A pointwise ergodic theorem for imprecise Markov chains

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My boon companions

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A discrete-time finite-state uncertain process

Uncertain variables $X_1, X_2, \ldots, X_n, \ldots$ assuming values in some finite set of states $\mathcal{X}$. 
A simple discrete-time finite-state uncertain process

\[ (a, a, a) \rightarrow (a, a, b) \rightarrow (a, b, a) \rightarrow (b, b, a) \rightarrow (b, b, b) \]
A simple discrete-time finite-state uncertain process

situations

(b, b) (b, b, b)
(b, a) (b, b, a) (b, a, b)
(a, b) (b, a, a) (a, b, b)
(a, a) (a, b, a) (a, a, b)
(b) (a, a, a)
A simple discrete-time finite-state uncertain process

A diagram of a simple discrete-time finite-state uncertain process with two states, $a$ and $b$, and transitions labeled $q(\cdot | a)$ and $q(\cdot | b)$.
A simple discrete-time finite-state uncertain process

\[ Q(\cdot | a) \]

- (a, b) \[ Q(\cdot | a, b) \]
  - (a, a) \[ Q(\cdot | a, a) \]
    - (a, a) \[ (a, a,a) \]
  - (a, a) \[ (a, a,a) \]
  - (a, b) \[ (a,b,a) \]
  - (a, b) \[ (a,b,a) \]
- (a, a) \[ (a, a,a) \]

\[ Q(\cdot | b) \]

- (b, b) \[ Q(\cdot | b,b) \]
  - (b, b) \[ (b, b,b) \]
  - (b, a) \[ (b,b,a) \]
- (b, a) \[ (b,a,a) \]
  - (b, a) \[ (b,a,a) \]
  - (b, b) \[ (b,b,a) \]
  - (b, b) \[ (b,b,a) \]
- (b, b) \[ (b, b,b) \]
  - (b, b) \[ (b, b,b) \]
Situations are nodes in the event tree.

\[ \text{situations } s = (x_1, x_2, \ldots, x_n) = x_{1:n} \]

The sample space \( \Omega \) is the set of all paths.

\[ \text{paths } \omega = (x_1, x_2, \ldots, x_n, \ldots) \in \mathcal{X}^\mathbb{N} \]

An event \( A \) is a subset of the sample space \( \Omega \): \( A \subseteq \Omega \).
A real process $\mathcal{F}$ is a real function defined on situations:

$s, a \quad \mathcal{F}(s, a) = -5$

$s, b \quad \mathcal{F}(s, b) = -3$

$s, c \quad \mathcal{F}(s, c) = 0$

$s, d \quad \mathcal{F}(s, d) = 9$

$s, e \quad \mathcal{F}(s, e) = -5$

and its process difference:

$\Delta \mathcal{F}(s) = \mathcal{F}(s \cdot) - \mathcal{F}(s) \in \mathcal{G}(D(s))$ for all situations $s$
Sub- and supermartingales

We can use the local models $Q(\cdot | s)$ to define sub- and supermartingales:

A submartingale $\mathcal{M}$ is a real process such that in all non-terminal situations $s$:

$$Q(\Delta \mathcal{M}(s) | s) \geq 0.$$ 

A supermartingale $\mathcal{M}$ is a real process such that in all non-terminal situations $s$:

$$\overline{Q}(\Delta \mathcal{M}(s) | s) \leq 0.$$
Lower and upper expectations

The most conservative coherent lower and upper expectations on $G(\Omega)$ that coincide with the local models and satisfy a number of additional continuity criteria (cut conglomerability and cut continuity):

Conditional lower expectations:

$$E(f|s) := \sup\left\{ M(s) : \limsup M(s \bullet) \leq f(s \bullet) \right\}$$

Conditional upper expectations:

$$\overline{E}(f|s) := \inf\left\{ \overline{M}(s) : \liminf \overline{M}(s \bullet) \geq f(s \bullet) \right\}$$
Test supermartingales and strictly null events

A test supermartingale $\mathcal{T}$
is a non-negative supermartingale with $\mathcal{T}(\square) = 1$.
(Very close to Ville’s definition of a martingale.)

An event $A$ is strictly null
if there is some test supermartingale $\mathcal{T}$ that converges to $+\infty$ on $A$:

$$\lim \mathcal{T}(\omega) = \lim_{n \to \infty} \mathcal{T}(\omega^n) = +\infty \text{ for all } \omega \in A.$$ 

If $A$ is strictly null then

$$\overline{P}(A) = \overline{E}(\mathbb{1}_A) = \inf\{\overline{M}(\square) : \liminf \overline{M} \geq \mathbb{1}_A\} = 0.$$
Consider any submartingale $\underline{M}$ such that its difference process
\[
\Delta \underline{M}(s) = \underline{M}(s \cdot) - \underline{M}(s) \in \mathcal{G}(D(s)) \text{ for all non-terminal } s
\]
is uniformly bounded. Then $\liminf \langle \underline{M} \rangle \geq 0$ strictly almost surely, or in other words
\[
\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta \underline{M}(X_1, \ldots, X_{k-1})(X_k) = \liminf_{n \to +\infty} \frac{1}{n} [\underline{M}(X_1, \ldots, X_n) - \underline{M}(\square)] \geq 0
\]
In particular, for any real function $f$ on $\mathcal{X}$:
\[
\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} [f(X_k) - \underline{Q}(f(X_k|X_{1:k-1})] \geq 0 \text{ strictly almost surely}
\]
Consider any submartingale \( \mathcal{M} \) such that its difference process
\[
\Delta \mathcal{M}(s) = \mathcal{M}(s \cdot) - \mathcal{M}(s) \in \mathcal{G}(D(s))
\]
for all non-terminal \( s \) is uniformly bounded. Then \( \liminf \langle \mathcal{M} \rangle \geq 0 \) strictly almost surely, or in other words
\[
\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \Delta \mathcal{M}(X_1, \ldots, X_{k-1})(X_k) = \liminf_{n \to +\infty} \frac{1}{n} \left[ \mathcal{M}(X_1, \ldots, X_n) - \mathcal{M}(\Box) \right] \geq 0
\]
In particular, for any real function \( f \) on \( \mathcal{X} \):
\[
\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \geq Q(f) \text{ strictly almost surely}
\]
Imprecise Markov chains
IMPRECISE MARKOV CHAINS AND THEIR LIMIT BEHAVIOR

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When the initial and transition probabilities of a finite Markov chain in discrete time are not well known, we should perform a sensitivity analysis. This can be done by considering as basic uncertainty models the so-called credal sets that these probabilities are known or believed to belong to and by allowing the probabilities to vary over such sets. This leads to the definition of an imprecise Markov chain. We show that
the time evolution of such a system can be studied very efficiently using so-called lower and upper expectations, which are equivalent mathematical representations of credal sets. We also study how the inferred credal set about the state at time $n$ evolves as $n \to \infty$: under quite unrestrictive conditions, it converges to a uniquely invariant credal set, regardless of the credal set given for the initial state. This leads to a non-trivial generalization of the classical Perron–Frobenius theorem to imprecise Markov chains.

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  author = {{d}e Cooman, Gert and Hermans, Filip and Quaegehebeur, Erik},
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A simple discrete-time finite-state stochastic process
An imprecise IID model
An imprecise Markov chain

\[ Q(\cdot \mid □) \]

\[ Q(\cdot \mid a) \]

\[ Q(\cdot \mid b) \]
Stationarity and ergodicity

The lower expectation $E_n$ for the state $X_n$ at time $n$:

$$E_n(f) = E(f(X_n))$$

The imprecise Markov chain is Perron–Frobenius-like if for all marginal models $E_1$ and all $f$:

$$E_n(f) \to E_\infty(f).$$

and if $E_1 = E_\infty$ then $E_n = E_\infty$, and the imprecise Markov chain is stationary.

In any Perron–Frobenius-like imprecise Markov chain:

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} E_n(f) = E_\infty(f)$$

and

$$E_\infty(f) \leq \liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \limsup_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} f(X_k) \leq \overline{E}_\infty(f) \text{ str. almost surely.}$$
The essence of the argument

From

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \left[ f(X_k) - Q(f(X_k) | X_{k-1}) \right] \geq 0 \text{ strictly almost surely}$$

to

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} \left[ f(X_k) - E_\infty(f) \right] \geq 0 \text{ strictly almost surely}$$