A pointwise ergodic theorem for imprecise Markov chains

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Abstract
Ergodicity theorem. Consider a Markov chain, with a finite state space $\mathcal{X}$. For such a system, we have proved a new Perron-Frobenius-like theorem. They provide necessary and sufficient conditions for the uncertainty model about the state $x_1$ to converge, as $n \to \infty$, to an uncertainty model independent of the initial state $x_0$. In Markov chains with precise probabilities, this convergence is sufficient for a pointwise ergodic theorem to hold:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_k(x_0) = E_0(x_0)$$

Result. Applying the theory of imprecise probabilities to stochastic processes, we can define so-called imprecise Markov chains as special cases of imprecise probability trees. We introduce and study submartingales and supermartingales in such trees, for which we are able to prove a strong law of large numbers for submartingale differences. Combining this result with the Perron-Frobenius-like character of our model we can prove the following theorems:

\begin{align*}
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(X_k) &\leq \mathbb{E}_0(f) \text{ almost surely,} \\
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_k(X_k) &\leq \mathbb{E}_0(E(X_0)) \text{ almost surely.}
\end{align*}

Imprecise probabilities

They have a subject about the approximates the value of a variable $X$ assumes in a finite set of possible values $\mathcal{X}$. His uncertainty is modelled by a lower bound $E_k(x)$ which is a real functional on the set $\mathcal{F}^{(x)}$ of all real-valued functions (gambles) $f : \mathcal{X} \to \mathbb{R}$, satisfying the following basic coherency axioms:

1. $E_k(x) \geq \inf \{ f(x) : f \in \mathcal{F}^{(x)} \}$
2. $E_k(x) \geq E_k(x) + \mathbb{E}E_k(x) + \mathbb{E}E_k(x)$
3. $E_k(x) = \mathbb{E}E_k(x) + \mathbb{E}E_k(x)$
4. $\mathbb{E}E_k(x) \geq \mathbb{E}E_k(x)$

The conjugate upper expectation $\mathbb{E}(f)$ is defined by $\mathbb{E}(f) = \inf \{ f(x) : f \in \mathcal{F}^{(x)} \}$ and follows from the coherence axioms 1-3 that:

$$\lim_{n \to \infty} \mathbb{E}_n(f) = \mathbb{E}(f) \text{ almost surely}$$

We denote the set of all submartingales for a given imprecise probability tree by $\mathcal{M}$, and $\mathcal{G}^+$ is the set of all real-valued functions (gambles) $f : \mathcal{X} \to \mathbb{R}$.

Theorem 1. Let $M$ be a supermartingale that is bounded below. Then $M$ converges almost surely to a real variable.

The intuition behind it is that there exists a test supermartingale which is bounded above. The process diverges. We were able to derive the following useful theorem:

Theorem 2 (Strong law of large numbers for imprecise Markov chains). Let $M$ be a supermartingale such that $M$ is uniformly bounded. Then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} E_k(X_k) = \mathbb{E}_0(E(X_0))$$

for some $x_0 \in \mathcal{X}$.

Theorem 3. Let $M$ be a supermartingale that is bounded below. Then $M$ converges almost surely to a real variable.

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