GHENT UNIVERSITY

Computable randomness is inherently imprecise

Gert de Cooman and Jasper De Bock

ISIPTA 2017

Lugano, 10 July 2017

FACULTY OF ENGINEERING AND ARCHITECTURE

A single forecast

The first player, *Forecaster*, specifies an interval bound $I = [\underline{p}, \overline{p}]$ for the expectation of an unknown outcome X in $\{0,1\}$. We interpret this *interval forecast* I as a commitment, on the part of Forecaster, to adopt $\underline{p}$ as a *supremum buying price* and $\overline{p}$ as a *infimum selling price* for the gamble (with reward function) X.

A single forecast

The first player, *Forecaster*, specifies an interval bound $I = [\underline{p}, \overline{p}]$ for the expectation of an unknown outcome X in $\{0,1\}$. We interpret this *interval forecast* I as a commitment, on the part of Forecaster, to adopt $\underline{p}$ as a *supremum buying price* and $\overline{p}$ as a *infimum selling price* for the gamble (with reward function) X.

The second player, *Sceptic*, can now in a second step take Forecaster up on any (combination) of the following commitments:

(i) for any $p \in [0,1]$ such that $p \leq \underline{p}$, and any $\alpha \geq 0$, Forecaster must accept the gamble $\alpha[X - p]$.

(ii) for any $q \in [0,1]$ such that $q \geq \overline{p}$, and any $\beta \geq 0$, Forecaster accepts the gamble $\beta[q - X]$.

A single forecast

The first player, *Forecaster*, specifies an interval bound $I = [\underline{p}, \overline{p}]$ for the expectation of an unknown outcome X in $\{0,1\}$. We interpret this *interval forecast* I as a commitment, on the part of Forecaster, to adopt $\underline{p}$ as a *supremum buying price* and $\overline{p}$ as a *infimum selling price* for the gamble (with reward function) X.

The second player, *Sceptic*, can now in a second step take Forecaster up on any (combination) of the following commitments:

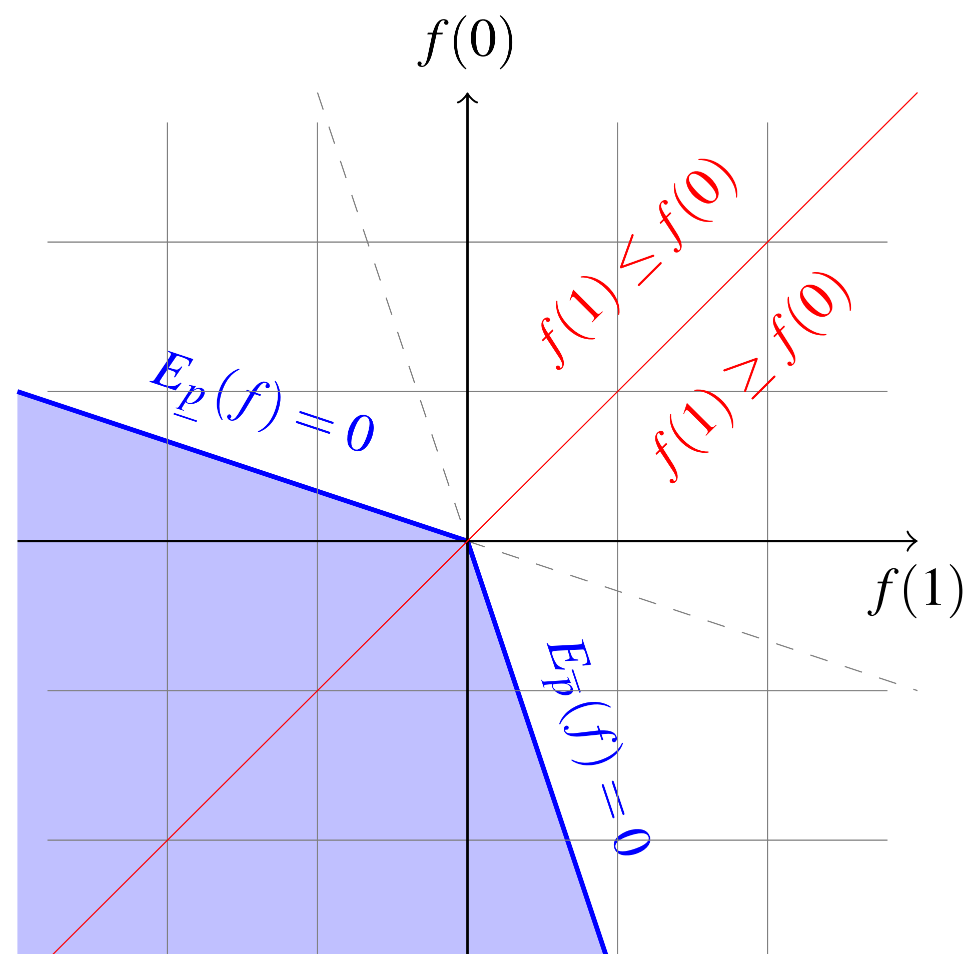
(i) for any $p \in [0,1]$ such that $p \leq \underline{p}$, and any $\alpha \geq 0$, Forecaster must accept the gamble $\alpha[X - p]$.

(ii) for any $q \in [0,1]$ such that $q \geq \overline{p}$, and any $\beta \geq 0$, Forecaster accepts the gamble $\beta[q - X]$.

Finally, in a third step, the third player, *Reality*, determines the value x of X in $\{0,1\}$.

Gambles available to Sceptic: interval forecast

$$f(X) = -\alpha[X - p] - \beta[q - X] \text{ with } \alpha \geq 0 \text{ and } \beta \geq 0 \text{ and } 0 \leq p \leq \underline{p} \text{ and } \overline{p} \leq q \leq 1$$



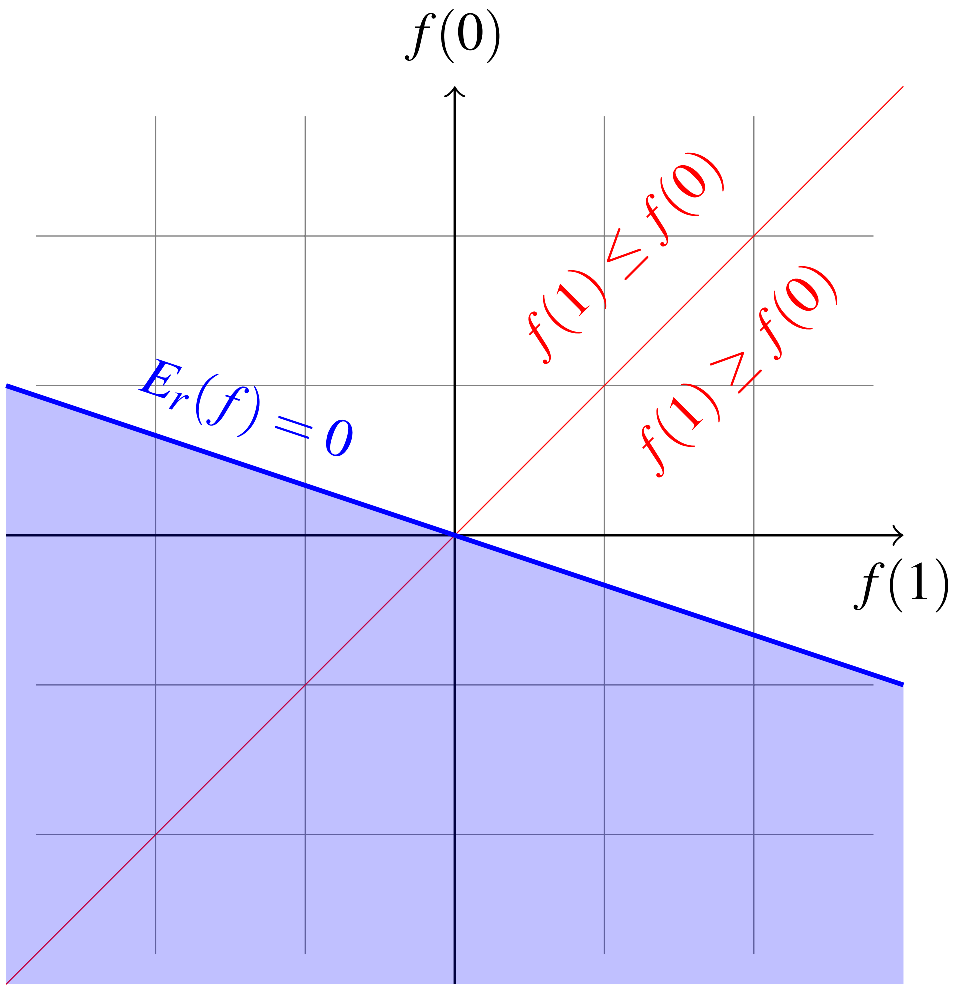
$$\overline{E}_I(f) \leq 0$$

$$\underline{E}_I(f) = \min_{p \in I} E_p(f) = \min_{p \in I} \left[p f(1) + (1-p) f(0) \right] = \begin{cases} E_{\underline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\overline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} \tag{1}$$

$$\overline{E}_I(f) = \max_{p \in I} E_p(f) = \begin{cases} E_{\overline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\underline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} = -\underline{E}_I(-f), \tag{2}$$

Gambles available to Sceptic: interval forecast

$$f(X) = -\alpha[X-p] - \beta[q-X] \text{ with } \alpha \geq 0 \text{ and } \beta \geq 0 \text{ and } 0 \leq p \leq \underline{p} \text{ and } \overline{p} \leq q \leq 1$$



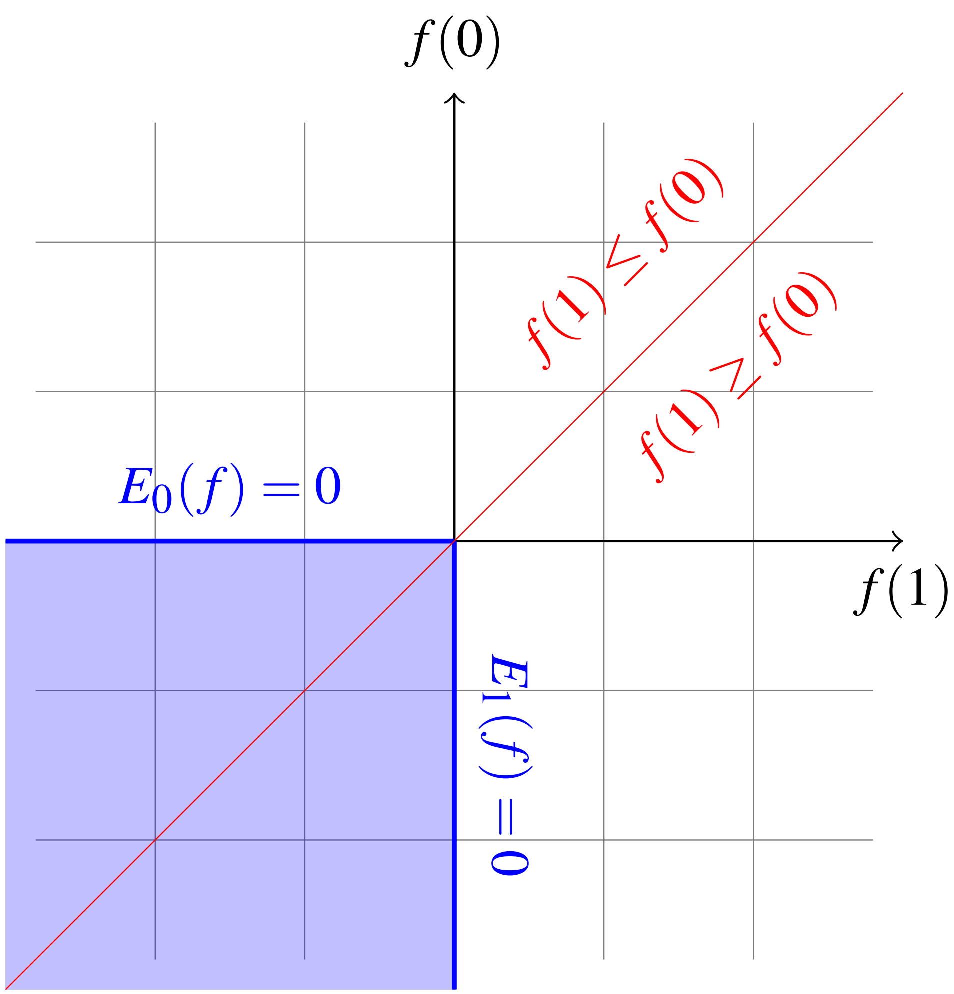
$$\overline{E}_I(f) \leq 0$$

$$\underline{E}_I(f) = \min_{p\in I} E_p(f) = \min_{p\in I}\left[pf(1) + (1-p)f(0)\right] = \begin{cases} E_{\underline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\overline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} \tag{1}$$

$$\overline{E}_I(f) = \max_{p\in I} E_p(f) = \begin{cases} E_{\overline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\underline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} = -\underline{E}_I(-f), \tag{2}$$

Gambles available to Sceptic: interval forecast

$f(X) = -\alpha[X - p] - \beta[q - X]$ with $\alpha \geq 0$ and $\beta \geq 0$ and $0 \leq p \leq \underline{p}$ and $\overline{p} \leq q \leq 1$

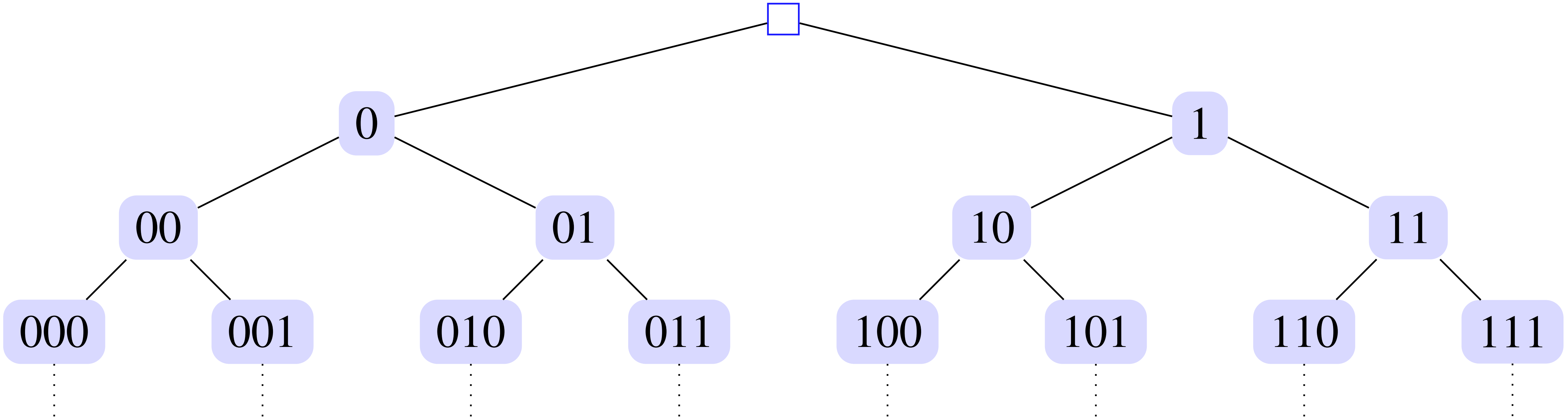


$$\overline{E}_I(f) \leq 0$$

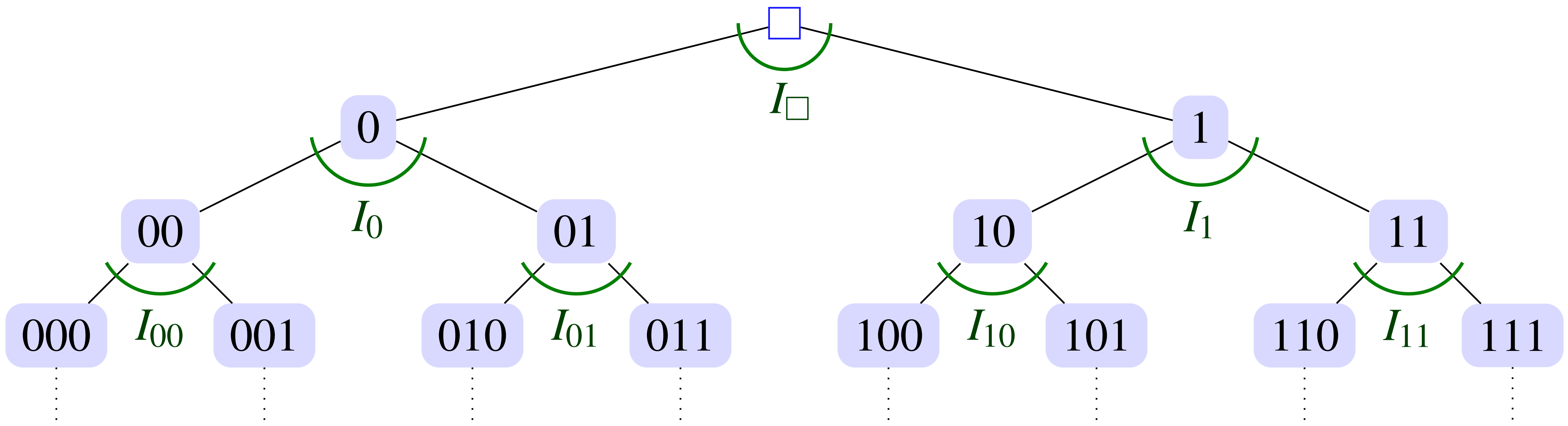
$$\underline{E}_I(f) = \min_{p \in I} E_p(f) = \min_{p \in I} \left[p f(1) + (1-p) f(0)\right] = \begin{cases} E_{\underline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\overline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} \tag{1}$$

$$\overline{E}_I(f) = \max_{p \in I} E_p(f) = \begin{cases} E_{\overline{p}}(f) & \text{if } f(1) \geq f(0) \\ E_{\underline{p}}(f) & \text{if } f(1) \leq f(0) \end{cases} = -\underline{E}_I(-f), \tag{2}$$

Event tree

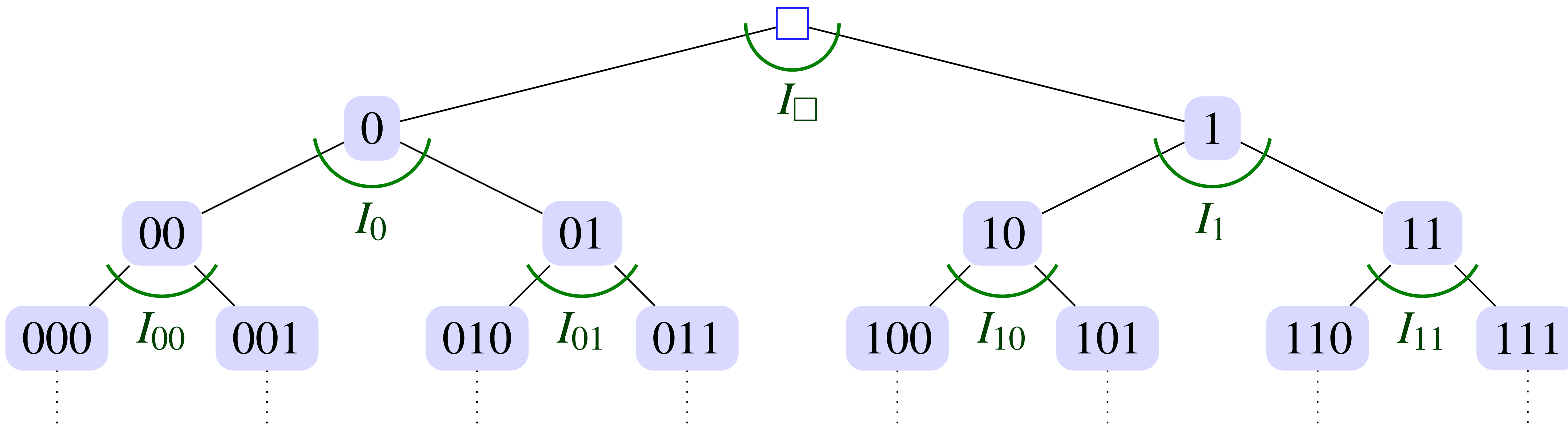


Forecasting system



A forecasting system γ associates with any *situation* $s = (x_1, \ldots, x_n)$ an *interval forecast* $\gamma(s) = I_s$.

Computable randomness of a sequence

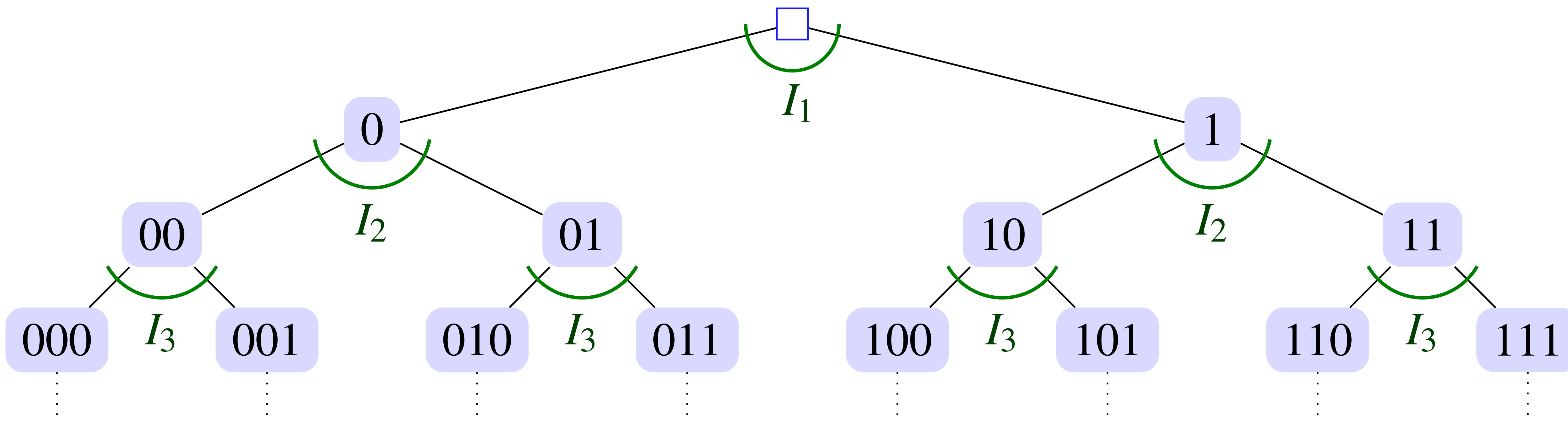


Definition 3 (Computable randomness) *Consider any forecasting system* $\gamma\colon \Omega^\Diamond \to \mathscr{C}$. *We call an outcome sequence* ω computably random for γ *if all computable non-negative supermartingales* T *remain bounded above on* ω, *meaning that there is some* $B \in \mathbb{R}$ *such that* $T(\omega^n) \leq B$ *for all* $n \in \mathbb{N}$. *We then also say that the forecasting system* γ makes ω computably random.

We denote by $\Gamma_C(\omega) := \{\gamma \in \Gamma\colon\ \omega \text{ is computably random for } \gamma\}$ the set of all forecasting systems for which the outcome sequence ω is computably random.

Consistency results

Theorem 6 *Consider any forecasting system $\gamma\colon \Omega^{\Diamond} \to \mathscr{C}$. Then (strictly) almost all outcome sequences are computably random for γ in the imprecise probability tree that corresponds to γ.*

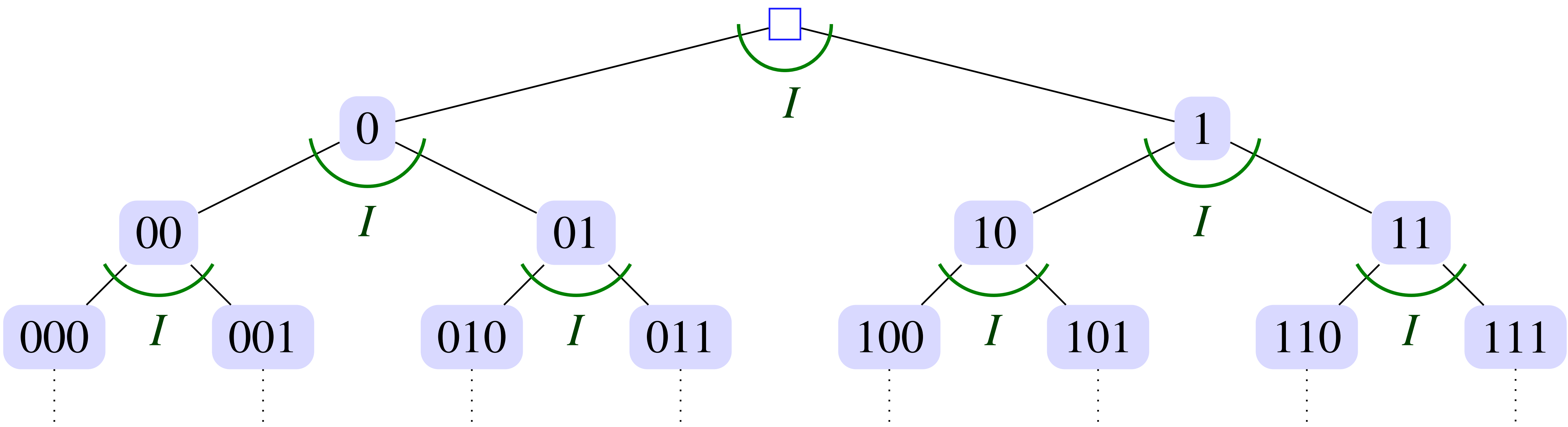


Corollary 7 *For any sequence of interval forecasts $(I_1,\dots,I_n,\dots)$ there is a forecasting system given by $\gamma(x_1,\dots,x_n) := I_{n+1}$ for all $(x_1,\dots,x_n) \in \{0,1\}^n$ and all $n \in \mathbb{N}_0$, and associated imprecise probability tree such that (strictly) almost all—and therefore definitely at least one—outcome sequences are computably random for γ in the associated imprecise probability tree.*

Constant interval forecasts

Stationary forecasting system γ_I:

$$\gamma_I(s) := I \text{ for all } s \in \Omega^\lozenge.$$



$$\mathscr{C}_\mathrm{C}(\omega) := \{I \in \mathscr{C} : \gamma_I \in \Gamma_\mathrm{C}(\omega)\} = \{I \in \mathscr{C} : \gamma_I \text{ makes } \omega \text{ computably random}\}.$$

Church randomness

Corollary 11 (Church randomness) *Consider any outcome sequence* $\omega = (x_1, \ldots, x_n, \ldots)$ *in* Ω *and any stationary interval forecast* $I = [\underline{p}, \overline{p}] \in \mathscr{C}_{\mathrm{C}}(\omega)$ *that makes* ω *computably random. Then for any computable selection process* $S\colon \Omega^{\Diamond} \to \{0,1\}$ *such that* $\sum_{k=0}^{n} S(x_1, \ldots, x_k) \to +\infty$*:*

$$\underline{p} \leq \liminf_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(x_1, \ldots, x_k) x_{k+1}}{\sum_{k=0}^{n-1} S(x_1, \ldots, x_k)} \leq \limsup_{n \to +\infty} \frac{\sum_{k=0}^{n-1} S(x_1, \ldots, x_k) x_{k+1}}{\sum_{k=0}^{n-1} S(x_1, \ldots, x_k)} \leq \overline{p}.$$

The set filter of forecasts that make a sequence random

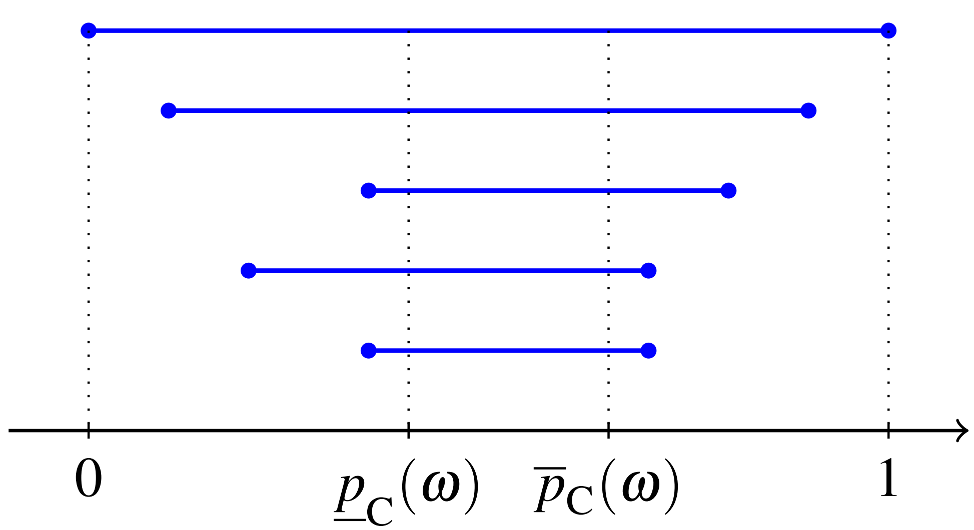
$$\mathscr{C}_{\mathrm{C}}(\omega) := \{I \in \mathscr{C} : \gamma_I \in \Gamma_{\mathrm{C}}(\omega)\} = \{I \in \mathscr{C} : \gamma_I \text{ makes } \omega \text{ computably random}\}.$$

Proposition 9 (Non-emptiness) *For all $\omega \in \Omega$, $[0,1] \in \mathscr{C}_{\mathrm{C}}(\omega)$, so any sequence of outcomes ω has at least one stationary forecast that makes it computably random: $\mathscr{C}_{\mathrm{C}}(\omega) \neq \emptyset$.*

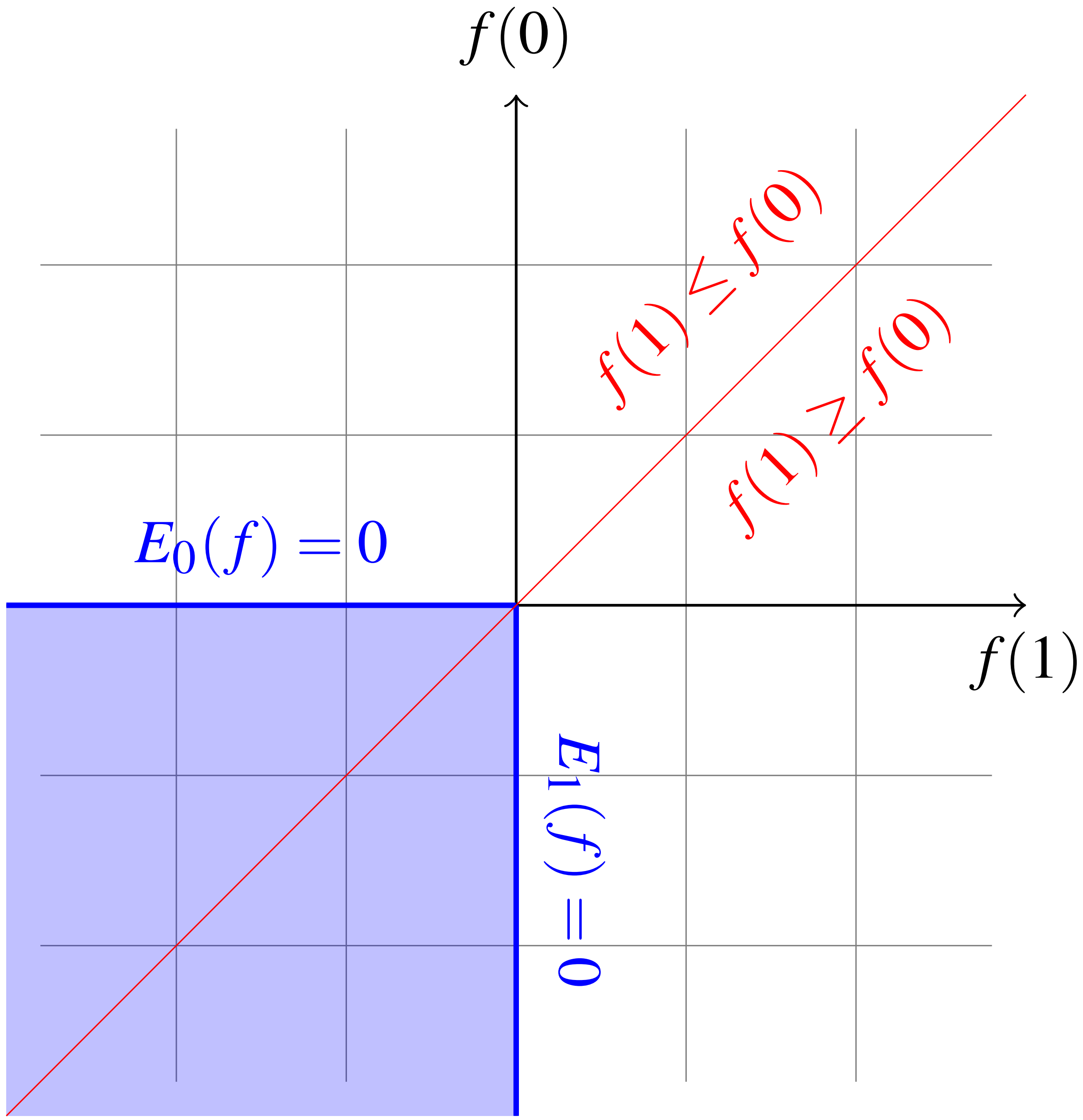
Proposition 10 (Increasingness) *Consider any $\omega \in \Omega$ and any $I, J \in \mathscr{C}$. If $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $I \subseteq J$, then also $J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

Proposition 12 *For any $\omega \in \Omega$ and any two interval forecasts I and J: if $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $J \in \mathscr{C}_{\mathrm{C}}(\omega)$ then $I \cap J \neq \emptyset$, and $I \cap J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

$$\emptyset \neq \bigcap \mathscr{C}_{\mathrm{C}}(\omega) = \left[\underline{p}_{\mathrm{C}}(\omega), \overline{p}_{\mathrm{C}}(\omega)\right].$$



Gambles available to Sceptic: interval forecast



The set filter of forecasts that make a sequence random

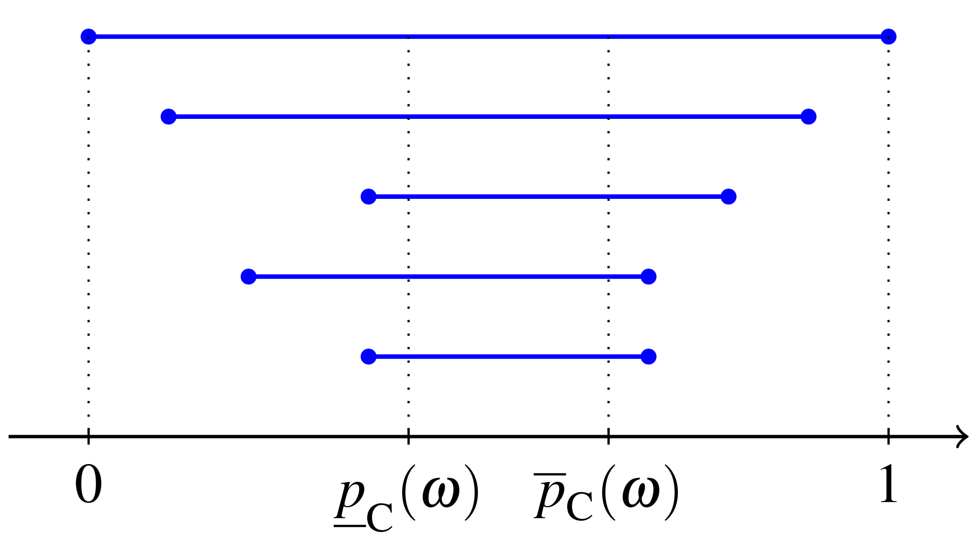
$$\mathscr{C}_{\mathrm{C}}(\omega) := \{I \in \mathscr{C} : \gamma_I \in \Gamma_{\mathrm{C}}(\omega)\} = \{I \in \mathscr{C} : \gamma_I \text{ makes } \omega \text{ computably random}\}.$$

Proposition 9 (Non-emptiness) *For all $\omega \in \Omega$, $[0,1] \in \mathscr{C}_{\mathrm{C}}(\omega)$, so any sequence of outcomes ω has at least one stationary forecast that makes it computably random: $\mathscr{C}_{\mathrm{C}}(\omega) \neq \emptyset$.*

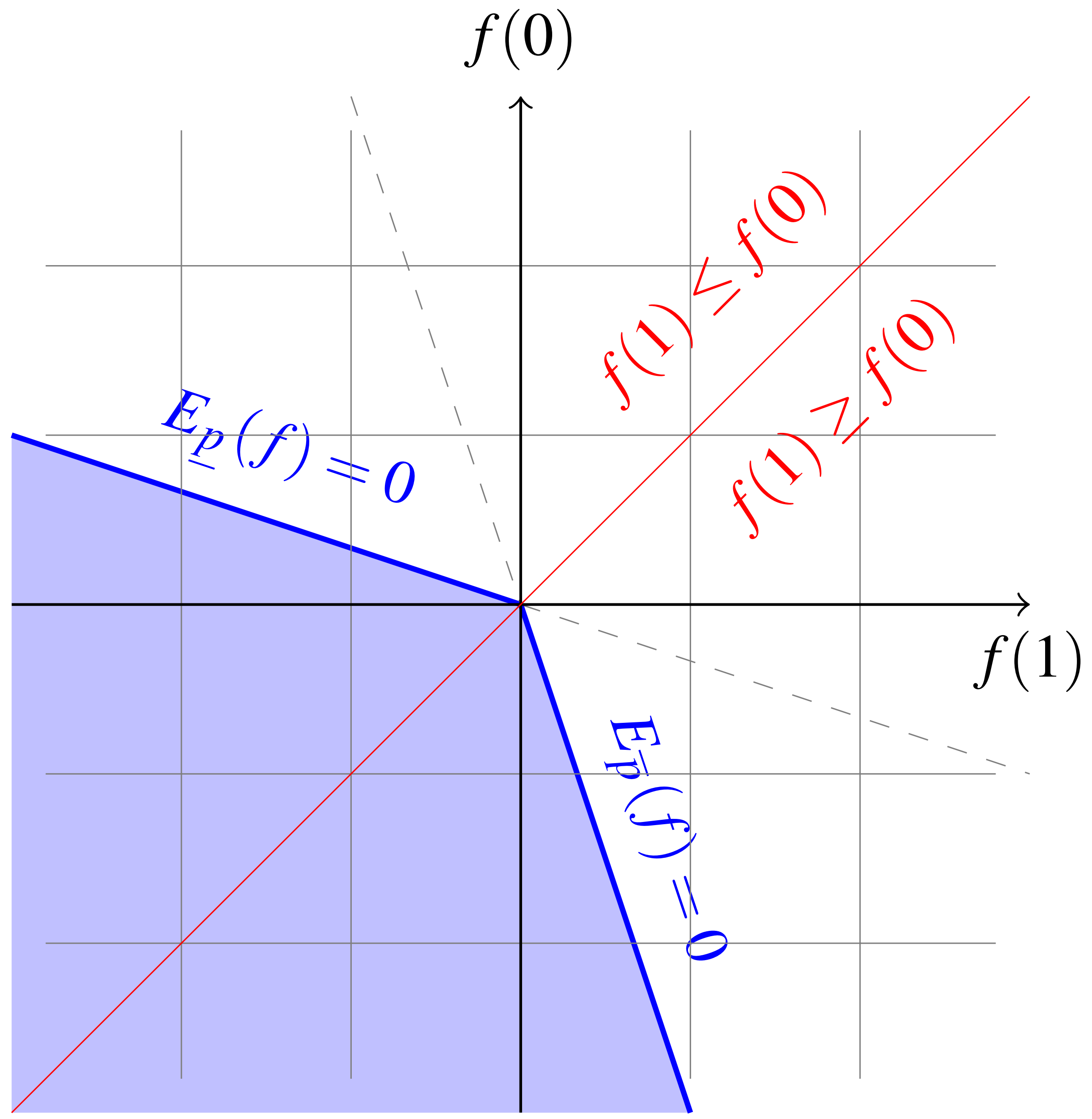
Proposition 10 (Increasingness) *Consider any $\omega \in \Omega$ and any $I, J \in \mathscr{C}$. If $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $I \subseteq J$, then also $J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

Proposition 12 *For any $\omega \in \Omega$ and any two interval forecasts I and J: if $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $J \in \mathscr{C}_{\mathrm{C}}(\omega)$ then $I \cap J \neq \emptyset$, and $I \cap J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

$$\emptyset \neq \bigcap \mathscr{C}_{\mathrm{C}}(\omega) = \left[\underline{p}_{\mathrm{C}}(\omega), \overline{p}_{\mathrm{C}}(\omega)\right].$$



Gambles available to Sceptic: interval forecast



The set filter of forecasts that make a sequence random

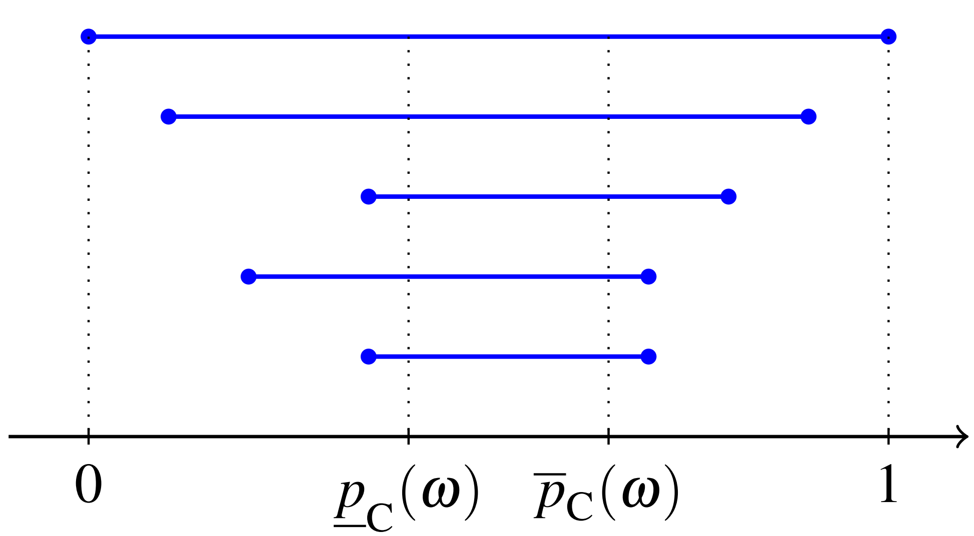
$$\mathscr{C}_{\mathrm{C}}(\omega) := \{I \in \mathscr{C} : \gamma_I \in \Gamma_{\mathrm{C}}(\omega)\} = \{I \in \mathscr{C} : \gamma_I \text{ makes } \omega \text{ computably random}\}.$$

Proposition 9 (Non-emptiness) *For all $\omega \in \Omega$, $[0,1] \in \mathscr{C}_{\mathrm{C}}(\omega)$, so any sequence of outcomes ω has at least one stationary forecast that makes it computably random: $\mathscr{C}_{\mathrm{C}}(\omega) \neq \emptyset$.*

Proposition 10 (Increasingness) *Consider any $\omega \in \Omega$ and any $I, J \in \mathscr{C}$. If $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $I \subseteq J$, then also $J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

Proposition 12 *For any $\omega \in \Omega$ and any two interval forecasts I and J: if $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $J \in \mathscr{C}_{\mathrm{C}}(\omega)$ then $I \cap J \neq \emptyset$, and $I \cap J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

$$\emptyset \neq \bigcap \mathscr{C}_{\mathrm{C}}(\omega) = \left[\underline{p}_{\mathrm{C}}(\omega), \overline{p}_{\mathrm{C}}(\omega)\right].$$



Interval randomness: a simple example

$$\gamma_{p,q}(z_1,\dots,z_n) := \begin{cases} p & \text{if } n \text{ is odd} \\ q & \text{if } n \text{ is even} \end{cases} \quad \text{for all } (z_1,\dots,z_n) \in \Omega^{\Diamond}.$$

Proposition 14 *Consider any* ω *that is computably random for the forecasting system* $\gamma_{p,q}$*. Then for all* $I \in \mathscr{C}$*,* $I \in \mathscr{C}_{\mathrm{C}}(\omega) \Leftrightarrow [p,q] \subseteq I$*.*

Point randomness, but not quite

$$p_n := \frac{1}{2} + (-1)^n \delta_n, \text{ with } \delta_n := e^{-\frac{1}{n+1}} \sqrt{e^{\frac{1}{n+1}} - 1} \text{ for all } n \in \mathbb{N},$$

$$\gamma_{\sim 1/2}(z_1, \dots, z_{n-1}) := p_n \text{ for all } n \in \mathbb{N} \text{ and } (z_1, \dots, z_{n-1}) \in \Omega^{\Diamond}.$$

Proposition 15 *Consider any ω that is computably random for the forecasting system $\gamma_{\sim 1/2}$. Then for all $I \in \mathscr{C}$, $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ if and only if $\min I < 1/2$ and $\max I > 1/2$.*

And where do we go from here?

1. Is it possible to use an equivalent Martin-Löf type approach, using randomness tests?

And where do we go from here?

1. Is it possible to use an equivalent Martin-Löf type approach, using randomness tests?

2. Can we take other notions of computability into account?

And where do we go from here?

1. Is it possible to use an equivalent Martin-Löf type approach, using randomness tests?

2. Can we take other notions of computability into account?

3. Are similar results possible on a prequential approach?

And where do we go from here?

1. Is it possible to use an equivalent Martin-Löf type approach, using randomness tests?

2. Can we take other notions of computability into account?

3. Are similar results possible on a prequential approach?

4. Our results seem to allow for an ontological interpretation of imprecise probabilities: how do we do statistics with them?