A pointwise ergodic theorem for imprecise Markov chains

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Imprecise probability tree

With a sequence of random variables (state variables) X_1, \ldots, X_n, \ldots assuming values in a finite state set \mathscr{X} , there corresponds an event tree with nodes (situations)

 $s = x_{1:k} = (x_1, \dots, x_k) \in \mathscr{X}^k \qquad k \in \mathbb{N}.$ A path is an infinite sequence of states

 $\boldsymbol{\omega} = (x_1,\ldots,x_n,\ldots) \in \Omega.$

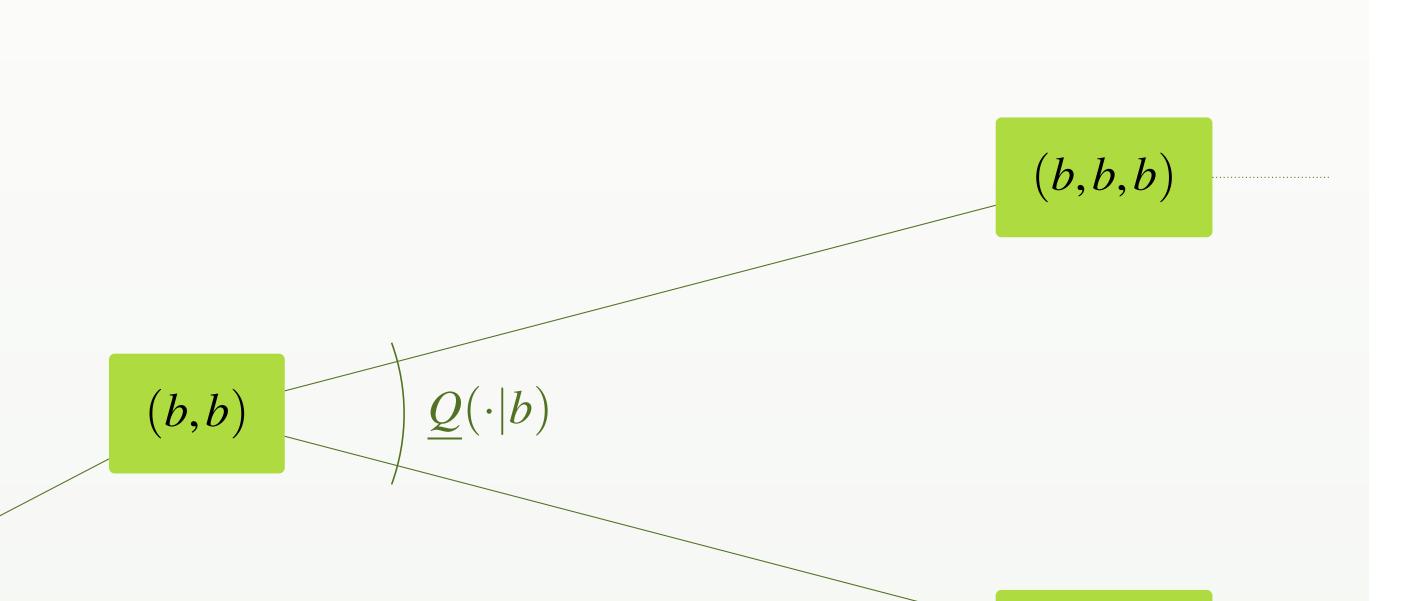
We get an imprecise probability tree when we add local uncertainty models: in each situa-

Global uncertainty models

A submartingale \mathcal{M} is a real process, for whose process difference

 $\Delta \mathscr{M}(s) = \mathscr{M}(s \cdot) - \mathscr{M}(s) \in \mathscr{G}(\mathscr{X})$ we have that

 $Q(\Delta \mathcal{M}(s)|s) \geq 0$ for all situations *s*. For the global uncertainty models on Ω , we have the (Ville–Shafer–Vovk) formula: $\underline{E}(f|s) \coloneqq \sup\{\mathscr{M}(s) \colon \limsup \mathscr{M}(s \bullet) \le f(s \bullet)\}.$

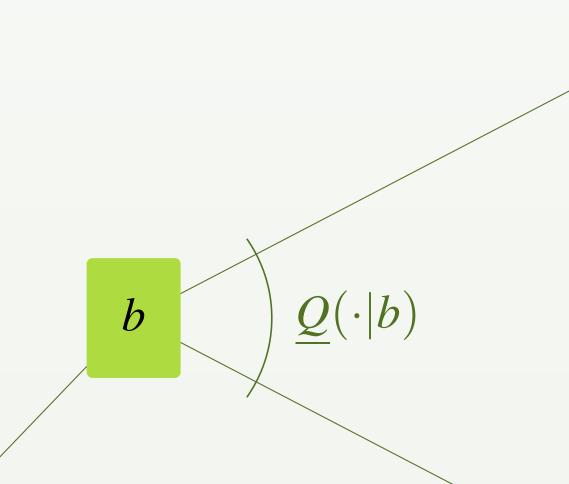


tion $s = x_{1:n}$ a (coherent) lower expectation

 $Q(\cdot|s)$ on $\mathscr{G}(\mathscr{X})$ for the next random variable X_{n+1} .

A process $\mathscr{F}(s)$ is a function on situations s.

A variable $f(\omega)$ is a function on paths ω .



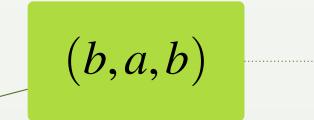
Law of large numbers

With a real process \mathscr{F} we associate its path average $\langle \mathscr{F} \rangle$, a real process defined in all situations $s = x_{1:n}$ by:

 $\langle \mathscr{F} \rangle(x_{1:n}) \coloneqq \begin{cases} \frac{1}{n} \sum_{k=0}^{n-1} [\mathscr{F}(x_{1:k+1}) - \mathscr{F}(x_{1:k})] & n > 0\\ 0 & n = 0. \end{cases}$

Strong law of large numbers for submartingale differences Let \mathcal{M} be a submartingale such that $\Delta \mathcal{M}$ is uniformly bounded. Then $liminf\langle \mathcal{M} \rangle \geq 0$ (strictly) almost surely.

 $Q(\cdot|a)$



(b,b,a)

Imprecise Markov chain

An imprecise probability tree is an imprecise Markov chain when its local models only depend on the last state:

> $Q(\cdot|x_{1:n}) = Q(\cdot|x_n)$ (Markov condition)

The VSV-formulas for the global models then lead in particular to the following expressions for the lower expectation

(b,a)

$\underline{E}_n(g) \coloneqq \underline{E}(g(X_n))$

 $Q(\cdot | \Box)$

Its local models are completely specified by fixing the initial model

 $\underline{E}_1(g) \coloneqq Q(g|\Box)$ for all $g \in \mathscr{G}(\mathscr{X})$ and the lower transition operator $\underline{T}: \mathscr{G}(\mathscr{X}) \to \mathscr{G}(\mathscr{X})$ defined by

 $\underline{\mathrm{T}}g(x) \coloneqq Q(g|x)$ for all $x \in \mathscr{X}$ and $g \in \mathscr{G}(\mathscr{X})$.

of real functions $g(X_n)$ of the state X_n at time n:

 $\underline{E}_n(g) = \underline{E}_1(\underline{T}^{n-1}g)$ or equivalently $\underline{E}_n = \underline{E}_1 \circ \underline{T}^{n-1}$.

We call the lower expectation $\underline{E}_1 \ \underline{T}$ -invariant when

(dual eigenvector with eigenvalue 1) $\underline{E}_1 = \underline{E}_1 \circ \underline{T}$

and then $\underline{E}_n = \underline{E}_1$ for all times $n \in \mathbb{N}$.

(a,b,a)

(a,a,b)

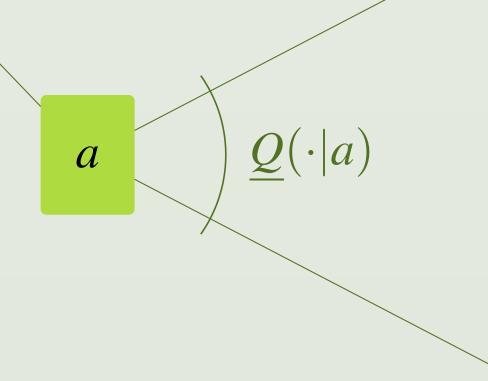
(b,a,a)

$\underline{Q}(\cdot|b)$ (a,b)

Perron–Frobenius

A lower transition operator \underline{T} is called Perron– Frobenius-like if for all $g \in \mathscr{G}(\mathscr{X})$

 $\underline{\mathrm{T}}^{n}g \rightarrow \text{some constant } \underline{E}_{\mathrm{PF}}(g) \in \mathbb{R}.$ Any lower transition operator \underline{T} has a coefficient of ergodicity



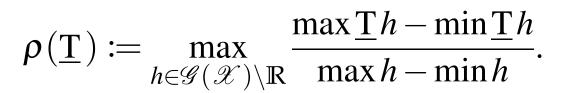
Ergodic theorem

Consider an imprecise Markov chain with initial model \underline{E}_1 and lower transition operator \underline{T} .

Pointwise ergodic theorem

If <u>T</u> is Perron–Frobenius-like, with <u>T</u>-invariant lower expectation \underline{E}_{∞} , then for all $f \in \mathscr{G}(\mathscr{X})$:

 $\liminf_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}f(X_k)\geq \underline{E}_{\infty}(f) \text{ strictly almost surely.}$



We know from work by De Cooman–Hermans (2009, 2012) and Škulj–Hable (2013) that:

Theorem

For any lower transition operator \underline{T} , the following are equivalent:

(i) \underline{T} is Perron–Frobenius-like;

(ii) there is some lower expectation \underline{E}_{∞} such that for any initial \underline{E}_1 and all $g \in \mathscr{G}(\mathscr{X})$: $\underline{E}_{n+1}(g) = \underline{E}_1(\underline{T}^n g) \to \underline{E}_{\infty}(g);$ (iii) $\rho(\underline{T}^r) < 1$ for some $r \in \mathbb{N}$.

In that case $\underline{E}_{PF} = \underline{E}_{\infty}$ is uniquely <u>T</u>-invariant.

Convergence results

In an imprecise Markov chain with initial model \underline{E}_1 and Perron–Frobenius-like lower transition operator \underline{T} :

 $\underline{E}_{\infty}(g) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \underline{T}^{n} g(X_{k})$ $= \lim_{n \to \infty} \frac{1}{n} \sum_{\ell=1}^{n} \underline{\mathrm{T}}^{\ell} g(X_n)$ $= \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \underline{E}_{k}(g) \text{ for any } g \in \mathscr{G}(\mathscr{X}).$

