A pointwise ergodic theorem for imprecise Markov chains

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Imprecise probability tree
With a sequence of random variables $X_1, \ldots, X_n, \ldots$ (state variables) assuming values in a finite state set $\mathcal{X}$, there corresponds an event tree with nodes (situations) $s = x_{1:k} = (x_1, \ldots, x_k) \in \mathcal{X}^k, k \in \mathbb{N}$. A path is an infinite sequence of states $\omega = (x_1, x_2, \ldots) \in \Omega$. We get an imprecise probability tree when we add local uncertainty models: in each situation $s = x_{1:k}$, a (coherent) lower expectation $Q(s)$ on $\mathcal{Y}(s)$ for the next random variable $X_{k+1}$. A process $\mathcal{Y}(s)$ is a function on situations $s$. A variable $f(\omega)$ is a function on paths $\omega$.

Global uncertainty models
A submartingale $\mathcal{M}$ is a real process, for whose process difference $\Delta \mathcal{M}(s) = \mathcal{M}(s) - \mathcal{M}(s) \in \mathcal{Y}(s)$ we have that $Q(\Delta \mathcal{M}(s)) \geq 0$ for all situations $s$. For the global uncertainty models on $\Omega$, we have the (Ville–Shapely–Vrott) formula: $E(f) := \sup(\mathcal{M}(s) \colon \limsup \mathcal{M}(s) \leq f(\omega))$.

Law of large numbers
With a real process $\mathcal{F}$ we associate its path average $\langle \mathcal{F} \rangle$, a real process defined in all situations $s = x_{1:k}$ by: 
$$\langle \mathcal{F} \rangle(s) = \left\{ \begin{array}{ll} \frac{1}{n} \sum_{i=1}^{n} \mathcal{F}(x_{i+1}) & \text{if } n > 0, \\ 0 & \text{if } n = 0. \end{array} \right.$$ 

Strong law of large numbers for submartingale differences
Let $\mathcal{M}$ be a submartingale such that $\Delta \mathcal{M}$ is uniformly bounded. Then $\limsup \langle \mathcal{M} \rangle \geq 0$ (strictly) almost surely.

Imprecise Markov chain
An imprecise probability tree is an imprecise Markov chain when its local models only depend on the last state: $Q(s_{1:k}) = Q(s_k)$ (Markov condition). Its local models are completely specified by fixing the initial model $Q(s_1) = Q(\omega)$ for all $s_1 \in \mathcal{Y}(s_1)$ and the lower transition operator $\mathbb{P}_L: \mathcal{Y}(X) \rightarrow \mathcal{Y}(X)$ defined by $\mathbb{P}_L(g(s)) = Q(s_1 = s_1)g(s)$ for all $s \in \mathcal{X}$ and $g \in \mathcal{Y}(X)$.

Perron–Frobenius
A lower transition operator $\mathbb{P}_L$ is called Perron–Frobenius-like if for all $g \in \mathcal{Y}(X)$ 
$$\mathbb{P}_L g = \text{ some constant } \mathbb{P}_L g(s) \in \mathbb{R}. $$

Any lower transition operator $\mathbb{P}_L$ has a coefficient of ergodicity $\rho(\mathbb{P}_L) := \max_{g \in \mathcal{Y}(X), \|g\| = 1} \|\mathbb{P}_L g\| = \min_{\mathbb{P}_L g \neq 0} \|\mathbb{P}_L g\|$. We know from work by De Cooman–Hermans (2009, 2012) and Škulj–Hable (2013) that:

Theorem
For any lower transition operator $\mathbb{P}_L$ the following are equivalent:
(i) $\mathbb{P}_L$ is Perron–Frobenius-like;
(ii) there is some lower expectation $\mathbb{E}_L$ such that for any initial $\mathbb{E}_L$ and all $g \in \mathcal{Y}(X)$: $E_L(g(s)) = E_L(\mathbb{P}_L g(s))$;
(iii) $\rho(\mathbb{P}_L) < 1$ for some $s \in \mathbb{N}$.
In that case $\mathbb{E}_L = \mathbb{E}_L$ is uniquely $\mathbb{P}_L$-invariant.

Ergodic theorem
Consider an imprecise Markov chain with initial model $\mathbb{E}_L$ and lower transition operator $\mathbb{P}_L$.

Pointwise ergodic theorem
If $\mathbb{P}_L$ is Perron–Frobenius-like, with $\mathbb{P}_L$-invariant lower expectation $\mathbb{E}_L$, then for all $f \in \mathcal{Y}(X)$:
$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) \geq \mathbb{E}_L(f)$$ strictly almost surely.

Convergence results
In an imprecise Markov chain with initial model $\mathbb{E}_L$ and Perron–Frobenius-like lower transition operator $\mathbb{P}_L$:
$$\mathbb{E}_L(g(s)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^{n} g(x_i) = \lim \frac{1}{n} \sum_{i=1}^{n} \mathbb{P}_L(g(x_i)) = \lim \frac{1}{n} \sum_{i=1}^{n} E_L(g)$$ for any $g \in \mathcal{Y}(X)$.