Computable randomness is inherently imprecise

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Single forecast game

Forecaster specifies an interval bound $I = [\underline{p}, \overline{p}]$ for the expectation of an unknown outcome X in $\{0,1\}$. $\mathscr{C}$ is the set of all closed intervals I in $[0,1]$.

This interval forecast $I = [\underline{p}, \overline{p}]$ is a commitment for Forecaster to adopt $\underline{p}$ as supremum buying price and $\overline{p}$ as infimum selling price for the gamble X.

The second player, Sceptic, can now in a second step take Forecaster up on any (combination) of the following commitments:

(i) for any $p \in [0,1]$ such that $p \leq \underline{p}$, and any $\alpha \geq 0$, Forecaster must accept the gamble $\alpha[X - p]$;

(ii) for any $q \in [0,1]$ such that $q \geq \overline{p}$, and any $\beta \geq 0$, Forecaster must accept the gamble $\beta[q - X]$.

Finally, the third player, Reality, determines the value x of X in $\{0,1\}$.

Sceptic's uncertain rewards

This leads to an uncertain reward (or capital increase) for Sceptic:

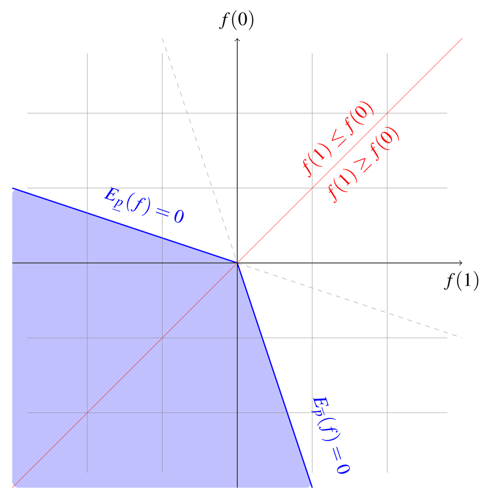
$$\Delta\mathscr{K} = -\alpha[X-p] - \beta[q-X]$$

characterised by

$$\overline{E}_I(\Delta\mathscr{K}) \leq 0$$

where the upper expectation $\overline{E}_I$ is defined by

$$\overline{E}_I(f) = \max_{p \in I} E_p(f)$$



Forecasting systems

We collect all possible outcome sequences $(x_1, x_2, \dots, x_n, \dots)$ in the set $\Omega := \{0,1\}^{\mathbb{N}}$. We collect the finite outcome sequences $(x_1, \dots, x_n)$ in the set $\Omega^\lozenge := \bigcup_{n \in \mathbb{N}_0} \{0,1\}^n$. Finite sequences s in $\Omega^\lozenge$ and infinite sequences ω in Ω are the nodes—called situations—and paths in an event tree with unbounded horizon.

A forecasting system is a map $\gamma\colon \Omega^\lozenge \to \mathscr{C}$, that associates with any situation s in the event tree an interval forecast $\gamma(s) = [\underline{\gamma}(s), \overline{\gamma}(s)] \in \mathscr{C}$. A forecasting system γ is called precise if $\underline{\gamma} = \overline{\gamma}$. Γ denotes the set $\mathscr{C}^{\Omega^\lozenge}$ of all forecasting systems.

Each interval forecast $I_s = \gamma(s)$ corresponds to a local upper expectation $\overline{E}_{I_s}$, with

$$\overline{E}_{\gamma(s)}(f) = \max_{p \in \gamma(s)} E_p(f) = \max_{p \in \gamma(s)} [pf(1) + (1-p)f(0)]$$

so the forecasting system γ turns the event tree into an imprecise probability tree.

Computable randomness

A map $M\colon \Omega^\lozenge \to \mathbb{R}$ is a supermartingale for γ if $\overline{E}_{\gamma(s)}(\Delta M(s)) \leq 0$ for all $s \in \Omega^\lozenge$. In other words, it is a possible capital process for Sceptic.

We call an event $A \subseteq \Omega$ strictly null if there is some non-negative supermartingale T for γ that converges to $+\infty$ on A, meaning that $\lim_{n\to+\infty} T(\omega^n) = +\infty$ for all $\omega \in A$. The complement A^c of a strictly null event A is never empty. A property that holds on A^c is said to hold strictly almost surely, or for strictly almost all outcome sequences.

An outcome sequence ω is computably random for γ if all computable non-negative supermartingales T for γ remain bounded above on ω, meaning that $\sup_{n \in \mathbb{N}} T(\omega^n) < +\infty$.

$$\Gamma_{\mathrm{C}}(\omega) := \{\gamma \in \Gamma\colon \omega \text{ is computably random for } \gamma\}$$

is the set of all forecasting systems for which ω is computably random.

Proposition 1. *All paths are computably random for the vacuous forecasting system: $\gamma_v \in \Gamma_{\mathrm{C}}(\omega)$ for all $\omega \in \Omega$, so $\Gamma_{\mathrm{C}}(\omega)$ is never empty.*

More conservative (or imprecise) forecasting systems have more computably random sequences.

Proposition 2. *Let ω be computably random for a forecasting system γ. Then ω is also computably random for any forecasting system γ^* such that $\gamma \subseteq \gamma^*$, meaning that $\gamma(s) \subseteq \gamma^*(s)$ for all $s \in \Omega^\lozenge$.*

Theorem 3. *Consider any forecasting system γ. Then strictly almost all outcome sequences are computably random for γ in the imprecise probability tree that corresponds to γ.*

Corollary 4. *For any sequence of interval forecasts $(I_1, \dots, I_n, \dots)$ there is a forecasting system given by $\gamma(x_1, \dots, x_n) := I_{n+1}$ for all $(x_1, \dots, x_n) \in \{0,1\}^n$ and all $n \in \mathbb{N}_0$, and associated imprecise probability tree such that strictly almost all—and therefore definitely at least one—outcome sequences are computably random for γ in the associated imprecise probability tree.*

Constant forecasts

The stationary forecasting system γ_I assigns the same interval forecast I to all nodes:

$$\gamma_I(s) := I \text{ for all } s \in \Omega^\lozenge$$

Consider all interval forecasts for which the corresponding stationary forecasting system makes ω computably random:

$$\mathscr{C}_{\mathrm{C}}(\omega) := \{I \in \mathscr{C}\colon \gamma_I \in \Gamma_{\mathrm{C}}(\omega)\}$$

Proposition 5 (Non-emptiness). *For all $\omega \in \Omega$, $[0,1] \in \mathscr{C}_{\mathrm{C}}(\omega)$, so any sequence of outcomes ω has at least one stationary forecast that makes it computably random: $\mathscr{C}_{\mathrm{C}}(\omega) \neq \emptyset$.*

Proposition 6 (Increasingness). *Consider any $\omega \in \Omega$ and any $I, J \in \mathscr{C}$. If $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $I \subseteq J$, then also $J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

Proposition 7 (Closure). *For any $\omega \in \Omega$ and any two interval forecasts I and J: if $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ and $J \in \mathscr{C}_{\mathrm{C}}(\omega)$ then $I \cap J \neq \emptyset$, and $I \cap J \in \mathscr{C}_{\mathrm{C}}(\omega)$.*

Hence, $\mathscr{C}_{\mathrm{C}}(\omega)$ is a set filter, and

$$\bigcap \mathscr{C}_{\mathrm{C}}(\omega) =: [\underline{p}_{\mathrm{C}}(\omega), \overline{p}_{\mathrm{C}}(\omega)]$$

is a non-empty closed interval. Also

$$[0,1] \cap [\underline{p}_{\mathrm{C}}(\omega) - \varepsilon_1, \overline{p}_{\mathrm{C}}(\omega) + \varepsilon_2] \in \mathscr{C}_{\mathrm{C}}(\omega)$$

for all $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$.

Examples

As a first example, fix any $p \leq q$ in $[0,1]$, and the forecasting system $\gamma_{p,q}$ with

$$\gamma_{p,q}(x_1, \dots, x_n) := \begin{cases} p & \text{if } n \text{ is odd} \\ q & \text{if } n \text{ is even} \end{cases}$$

Proposition 9. *Consider any ω that is computably random for the precise forecasting system $\gamma_{p,q}$. Then for all $I \in \mathscr{C}$, $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ if and only if $[p,q] \subseteq I$.*

Hence, $\underline{p}_{\mathrm{C}}(\omega) = p$ and $\overline{p}_{\mathrm{C}}(\omega) = q$.

As a second example, consider the sequence $\{p_n\}_{n \in \mathbb{N}}$ in $[0,1]$ converging to $1/2$:

$$p_n := \frac{1}{2} + (-1)^n \delta_n, \text{ with}$$

$$\delta_n := e^{-\frac{1}{n+1}} \sqrt{e^{\frac{1}{n+1}} - 1} \text{ for all } n \in \mathbb{N}$$

and the forecasting system $\gamma_{\sim 1/2}$ with

$$\gamma_{\sim 1/2}(x_1, \dots, x_{n-1}) := p_n \text{ for all } n \in \mathbb{N}$$

Proposition 10. *Consider any ω that is computably random for the precise forecasting system $\gamma_{\sim 1/2}$. Then for all $I \in \mathscr{C}$, $I \in \mathscr{C}_{\mathrm{C}}(\omega)$ if and only if $\min I < 1/2$ and $\max I > 1/2$.*

Hence $\underline{p}_{\mathrm{C}}(\omega) = \overline{p}_{\mathrm{C}}(\omega) = 1/2$.

Church randomness

Computable randomness implies an intuitive limiting frequencies result:

Theorem 8 (Church randomness). *Consider any outcome sequence $\omega = (x_1, \dots, x_n, \dots)$ in Ω and any stationary interval forecast $I = [\underline{p}, \overline{p}] \in \mathscr{C}_{\mathrm{C}}(\omega)$ that makes ω computably random. Then for any computable selection process $S\colon \Omega^\lozenge \to \{0,1\}$ such that $\sum_{k=0}^{n} S(x_1, \dots, x_k) \to +\infty$:*

$$\underline{p} \leq \liminf_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(x_1, \dots, x_k) x_{k+1}}{\sum_{k=0}^{n-1} S(x_1, \dots, x_k)} \leq \limsup_{n\to+\infty} \frac{\sum_{k=0}^{n-1} S(x_1, \dots, x_k) x_{k+1}}{\sum_{k=0}^{n-1} S(x_1, \dots, x_k)} \leq \overline{p}$$

$\square$ $\gamma(\square) = I_\square$

1 $\gamma(1) = I_1$

11 $\gamma(11) = I_{11}$

10 $\gamma(10) = I_{10}$

0 $\gamma(0) = I_0$

01 $\gamma(01) = I_{01}$

00 $\gamma(00) = I_{00}$

111

110

101

100

011

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001

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