Randomness and Imprecision: A Discussion of Recent Results

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Abstract

We discuss our recent work on incorporating imprecision in the field of algorithmic randomness, based on the martingale-theoretic approach of game-theoretic probability. We consider several notions of randomness associated with interval, rather than precise, forecasting systems. We study their properties and argue that there are quite a number of reasons for wanting to do so. First, the richer mathematical structure in this generalisation provides a useful backdrop for a better understanding of precise randomness. Second, randomness associated with non-stationary precise forecasting systems can be captured by a constant but less precise interval forecast: greater model simplicity requires more imprecision. Third, imprecise randomness can’t always be explained away as a result of (over)simplification: there are sequences that are random for a constant interval forecast, but never random for any computable (more) precise forecasting system. Incorporating imprecision into randomness therefore allows us to do more than was hitherto possible. Finally, the random sequences for a non-vacuous interval forecast constitute a meagre set, as they do for precise forecasts: imprecise random sequences are equally rare from a topological point of view, and are, in that sense, equally interesting.

Keywords: Martin-Löf randomness, computable randomness, Schnorr randomness, computable stochasticity, imprecise probabilities, game-theoretic probability, interval forecast, supermartingale, computability, meagre set.

1. Introduction

This paper presents an overview of our work on incorporating imprecision into the study of randomness, where we aim at giving a precise mathematical meaning to, and study the mathematical consequences of, associating randomness with interval rather than precise probabilities. We believe it can provide a satisfactory answer to questions raised by a number of researchers [19, 20, 21, 50] about frequentist and ‘objective’ aspects of interval, or imprecise, probabilities. There are many notions of randomness [1, 4], but we focus here essentially on Martin-Löf, computable, and Schnorr randomness. We refer to the preprint [12] for a much more extensive and detailed version of this paper, with proofs for what we claim below, and to Ref. [13] for a much more limited report on our earlier efforts in this direction.

We consider an infinite sequence \( \omega = (z_1, \ldots, z_n, \ldots) \), whose components \( z_n \) are either 0 or 1, and are considered as successive outcomes of some experiment. In the literature, the randomness of such a sequence \( \omega \) is typically associated with a forecasting system \( \varphi \) that associates with each finite sequence of outcomes \( (x_1, \ldots, x_n) \) the (conditional) expectation \( \varphi(x_1, \ldots, x_n) = E(X_{n+1} | x_1, \ldots, x_n) \) for the next, as yet unknown, outcome \( X_{n+1} \). This \( \varphi(x_1, \ldots, x_n) \) is a (precise) forecast for the value of \( X_{n+1} \) after observing the values \( x_1, \ldots, x_n \) of the earlier outcomes \( X_1, \ldots, X_n \), and can be seen as a fair price for—and therefore a commitment to bet on—the unknown next outcome \( X_{n+1} \) after observing the first \( n \) outcomes \( x_1, \ldots, x_n \). The sequence \( \omega \) is then ‘random’ when there is no ‘allowable’ strategy for getting infinitely rich by exploiting the bets made available by the forecasting system \( \varphi \) along the sequence, without borrowing. Betting strategies that are made available by the forecasting system \( \varphi \) are called supermartingales. Which supermartingales are considered ‘allowable’ differs in various approaches [1, 4, 18, 24, 33], but typically involves some (semi)computability requirement.

This martingale-theoretic, or algorithmic randomness, approach lends itself elegantly to allowing for interval rather than precise forecasts, and therefore to allowing for ‘imprecision’ in the definition of randomness. As we explain in Section 2, an ‘imprecise’ forecasting system \( \varphi \) associates with each finite sequence of outcomes \( (x_1, \ldots, x_n) \) a (conditional) expectation interval \( \varphi(x_1, \ldots, x_n) \) for the next outcome \( X_{n+1} \). The lower bound of this interval forecast represents a supremum acceptable buying price, and its upper bound an infimum acceptable selling price, for the next outcome \( X_{n+1} \) [2, 43, 49]. This idea allows us to associate supermartingales with an interval forecasting system, and therefore in Section 3 to extend a number of existing notions of randomness to allow for interval, rather than precise, forecasts: we include in particular Martin-Löf, computable, and Schnorr randomness [1, 4, 18, 33]. We discuss interesting properties of these randomness notions in Section 4. In Section 5, we restrict our attention to stationary interval forecasts, as an extension of the more classical accounts of randomness, which typically consider a forecasting system with constant forecast \( \frac{1}{2} \) — corresponding to flipping a fair coin. In the precise case, a given sequence
may not be random for any stationary forecast, but as we will see, for interval forecasting there typically is a filter of intervals that a sequence is random for. We show in Section 6 by means of a few examples that this filter may not have a smallest element, and even when it does, this smallest element may be a non-vanishing interval. These examples involve sequences that are random for some computable non-stationary precise forecast, but cannot be random for a stationary forecast unless it becomes interval-valued, or imprecise. This might lead to the suspicion that this imprecision is perhaps only an artefact, which results from looking at non-stationary phenomena through an imperfect stationary lens. We continue the argument by showing that this suspicion is unfounded: there are sequences that are random for a stationary interval forecast, but not random for any computable (more) precise forecast, be it stationary or not. This serves to corroborate our claim that randomness is inherently imprecise. Finally, we argue in Section 7 that ‘imprecise’ randomness is an interesting extension of the existing notions of ‘precise’ randomness, because it is equally rare: just as for precise stationary forecasts, the set of all sequences that are random for a non-vacuous stationary interval forecast is meagre.

2. Preliminaries

We begin by introducing the preliminary notions needed to define and study randomness in the following sections.

2.1. The Forecasting Game

The dynamics of forecasting can be made clear, after the fashion first introduced by Shafer and Vovk [35, 36], by considering a game amongst three players, Forecaster, Sceptic and Reality. It involves a sequence of initially unknown outcomes \(X_1, X_2, \ldots, X_n, \ldots\) in the set of possible outcomes \(\{0, 1\}\). To stress that they are unknown, we call them variables, and use upper-case notation.

Each successive stage \(n \in \mathbb{N}\) of the game consists of three steps. Here and in what follows, \(\mathbb{N}\) is the set of all natural numbers, without zero, and \(\mathbb{N}_0 := \mathbb{N} \cup \{0\}\).

In a first step, Forecaster specifies an interval \(I_n = [\tilde{p}_n, \tilde{p}_n] \subseteq [0, 1]\) for the expectation of the as yet unknown outcome \(X_n\) in \(\{0, 1\}\)—or equivalently, for the probability that \(X_n = 1\). We interpret this so-called interval forecast \(I_n\) as a commitment for Forecaster to adopt \(\tilde{p}_n\) as his supremum acceptable buying price and \(\tilde{p}_n\) as his infimum acceptable selling price for the gamble (with reward function) \(X_n\). This means that Sceptic can now in a second step take Forecaster up on any (combination) of the following commitments, whose (possibly negative) uncertain pay-offs are expressed in units of a linear utility: (i) for all real \(q \leq \tilde{p}_n\) and \(\alpha \geq 0\), Forecaster is committed to accepting the gamble \(\alpha[X_n - q]\), leading to an uncertain reward \(-\alpha[X_n - q]\) for Sceptic; and (ii) for all real \(r \geq \tilde{p}_n\) and \(\beta \geq 0\), Forecaster is committed to accepting the gamble \(\beta[r - X_n]\), leading to an uncertain reward \(-\beta[r - X_n]\) for Sceptic. Finally, in a third step, Reality determines the value \(x_n\) of \(X_n\) in \(\{0, 1\}\), and the corresponding rewards \(-\alpha[x_n - q]\) or \(-\beta[r - x_n]\) are paid by Forecaster to Sceptic, who adds them to his current capital.

Elements \(x\) of \(\{0, 1\}\) are called outcomes, and elements \(p\) of the real unit interval \([0, 1]\) will serve as (precise) forecasts. We denote by \(\mathcal{F}\) the set of non-empty closed subintervals of the real unit interval \([0, 1]\). Any element \(I\) of \(\mathcal{F}\) will serve as an interval forecast. We will use the generic notation \(I = [p, \bar{p}]\) for such an interval forecast, and \(p := \min I\) and \(\bar{p} := \max I\) for its lower and upper bounds, respectively. An interval forecast \(I = [p, \bar{p}]\) is of course precise when \(p = \bar{p} = p\), and we will then make no distinction between the singleton interval forecast \(I = \{p\} \in \mathcal{F}\) and the corresponding precise forecast \(p \in [0, 1]\).

When Forecaster announces an interval forecast \(I_n\), Sceptic can try and increase her capital by taking a gamble on the unknown outcome \(X_n\). Any such gamble can be identified with a map \(f_n: \{0, 1\} \rightarrow \mathbb{R}\), and can therefore be represented as a point or vector \((f_n(1), f_n(0))\) in the two-dimensional vector space \(\mathbb{R}^2\). \(f_n(X_n)\) is then the (possibly negative) increase in Sceptic’s capital in stage \(n\) of the game, as a function of the outcome variable \(X_n\). Not every gamble \(f_n(X_n)\) on the unknown outcome \(X_n\) will be available to Sceptic: which gambles she can take is determined by Forecaster’s interval forecast \(I_n\). As indicated above, they have the form \(f_n(X_n) = -\alpha[X_n - q] - \beta[r - X_n]\), where \(\alpha\) and \(\beta\) are non-negative real numbers, \(q \leq \tilde{p}_n\) and \(r \geq \tilde{p}_n\). They constitute a closed convex cone \(\mathcal{A}_n\) in \(\mathbb{R}^2\).

If we associate with any precise forecast \(p \in [0, 1]\) the expectation \(E_p\), defined by \(E_p(f) := pf(1) + (1 - p)f(0)\) for any gamble \(f: \{0, 1\} \rightarrow \mathbb{R}\), and also consider the so-called upper expectation \(E_U\) associated with an interval forecast \(I \in \mathcal{F}\), defined by

\[
E_U(f) := \max_{p \in I} E_p(f) = \begin{cases} E_p(f) & \text{if } f(1) \geq f(0) \\ E_p(f) & \text{if } f(1) \leq f(0) \end{cases}
\]

for any gamble \(f: \{0, 1\} \rightarrow \mathbb{R}\),

then the closed convex cone \(\mathcal{A}_n\) of all gambles \(f_n(X_n)\) on the outcome \(X_n\) that are available to Sceptic at stage \(n\), after Forecaster announces his interval forecast \(I_n\), is completely determined by the condition \(E_U(f_n) \leq 0\). When Reality then chooses a value \(x_n\) for \(X_n\), this results in a (possibly negative) gain in capital \(f_n(x_n)\) for Sceptic.

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1. Because we allow \(q \leq \tilde{p}_n\) rather than \(q < \tilde{p}_n\), we actually see \(\tilde{p}_n\) as a maximum acceptable buying price, rather than a supremum one. We do this because it doesn’t affect the conclusions, but it does simplify the mathematics and the discussion somewhat. Similarly for \(r \geq \tilde{p}_n\).
2.2. The Event Tree and Its Forecasting Systems

We call \((x_1, x_2, \ldots, x_n, \ldots)\) an outcome sequence, and we collect all possible outcome sequences in the set \(\Omega := \{0, 1\}^\mathbb{N}\). We collect the finite outcome sequences \(x_{1:n} := (x_1, \ldots, x_n)\) in the set \(\mathcal{S} := \{0, 1\}^n \cup \bigcup_{n \in \mathbb{N}} \{0, 1\}^n\). The finite outcome sequences \(s\) in \(\mathcal{S}\) and infinite outcome sequences \(\omega\) in \(\Omega\) constitute the nodes—also called situations—and paths in an event tree with unbounded horizon, part of which is depicted below. The empty sequence \(x_{1:0} := \emptyset\) is also called the initial situation.

![Event Tree](image)

In the repeated game described above, Forecaster will only provide interval forecasts \(I_s\) after observing the actual sequence \((x_1, \ldots, x_{n-1})\) that Reality has chosen, and the corresponding sequence of gambles \((f_1, f_2, \ldots, f_{n-1})\) that Sceptic has chosen. This is the essence of so-called prequential forecasting [6, 7, 10]. For the present discussion, it will be more advantageous to consider an alternative, and in some aspects more involved, setting where a forecast \(I_s\) is specified in each of the possible situations \(s\) in the event tree \(\mathcal{S}\); see the figure below.

![Event Tree](image)

We can use this idea to extend the notion of a forecasting system in Refs. [8, 47] from precise to interval forecasts.

**Definition 1 (Forecasting System)**

A forecasting system is a map \(\varphi : \mathcal{S} \to \mathcal{I}\), that associates an interval forecast \(\varphi(s) \in \mathcal{I}\) with any situation \(s\) in the event tree \(\mathcal{S}\). With any forecasting system \(\varphi\) we can associate two real processes \(\varphi^+\) and \(\varphi^-\), defined by \(\varphi^+(s) := \min \varphi(s)\) and \(\varphi^-(s) := \max \varphi(s)\) for all \(s \in \mathcal{S}\). A forecasting system \(\varphi\) is called precise if \(\varphi = \overline{\varphi}\).

Specifying a forecasting system \(\varphi\) requires that Forecaster should imagine in advance all the moves that Reality (and Sceptic) could make, and that he should devise in advance what forecast \(\varphi(s)\) to give in each situation \(s \in \mathcal{S}\).

We denote by \(\Phi\) the set \(\mathcal{I}^\Phi\) of all forecasting systems, and use the notation \(\varphi \subseteq \varphi^+\) to mean that the forecasting system \(\varphi^+\) is at least as conservative as \(\varphi\), meaning that \(\varphi(s) \leq \varphi^+(s)\) for all \(s \in \mathcal{S}\).

2.3. Imprecise Probability Trees

Since in each situation \(s\) the interval forecast \(I_s = \varphi(s)\) corresponds to a so-called local upper expectation \(E_{I_s}\), we can use the argumentation in our earlier papers [14, 16, 17] on imprecise stochastic processes to help \(\varphi\) turn the event tree into an imprecise probability tree, with an associated global upper expectation on paths, and a corresponding notion of ‘almost surely’ [14, 16, 17, 35, 36, 37, 46].

For any path \(\omega \in \Omega\), the initial sequence that consists of its first \(n\) elements is a situation in \(\{0, 1\}^n\), denoted by \(\omega_{1:n}\). Its \(n\)-th element belongs to \(\{0, 1\}\) and is denoted by \(\omega_n\). As a convention, we let its 0-th element be the initial situation \(\omega_{1:0} = \omega_0 = \emptyset\).

For any situation \(s \in \mathcal{S}\) and path \(\omega \in \Omega\), \(\omega\) goes through \(s\) if there is some \(n \in \mathbb{N}_0\) such that \(\omega_{1:n} = \omega\). We denote by \(\Gamma(s)\) the so-called cylinder set of all paths \(\omega \in \Omega\) that go through \(s\). We write \(s \subseteq t\), and say that the situation \(s\) precedes the situation \(t\), when every path that goes through \(t\) also goes through \(s\): \(\Gamma(t) \subseteq \Gamma(s)\). We say that the situation \(s\) strictly precedes the situation \(t\), and write \(s \subset t\), when \(s \subseteq t\) and \(s \neq t\), or equivalently, when \(\Gamma(t) \subset \Gamma(s)\).

For any situation \(s = (x_1, \ldots, x_n) \in \mathcal{S}\), we call \(n = |s|\) its depth in the tree. Of course, \(|s| \geq |\emptyset| = 0\). Also, for any \(x \in \{0, 1\}\), we denote by \(sx\) the situation \((x_1, \ldots, x_n, x)\).

A process \(F\) is a map defined on \(\mathcal{S}\). A real process associates a real number \(F(s) \in \mathbb{R}\) with every situation \(s \in \mathcal{S}\). With any real process \(F\), we can always associate a process \(\Delta F\), called the process difference. For every \(s \in \mathcal{S}\), \(\Delta F(s)\) is the gamble on \(\{0, 1\}\) defined by

\[
\Delta F(s)(x) := F(sx) - F(s)
\]

for all \(x \in \{0, 1\}\).

The initial value of a process \(F\) is its value \(F(\emptyset)\) in the situation \(\emptyset\). We call a real process non-negative if it is non-negative in all situations. Similarly, a positive real process is (strictly) positive in all situations. We call a test process any non-negative real process \(F\) with \(F(\emptyset) = 1\).

We now look at a number of special real processes. In the imprecise probability tree associated with a given forecasting system \(\varphi\), a supermartingale \(M\) for \(\varphi\) is a real process such that

\[
\overline{E}_F(\varphi)(\varphi^+(s)) \leq 0 \text{ for all } s \in \mathcal{S}.
\]

In other words, all supermartingale differences have non-positive upper expectation: supermartingales are real processes that Forecaster expects to decrease. We denote the set of all supermartingales for a given forecasting system \(\varphi\) by \(\mathcal{M}_\varphi\)—whether a real process is a supermartingale depends on the forecasts in the situations.

The supermartingales for \(\varphi\) are effectively all the possible capital processes \(M\) for a Sceptic who starts with an initial capital \(M(\emptyset)\), and in each possible subsequent situation \(s\) selects a gamble \(f_s = \Delta M(s)\) that is available there because of Forecaster’s specification of the interval forecast \(I_s = \varphi(s)\): \(E_{I_s}(f_s) \leq 0\). If Reality chooses the successive outcomes \(x_1, \ldots, x_n\), then Sceptic will end up in the
corresponding situation $s = (x_1, \ldots, x_n)$ with a capital

$$M(x_1, \ldots, x_n) = M(\square) + \sum_{k=0}^{n-1} \Delta M(x_1, \ldots, x_k)(x_{k+1}).$$

We call test supermartingale for $\varphi$ any test process that is also a supermartingale for $\varphi$, or in other words, any non-negative supermartingale $M$ for $\varphi$ with initial value $M(\square) = 1$. It corresponds to Sceptic starting with unit capital and never borrowing. We collect all test supermartingales for $\varphi$ in the set $\mathcal{P}$.\[38\]

We also pay attention to a particular way of constructing test supermartingales. Define a multiplier process as a map $D$ from $\mathbb{S}$ to non-negative gambles on $\{0, 1\}$. Given such a multiplier process $D$, we can construct a test process $D^\varphi$ by the recursion equation

$$D^\varphi(sx) := D^\varphi(s)D(s)(x) \text{ for all } s \in \mathbb{S} \text{ and } x \in \{0, 1\},$$

with $D^\varphi(\square) := 1$. We call $D^\varphi$ the test process generated by the multiplier process $D$. Any multiplier process $D$ that satisfies the additional condition that $\mathcal{E}_{\varphi(s)}(D(s)) \leq 1$ for all $s \in \mathbb{S}$, is called a supermartingale multiplier for the forecasting system $\varphi$. The test process $D^\varphi$ generated by $D$ is then a test supermartingale for $\varphi$.

### 2.4. Upper Expectations and Null Events

In the context of (imprecise) probability trees, any bounded real-valued map defined on the sample space $\Omega$ is called a gamble on $\Omega$, or also a global gamble. An event $A$ is a subset of $\Omega$, and its indicator $\mathbb{I}_A$ is the gamble on $\Omega$ that assumes the value 1 on $A$ and 0 elsewhere.

The supermartingales for a forecasting system $\varphi$ allow us to associate a global upper expectation $\mathcal{E}^\varphi$ with $\varphi$:

$$\mathcal{E}^\varphi(g) := \inf \{M(\square) : M \in \mathbb{M}^\varphi \text{ and } \liminf M \geq g\}$$

for all gambles $g$ on $\Omega$, \(1\)

where $\liminf M(\omega) := \liminf_{n \to \infty} M(\omega_1, \ldots, \omega_n)$ for all $\omega \in \Omega$.

For extensive discussion about why the expression (1) is interesting and useful, we refer to Refs. \[14, 17, 35, 36, 38, 39, 40, 42\]. For our present purposes, it may suffice to mention that for precise forecasts, it leads to a model that coincides with the one found in measure-theoretic probability theory; see Refs. \[35, Chapter 8\] and \[36, Chapter 9\], as well as Ref. \[42\]. In particular, when all $I = \{1/2\}$, it coincides on all measurable global gambles with the usual uniform (Lebesgue) expectation. More generally, for an imprecise forecast $\varphi \in \Phi$, the upper expectation $\mathcal{E}^\varphi$ provides a tight upper bound on the measure-theoretic expectation of every precise forecasting system $\varphi'$ that is compatible with $\varphi$ in the sense that $\varphi' \subseteq \varphi$ \[38\].

For an event $A \subseteq \Omega$, the corresponding upper probability is defined by $\mathcal{P}^\varphi(A) := \mathcal{E}^\varphi(1_A)$. We call an event $A \subseteq \Omega$ null for a forecasting system $\varphi$ if $\mathcal{P}^\varphi(A) = 0$. As usual, any property that holds, except perhaps on a null event, is said to hold almost surely for the forecasting system $\varphi$. We will then also say that almost all paths have that property in the imprecise probability tree corresponding to $\varphi$.

### 2.5. Computability

A recursive map $\psi : \mathbb{N}_0 \to \mathbb{N}_0$ is a map that can be computed by a Turing machine. By the Church–Turing (hypo)thesis, this is equivalent to the existence of an algorithm that, upon input of a number $n \in \mathbb{N}_0$, outputs the number $\psi(n) \in \mathbb{N}_0$. All notions of computability that we need are based on this notion, and we use the equivalent condition consistently. It is clear that in this definition, we can replace any of the $\mathbb{N}_0$ with any other countable set that is linked with $\mathbb{N}_0$ through a recursive bijection whose inverse is also recursive.

In what follows, we will need a notion of computable real processes, or in other words, computable real-valued maps $F : \mathbb{S} \to \mathbb{R}$ defined in the set $\mathbb{S}$ of all situations. Because there is an obvious recursive bijection between $\mathbb{N}_0$ and $\mathbb{S}$, whose inverse is also recursive, we can identify real processes and real sequences, and simply import, *mutatis mutandis*, the definitions for computable real sequences common in the literature \[30, Chapter 0, Definition 5\].

We call a net of rational numbers $r_{s,n}$ recursive if there are three recursive maps $a, b, \varsigma$ from $\mathbb{S} \times \mathbb{N}_0$ to $\mathbb{N}_0$ such that

$$b(s,n) > 0 \text{ and } r_{s,n} = (-1)^{\varsigma(s,n)} a(s,n) b(s,n)$$

for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$.

We call a real process $F : \mathbb{S} \to \mathbb{R}$ computable if there is a recursive net of rational numbers $r_{s,n}$ and a recursive map $e : \mathbb{S} \times \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$n \geq e(s,N) \Rightarrow |r_{s,n} - F(s)| \leq 2^{-N}$$

for all $s \in \mathbb{S}$ and $n, N \in \mathbb{N}_0$.

A forecasting system $\varphi$ is computable if the processes $\varphi$ and $\overline{\varphi}$ are.

A real process $F$ is lower semicomputable \[33, 25\] if it can be approximated from below by a recursive net of rational numbers, meaning that there is some recursive net of rational numbers $r_{s,n}$ such that

(i) $r_{s,n} \geq r_{s,n+1}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$;

(ii) $F(s) = \lim_{n \to \infty} r_{s,n}$ for all $s \in \mathbb{S}$.

We say that $F$ is *upper semicomputable* if $-F$ is lower semicomputable. Computability can be related to lower and upper computability: a real process $F$ is computable if and only if it is both lower and upper semicomputable. The set of all (semi)computable processes is countable; see for instance Ref. \[47, Lemma 13\]. The (semi)computability of multiplier processes is defined similarly, by replacing the domain $\mathbb{S}$ by $\mathbb{S} \times \{0, 1\}$.
3. Several Notions of Randomness

We denote by $\mathcal{A}$ any countable set of test processes that includes the countable set of all computable positive test processes, which we denote by $\mathcal{A}_C$. Examples of such sets $\mathcal{A}$ are:

- $\mathcal{A}_C^+$: all computable positive test processes
- $\mathcal{A}_C$: all computable test processes
- $\mathcal{A}_{\text{ML}}$: all lower semicomputable test processes
- $\mathcal{A}_{\text{ML}}^*$: all test processes generated by lower semicomputable multiplier processes.

We call such test processes in $\mathcal{A}$ allowable. It holds that

$$\mathcal{A}_C^+ \subseteq \mathcal{A}_C \subseteq \mathcal{A}_{\text{ML}} \subseteq \mathcal{A}_{\text{ML}}^*.$$  \hspace{1cm} (2)

The test supermartingales for $\varphi$ in this set $\mathcal{A}$ are called allowable test supermartingales, and collected in the set $\mathcal{T}_\mathcal{A} := \mathcal{A} \cap \mathcal{T}^\varphi$. In particular, $\mathcal{T}_C^\varphi := \mathcal{A}_C^+ \cap \mathcal{T}^\varphi$, $\mathcal{T}_C := \mathcal{A}_C \cap \mathcal{T}^\varphi$, $\mathcal{T}_{\text{ML}}^\varphi := \mathcal{A}_{\text{ML}} \cap \mathcal{T}^\varphi$ and $\mathcal{T}_{\text{ML}} := \mathcal{A}_{\text{ML}}^* \cap \mathcal{T}^\varphi$. Hereafter, unless explicitly stated to the contrary, $\mathcal{A}$ is an arbitrary but fixed set of allowable test processes.

We introduce several versions of randomness, each connected with a particular class of test supermartingales.

**Definition 2 (Randomness)**

Consider any forecasting system $\varphi: \mathcal{S} \to \mathcal{I}$ and any path $\omega \in \Omega$. We call $\omega$ $\mathcal{A}$-random for $\varphi$ if all (allowable) test supermartingales $T$ in $\mathcal{T}_\mathcal{A}$ remain bounded above on $\omega$, meaning that $\sup_{n \in \mathbb{N}} T(\omega_{[n:n]}) < \infty$. We then also say that the forecasting system $\varphi$ makes $\omega$ $\mathcal{A}$-random.

In other words, $\mathcal{A}$-randomness of a path means that there is no allowable strategy that starts with unit capital and avoids borrowing, and allows Sceptic to increase her capital without bounds by exploiting the bets on the outcomes along the path that are made available to her by Forecaster’s specification of the forecasting system $\varphi$.

We let $\Phi_\mathcal{A}(\omega) := \{ \varphi \in \Phi: \omega$ is $\mathcal{A}$-random for $\varphi \}$ denote the set of all forecasting systems that make the path $\omega$ $\mathcal{A}$-random. We will also use the special notations $\Phi_C(\omega)$, $\Phi_C^\omega(\omega)$, $\Phi_{\text{ML}}(\omega)$ and $\Phi_{\text{ML}}(\omega)$ in the cases that $\mathcal{A}$ is equal to $\mathcal{A}_C^+$, $\mathcal{A}_C$, $\mathcal{A}_{\text{ML}}$ and $\mathcal{A}_{\text{ML}}^*$, respectively.

When the forecasting system $\varphi$ is precise and computable, and $\mathcal{A}$ is the set $\mathcal{A}_{\text{ML}}$ of all lower semicomputable test processes, our definition reduces to that of Martin-Löf randomness on the Schnorr-Levin (martingale-theoretic) account [1, 4, 33, 34, 47]. We continue to call $\mathcal{A}_{\text{ML}}$-randomness Martin-Löf randomness when the forecasting system $\varphi$ is no longer precise or computable. Because $\mathcal{A}_{\text{ML}}$-randomness is weaker than Martin-Löf randomness, but has a similar flavour, we will also call it weak Martin-Löf randomness. When the forecasting system $\varphi$ is precise and computable, and $\mathcal{A}$ is the set $\mathcal{A}_C$ of all computable test processes, our definition reduces to that of computable randomness [1, 4]. We continue to call $\mathcal{A}_C$-randomness computable randomness when the forecasting system $\varphi$ is no longer precise or computable.

We also extend the notion of Schnorr randomness [33, 34] to our present context. To this end, we call a map $\rho: \mathbb{N} \to \mathbb{N}$ a growth function if it is recursive, non-decreasing and unbounded, and call a real-valued map $\mu: \mathbb{N} \to \mathbb{R}$ computably unbounded if there is some growth function $\rho$ such that $\lim \sup_{n \to \infty} |\mu(n) - \rho(n)| > 0$.

**Definition 3 (Schnorr Randomness)**

Consider any forecasting system $\varphi: \mathcal{S} \to \mathcal{I}$. We call a path $\omega \in \Omega$ Schnorr random if any computable test supermartingale $T \in \mathcal{T}_C^\varphi$ for $\varphi$ is computably unbounded on $\omega$. We then also say that the forecasting system $\varphi$ makes $\omega$ Schnorr random, and we collect all such forecasting systems in the set $\Phi_\Omega(\omega)$.

4. Properties

The more conservative—imprecise—a forecasting system, the less stringent is the corresponding randomness notion.

**Proposition 4** Let $\omega$ be $\mathcal{A}$-random (respectively Schnorr random) for a forecasting system $\varphi$. Then $\omega$ is also $\mathcal{A}'$-random (respectively Schnorr random) for any forecasting system $\varphi'$ such that $\varphi \subseteq \varphi'$.

The larger a set $\mathcal{A}$ of allowable test processes, the more stringent is the corresponding randomness notion, and the ‘fewer’ $\mathcal{A}$-random paths there are. And Schnorr randomness is the weakest form of randomness considered here.

**Proposition 5** Consider two sets $\mathcal{A}, \mathcal{A}'$ of allowable test processes such that $\mathcal{A}' \subseteq \mathcal{A}$. If $\omega$ is $\mathcal{A}$-random for a forecasting system $\varphi$, then $\omega$ is also $\mathcal{A}'$-random for $\varphi$ as well as Schnorr random, so $\Phi_\mathcal{A}(\omega) \subseteq \Phi_{\mathcal{A}'}(\omega) \subseteq \Phi_\Omega(\omega)$.

As a consequence of Equation (2), we can infer from Proposition 5 that

$$\Phi_{\text{ML}}(\omega) \subseteq \Phi_{\text{ML}}^\omega(\omega) \subseteq \Phi_C(\omega) = \Phi_C^\omega(\omega) \subseteq \Phi_\Omega(\omega).$$  \hspace{1cm} (3)

As a special case, the (computable) vacuous forecasting system $\varphi$, assigns the vacuous forecast $\varphi_s(s) := [0, 1]$ to all situations $s \in \mathcal{S}$. Clearly $\varphi \subseteq \varphi_s$ for all $\varphi \in \Phi$, so $\varphi_s$ is the most conservative forecasting system. It corresponds to Forecaster making no actual commitments. This vacuous forecasting system can be used to conclude that for any path $\omega$ there are forecasting systems that make it random.

**Proposition 6** All paths are $\mathcal{A}$-random and Schnorr random for the vacuous forecasting system, so $\varphi_s \in \Phi_\mathcal{A}(\omega) \subseteq \Phi_\Omega(\omega)$ for all $\omega \in \Omega$.

We now turn to a number of important consistency results for the randomness notions we have introduced. We first
show that any Forecaster who specifies a forecasting system is consistent in the sense that he believes himself to be well-calibrated: in the imprecise probability tree generated by his own forecasts, almost all paths will be random, so he is ‘almost sure’ that Sceptic won’t be able to become infinitely rich by exploiting his—Forecaster’s—forecasts.

**Theorem 7** Consider any forecasting system \( \varphi : \mathbb{S} \rightarrow \mathcal{I} \). Then almost all paths are \( k \)-random, and therefore also Schnorr random, for \( \varphi \) in the imprecise probability tree that corresponds to \( \varphi \).

This result guarantees in particular that there always are random paths, for any forecasting system, and leads to the following ‘converse’ to Proposition 6.

**Corollary 8** For any forecasting system \( \varphi \) there is at least one path that is \( k \)-random, and therefore also Schnorr random, for \( \varphi \).

Theorem 9 below shows that if we concentrate on a specific path that is random, then the limsup average gain for Sceptic along that path for betting on a fixed gamble \( h : \{0, 1\} \rightarrow \mathbb{R} \) with rates provided by Forecaster is non-positive. In this result, the average can be taken over any recursive selection of situations. To formalise this, we call any process that assumes values in \( \{0, 1\} \) a selection process. For any \( k \in \mathbb{N} \), the situation \( \omega_{1:k} \) is then selected on the path \( \omega \) only if \( S(\omega_{1:k}) = 1 \).

**Theorem 9 (Average Gains: Selection Processes)** Consider any computable forecasting system \( \varphi : \mathbb{S} \rightarrow \mathcal{I} \), any path \( \omega \in \Omega \) that is \( k \)-random for \( \varphi \), and the corresponding sequence \( (I_1, I_2, \ldots) \) of interval forecasts \( I_k = [p_k, \bar{p}_k] := (\varphi (\omega_{1:n-1}) \) for the path \( \omega \). If \( S : \mathbb{S} \rightarrow \{0, 1\} \) is a recursive selection process such that \( \lim_{n \to \infty} \sum_{k=0}^{n} S(\omega_{1:k}) = \infty \), then

\[
\liminf_{n \to \infty} \frac{\sum_{k=0}^{n} h(\omega_{1:k}) - E_{I_{k+1}}(h)}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0
\]

for any gamble \( h \) on \( \{0, 1\} \).

For Schnorr randomness we can only prove a weaker result, involving the simpler notion of a selection function \( \sigma : \mathbb{N} \rightarrow \{0, 1\} \); at any ‘time point’ \( k \in \mathbb{N} \), the outcome \( \sigma_k \) is selected along the path \( \omega \) only if \( \sigma(k) = 1 \).

**Theorem 10 (Average Gains: Selection Functions)** Consider any computable forecasting system \( \varphi : \mathbb{S} \rightarrow \mathcal{I} \), any path \( \omega \in \Omega \) that is \( k \)-random for \( \varphi \), and the corresponding sequence \( (I_1, I_2, \ldots) \) of interval forecasts \( I_k = [p_k, \bar{p}_k] := (\varphi (\omega_{1:n-1}) \) for the path \( \omega \). If \( \sigma \) is a recursive selection function such that \( \lim_{n \to \infty} \sum_{k=0}^{n} \sigma(k) = \infty \), then

\[
\liminf_{n \to \infty} \frac{\sum_{k=0}^{n} h(\omega_{1:k}) - E_{I_{k+1}}(h)}{\sum_{k=0}^{n-1} \sigma(k)} \geq 0
\]

for any gamble \( h \) on \( \{0, 1\} \). The same conclusion continues to hold when \( \omega \) is Schnorr random for \( \varphi \).

5. **Stationary Forecasting Systems**

We now turn to the special case where the interval forecasts \( I \in \mathcal{I} \) are constant, and don’t depend on the already observed outcomes. This leads to a generalisation of the classical case of randomness associated with a fair coin, which corresponds to \( I = \{1/2\} \). For any interval \( I \in \mathcal{I} \), we denote by \( \gamma : \mathbb{S} \rightarrow \mathcal{I} \) the corresponding so-called stationary forecasting system that assigns the same interval forecast \( I \) to all situations: \( \gamma(s) := I \) for all \( s \in \mathbb{S} \).

In order to investigate the mathematical properties of imprecise randomness, we associate, with any path \( \omega \), the collection of all interval forecasts that make \( \omega \) \( k \)-random:

\[ I_{K}(\omega) := \{ I \in \mathcal{I} : \gamma \in \Phi_{K}(\omega) \} \]

We use the special notations \( \mathcal{I}_{C}(\omega) \), \( \mathcal{I}_{C}(\omega) \), \( \mathcal{I}_{M}(\omega) \) and \( \mathcal{I}_{M}(\omega) \) in the cases that \( K \) is equal to \( K_{C} \), \( K_{C} \), \( K_{M} \) and \( K_{M} \), respectively. Similarly, \( \mathcal{I}_{S}(\omega) := \{ I \in \mathcal{I} : \gamma \in \Phi_{S}(\omega) \} \).

5.1. **Computable Stochasticity**

We begin our study of the randomness associated with stationary forecasting systems by considering the behaviour of relative frequencies along random paths. Theorem 9 implies the consistency property in Corollary 11 below, which is a counterpart in our more general context of the notion of computable stochasticity or Church randomness in the precise fair-coin case where \( I = \{1/2\} \) [1]. Interestingly, this corollary does not impose any computability requirements on the interval forecast \( I \).

We define stochasticity, or Church randomness, goes back to Alonzo Church’s account of randomness [5]. He required of a random path \( \omega \) that for any recursive selection process \( S \) such that \( \sum_{k=0}^{n} S(\omega_{1:k}) \to \infty \),

\[
\lim_{n \to \infty} \frac{\sum_{k=0}^{n} S(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \to \frac{1}{2}
\]

In other words, the relative frequencies of the ones—the successes—in the outcomes that \( S \) selects along the random path \( \omega \) should converge to the constant probability \( 1/2 \) of a success. It is well-known that all paths that are computable random—and therefore also all Martin-Löf random paths—for a stationary forecast \( I = \{1/2\} \) are also Church random; see for instance Refs. [1, 51].

Our generalised notions of randomness no longer imply such convergence, but we are still able to conclude that the limits inferior and superior of the relative frequencies of the successes in the selected outcomes of a random path must lie in the forecast interval.

**Corollary 11 (Church Randomness)** For any path \( \omega \in \Omega \), any constant interval forecast \( [p, \bar{p}] \in \mathcal{I}_{K}(\omega) \) that makes \( \omega \) \( k \)-random, and any recursive selection process \( S : \mathbb{S} \rightarrow \{0, 1\} \) such that \( \sum_{k=0}^{n} S(\omega_{1:k}) \to \infty \),

\[
\liminf_{n \to \infty} \frac{\sum_{k=0}^{n} S(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \frac{1}{2}
\]

\[
\limsup_{n \to \infty} \frac{\sum_{k=0}^{n} S(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq \frac{1}{2}
\]
\[ p \leq \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{t:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S'(\omega_{t:k})} \leq \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{t:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S'(\omega_{t:k})} \leq P. \]

That Corollary 11 needn’t hold for Schnorr randomness, is in accordance with the fact that, in the particular fair-coin case where \( I = \{1/2\} \), Schnorr randomness was shown by Wang [51] not to imply computable stochasticity either [51]. We can prove a weaker result though for paths that are (only) Schnorr random, now based on Theorem 10.

**Corollary 12 (Weak Church Randomness)**

For any path \( \omega \in \Omega \), any constant interval forecast \( [p, P] \in \mathcal{I}_h(\omega) \) that makes \( \lambda \)-random, and any recursive selection function \( \sigma \) such that \( \lim_{n \to \infty} \sum_{k=0}^{n} \sigma(k) = \infty \):

\[ p \leq \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} \sigma(k) \omega_{k+1}}{\sum_{k=0}^{n-1} \sigma(k)} \leq \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} \sigma(k) \omega_{k+1}}{\sum_{k=0}^{n-1} \sigma(k)} \leq P. \]

The same conclusion continues to hold when the interval forecast \([p, P]\) makes \( \omega \) Schnorr random.

If we were to strengthen the requirements on the selection processes \( S \) in Theorem 9 and Corollary 11 from ‘being recursive’ to ‘being recursive and displaying recursive behaviour on the path \( \omega \) under consideration’, then the corresponding (weaker) computable stochasticity result would still hold for all Schnorr random paths. This is essentially what we do in Theorem 10 and Corollary 12. Any criticism of Schnorr randomness along the lines of Wang’s argument [51] will therefore have to include an argumentation for why such a strengthening of the requirements on the selection processes is not reasonable, or undesirable, or alternatively, why selection processes rather than selection functions appear in the requirements.

### 5.2. The Structure of the Interval Forecasts That Make a Path Random

It is guaranteed by Proposition 5 and Equation (3) that \( \mathcal{I}_h(\omega) \subseteq \mathcal{I}_S(\omega) \) and

\[ \mathcal{I}_{ML}(\omega) \subseteq \mathcal{I}_{ML}(\omega) \subseteq \mathcal{I}_C(\omega) = \mathcal{I}_C^+(\omega) \subseteq \mathcal{I}_S(\omega). \]  

Most of our efforts here will be devoted to investigating the mathematical structure of these sets of interval forecasts.

As immediate consequences of the results in Section 4, we find that all these sets of interval forecasts associated with a random path are non-empty and increasing.

**Proposition 13 (Non-emptiness)** For all \( \omega \in \Omega \), \([0, 1] \in \mathcal{I}_h(\omega) \subseteq \mathcal{I}_S(\omega) \), so any sequence of outcomes \( \omega \) has at least one stationary forecast that makes it \( \lambda \)-random and therefore also Schnorr random.

**Proposition 14 (Increasingness)** For all \( \omega \in \Omega \) and any \( I, J \in \mathcal{I} \):

- if \( I \subseteq J \), then \( J \subseteq \mathcal{I}_h(\omega) \);
- if \( J \subseteq \mathcal{I}_C(\omega) \) and \( I \subseteq J \), then \( J \subseteq \mathcal{I}_S(\omega) \).

Corollary 12 allows us to derive the following consistency result: any collection of interval forecasts that make some path random must have a non-empty intersection.

**Proposition 15** For any \( \omega \in \Omega \), \( \mathcal{I}_h(\omega) \) and \( \mathcal{I}_S(\omega) \) have the intersection property: any of their subsets has a non-empty intersection. In fact,

\[ \left[ \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k, \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k \right] \subseteq \bigcap \mathcal{I}_h(\omega) \subseteq \bigcap \mathcal{I}_S(\omega). \]  

Proposition 16 below guarantees, together with Proposition 14, that \( \mathcal{I}_C(\omega) \), \( \mathcal{I}_{ML}(\omega) \) and \( \mathcal{I}_S(\omega) \) are set filters: increasing sets that are closed under finite intersections. We have no proof for a corresponding result for Martin-Löf randomness; it is an open problem whether the set of constant interval forecasts \( \mathcal{I}_{ML}(\omega) \) is closed under finite intersections, and therefore a set filter.

**Proposition 16** For any \( \omega \in \Omega \), the sets of interval forecasts \( \mathcal{I}_{ML}(\omega) \), \( \mathcal{I}_C(\omega) \) and \( \mathcal{I}_S(\omega) \) are closed under finite intersections.

In these specific cases, any interval forecast that includes the non-empty closed intervals \( \bigcap \mathcal{I}_h(\omega) = [p_h(\omega), \bar{p}_h(\omega)] \) and \( \bigcap \mathcal{I}_S(\omega) = [p_S(\omega), \bar{p}_S(\omega)] \) strictly on both sides will make the path \( \omega \) \( \lambda \)-random, respectively Schnorr random. We will see that it may depend on the case at hand whether the interval forecasts \([p_h(\omega), \bar{p}_h(\omega)]\) and \([p_S(\omega), \bar{p}_S(\omega)]\) themselves do the job: in the following sections, we will come across a number of examples where they do, and another example where they don’t.

### 5.3. Examples at the Extreme Ends

We conclude the discussion in this section with a few immediate examples of possible sets of interval forecasts.

For any precise forecast \( p \in [0, 1] \), there always are paths \( \omega \) that are \( \lambda \)-random, and at least as many that are Schnorr random, for the precise stationary forecasting system \( \gamma_p \); see Corollary 8. A constant interval forecast \( I \) will make any such path \( \omega \) \( \lambda \)-random if and only if it contains the precise forecast \( p : \mathcal{I}_h(\omega) = \{I \in \mathcal{I} : p \in I\} \); and similarly for Schnorr random paths.

On the other hand, any recursive path with infinitely many zeroes and ones will only be random for the vacuum interval forecast.

**Proposition 17** If a path \( \omega \in \Omega \) is recursive and has infinitely many zeroes and infinitely many ones, then the only interval forecast that makes \( \omega \) \( \lambda \)-random, or Schnorr random, is the vacuum one: \( \mathcal{I}_h(\omega) = \mathcal{I}_S(\omega) = \{[0, 1]\} \).
The examples in the next section will show that, in between these extremes of total imprecision and maximal precision, there lies an uncharted realm of paths whose unpredictability is ‘similar’ to that of the ones traditionally called ‘random’, but for which $0 < p_n(\omega) < \mathcal{P}_n(\omega) < 1$, and similarly, $0 < p_n(\omega) < \mathcal{P}_n(\omega) < 1$.

6. Why We Claim That Randomness Is Inherently Imprecise

We have learnt from our work on imprecise Markov chains [11, 15, 17, 23, 41] that we can often compute tight bounds on expectations in non-stationary precise Markov chains very efficiently by replacing them with stationary but imprecise versions. Similarly, in statistical modelling, when learning from data sampled from a distribution with a varying (non-stationary) parameter, it is quite a challenge to estimate the time sequence of its values, but we may be more successful in learning about its (stationary) interval range. Such ideas also lie behind the proposal by Fierens et al. [20] of a frequentist interpretation for imprecise probability models, based on non-stationarity.

Here, we explore this idea in the context of our study of imprecise randomness, and illustrate in a number of interesting examples that randomness associated with non-stationary precise forecasting systems can be captured by a stationary forecasting system, which must then be less precise: we gain simplicity of representation by going from a non-stationary to a stationary one, but we must then pay for it by losing precision.

We start with a simple example to introduce the basic idea. Fix any $p, q$ in $[0, 1]$ with $p < q$, and any path $\omega$ that is $\mathcal{A}$-random for the forecasting system $\mathcal{P}_{p,q}$, defined by

$$\mathcal{P}_{p,q}(s) := \begin{cases} p & \text{if } |s| \text{ is odd} \\ q & \text{if } |s| \text{ is even} \end{cases} \quad \forall s \in \mathbb{S}.$$  

Corollary 8 guarantees that there is at least one such path. Then $\mathcal{S}_\mathcal{A}(\omega) = \mathcal{P}_\mathcal{A}(\omega) = \{I \in \mathcal{F} : [p, q] \subseteq I\}$.

We next look at sequences that are ‘nearly’ random for the constant precise forecast $1/2$, but not quite. Consider the following sequence $\{p_n\}_{n \in \mathbb{N}_0}$ of precise forecasts:

$$p_n := \frac{1}{2} + (-1)^n \delta_n \quad \text{with } \delta_n := \sqrt{\frac{8}{n + 33}} \quad \forall n \in \mathbb{N}_0.$$  

We see that $p_n \to 1/2$ and that $p_n \in (0, 1)$ for all $n \in \mathbb{N}_0$. Focus on an arbitrary but fixed path $\omega$ that is $\mathcal{A}_{\mathcal{M}_0}$-random for the computable precise forecasting system $\mathcal{P}_{1/2}$ with

$$\mathcal{P}_{1/2}(s) := p[s] \quad \forall s \in \mathbb{S}.$$  

There is at least one such path, by Corollary 8. Then for all $\mathcal{A}$ such that $\mathcal{A}_{\mathcal{M}_0} \subseteq \mathcal{A} \subseteq \mathcal{A}_{\mathcal{M}_0}$:

$$\mathcal{S}_\mathcal{A}(\omega) = \mathcal{P}_\mathcal{A}(\omega) = \left\{ [p, q] \in \mathcal{F} : p < 1/2 < q \right\}.$$  

These two examples indicate that randomness associated with a non-stationary precise forecasting system can also be ‘described’ as randomness for a simpler, stationary but then necessarily imprecise, forecasting system. They might lead us to suspect that all stationary imprecise forms of randomness could be ‘explained away’ as such simpler representations of non-stationary but precise forms of randomness. This would imply that the imprecision in the stationary forecasts is not essential, and can always be dismissed as a necessary consequence of using a stationary representation that is not powerful enough to allow for the ideal, precise but non-stationary, representation.

We will now argue that this suspicion is misguided, and in fact wrong when we focus on computable forecasting systems: we outline in the theorem below that there are paths that are random for a (computable) stationary interval forecasting system but never for any computable precise forecasting system, be it stationary or not. This serves to corroborate our claim that randomness is inherently imprecise, as its imprecision cannot be explained away as an effect of oversimplification. The imprecision involved is furthermore non-negligible, and can be made arbitrarily large, because besides excluding the possibility of randomness of such paths for precise computable forecasting systems, we also show they can’t be random for any computable forecasting system whose highest imprecision is smaller than that of the original, stationary one.

**Theorem 18 (Randomness is inherently imprecise)**

Consider any set of allowable test processes $\mathcal{A}$, and any interval forecast $[p, q] \in \mathcal{F}$. Then there is path $\omega \in \mathcal{W}$ that is $\mathcal{A}$-random—and therefore also Schnorr random—for the stationary interval forecast $[p, q]$, but that is never Schnorr random—and therefore never $\mathcal{A}$-random—for any computable forecasting system $\mathcal{P}$ whose imprecision is smaller than that of $[p, q]$, in the specific sense that $\sup_{s \in \mathcal{S}} [\mathcal{P}(s) - \mathcal{Q}(s)] < \mathcal{P} - \mathcal{P}.$

For an example showing that the computability condition in this result cannot be dropped, and a discussion on the theoretical and practical relevance of this condition, we refer to recent work by Persiau and us [29].

7. Why Random Sequences Are Topologically Rare

Theorem 7 tells us that the set of all random paths for a forecasting system has lower probability one—since its complement has upper probability zero—so there are many such random paths in a ‘measure-theoretic’ sense. But we will see presently that, in a topological sense, random paths are few, as they typically constitute only a meagre set. This is a known result for precise randomness, that was, as far as we can judge, first formulated in the context of a much more encompassing discussion on the nature of randomness by
We call \( \omega \) a path lawful if there is some algorithm that, given as input any situation \( s \) on the path \( \omega \), outputs a non-trivial finite set \( R(s) \) of situations \( t \subseteq s \) such that one of these ‘extensions’ \( t \) is also on the path—meaning that \( \omega \in \Gamma(t) \). By ‘non-trivial’, they mean that \( R(s) \) is restrictive: it actually eliminates possible extensions. They then go on to show that the set of all lawful paths is meagre, and finally, that random paths, because they satisfy the law of large numbers, are lawful.

We now show that we can extend this argument to imprecise stationary forecasts. First of all, let us give a definition of lawfulness that makes the formulation above more precise. A partial function on a domain \( D \) is a function that need not be defined on all elements of \( D \).

**Definition 19 (Lawfulness [27, Definition 2.1])**

We call algorithm any recursive (partial) function \( R \) from \( S \) to the collection of finite subsets of \( S \). A path \( \omega \in \Omega \) is called lawful for an algorithm \( R \) if for all \( m \in \mathbb{N}_0 \):

(i) \( R \) is defined in the situation \( \omega|_m \);

(ii) \( R(\omega|_m) \) is a non-empty finite subset of \( S \) such that \( \omega|_m \sqsubseteq t \) for all \( t \in R(\omega|_m) \);

(iii) \( R(\omega|_m) \) is non-trivial: \( \bigcup_{t \in R(\omega|_m)} \Gamma(t) \subseteq \Gamma(\omega|_m) \);

(iv) there is some \( t \in R(\omega|_m) \) such that \( \omega \in \Gamma(t) \).

A path \( \omega \in \Omega \) is called lawful if it is lawful for some algorithm \( R \). A path that is not lawful is called lawless.

A set of paths \( A \subseteq \Omega \) is nowhere dense in \( \Omega \) [27] if for every \( s \in A \), there is some \( t \in S \) such that \( s \sqsupseteq t \) and \( A \cap \Gamma(t) = \emptyset \).

A set of paths \( B \subseteq \Omega \) is then called meagre, or first category, it is a countable union of nowhere dense sets. We rely on the following central result in Ref. [27].

**Theorem 20 ([27, Corollary 2.3])** Any subset of \( \Omega \) containing only lawful paths is meagre.

To see that a set of random paths is meagre, it therefore suffices to prove that these random paths are all lawful. This turns out to be not too difficult, because relative frequencies along lawless paths behave very ‘wildly’.

**Proposition 21** Let \( \omega \in \Omega \) be a lawless path. Then

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k = 1.
\]

So, in order to derive our result, it now suffices to consider that relative frequencies along random paths can’t behave so wildly, because they are constrained by our ‘weak computable stochasticity’ result in Corollary 12. Random paths are typically lawful.

**Theorem 22** Let \( I = [p, \overline{p}] \in \mathcal{I} \) be any closed subinterval of \([0, 1]\) strictly included in \([0, 1] \), so \( p > 0 \) or \( \overline{p} < 1 \). Then the set of all paths that are \( \lambda \)-random for the stationary forecasting system \( \gamma_I \) is meagre. Similarly, the set of all Schnorr random paths for \( \gamma_I \) is meagre.

This result shows that the important distinction for random paths lies not between precise and imprecise stationary forecasts, but rather between vacuous and non-vacuous ones: for any non-vacuous stationary forecast, the set of random paths is meagre, whereas for the vacuous stationary forecast, all paths are random, and therefore the corresponding set of random paths is co-meagre—the complement of a meagre set.

It also suggests that the paths that are random for non-vacuous interval forecasts are ‘equally rare’ as those that are random for precise forecasts, which, we believe, only adds to their mathematical interest.

8. Conclusion

There have been a number of papers [19, 20, 21, 48] that aim to introduce imprecision for probabilities that have a physical, or frequentist, interpretation. The present paper tries to continue that tradition.

We believe the work described here is a first systematic attempt at reconciling imprecision with the study of algorithmic randomness along the lines of von Mises [45], Church [5], Kolmogorov [22], Ville [44], Martin-Löf [26], Levin [24] and Schnorr [33, 34]. Our results indicate that this is possible and interesting.

Besides the sequences that are random for precise forecasts, new realms of sequences arise that are random for interval forecasts, and have interesting properties. They are as rare as their precise counterparts, as they also constitute meagre sets. Our examples show that incorporating imprecision into the study of randomness allows for a richer mathematical structure to arise, and our treatment allows us to better understand, as special cases, the existing results in the precise limit.

On the one hand, ‘imprecise randomness’ arises as a useful stationary model simplification when dealing with non-stationarity. But, we have also shown that it has a more fundamental role, as there are sequences that are random for a given computable interval forecast, but not for any computable (more) precise forecast.

All this leads us to the conviction that there is more to randomness than the classical account for precise forecasts seems to suggest.

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Author Contributions

As with most of our joint work, there is no telling, after a while, which of us had what idea, or did what, exactly. We have both contributed equally to this paper. But since a paper must have a first author, we decided it should be the one who took the first significant steps: Gert, in this case.

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