

We'll say that You consider a *set of things S* to be *desirable* if You consider at least one thing in S to be. In other words, in a set of desirable things, all things are desirable, whereas in a desirable set of things, at least one thing is. As with the desirability of things, You can make many desirability statements for sets of things, and we then collect all of these in a *set of desirable sets of things*—or for short SDS, plural SDSes—$W \subseteq \mathscr{P}(T)$. So W is an SDS for You if all sets of things $S \in W$ are desirable to You, in the sense that each of them contains at least one desirable thing.

Sets of desirable sets of things can be ordered by set inclusion too. We take $W_1 \subseteq W_2$ to mean that W_1 is *less informative*, or *more conservative*, than W_2, simply because a subject with an SDS W_1 finds fewer sets of things desirable than a subject with SDS W_2.

The inference mechanism for the desirability of things also has its consequences for the desirability of sets of things, as we'll now make clear. Consider any set of sets of things $W \subseteq \mathscr{P}(T)$, then we denote by Φ_W the set of all so-called *selection maps*

$$\sigma\colon W \to T\colon S \mapsto \sigma(S) \text{ such that } \sigma(S) \in S \text{ for all } S \in W.$$

Each such selection map $\sigma \in \Phi_W$ selects a thing $\sigma(S)$ from each set of things S in W, and we use the notation $\sigma(W) := \{\sigma(S)\colon S \in W\} \in \mathscr{P}(T)$ for the corresponding set of all these selected things.

We now call an SDS $K \subseteq \mathscr{P}(T)$ *coherent* if it satisfies the following conditions:

K_1. $\emptyset \notin K$;

K_2. if $S_1 \in K$ and $S_1 \subseteq S_2$ then $S_2 \in K$, for all $S_1, S_2 \in \mathscr{P}(T)$;

K_3. if $S \in K$ then $S \setminus T_- \in K$, for all $S \in \mathscr{P}(T)$;

K_4. $\{t_+\} \in K$ for all $t_+ \in T_+$;

K_5. if $t_\sigma \in \mathrm{Cl}_{\mathbf{D}}(\sigma(W))$ for all $\sigma \in \Phi_W$, then $\{t_\sigma\colon \sigma \in \Phi_W\} \in K$, for all $\emptyset \neq W \subseteq K$.

The first condition K_1 takes into account that the empty set of things can't be desirable, as it contains no desirable thing. The second condition K_2 reminds us that if a set of things contains a desirable thing, then of course so do all its supersets. The third condition K_3 reflects that things in T_- can never be desirable, by D_2, and can therefore safely be removed from any set of things without affecting the latter's desirability. And, to conclude, we'll see further on that the last two conditions K_4 and K_5 do a very fine job of lifting the effects of inferential closure from the desirability of things to the desirability of sets of things. They can be justified as follows. For K_4, recall from the discussion above that any element of T_+ is always implicitly desirable, and so therefore will be any set that contains it. For K_5, recall that $\emptyset \neq W \subseteq K$ means that each set of things $S \in W$ contains at least one desirable thing, and

[2] We'll use the notation $(\lambda, \mu) > 0$ to mean that $\lambda \geq 0$ and $\mu \geq 0$ and $\lambda + \mu > 0$.

therefore there must be some selection map $\sigma_o \in \Phi_W$ such that $\sigma_o(S)$ is a desirable thing for all $S \in W$. This implies that all things in $\sigma_o(W)$ are desirable, and therefore so are all things in $\mathrm{Cl}_{\mathbf{D}}(\sigma_o(W))$, by D_1. Whatever t_{σ_o} we choose in $\mathrm{Cl}_{\mathbf{D}}(\sigma_o(W))$ will therefore be desirable, which guarantees that the set of things $\{t_\sigma : \sigma \in \Phi_W\}$ must also be desirable, because it contains the desirable thing t_{σ_o}.

We denote the set of all coherent SDSes by $\overline{\mathbf{K}}$, and we let $\mathbf{K} := \overline{\mathbf{K}} \cup \{\mathscr{P}(T)\}$. Observe that $\mathscr{P}(T)$ is never coherent, by K_1. Since each of the axioms K_1–K_5 is preserved under taking arbitrary non-empty intersections, the set $\overline{\mathbf{K}}$ of all coherent SDSes constitutes an intersection structure: the intersection of any non-empty family of coherent SDSes is still coherent, or in other words, for any non-empty family K_i, $i \in I$ of elements of $\overline{\mathbf{K}}$, we see that still $\bigcap_{i\in I} K_i \in \overline{\mathbf{K}}$. As explained in Ref. [4, Chapter 7], this allows us to capture the inferential aspects of desirability at this level using the closure operator $\mathrm{Cl}_{\mathbf{K}} : \mathscr{P}(\mathscr{P}(T)) \to \mathbf{K}$ associated with the collection $\mathbf{K}$ of *closed* SDSes, defined by

$$\mathrm{Cl}_{\mathbf{K}}(W) := \bigcap\{K \in \mathbf{K} : W \subseteq K\} \text{ for all } W \subseteq \mathscr{P}(T).$$

If we call an SDS W *consistent* if it can be extended to some coherent SDS, or equivalently, if $\mathrm{Cl}_{\mathbf{K}}(W) \neq \mathscr{P}(T)$, then we find that $\mathrm{Cl}_{\mathbf{K}}(W)$ is the smallest, or most conservative, coherent SDS that includes W, for any consistent W. Of course, $W = \mathrm{Cl}_{\mathbf{K}}(W) \Leftrightarrow W \in \mathbf{K}$ for all $W \subseteq \mathscr{P}(T)$, so $\mathbf{K} = \mathrm{Cl}_{\mathbf{K}}(\mathscr{P}(\mathscr{P}(T)))$. The following result's then again a standard conclusion in order theory [4, Chapter 7].

Proposition 2 *The partially ordered set $\langle \mathbf{K}, \subseteq\rangle$ is a complete lattice with top $1_{\mathbf{K}} = \mathscr{P}(T)$ and bottom $0_{\mathbf{K}} = \mathrm{Cl}_{\mathbf{K}}(\emptyset)$. For any non-empty family W_i, $i \in I$ of elements of $\mathbf{K}$, we have for its infimum and its supremum that, respectively, $\inf_{i\in I} W_i = \bigcap_{i\in I} W_i$ and $\sup_{i\in I} W_i = \mathrm{Cl}_{\mathbf{K}}(\bigcup_{i\in I} W_i)$.*

Interestingly, the smallest coherent SDS $0_{\mathbf{K}}$ is easy to identify: $0_{\mathbf{K}} = \bigcap \overline{\mathbf{K}} = \{S \in \mathscr{P}(T) : S \cap T_+ \neq \emptyset\}$.

2.3. Desirable Sets of Things: The Finitary Case

We call a subset K of $\mathscr{P}(T)$ a *finitely* coherent SDS if it satisfies conditions K_1–K_4, together with the following finitary version of K_5:

K_5^{fin}. if $t_\sigma \in \mathrm{Cl}_{\mathbf{D}}(\sigma(W))$ for all $\sigma \in \Phi_W$, then $\{t_\sigma : \sigma \in \Phi_W\} \in K$, for all $\emptyset \neq W \Subset K$;

We denote by $\overline{\mathbf{K}}_{\mathrm{fin}}$ the set of all finitely coherent SDSes, and we let $\mathbf{K}_{\mathrm{fin}} := \overline{\mathbf{K}}_{\mathrm{fin}} \cup \{\mathscr{P}(T)\}$.

For this finitary version, the discussion, definitions and the ensuing results about the intersection structure $\overline{\mathbf{K}}_{\mathrm{fin}}$, the complete lattice $\langle \mathbf{K}_{\mathrm{fin}}, \subseteq\rangle$, and the associated closure operator $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}$ are completely similar, and we'll refrain from repeating them here. Observe nevertheless that $\overline{\mathbf{K}} \subseteq \overline{\mathbf{K}}_{\mathrm{fin}}$ and therefore also $\mathbf{K} \subseteq \mathbf{K}_{\mathrm{fin}}$: since K_5 clearly implies K_5^{fin}, any coherent K is also finitely coherent, so finite coherence is the weaker requirement. As a consequence, we also find that $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}(W) \subseteq \mathrm{Cl}_{\mathbf{K}}(W)$ for all $W \subseteq \mathscr{P}(T)$.

Running Example We now can lift the framework of sets of desirable gambles to *sets of desirable gamble sets*: rather than use gambles as things that are potentially desirable, we now turn to *gamble sets*, instead. In doing so, we move from binary preferences between gambles to more general preferences that aren't necessarily binary.

We'll allow You to state for a gamble set $S \in \mathscr{P}(\mathscr{G})$ that at least one of its elements is desirable to You, but without Your needing to specify which; we'll then say that S is *desirable* to You, and call S a *desirable gamble set*.

A set of desirable gamble sets K is called *coherent* when

OK_1. $\emptyset \notin K$;

OK_2. if $S_1 \in K$ and $S_1 \subseteq S_2$ then $S_2 \in K$, for all $S_1, S_2 \in \mathscr{P}(\mathscr{G})$;

OK_3. if $S \in K$ then $S \setminus \mathscr{G}_{\leq 0} \in K$, for all $S \in \mathscr{P}(\mathscr{G})$;

OK_4. $\{f_+\} \in K$ for all $f_+ \in \mathscr{G}_{>0}$;

OK_5. if, with $n \in \mathbb{N}$,[3] $S_1, \dots, S_n \in K$ then also

$$\left\{\sum_{k=1}^{n} \lambda^k_{f_1,\dots,f_n} f_k : f_k \in S_k, k = 1, \dots, n\right\} \in K,$$

with $\lambda^k_{f_1,\dots,f_n} \geq 0$ and $\sum_{k=1}^{n} \lambda^k_{f_1,\dots,f_n} > 0$.

These coherence requirements can be reinterpreted as, essentially, Axioms K_1–K_5^{fin}, after a proper identification of the relevant concepts here with those in the abstract treatment of desirable SDTs.

2.4. Conjunctive Models

We can order embed the structure $\langle \overline{\mathbf{D}}, \subseteq\rangle$ into the structure $\langle \overline{\mathbf{K}}, \subseteq\rangle$, and therefore also into the structure $\langle \overline{\mathbf{K}}_{\mathrm{fin}}, \subseteq\rangle$, in a straightforward and natural manner. Let's show how.

If we consider any set of things S that's an element of the coherent SDS K, then we know from the coherence condition K_2 that all its supersets are also in K. But, of course, not all of its subsets will be, as is made clear by the coherence condition K_1. This observation brings us to the following idea. Consider any SDS W—not necessarily coherent—and any element $S \in W$. If there's some finite subset $\hat{S}$ of S such that $\hat{S} \in W$, then we'll call S *finitary* (in W). If, moreover, all the elements S of the SDS W are finitary, then we'll call W *finitary* as well; so any desirable set in a finitary W has a desirable finite subset. The *(finitely) coherent* finitary SDSes will be studied in much more detail in Section 8. They are special because they are completely determined by their finite elements.

For the present discussion, however, we restrict our attention to an important special case of such finitary SDSes, where each desirable set has a desirable *singleton* subset:

[3] $\mathbb{N}$ is the set of natural numbers (zero excluded).

Definition 3 (Conjunctivity) *We call an SDS $W \subseteq \mathscr{P}(T)$* conjunctive *if* $(\forall S \in W)(\exists t \in S)\{t\} \in W$.

In the remainder of this section, we'll spend some effort on identifying the conjunctive *coherent* SDSes. We begin by introducing ways to turn an SDT into an SDS, and vice versa. Consider any $S_o \subseteq T$ and any $W_o \subseteq \mathscr{P}(T)$, and let

$$D_{W_o} := \{t \in T \colon \{t\} \in W_o\} \subseteq T \tag{1}$$

$$K_{S_o} := \{S \subseteq T \colon S \cap S_o \neq \emptyset\} \subseteq \mathscr{P}(T). \tag{2}$$

Let us first investigate some conditions under which D_{W_o} is a coherent SDT, and K_{S_o} is a (finitely) coherent SDS.

Proposition 4 *Consider any SDS K. If K is (finitely) coherent, then $K_{D_K} \subseteq K$. Moreover,*
(i) *if K is coherent, then D_K is coherent;*
(ii) *if K is finitely coherent and the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then D_K is coherent.*

Proposition 5 *Consider any set of things $D \in \mathscr{P}(T)$, then $D_{K_D} = D$. Moreover, consider the statements:*
(i) *D is a coherent SDT;*
(ii) *K_D is a coherent SDS;*
(iii) *K_D is a finitely coherent SDS.*
Then (i)⇔(ii) and (ii)⇒(iii), so (i)⇒(iii). Moreover, if the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then also (i)⇔(iii).

So, if we start out with a coherent SDS K, then the corresponding coherent and conjunctive SDS K_{D_K} is a conservative approximation of K: going from a model K to its *conjunctive part* $K_{D_K} = \{S \in K \colon (\exists t \in S)\{t\} \in K\}$ typically results in a loss of information.

On the other hand, going from a coherent SDT D to the corresponding coherent and conjunctive SDS K_D doesn't result in a loss of information: it's easy to see that it results in an order embedding $K_\bullet$ of the (intersection) structure $\langle \overline{\mathbf{D}}, \subseteq \rangle$ into the (intersection) structure $\langle \overline{\mathbf{K}}, \subseteq \rangle$.

We're now in a position to find out what the conjunctive and (finitely) coherent SDSes look like.

Proposition 6 (Conjunctivity)
(i) *A coherent SDS K is conjunctive iff there's some coherent SDT $D \in \overline{\mathbf{D}}$ such that $K = K_D$.*
(ii) *When the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then a finitely coherent SDS K is conjunctive iff there's some coherent SDT $D \in \overline{\mathbf{D}}$ such that $K = K_D$.*
In both these cases then necessarily $D = D_K$.

3. Towards a Representation With Filters

It's a well-established consequence of Stone's Representation Theorem [4, Chapters 5, 10 and 11] that filters of subsets of a space constitute abstract ways of dealing with deductively closed sets of propositions about elements of that space. Very simply put, they allow us to do propositional logic with statements about elements of the space.

Recall that a *filter* of subsets of a space $\mathscr{X}$—also called a filter on $\langle \mathscr{P}(\mathscr{X}), \subseteq \rangle$—is a non-empty subset $\mathscr{F}$ of the power set $\mathscr{P}(\mathscr{X})$ of $\mathscr{X}$ such that for all $A, B \in \mathscr{P}(\mathscr{X})$:
F_1. if $A \in \mathscr{F}$ and $A \subseteq B$ then also $B \in \mathscr{F}$;
F_2. if $A \in \mathscr{F}$ and $B \in \mathscr{F}$ then also $A \cap B \in \mathscr{F}$.
We call a filter *proper* if $\mathscr{F} \neq \mathscr{P}(\mathscr{X})$.

One particular space we'll be considering in this paper, is the set $\overline{\mathbf{D}}$ of all coherent SDTs. To guide the interpretation, we'll assume that there's an *actual* (but unknown) SDT D_{T}. It's assumed to be coherent, and therefore a specific element of the set $\overline{\mathbf{D}}$. The elements of D_{T} are the things that *actually* are desirable, and all other things in T aren't. Moreover, each coherent SDT $D \in \overline{\mathbf{D}}$ is a *possible* identification of this actual set D_{T}.

Any non-contradictory propositional statement about this D_{T} corresponds to some non-empty subset $A \subseteq \overline{\mathbf{D}}$ of coherent SDTs for which the statement holds true, and this subset A represents the remaining possible identifications of D_{T} after the statement has been made. We'll call such subsets *events*. The empty subset of $\overline{\mathbf{D}}$ represents contradictory propositional statements.

Any proper filter $\mathscr{F}$ of such events $A \subseteq \overline{\mathbf{D}}$ then corresponds to a deductively closed collection of propositional statements—a so-called theory—about D_{T}, where intersection of events represents the conjunction of propositional statements, and inclusion of events represents implication of propositional statements. The only improper filter $\mathscr{P}(\overline{\mathbf{D}})$, which contains the empty event, then represents logical contradiction at this level.

We can interpret the desirability statements studied in Section 2 as statements about such an actual D_{T}. Stating that a '*set of things S is desirable*' corresponds to the event

$$\overline{\mathbf{D}}_S := \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\} \subseteq \overline{\mathbf{D}},$$

as this amounts to requiring that at least one element of S must be actually desirable, and must therefore belong to D_{T}: the desirability statement is indeed equivalent to '$D_{\mathrm{T}} \in \overline{\mathbf{D}}_S$'. As a special case, stating that a thing t is desirable corresponds to the event $\overline{\mathbf{D}}_{\{t\}} := \{D \in \overline{\mathbf{D}} \colon t \in D\}$, as it amounts to requiring that t must belong to D_{T}.

More generally, working with SDSes W, as we did in Section 2, therefore corresponds to dealing with a conjunction of the desirability statements 'the set of things S is desirable' for all $S \in W$, so with events of the type

$$\mathscr{E}(W) := \bigcap_{S \in W} \overline{\mathbf{D}}_S = \bigcap_{S \in W} \{D \in \overline{\mathbf{D}} \colon S \cap D \neq \emptyset\}, W \subseteq \mathscr{P}(T).$$

As a special case, the vacuous assessment $W = \emptyset$ leads to no restrictions on D_{T}: $\mathscr{E}(\emptyset) = \overline{\mathbf{D}}$.[4]

[4] The empty intersection of subsets of $\overline{\mathbf{D}}$ is $\overline{\mathbf{D}}$ itself.

Working with the filters of subsets of $\overline{\mathbf{D}}$—filters of events—that are generated by such collections, then represents doing propositional logic with basic statements of the type 'the set of things S is desirable', for $S \in \mathscr{P}(T)$. We might therefore suspect that the language of such filters could be able to represent, explain, and perhaps also refine the relationships between the inference mechanisms that lie behind the intersection structures and closure operators in Section 2. Investigating this type of representation in terms of filters of events is the main aim of this paper.

There is, however, a particular aspect of the inference mechanisms at hand that tends to complicate—or is it simplify?— matters somewhat. Not all events in $\mathscr{P}(\overline{\mathbf{D}})$ are relevant to our problem; only the ones that are intersections (and, as we'll see further on, unions) of the basic events of the type $\overline{\mathbf{D}}_S$, $S \subseteq T$ seem to require attention. We'll therefore restrict our focus to these, and as a result, the representing collection of events will no longer constitute a Boolean lattice, but only a specific *distributive sublattice*. As we'll see in Sections 6 and 7, the effect will be two-fold: we'll broadly speaking be led to a more general *prime filter* rather than an *ultrafilter representation*, and this representation will be an isomorphism rather than an endomorphism.

4. The Basic Representation Lattices

Since we expect the events $\mathscr{E}(W)$ to become important in what follows, let's study them a bit closer.

The following propositions are easy to prove, but will be instrumental in our discussion further on, so we've gathered them here for easy reference. It's on these two simple properties that all our results about filter representation essentially rest.

The propositional statement that W is an SDS is never contradictory as soon as W is some part of a coherent SDS.

Proposition 7 (Consistency) *For any coherent SDS $K \in \overline{\mathbf{K}}$, $\mathscr{E}(W) \neq \emptyset$ for all $W \subseteq K$. Similarly, for any finitely coherent SDS $K \in \overline{\mathbf{K}}_{\text{fin}}$, $\mathscr{E}(W) \neq \emptyset$ for all $W \Subset K$.*

The following result is formulated for finitely coherent SDSes, but the infinitary version holds *mutatis mutandis* for their coherent counterparts as well.

Proposition 8 *Consider any finitely coherent SDS $K \in \overline{\mathbf{K}}_{\text{fin}}$, then for all $W_1, W_2 \Subset \mathscr{P}(T)$:*

(i) *if $\mathscr{E}(W_1) \subseteq \mathscr{E}(W_2)$ and $W_1 \Subset K$ then also $W_2 \Subset K$;*
(ii) *if $\mathscr{E}(W_1) = \mathscr{E}(W_2)$ then $W_1 \Subset K \Leftrightarrow W_2 \Subset K$.*

We're now ready to introduce the particular sets of events we'll build our further discussion on. Let's consider the sets $\mathbf{E} := \{\mathscr{E}(W) : W \subseteq \mathscr{P}(T)\}$ and $\mathbf{E}_{\text{fin}} := \{\mathscr{E}(W) : W \Subset \mathscr{P}(T)\}$, and order them by set inclusion $\subseteq$.

Proposition 9 *The partially ordered set $\langle \mathbf{E}, \subseteq \rangle$ is a completely distributive complete lattice, with union as join and intersection as meet, $\emptyset$ as bottom and $\overline{\mathbf{D}}$ as top. Similarly, $\langle \mathbf{E}_{\text{fin}}, \subseteq \rangle$ is a bounded distributive lattice, with union as join and intersection as meet, $\emptyset$ as bottom and $\overline{\mathbf{D}}$ as top.*

5. A Brief Primer on (Inference With) Filters

The discussion in Section 3 led us to try and represent inference about desirability statements using filters on appropriate lattices of events. After spending some effort on identifying these lattices in Section 4, we're now ready to start looking at how to do inference with filters, and how to use that inference mechanism to represent reasoning about desirability statements. Here, we'll summarise those aspects of filters and filter inference on (bounded distributive) lattices that are relevant to our representation effort.

We begin by recalling the definition of a filter on a bounded lattice $\langle L, \leq \rangle$ with meet $\frown$ and join $\smile$. It's an immediate generalisation of the definition of a filter of subsets we gave near the beginning of Section 3.

Definition 10 (Filters) *A non-empty subset $\mathscr{F}$ of the set L is called a* filter *on $\langle L, \leq \rangle$ if it satisfies the properties:*

LF_1. *if $a \in \mathscr{F}$ and $a \leq b$ then also $b \in \mathscr{F}$, for all $a, b \in L$;*
LF_2. *if $a \in \mathscr{F}$ and $b \in \mathscr{F}$ then also $a \frown b \in \mathscr{F}$, for all $a, b \in L$.*

We call a filter $\mathscr{F}$ proper *if $\mathscr{F} \neq L$. We denote the set of all proper filters of $\langle L, \leq \rangle$ by $\overline{\mathbb{F}}(L)$, and the set of all filters by $\mathbb{F}(L) = \overline{\mathbb{F}}(L) \cup \{L\}$.*

The inference mechanism associated with filters is, as are all such mechanisms, based on the idea of *closure* and *intersection structures*, which we already brought to the fore in Section 2. Here too, it's easy to see that the set $\overline{\mathbb{F}}(L)$ of all proper filters on L is indeed an intersection structure, meaning that it's closed under arbitrary non-empty intersections: for any non-empty family $\mathscr{F}_i$, $i \in I$ of elements of $\overline{\mathbb{F}}(L)$, we see that still $\bigcap_{i \in I} \mathscr{F}_i \in \overline{\mathbb{F}}(L)$.

We associate with this intersection structure the following *closure operator* $\mathrm{Cl}_{\mathbb{F}(L)} : \mathscr{P}(L) \to \mathbb{F}(L) : H \mapsto \mathrm{Cl}_{\mathbb{F}(L)}(H)$ with

$$\mathrm{Cl}_{\mathbb{F}(L)}(H) := \bigcap \{\mathscr{F} \in \mathbb{F}(L) : H \subseteq \mathscr{F}\}.$$

In this language, the filters are the deductively closed subsets of the bounded lattice L, and closure can be used to extend any set of lattice elements to the smallest deductively closed set that includes it. If we call a set H *filterisable* if it's included in some proper filter, or equivalently, if $\mathrm{Cl}_{\mathbb{F}(L)}(H) \neq L$, then $\mathrm{Cl}_{\mathbb{F}(L)}(H)$ is the smallest proper filter that includes H, for any filterisable set H. Of course, $H = \mathrm{Cl}_{\mathbb{F}(L)}(H) \Leftrightarrow H \in \mathbb{F}(L)$, for all $H \subseteq L$, and therefore also $\mathbb{F}(L) = \mathrm{Cl}_{\mathbb{F}(L)}(\mathscr{P}(L))$. The following result is then a standard conclusion in order theory [4, Chapter 7].

Proposition 11 *The partially ordered set* $\langle \mathbb{F}(L), \subseteq \rangle$ *is a complete lattice with top* L *and bottom* $\mathrm{Cl}_{\mathbb{F}(L)}(\emptyset) = \bigcap \overline{\mathbb{F}}(L)$*. For any non-empty family* $\mathscr{F}_i$*,* $i \in I$ *of elements of* $\mathbb{F}(L)$*, we have for its infimum and its supremum that, respectively,* $\inf_{i \in I} \mathscr{F}_i = \bigcap_{i \in I} \mathscr{F}_i$ *and* $\sup_{i \in I} \mathscr{F}_i = \mathrm{Cl}_{\mathbb{F}(L)}(\bigcup_{i \in I} \mathscr{F}_i)$.

Two special types of filters deserve more attention in the light of what's to come.

A *prime filter* $\mathscr{G}$ on $\langle L, \leq \rangle$ is a proper filter that also satisfies the following condition:

LPF. if $a \smile b \in \mathscr{G}$ then $a \in \mathscr{G}$ or $b \in \mathscr{G}$, for all $a, b \in L$.

We denote the set of all prime filters on $\langle L, \leq \rangle$ by $\overline{\mathbb{F}}_{\mathrm{p}}(L)$. When the bounded lattice $\langle L, \leq \rangle$ is distributive, then any proper filter can be represented by prime filters, as it's the intersection of all the prime filters that include it; see Ref. [4, Sections 10.7–21] for more details.

Theorem 12 (Prime Filter Representation) *Let* $\langle L, \leq \rangle$ *be a bounded distributive lattice. Then any non-empty* $\mathscr{F} \subseteq L$ *is a proper filter iff* $\mathscr{F} = \bigcap \{\mathscr{G} \in \overline{\mathbb{F}}_{\mathrm{p}}(L) : \mathscr{F} \subseteq \mathscr{G}\}$.

In the special case that $\langle L, \leq \rangle$ is a *complete* lattice, we can replace the finite meets in LF_2 by arbitrary, possibly infinite ones, as in

$\mathrm{LF}_2^{\mathrm{p}}$. if $A \subseteq \mathscr{F}$ then also $\inf A \in \mathscr{F}$, for all $\emptyset \neq A \subseteq L$.

We then find that $\inf \mathscr{F} \in \mathscr{F}$, and that $\mathscr{F}$ consists of all elements of L that dominate $\inf \mathscr{F}$. Such a so-called *principal filter* is clearly proper iff $\inf \mathscr{F} \neq 0_L$. The set of all principal filters, ordered by set inclusion, is trivially order-isomorphic to the complete lattice $\langle L, \leq \rangle$ itself.

Taking Stock Now that we know what the inference mechanism underlying filters is, we can make clearer what we mean by *filter representation* of other inference mechanisms. Axioms K_1–K_5 govern the inference mechanism behind the desirability of sets of things, and we've seen in Section 2.2, and in particular in Proposition 2, that its mathematical essence can be condensed into the complete lattice $\langle \mathbf{K}, \subseteq \rangle$ and the closure operator $\mathrm{Cl}_{\mathbf{K}}$. Similarly, the finitary version of this inference mechanism is laid down in Axioms K_1–$\mathrm{K}_5^{\mathrm{fin}}$, and is captured by the complete lattice $\langle \mathbf{K}_{\mathrm{fin}}, \subseteq \rangle$ and the closure operator $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}$.

The question raised in Section 3 is then, in its purest form: can we find bounded lattices $\langle L, \leq \rangle$ such that the complete lattice $\langle \mathbb{F}(L), \subseteq \rangle$ and the closure operator $\mathrm{Cl}_{\mathbb{F}(L)}$ can be identified through an order isomorphism with the complete lattice $\langle \mathbf{K}, \subseteq \rangle$ and the closure operator $\mathrm{Cl}_{\mathbf{K}}$; or in the finitary case, identified through an order isomorphism with the complete lattice $\langle \mathbf{K}_{\mathrm{fin}}, \subseteq \rangle$ and the closure operator $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}$? We'll show in Sections 6 and 7 that, indeed, we can find such lattices: the completely distributive complete lattice of events $\langle \mathbf{E}, \subseteq \rangle$ and the bounded distributive lattice of events $\langle \mathbf{E}_{\mathrm{fin}}, \subseteq \rangle$, respectively.

Why bother? What's so special about such representations in terms of filters of events? The answer is twofold.

First of all, there's the issue of interpretation we've already drawn attention to in Section 3. The events in $\mathbf{E}$ and $\mathbf{E}_{\mathrm{fin}}$ represent propositional statements about the desirability of things, and filters of such events represent collections of such propositional statements that are closed under logical deduction—conjunction and modus ponens. The order isomorphisms that we'll identify below then simply tell us that making inferences about desirable things and desirable sets of things based on the Axioms D_1–D_3 and K_1–K_5/$\mathrm{K}_5^{\mathrm{fin}}$ is mathematically equivalent to doing propositional logic with propositional statements about the desirability of things.

The second reason has a more mathematical flavour. Since the sets of events $\mathbf{E}$ and $\mathbf{E}_{\mathrm{fin}}$ are bounded distributive lattices when ordered by set inclusion, we can use the Prime Filter Representation Theorem on such bounded distributive lattices, which states that any filter can be written as the intersection of all the prime filters it's included in. The order isomorphisms we're about to identify in the following sections will then allow us to transport this theorem to the context of (finitely) coherent SDSes, and write these as intersections of special types of them, namely the conjunctive ones. This will lead us to the so-called conjunctive representation results for coherent SDSes in Theorem 19 and for finitely coherent SDSes in Theorem 23.

6. Filter Representation for Finitely Coherent SDSes

We're now first going to consider *finitely* coherent SDSes, and try to relate them to the filters on the distributive lattice $\langle \mathbf{E}_{\mathrm{fin}}, \subseteq \rangle$. This will lead to a so-called conjunctive representation result of finitely coherent SDSes in terms of conjunctive ones. We'll then see in the next section that coherent SDSes also have a conjunctive representation result, that turns out to be formally simpler. We adapt the generic notations and definitions from Section 5 to the specific bounded distributive lattice $\langle \mathbf{E}_{\mathrm{fin}}, \subseteq \rangle$.

To establish an order isomorphism between the complete lattices $\langle \mathbf{K}_{\mathrm{fin}}, \subseteq \rangle$ and $\langle \mathbb{F}(\mathbf{E}_{\mathrm{fin}}), \subseteq \rangle$, we consider the maps $\varphi_{\mathbf{D}}^{\mathrm{fin}} : \mathscr{P}(\mathscr{P}(T)) \to \mathscr{P}(\mathbf{E}_{\mathrm{fin}}) : K \mapsto \varphi_{\mathbf{D}}^{\mathrm{fin}}(K)$, with

$$\varphi_{\mathbf{D}}^{\mathrm{fin}}(K) \coloneqq \{\mathscr{E}(W) : W \Subset K\},$$

and $\kappa_{\mathbf{D}}^{\mathrm{fin}} : \mathscr{P}(\mathbf{E}_{\mathrm{fin}}) \to \mathscr{P}(\mathscr{P}(T)) : \mathscr{F} \mapsto \kappa_{\mathbf{D}}^{\mathrm{fin}}(\mathscr{F})$, with

$$\kappa_{\mathbf{D}}^{\mathrm{fin}}(\mathscr{F}) \coloneqq \{S \in \mathscr{P}(T) : \overline{\mathbf{D}}_S \in \mathscr{F}\}.$$

Theorem 13 $\varphi_{\mathbf{D}}^{\mathrm{fin}}$ *is an order isomorphism between* $\langle \mathbf{K}_{\mathrm{fin}}, \subseteq \rangle$ *and* $\langle \mathbb{F}(\mathbf{E}_{\mathrm{fin}}), \subseteq \rangle$*, with inverse order isomorphism* $\kappa_{\mathbf{D}}^{\mathrm{fin}}$*. Moreover, if the proper filter* $\mathscr{F} = \varphi_{\mathbf{D}}^{\mathrm{fin}}(K)$ *and the finitely coherent SDS* $K = \kappa_{\mathbf{D}}^{\mathrm{fin}}(\mathscr{F})$ *are related by* *this order isomorphism, then $\mathscr{F}$ is a prime filter on $\langle \mathbf{E}_{\mathrm{fin}}, \subseteq\rangle$ iff K satisfies the* completeness condition

$$(\forall S_1, S_2 \subseteq T)\big(S_1 \cup S_2 \in K \Rightarrow (S_1 \in K \text{ or } S_2 \in K)\big). \quad (3)$$

An important consequence of the existence of the order isomorphism in Theorem 13 is that it allows us to represent any finitely coherent SDS in terms of coherent but conjunctive models. This is interesting, because by Proposition 6 such coherent conjunctive SDSes are conceptually much simpler, as they represent SDTs—they only represent *conjunctive* desirability statements.

To see how this representation in terms of conjunctive models comes about, we begin by recalling that the events $\mathscr{E}(W)$ for $W \Subset \mathscr{P}(T)$ are sets of coherent SDTs. They are completely determined by the following argument: consider any $D \in \overline{\mathbf{D}}$, then

$$\begin{aligned} D \in \mathscr{E}(W) &\Leftrightarrow D \in \bigcap_{S \in W} \overline{\mathbf{D}}_S \Leftrightarrow (\forall S \in W) S \cap D \neq \emptyset \\ &\Leftrightarrow (\forall S \in W) S \in K_D \Leftrightarrow W \Subset K_D, \end{aligned}$$

where we recall that $K_D \coloneqq \{S \in \mathscr{P}(T) \colon S \cap D \neq \emptyset\}$. Hence, $\mathscr{E}(W) = \{D \in \overline{\mathbf{D}} \colon W \Subset K_D\}$ for all $W \Subset \mathscr{P}(T)$.

Let us now consider any proper filter $\mathscr{F} \in \overline{\mathbb{F}}(\mathbf{E}_{\mathrm{fin}})$ and any finitely coherent SDS $K \in \overline{\mathbf{K}}_{\mathrm{fin}}$ that correspond, in the sense that $K = \kappa_{\mathbf{D}}^{\mathrm{fin}}(\mathscr{F})$ and $\mathscr{F} = \varphi_{\mathbf{D}}^{\mathrm{fin}}(K)$. On the one hand, we infer from $K = \kappa_{\mathbf{D}}^{\mathrm{fin}}(\mathscr{F})$ that for any $S \in \mathscr{P}(T)$:

$$\begin{aligned} S \in K \Leftrightarrow \overline{\mathbf{D}}_S \in \mathscr{F} &\Leftrightarrow (\exists V \in \mathscr{F}) V \subseteq \overline{\mathbf{D}}_S \\ &\Leftrightarrow (\exists V \in \mathscr{F})(\forall D \in V) S \cap D \neq \emptyset \\ &\Leftrightarrow (\exists V \in \mathscr{F})(\forall D \in V) S \in K_D, \end{aligned}$$

which tells us that $K = \bigcup_{V \in \mathscr{F}} \bigcap_{D \in V} K_D$. On the other hand, we infer from $\mathscr{F} = \varphi_{\mathbf{D}}^{\mathrm{fin}}(K)$ and the argumentation above that $\mathscr{F} = \big\{\{D \in \overline{\mathbf{D}} \colon W \Subset K_D\} \colon W \Subset K\big\}$. We're thus led to the following representation result for finite consistency, finite coherence, and the corresponding closure operator $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}$ in terms of the conjunctive models K_D.

Theorem 14 (Conjunctive Representation) *Consider any SDS $K \subseteq \mathscr{P}(T)$, then the following statements hold:*

(i) *K is finitely consistent iff $\mathscr{E}(W) = \{D \in \overline{\mathbf{D}} \colon W \Subset K_D\} \neq \emptyset$ for all $W \Subset K$;*
(ii) $\mathrm{Cl}_{\mathbf{K}_{\mathrm{fin}}}(K) = \bigcup_{W \Subset K} \bigcap_{D \in \overline{\mathbf{D}} \colon W \Subset K_D} K_D$*;*
(iii) *K is finitely coherent iff K is finitely consistent and $K = \bigcup_{W \Subset K} \bigcap_{D \in \overline{\mathbf{D}} \colon W \Subset K_D} K_D$.*

We thus find that an SDS is finitely consistent iff any of its finite subsets is included in some conjunctive model, and that any finitely coherent SDS can be written also as a limit inferior of conjunctive models. Even if the representation in terms of such limits inferior is formally somewhat complicated, it has the advantage that the basic representing models are the conjunctive ones, which are easy to identify and 'construct'.

There is, however, another representation result that's formally simpler, but where the representing models are now less easy to 'construct': a representation that's based on the representing role that prime filters play in distributive lattices; see the discussion in Ref. [4, Sections 10.7–21] and Section 5 for more details. Let us now, in the rest of this section, explain how it comes about.

Definition 15 (Completeness) *We call an SDS $W \subseteq \mathscr{P}(T)$* complete *if it satisfies the completeness condition (3), and we denote by $\overline{\mathbf{K}}_{\mathrm{fin,c}}$ the set of all complete and finitely coherent SDSes, and by $\overline{\mathbf{K}}_{\mathrm{c}}$ the set of all complete and coherent SDSes.*

Theorem 13 tells us that the complete finitely coherent SDSes are in a one-to-one relationship with the prime filters on the distributive lattice $\langle \mathbf{E}_{\mathrm{fin}}, \subseteq\rangle$, and the order isomorphism $\kappa_{\mathbf{D}}^{\mathrm{fin}}$ identified in that theorem allows us to easily transform the prime filter representation result of Theorem 12 into the following alternative representation theorem for finitely coherent SDSes.

Theorem 16 (Prime Filter Representation) *A finitely consistent SDS $K \subseteq \mathscr{P}(T)$ is finitely coherent iff $K = \bigcap\{K' \in \overline{\mathbf{K}}_{\mathrm{fin,c}} \colon K \subseteq K'\}$.*

As suggested above, a disadvantage of this type of representation is that the complete SDSes are—much like their prime filter counterparts—hard if not impossible to identify 'constructively'. They include, however, all conjunctive models, which will be very helpful in our discussion of finitary models in Section 8 further on.

Proposition 17 *Consider any coherent SDT $D \in \overline{\mathbf{D}}$, then the (finitely) coherent conjunctive SDS K_D is complete.*

7. Filter Representation for Coherent SDSes

We now turn to representation for coherent, rather than merely finitely coherent, SDSes. As expected by now, we'll focus on the filters of the set $\mathbf{E}$ in order to achieve that.

Our representation will involve the proper *principal* filters of this set $\mathbf{E}$. We denote the set of all principal filters by $\mathbb{P}(\mathbf{E})$, and the set of all proper principal filters on $\mathbf{E}$ by $\overline{\mathbb{P}}(\mathbf{E})$, where $\overline{\mathbb{P}}(\mathbf{E}) = \mathbb{P}(\mathbf{E}) \setminus \{\mathbf{E}\}$. It's easy to see that $\mathbb{P}(\mathbf{E})$ is closed under arbitrary intersections, and therefore $\langle \mathbb{P}(\mathbf{E}), \subseteq\rangle$ is a complete lattice, with intersection as infimum, and with bottom $0_{\mathbb{P}(\mathbf{E})} = \{\overline{\mathbf{D}}\}$ and top $1_{\mathbb{P}(\mathbf{E})} = \mathbf{E}$.

To establish an order isomorphism between the complete lattices $\langle \mathbf{K}, \subseteq\rangle$ and $\langle \mathbb{P}(\mathbf{E}), \subseteq\rangle$, we now consider the maps $\varphi_{\mathbf{D}} \colon \mathscr{P}(\mathscr{P}(T)) \to \mathscr{P}(\mathbf{E}) \colon K \mapsto \varphi_{\mathbf{D}}(K)$, with

$$\varphi_{\mathbf{D}}(K) \coloneqq \{\mathscr{E}(W) \colon W \subseteq K\}.$$

and $\kappa_{\mathbf{D}} \colon \mathscr{P}(\mathbf{E}) \to \mathscr{P}(\mathscr{P}(T)) \colon \mathscr{F} \mapsto \kappa_{\mathbf{D}}(\mathscr{F})$, with

$$\kappa_{\mathbf{D}}(\mathscr{F}) \coloneqq \{S \in \mathscr{P}(T) \colon \overline{\mathbf{D}}_S \in \mathscr{F}\}.$$

Theorem 18 *$\varphi_{\mathbf{D}}$ is an order isomorphism between $\langle \mathbf{K}, \subseteq \rangle$ and $\langle \overline{\mathbb{P}}(\mathbf{E}), \subseteq \rangle$, with inverse order isomorphism $\kappa_{\mathbf{D}}$.*

Similarly to what we did for finitely coherent SDSes in Section 6, we now consider any proper principal filter $\mathscr{F} \in \overline{\mathbb{P}}(\mathbf{E})$ and any coherent SDS $K \in \overline{\mathbf{K}}$ that correspond, in the sense that $K = \kappa_{\mathbf{D}}(\mathscr{F})$ and $\mathscr{F} = \varphi_{\mathbf{D}}(K)$. The principal filter $\mathscr{F} = \varphi_{\mathbf{D}}(K)$ is completely determined by its smallest element $\bigcap \mathscr{F}$, which is the subset of $\overline{\mathbf{D}}$ given by:

$$\bigcap \mathscr{F} = \bigcap \varphi_{\mathbf{D}}(K) = \bigcap \{\mathscr{E}(W) \colon W \subseteq K\} = \mathscr{E}(K).$$

This leads to the following chain of equivalences, for any $D \in \overline{\mathbf{D}}$:

$$D \in \bigcap \varphi_{\mathbf{D}}(K) \Leftrightarrow D \in \bigcap_{S \in K} \overline{\mathbf{D}}_S \Leftrightarrow (\forall S \in K) D \in \overline{\mathbf{D}}_S \Leftrightarrow (\forall S \in K) S \in K_D \Leftrightarrow K \subseteq K_D.$$

This tells us that, on the one hand, $\mathscr{E}(K) = \bigcap \varphi_{\mathbf{D}}(K) = \{D \in \overline{\mathbf{D}} \colon K \subseteq K_D\}$. On the other hand, we infer from $K = \kappa_{\mathbf{D}}(\mathscr{F})$ that for any $S \in \mathscr{P}(T)$, since $\mathscr{E}(\{S\}) = \overline{\mathbf{D}}_S$ and taking into account the principal character of $\mathscr{F}$,

$$S \in K \Leftrightarrow \bigcap \mathscr{F} \subseteq \overline{\mathbf{D}}_S \Leftrightarrow (\forall D \in \mathscr{E}(K)) D \cap S \neq \emptyset \Leftrightarrow (\forall D \in \mathscr{E}(K)) S \in K_D,$$

so we can conclude that $K = \bigcap_{D \in \overline{\mathbf{D}} \colon K \subseteq K_D} K_D$. We're thus led to the following representation result for coherent SDSes in terms of the conjunctive models K_D, providing an elegant alternative proof strategy for a similar result in Ref. [7].

Theorem 19 (Conjunctive Representation) *Consider any SDS $K \subseteq \mathscr{P}(T)$, then the following statements hold:*

(i) *K is consistent iff $\mathscr{E}(K) = \{D \in \overline{\mathbf{D}} \colon K \subseteq K_D\} \neq \emptyset$;*
(ii) *$\mathrm{Cl}_{\mathbf{K}}(K) = \bigcap_{D \in \overline{\mathbf{D}} \colon K \subseteq K_D} K_D$;*
(iii) *K is coherent iff K is consistent and $K = \bigcap_{D \in \overline{\mathbf{D}} \colon K \subseteq K_D} K_D$.*

8. Finitary SDSes

We conclude from the discussion above that the conjunctive representation for coherent SDSes is remarkably simpler than the one for merely finitely coherent SDSes. But, as we'll explain presently, we can recover the simpler conjunctive representation also for finitely coherent SDSes, provided that we focus on finite sets of things. This has recently also been proved by De Bock [7], but we intend to derive this remarkable result here using our filter representation approach, which allows for an alternative and arguably simpler proof, based on the Prime Filter Representation Theorem we recalled in Theorem 12.

Let us denote by $\mathscr{Q}(T)$ the set of all finite sets of things:

$$\mathscr{Q}(T) \coloneqq \{S \in \mathscr{P}(T) \colon S \Subset T\}.$$

As already mentioned earlier, we follow the convention that the empty set's finite, so $\emptyset \in \mathscr{Q}(T)$. If $W \subseteq \mathscr{P}(T)$ is an SDS, then we call $W \cap \mathscr{Q}(T)$ its *finite part* and

$$\mathrm{fin}(W) \coloneqq \{S \in \mathscr{P}(T) \colon (\exists \hat{S} \in W \cap \mathscr{Q}(T)) \hat{S} \subseteq S\}$$

its *finitary part*. We call the SDS W *finitary* if each of its desirable sets has a finite desirable subset, meaning that

$$(\forall S \in W)(\exists \hat{S} \in W \cap \mathscr{Q}(T)) \hat{S} \subseteq S,$$

or equivalently, $W \subseteq \mathrm{fin}(W)$.

Interestingly, for any (finitely) coherent SDS K, the coherence condition K_2 guarantees that, since $K \cap \mathscr{Q}(T) \subseteq K$, also $\mathrm{fin}(K) \subseteq K$. This tells us that a (finitely) coherent SDS K is finitary if and only if $K = \mathrm{fin}(K)$: *a (finitely) coherent finitary SDS K is equal to its finitary part* $\mathrm{fin}(K)$, *and therefore completely determined by its finite part $K \cap \mathscr{Q}(T)$.*

Moreover, it's easy to see that $\mathrm{fin}(\mathrm{fin}(K)) = \mathrm{fin}(K)$ for any (finitely) coherent SDS K, implying that *its finitary part* $\mathrm{fin}(K)$ *is always finitary.*[5]

Does the (finite) coherence of an SDS imply the coherence (finite or otherwise) of its finitary part? The following proposition provides the beginning of an answer, which we'll be able to complete further on in Corollary 24.

Proposition 20 *If an SDS K is (finitely) coherent, then its finitary part* $\mathrm{fin}(K)$ *is finitely coherent.*

Let us now find out more about how, for a (finitely) coherent SDS, being finitary relates to being complete, and in particular to being conjunctive.

All coherent and conjunctive SDSes are finitary.

Proposition 21 *Consider any coherent SDT $D \in \overline{\mathbf{D}}$, then the (finitely) coherent conjunctive SDS K_D is finitary:* $\mathrm{fin}(K_D) = K_D$.

Coherent SDSes that are conjunctive are always complete; see also Propositions 6 and 17. On the other hand, complete coherent SDSes are not necessarily conjunctive, but we'll see below that they necessarily have a conjunctive finitary part. Consequently, the (finitely) coherent conjunctive SDSes are exactly the complete and coherent SDSes that are finitary. The following proposition gives a more detailed statement.

[5] This also allows us to see $\mathrm{fin}(\bullet)$ as an *interior operator* on the set of all (finitely) coherent SDSes.

Proposition 22 *For any complete and coherent SDS $K \in \overline{\mathbf{K}}_{\mathrm{c}}$, there's some $D \in \overline{\mathbf{D}}$ such that $K \cap \mathscr{Q}(T) = K_D \cap \mathscr{Q}(T)$, and therefore also $\operatorname{fin}(K) = \operatorname{fin}(K_D) = K_D$, namely $D = D_K$. Moreover, if the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then for any complete and finitely coherent SDS $K \in \overline{\mathbf{K}}_{\mathrm{fin,c}}$, there's some $D \in \overline{\mathbf{D}}$ such that $K \cap \mathscr{Q}(T) = K_D \cap \mathscr{Q}(T)$, and therefore also $\operatorname{fin}(K) = \operatorname{fin}(K_D) = K_D$, namely $D = D_K$.*

Since the finitary part of a complete and (finitely) coherent SDS is conjunctive, the Prime Filter Representation Theorem results in a representation with conjunctive models.

Theorem 23 (Conjunctive Representation) *If the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then a finitary and finitely consistent SDS $K \subseteq \mathscr{P}(T)$ is finitely coherent iff $K = \bigcap\{K_D : D \in \overline{\mathbf{D}} \text{ and } K \subseteq K_D\}$.*

This leads to the remarkable conclusion that for finitary SDSes there's no difference between finite coherence and coherence, as long as the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary.

Corollary 24 *If the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then any finitary SDS is finitely coherent iff it's coherent.*

Corollary 25 *If the closure operator $\mathrm{Cl}_{\mathbf{D}}$ is finitary, then the finitary part $\operatorname{fin}(K)$ of any (finitely) coherent SDS K is coherent.*

Running Example Since the closure operator associated with the desirability of gambles in earlier instalments of this running example is finitary, all the results in this section apply in particular also to the coherent sets of desirable gamble sets, that is, to the $K \subseteq \mathscr{P}(\mathscr{G})$ satisfying the finite coherence axioms OK_1–OK_5. In particular, Theorem 23 shows that there is a representation for a finitary $K = \operatorname{fin}(K)$ as an intersection of the conjunctive K_D that include it.

Since these finitary K are completely determined by their *finite* parts $K \cap \mathscr{Q}(\mathscr{G})$, we are led to wonder whether these ideas and results can be connected with our earlier work on sets of desirable gamble sets [7, 8, 9, 10], where You're only allowed to state for *finite* gamble sets $S \in \mathscr{Q}(\mathscr{G})$ that at least one of its elements is desirable to You. A set of finite desirable gamble sets $F \subseteq \mathscr{Q}(\mathscr{G})$ is called *coherent* there when it satisfies the (finite) coherence requirements OK_1–OK_5, with $\mathscr{P}(\mathscr{G})$ replaced by $\mathscr{Q}(\mathscr{G})$. Such coherent sets of finite desirable gamble sets F have also been shown to have a representation in terms of the conjunctive models F_D, as $F = \bigcap\{F_D : F \subseteq F_D\}$. This suggests that every coherent set of finite desirable gamble sets F will also be representable by a principal filter of events, whose smallest element $\{D \in \overline{\mathbf{D}} : F \subseteq F_D\}$ can then be interpreted as the set of remaining possible identifications for D_{T} after making all the desirability statements corresponding to all the finite desirable gamble sets in F. Whether such a connection exists, and what form it then takes, is a subject for further research.

Now, as discussed in detail in Refs. [9, 10], it's possible to impose additional (rationality) requirements on finitary sets of desirable gamble sets F, besides coherence, and it will be interesting to mention a few of them here, even if a lack of space prevents us from going into any detail. If we formulate appropriate *Archimedeanity* and *mixingness* conditions for F, a coherent set of finite desirable gamble sets F will satisfy them iff $F = \bigcap\{F_p : p \in \mathbf{P} \text{ and } F \subseteq F_p\}$, where $\mathbf{P}$ is the set of all probability mass functions p on the set $\mathscr{X}$, and $F_p := \{S \in \mathscr{Q}(\mathscr{G}) : (\exists f \in S) E_p(f) > 0\}$ is the set of desirable gamble sets that corresponds to Your having this precise probability model p.[6] Rather than saying something about an actual model $D_{\mathrm{T}} \in \overline{\mathbf{D}}$, the desirability statements present in an Archimedean and mixing F can therefore be interpreted as propositional statements about an actual model p_{T} in a set of possible identifications $\mathbf{P}$. This suggests a representation in terms of (principal) filters of probability mass functions p on $\mathscr{X}$. We therefore recover, as a special case, the filter representation results proved in the seminal work by Catrin Campbell–Moore [2].

9. Conclusion

Laying bare the exact nature of the conservative inference mechanism behind coherent SDSes has allowed us to prove powerful representation results for such coherent SDSes in terms of the simpler, conjunctive, models which are essentially coherent SDTs.

These representation results, in their simplest form (Theorems 19 and 23), are reminiscent of—are formal generalisations of—decision making using Levi's E-admissibility Rule [15]. This connection with E-admissibility is also briefly mentioned in the final instalment of our Running Example.

Interestingly, in another interesting special case, where the desirable things are asserted propositions in propositional logic, the additional layer of working with asserted sets of propositions—desirable sets of things—does not add anything new: all coherent sets of desirable sets of things are conjunctive there. This is, of course, not really surprising, as desirable *sets of things* are introduced to deal with disjunctive statements, which are already present in the language of *things themselves* as propositions in propositional logic. The case of things as gambles, on the other hand, shows that in other inference contexts where disjunctive statements are not already part of the language of things, going from desirable things to desirable sets of things is indeed meaningful and useful.

A more detailed and comprehensive study of these and other special cases is the topic of current research.

[6] This is, essentially, Levi's E-admissibility Rule [15].

Acknowledgments

The authors are grateful to Catrin Campbell–Moore for quite a number of animated discussions about this topic. Her ideas about using filters of sets of probabilities provided the initial inspiration for some of the developments here. Gert's research on this paper was partly supported by a sabbatical grant from Ghent University, and from the FWO, reference number K801523N. He wishes to express his gratitude to Jason Konek for funding several research stays at University of Bristol's Department of Philosophy, which allowed him to work more closely together with Arthur, and to discuss results with Catrin and Jason. Work by Gert and Jasper on this paper was also partly supported by the Research Foundation – Flanders (FWO), project number 3G028919. Jasper's work was furthermore supported by his BOF Starting Grant "Rational decision making under uncertainty: a new paradigm based on choice functions", number 01N04819. Arthur Van Camp's research was funded by Jason Konek's ERC Starting Grant "Epistemic Utility for Imprecise Probability" under the European Union's Horizon 2020 research and innovation programme (grant agreement no. 852677). This paper has also benefitted from the constructive feedback of three anonymous reviewers.

Author Contributions

The main mathematical developments were done by Gert, based on many discussions with Arthur and Jasper. Gert also wrote the first draft, which was subsequently revised and commented on by Arthur and Jasper.

References

[1] Thomas Augustin, Frank P. A. Coolen, Gert de Cooman, and Matthias C. M. Troffaes, editors. *Introduction to Imprecise Probabilities*. John Wiley & Sons, 2014.

[2] Catrin Campbell–Moore. Probability filters as a model of belief; comparisons to the framework of desirable gambles. *Proceedings of Machine Learning Research*, 147:42—-50, 2021. Proceedings of the ISIPTA 2021.

[3] Inés Couso and Serafín Moral. Sets of desirable gambles: conditioning, representation, and precise probabilities. *International Journal of Approximate Reasoning*, 52(7), 2011. doi:10.1016/j.ijar.2011.08.001.

[4] B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, Cambridge, second edition, 2002.

[5] Jasper De Bock. *Credal Networks under Epistemic Irrelevance: Theory and algorithms*. PhD thesis, Ghent University, 2015.

[6] Jasper De Bock. Archimedean choice functions: an axiomatic foundation for imprecise decision making. *arXiv*, 2020. doi:10.48550/ARXIV.2002.05196. URL https://arxiv.org/abs/2002.05196.

[7] Jasper De Bock. A theory of desirable things. *arXiv*, 2023. doi:10.48550/ARXIV.2302.07412. URL https://arxiv.org/abs/2302.07412. A shortened version has been accepted for publication in the Proceedings of ISIPTA 2023.

[8] Jasper De Bock and Gert de Cooman. A desirability-based axiomatisation for coherent choice functions. In S. Destercke, T. Denoeux, M.A. Gil, P. Grzergorzewski, and O. Hryniewicz, editors, *Uncertainty Modelling in Data Science: Proceedings of SMPS 2018*, pages 46–53, 2018.

[9] Jasper De Bock and Gert de Cooman. Interpreting, axiomatising and representing coherent choice functions in terms of desirability. In Jasper De Bock, Cassio P. de Campos, Gert de Cooman, Erik Quaeghebeur, and Gregory Wheeler, editors, *Proceedings of the Eleventh International Symposium on Imprecise Probabilities: Theories and Applications*, volume 103 of *Proceedings of Machine Learning Research*, pages 125–134, Thagaste, Ghent, Belgium, 3–6 July 2019.

[10] Gert de Cooman. Coherent and Archimedean choice in general Banach spaces. *International Journal of Approximate Reasoning*, 140:255–281, 2022. doi:10.1016/j.ijar.2021.09.005.

[11] Gert de Cooman and Erik Quaeghebeur. Exchangeability and sets of desirable gambles. *International Journal of Approximate Reasoning*, 53(3):363–395, 2012. Special issue in honour of Henry E. Kyburg, Jr.

[12] Gert de Cooman, Jasper De Bock, and Márcio Alves Diniz. Coherent predictive inference under exchangeability with imprecise probabilities. *Artificial Intelligence Journal*, 52:1–95, 2015.

[13] Gert de Cooman, Arthur Van Camp, and Jasper De Bock. The logic behind desirable sets of things, and its filter representation. *arXiv*, 2023. doi:10.48550/ARXIV.2302.08176. URL https://arxiv.org/abs/2302.08176.

[14] Joseph B. Kadane, Mark J. Schervish, and Teddy Seidenfeld. A Rubinesque theory of decision. *Institute of Mathematical Statistics Lecture Notes-Monograph Series*, 45:45–55, 2004.

doi:10.1214/lnms/1196285378. URL http://www.jstor.org/stable/4356297.

[15] Isaac Levi. *The Enterprise of Knowledge*. MIT Press, London, 1980.

[16] Enrique Miranda and Marco Zaffalon. Nonlinear desirability theory. *International Journal of Approximate Reasoning*, 154:176–199, 2023. doi:10.1016/j.ijar.2022.12.015.

[17] Erik Quaeghebeur. Introduction to imprecise probabilities. chapter Desirability. John Wiley & Sons, 2014.

[18] Erik Quaeghebeur, Gert de Cooman, and Filip Hermans. Accept & reject statement-based uncertainty models. *International Journal of Approximate Reasoning*, 57:69–102, 2015.

[19] Herman Rubin. A weak system of axioms for “rational" behavior and the nonseparability of utility from prior. *Statistics & Risk Modeling*, 5(1-2):47–58, 1987. doi:10.1524/strm.1987.5.12.47.

[20] Teddy Seidenfeld, Mark J. Schervish, and Jay B. Kadane. A representation of partially ordered preferences. *The Annals of Statistics*, 23:2168–2217, 1995. Reprinted in [21], pp. 69–129.

[21] Teddy Seidenfeld, Mark J. Schervish, and Jay B. Kadane. *Rethinking the Foundations of Statistics*. Cambridge University Press, Cambridge, 1999.

[22] Teddy Seidenfeld, Mark J. Schervish, and Joseph B. Kadane. Coherent choice functions under uncertainty. *Synthese*, 172(1):157–176, 2010. doi:10.1007/s11229-009-9470-7.

[23] Matthias C. M. Troffaes and Gert de Cooman. *Lower Previsions*. Wiley, 2014.

[24] Arthur Van Camp. *Choice functions as a tool to model uncertainty*. PhD thesis, Universiteit Gent, 2018.

[25] Peter Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[26] Peter Walley. Towards a unified theory of imprecise probability. *International Journal of Approximate Reasoning*, 24:125–148, 2000.