A COMPARATIVE STUDY OF THE SMALLEST PROBABILITY INTERVALS FOR WHICH A BINARY SEQUENCE IS RANDOM

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ABSTRACT. There are many randomness notions. Classically, many of them are about whether a given infinite binary sequence is random for some given probability, meaning that it is random for the i.i.d. process that assigns this probability to the outcome 1. In that case, if a sequence is random according to several randomness notions, then the probability for which it is random is the same for all these notions. Comparing randomness notions then amounts to finding out according to which of them a given sequence is random. This changes dramatically when we consider randomness for probability intervals, because here, a sequence is always random for at least one such interval, so the question is not if, but rather for which intervals, a sequence is random. We show that for many randomness notions thus generalised, every sequence has a smallest interval for which it is (almost) random. We study such smallest intervals and then use them as a way to compare the corresponding randomness notions. We establish conditions under which such smallest intervals coincide, and provide examples where they do not.

1. INTRODUCTION

The field of algorithmic randomness studies what it means for an infinite binary sequence, such as $\omega = 0100110100\ldots$, to be random for an uncertainty model. Classically, this uncertainty model is often (the i.i.d. process that corresponds to) a single (so-called *precise*) probability $p \in [0,1]$. Some of the best-studied precise randomness notions are Martin-Löf randomness, computable randomness, Schnorr randomness and Church randomness. They are increasingly weaker; for example, if a sequence ω is Martin-Löf random for a probability p, then it is also computably random, Schnorr random and Church random for p. Meanwhile, they do not coincide; it is for example possible that a sequence ω is Church random but not computably random for $1/2$. From a traditional perspective, this is how we can typically differentiate between various randomness notions [1–4].

As shown by De Cooman and De Bock [5–7], these now more or less traditional randomness notions can be generalised by allowing for imprecise-probabilistic uncertainty models, such as closed probability intervals $I \subseteq [0,1]$.[1] These more general randomness notions, and their corresponding properties, allow for more detail to emerge in their comparison. Indeed, every infinite binary sequence ω is for example random for at least one closed probability interval (and then also for all the ones it is included in). And for the imprecise generalisations of many of the aforementioned precise randomness notions, we will see further on that for every (or in some cases many) ω, there is some smallest probability interval for which ω is (almost) random—we will explain the modifier 'almost' further on. It is these *smallest* probability intervals that we will use to compare a number of different randomness notions.

Our endeavour to associate a unique (smallest) probability interval with an infinite binary sequence can be seen as a continuation of work by Richard von Mises [11], and Pablo

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[1] We have discussed in a recent paper [8] how this approach can be connected mathematically to earlier work on uniform randomness by Levin [9, 10].

Fierens, Terrence Fine and Adrian Papamarcou [12, 13], who, on the one hand, aimed at providing a frequentist interpretation for (imprecise) probabilities by identifying them with relative frequencies along infinite sequences of zeroes and ones, and, on the other hand, tried to establish methods for learning such uncertainty models from data. In this paper, we want to provide new insights into the first aspiration from the vantage point of algorithmic randomness.

We will focus on the following three questions: (i) when is there a well-defined smallest probability interval for which an infinite binary sequence ω is (almost) random; (ii) are there alternative expressions for these smallest intervals; and (iii) for a given sequence ω, how do these smallest intervals compare for different randomness notions? For all randomness notions that we will consider, except for Martin-Löf randomness, we will answer the first question conclusively by associating a unique probability interval with every infinite sequence. Once these smallest probability intervals are defined, the second and third questions explore, respectively, what they look like, and how robust they are with respect to the adopted randomness notion. We will derive conditions for such smallest probability intervals to not depend on the adopted randomness notion, but we will also provide examples where such smallest intervals do not coincide. Thus, by looking at algorithmic randomness from an imprecise probabilities perspective, we will be able to do more than merely confirm the known differences between several randomness notions. Extending existing randomness notions to allow for probability intervals will also allow us to explore *to what extent* these randomness notions are different, in the sense that we can compare the smallest probability intervals for which infinite binary sequences are (almost) random.

Our contribution is structured as follows.

In Section 2, we introduce (non-)stationary (im)precise uncertainty models for infinite binary sequences—so-called forecasting systems—that associate with every finite binary sequence a possibly different probability interval; a probability interval thus corresponds to a stationary forecasting system. We also introduce a generic definition of randomness that allows us to formally define what it means for a sequence to have a smallest interval for which it is (almost) random.

In Section 3, we provide the mathematical background on supermartingales that we need in order to introduce a number—six in all—of randomness notions in Section 4: (weak) Martin-Löf randomness, computable randomness, Schnorr randomness, and (weak) Church randomness. The material from computability theory that we need to define these randomness notions, is summarised in Appendix A.

In the subsequent sections, we tackle our three main questions (i)–(iii).

We study the existence of the smallest intervals for which an infinite binary sequence ω is (almost) random in Section 5.

In Section 6, we provide outer and inner bounds for these smallest intervals. When these bounds coincide, which we will show to be the case if ω is random for a computable precise forecasting system, we obtain alternative expressions for such smallest intervals. We also provide examples that indicate that our bounds are not tight in general.

In Section 7, we compare these smallest intervals for various randomness notions. Again, if ω is random for a computable precise forecasting system, we show that such smallest intervals are reasonably robust with respect to the adopted notion of randomness in the sense that many of them coincide. If ω is (only) random for a forecasting system that is not computable or is not precise, our approach allows for a better differentiation between several randomness notions. We show how this can be achieved by providing examples where the smallest intervals do not coincide.

All novel results—that is, the ones that are not merely borrowed from other publications—are provided with a proof. Shorter proofs (that don't make use of computability theory) usually appear directly after the result. For narrative clarity, the ones that do make use of computability theory are relegated to Appendix B.

2. Forecasting systems and randomness

Consider an infinite sequence of binary variables $X_1, \dots, X_n, \dots$, where each variable X_n takes values in the binary *sample space* $\mathscr{X} := \{0,1\}$; we generically denote such values by x_n. We are interested in the corresponding infinite outcome sequences $(x_1, \dots, x_n, \dots)$, and, in particular, in whether they are random. We denote such a sequence generically by ω and call it a *path*. All such paths are collected in the set $\Omega := \mathscr{X}^{\mathbb{N}}$.[2] For any path $\omega = (x_1, \dots, x_n, \dots) \in \Omega$, we let $\omega_{1:n} := (x_1, \dots, x_n)$ and $\omega_n := x_n$ for all $n \in \mathbb{N}$. For $n = 0$, the empty sequence $\omega_{1:0} := \omega_0 := ()$ is called the *initial situation*, and we also denote it by $\square$. For any $n \in \mathbb{N}_0$, a finite outcome sequence $(x_1, \dots, x_n) \in \mathscr{X}^n$ is called a *situation*, also generically denoted by s, and its length is then denoted by $|s| := n$. All situations are collected in the set $\mathbb{S} := \bigcup_{n \in \mathbb{N}_0} \mathscr{X}^n$. For any $s = (x_1, \dots, x_n) \in \mathbb{S}$ and $x \in \mathscr{X}$, we use sx to denote the concatenation $(x_1, \dots, x_n, x)$.

For any situation $s \in \mathbb{S}$ and any path $\omega \in \Omega$, we write $s \sqsubseteq \omega$ when $\omega_{1:|s|} = s$, or in other words when ω extends s, and we then say that the path ω *goes through* the situation s. For any two situations $s, t \in \mathbb{S}$, we write $s \sqsubseteq t$ when every path that goes through t also goes through s, and we then say that the situation s *precedes* the situation t; so s is a precursor of t. We say that s *strictly precedes* t, and write $s \sqsubset t$, when $s \sqsubseteq t$ and $s \neq t$. For any situation $s \in \mathbb{S}$, we denote the set $\{t \in \mathbb{S} \colon s \sqsubseteq t\}$ of all situations that extend s by $[s]$, and the set $\{\omega \in \Omega \colon s \sqsubseteq \omega\}$ of all paths that go through s by $[\![s]\!]$.

The randomness of a path $\omega \in \Omega$ is always defined with respect to some uncertainty model. Classically, this uncertainty model is often simply a real number $p \in [0,1]$, interpreted as the probability that X_n equals 1, for any $n \in \mathbb{N}$. As explained in the Introduction, we can generalise this by considering a closed probability interval $I \subseteq [0,1]$ instead.[4] Such more general uncertainty models will be called *interval forecasts*, and we collect all such closed probability intervals in the set $\mathcal{I}$. Another generalisation of the classical approach consists in allowing for non-stationary probabilities that X_n equals 1, which may depend on the already observed outcomes $s = (x_1, \dots, x_{n-1})$ or only on $|s| = n$. Each of these generalisations can themselves be seen as special cases of an even more general approach, which consists in providing every situation $s \in \mathbb{S}$ with a (possibly different) interval forecast in $\mathcal{I}$, denoted by $\varphi(s)$. This interval forecast $\varphi(s) \in \mathcal{I}$ then describes the uncertainty about the *a priori* unknown outcome of $X_{|s|+1}$, given that the situation s has been observed. We call such general uncertainty models *forecasting systems*.

Definition 1. A *forecasting system* is a map $\varphi \colon \mathbb{S} \to \mathcal{I}$ that associates an interval forecast $\varphi(s) \in \mathcal{I}$ with every situation $s \in \mathbb{S}$. We denote by Φ the set of all forecasting systems.

With any forecasting system $\varphi \in \Phi$, we associate two real processes $\underline{\varphi}$ and $\overline{\varphi}$, defined by $\underline{\varphi}(s) := \min \varphi(s)$ and $\overline{\varphi}(s) := \max \varphi(s)$, for all $s \in \mathbb{S}$. A forecasting system $\varphi \in \Phi$ is called *precise* if $\underline{\varphi} = \overline{\varphi}$. A forecasting system $\varphi \in \Phi$ is called *stationary* if there is some interval forecast $I \in \mathcal{I}$ such that $\varphi(s) = I$ for all $s \in \mathbb{S}$; for ease of notation, we will then denote this forecasting system simply by I; the case of a single probability p corresponds to a stationary forecasting system with $I = \{p\}$. A forecasting system $\varphi \in \Phi$ is called *temporal* if its interval forecasts $\varphi(s)$ only depend on the situations $s \in \mathbb{S}$ through their length $|s|$, meaning that $\varphi(s) = \varphi(t)$ for any two situations $s, t \in \mathbb{S}$ that have the same length $|s| = |t|$. Allowing ourselves a slight abuse of notation, we will also consider a temporal forecasting system $\varphi \colon \mathbb{S} \to \mathcal{I}$ to be a map from the non-negative integers to the set of interval forecasts $\mathcal{I}$, thus enabling us to write $\varphi(n)$ instead of $\varphi(s)$ for all $n \in \mathbb{N}_0$ and $s \in \mathbb{S}$ with $|s| = n$.

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[2] $\mathbb{N}$ denotes the set of all natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ the set of all non-negative integers.[3]

[3] A real number $x \in \mathbb{R}$ is called negative, positive, non-negative and non-positive, respectively, if $x < 0$, $x > 0$, $x \geq 0$ and $x \leq 0$.

[4] We will make no distinction between a precise forecast $p \in [0,1]$ and the corresponding (degenerate) interval forecast $\{p\} = [p,p] \in \mathcal{I}$. This will allow us to treat precise forecasts as a special case of interval forecasts.

In some of our results, we will consider forecasting systems that are computable. The following intuitive description will suffice to be able to follow the argumentation and understand our results: a forecasting system $\varphi \in \Phi$ is *computable* if there is some (necessarily finite) algorithm that, given any $s \in \mathbb{S}$ and any $N \in \mathbb{N}_0$ as input, can compute the real numbers $\underline{\varphi}(s)$ and $\overline{\varphi}(s)$ to within a precision of 2^{-N}. For a formal definition of computability, which we use in our proofs, we refer the reader to Appendix A.

So what does it mean for a path $\omega \in \Omega$ to be random for a forecasting system $\varphi \in \Phi$? Since there are many different definitions of randomness, and since we intend to compare them, we now introduce a general abstract definition and a number of *natural* properties of such randomness notions that will allow us to do so.

Definition 2. A notion of *randomness* R associates with every forecasting system $\varphi \in \Phi$ a set of paths $\Omega_{\mathrm{R}}(\varphi)$. A path $\omega \in \Omega$ is called R-*random* for φ if $\omega \in \Omega_{\mathrm{R}}(\varphi)$. In particular, for any $I \in \mathcal{I}$, we will also call any path $\omega \in \Omega_{\mathrm{R}}(I)$ R-*random* for the interval forecast I.

The randomness notions we will be considering further on satisfy additional properties. The first one is a monotonicity property, which we can describe as follows: if a path $\omega \in \Omega$ is R-random for a forecasting system $\varphi \in \Phi$, it is also R-random for any forecasting system $\varphi' \in \Phi$ that is *less precise*, or *more conservative*, in the sense that $\varphi \subseteq \varphi'$, meaning that $\varphi(s) \subseteq \varphi'(s)$ for all $s \in \mathbb{S}$. Consequently, monotonicity requires that the more precise a forecasting system is, the fewer R-random paths it has.

Property 1. *If $\varphi \subseteq \varphi'$ then also $\Omega_{\mathrm{R}}(\varphi) \subseteq \Omega_{\mathrm{R}}(\varphi')$, for any forecasting systems $\varphi, \varphi' \in \Phi$.*

It will also prove useful to consider the property stating that all paths $\omega \in \Omega$ are R-random for the (maximally imprecise) so-called *vacuous forecasting system* $\varphi_v \in \Phi$, which is the stationary forecasting system defined by $\varphi_v(s) := [0,1]$ for all $s \in \mathbb{S}$.

Property 2. $\Omega_{\mathrm{R}}([0,1]) = \Omega$.

Thus, concentrating on stationary forecasting systems in particular, if Properties 1 and 2 hold for a randomness notion R, every path $\omega \in \Omega$ will in particular be R-random for at least one interval forecast—the vacuous forecast $I = [0,1]$—and if a path $\omega \in \Omega$ is R-random for an interval forecast $I \in \mathcal{I}$, then it will also be R-random for any interval forecast $I' \in \mathcal{I}$ for which $I \subseteq I'$.

It is therefore natural to wonder whether every path $\omega \in \Omega$ has some *smallest* interval forecast I such that $\omega \in \Omega_{\mathrm{R}}(I)$. As a step towards answering this question, we consider yet another property that seems natural to require of a notion of randomness: if a path $\omega \in \Omega$ is R-random for two interval forecasts $I, I' \in \mathcal{I}$, then it should also be R-random for the intersection $I \cap I'$.

Property 3. *For any two interval forecasts $I, I' \in \mathcal{I}$ and any path $\omega \in \Omega$, if $\omega \in \Omega_{\mathrm{R}}(I) \cap \Omega_{\mathrm{R}}(I')$ then $\omega \in \Omega_{\mathrm{R}}(I \cap I')$.*

Note that, in the above property, it need not be guaranteed that the intersection $I \cap I'$ is non-empty. To guarantee that it will be—and as an imprecise generalisation of the law of large numbers—it suffices to consider the additional property that if a path $\omega \in \Omega$ is R-random for an interval forecast $I \in \mathcal{I}$, then this I should provide outer bounds for the limiting relative frequency of ones along ω:

Property 4. *For any interval forecast $I \in \mathcal{I}$ and any path $\omega \in \Omega_{\mathrm{R}}(I)$, it holds that*

$$\min I \leq \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k \leq \max I. \tag{1}$$

Indeed, if a randomness notion R has Property 4, then

$$\omega \in \Omega_{\mathrm{R}}(I) \cap \Omega_{\mathrm{R}}(I') \Rightarrow \emptyset \neq \left[\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k, \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \omega_k\right]$$

$$\subseteq [\max\{\min I, \min I'\}, \min\{\max I, \max I'\}] = I \cap I',$$

for all $I, I' \in \mathcal{I}$ and all $\omega \in \Omega$.

When the above 4 properties hold for a randomness notion R, we come very close to answering the question whether every path $\omega \in \Omega$ has some smallest interval forecast for which it is random. To this end, consider for any given path $\omega \in \Omega$ the set $\mathcal{I}_\mathrm{R}(\omega)$ that contains all interval forecasts $I \in \mathcal{I}$ for which ω is R-random [this set is non-empty by Property 2 and increasing by Property 1]:

$$\mathcal{I}_\mathrm{R}(\omega) := \big\{I \in \mathcal{I} \colon \omega \in \Omega_\mathrm{R}(I)\big\}.$$

If there is a smallest such interval forecast, then it is necessarily given by

$$I_\mathrm{R}(\omega) := \bigcap \mathcal{I}_\mathrm{R}(\omega) = \bigcap \{I \in \mathcal{I} \colon \omega \text{ is R-random for } I\}.$$

If a randomness notion R has Property 4, then $I_\mathrm{R}(\omega)$ is guaranteed to be non-empty:

$$I \in \mathcal{I}_\mathrm{R}(\omega) \Rightarrow \emptyset \neq \left[\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \omega_k, \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \omega_k\right] \subseteq I, \text{ for all } I \in \mathcal{I} \text{ and all } \omega \in \Omega,$$

and therefore, indeed, $I_\mathrm{R}(\omega) \neq \emptyset$. For all randomness notions R that we will consider, Properties 1, 2 and 4 are satisfied, which implies that $I_\mathrm{R}(\omega)$ will always be well-defined in our context.

As we will see, for some randomness notions R that satisfy Properties 1–4, $I_\mathrm{R}(\omega)$ will indeed be the smallest interval forecast for which any ω is random. Consequently, for these notions, and for any $\omega \in \Omega$, the set $\mathcal{I}_\mathrm{R}(\omega)$ is a so-called *principal set filter*, completely characterised by the interval forecast $I_\mathrm{R}(\omega)$ in the sense that ω will be R-random for an interval forecast $I \in \mathcal{I}$ if and only if $I_\mathrm{R}(\omega) \subseteq I$. For the other randomness notions that satisfy Properties 1–4, the set $\mathcal{I}_\mathrm{R}(\omega)$ will only be a set filter, and then $I_\mathrm{R}(\omega)$ is not the smallest interval forecast for which ω is R-random, and $I_\mathrm{R}(\omega)$ does not fully characterise the set $\mathcal{I}_\mathrm{R}(\omega)$. However, in these cases, we do have that $I_\mathrm{R}(\omega)$ is the smallest interval forecast for which ω is *almost* R-random.

Definition 3. A path $\omega \in \Omega$ is called *almost* R*-random* for an interval forecast $I \in \mathcal{I}$ if it is R-random for all interval forecasts $I' \in \mathcal{I}$ of the form

$$I' = [\min I - \varepsilon_1, \max I + \varepsilon_2] \cap [0,1], \text{ with } \varepsilon_1, \varepsilon_2 > 0.$$

Proposition 1. *If a notion of randomness* R *satisfies Properties 1–4, then* $I_\mathrm{R}(\omega) \in \mathcal{I}$ *is the smallest interval forecast for which a path* $\omega \in \Omega$ *is almost* R*-random.*

Proof. Recall that $I_\mathrm{R}(\omega)$ is well-defined and non-empty by Properties 1, 2 and 4. We now start by proving that ω is almost R-random for $I_\mathrm{R}(\omega)$. That is, we fix any $\varepsilon_1, \varepsilon_2 > 0$, consider the interval forecast $I := [\min I_\mathrm{R}(\omega) - \varepsilon_1, \max I_\mathrm{R}(\omega) + \varepsilon_2] \cap [0,1]$, and show that $I \in \mathcal{I}_\mathrm{R}(\omega)$. Observe to this end that, since $\emptyset \neq I_\mathrm{R}(\omega) = \bigcap \mathcal{I}_\mathrm{R}(\omega)$, there are two interval forecasts $I_1, I_2 \in \mathcal{I}_\mathrm{R}(\omega)$ such that $\min I \leq \min I_1 \leq \min I_\mathrm{R}(\omega)$ and $\max I_\mathrm{R}(\omega) \leq \max I_2 \leq \max I$. Since Property 3 is satisfied, we infer that then also $I_1 \cap I_2 \in \mathcal{I}_\mathrm{R}(\omega)$, and since $I_1 \cap I_2 \subseteq I$, it follows from Property 1 that, indeed, $I \in \mathcal{I}_\mathrm{R}(\omega)$.

It only remains to prove that $I_\mathrm{R}(\omega)$ is the smallest interval forecast for which ω is almost R-random. Consider any interval forecast I for which ω is almost R-random, meaning that $[\min I - \varepsilon_1, \max I + \varepsilon_2] \cap [0,1] \in \mathcal{I}_\mathrm{R}(\omega)$, and therefore also $I_\mathrm{R}(\omega) \subseteq [\min I - \varepsilon_1, \max I + \varepsilon_2] \cap [0,1]$, for all $\varepsilon_1, \varepsilon_2 > 0$. But then obviously also $I_\mathrm{R}(\omega) \subseteq I$, and we are done. □

If a path $\omega \in \Omega$ is almost R-random for the interval forecast $I_\mathrm{R}(\omega)$, then $I_\mathrm{R}(\omega)$ almost completely characterises the set $\mathcal{I}_\mathrm{R}(\omega)$: the only case where we cannot immediately decide whether a path ω is R-random for an interval forecast $I \in \mathcal{I}$ or not, occurs when $\min I = \min I_\mathrm{R}(\omega)$ or $\max I = \max I_\mathrm{R}(\omega)$. Moreover, as our terminology also suggests, because Property 1 holds, $\omega \in \Omega$ will be *almost* R-random for every interval forecast $I \in \mathcal{I}$ for which it is random. To clarify this, we provide a graphical representation in Figure 1.

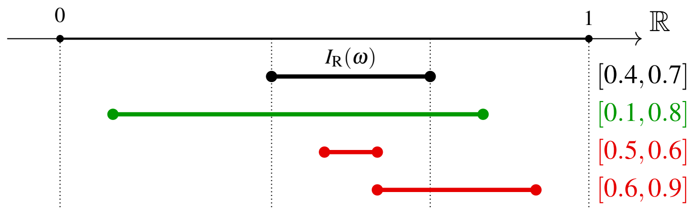

FIGURE 1. Consider a path $\omega \in \Omega$ that is almost R-random for the interval forecast $I_{\mathrm{R}}(\omega) := [0.4, 0.7]$, where the randomness notion R satisfies Properties 1–4. The green interval corresponds to an interval forecast for which ω is R-random, whereas the red intervals correspond to interval forecasts for which ω is not R-random.

Finally, although we do not deem this desirable, there could be some randomness notions R for which $\mathcal{I}_{\mathrm{R}}(\omega)$ is not a (principal) set filter, and for which Properties 1–4 then do not hold. For example, for the notion of ML-randomness that we will consider, the question of whether it satisfies Property 3 will remain open. If this (or any other) notion of randomness does not satisfy Property 3, we could then for instance have the situation depicted in Figure 2, where there is a path $\omega' \in \Omega$ that is R-random for all interval forecasts of the form $[p, 1]$ and $[0, q]$, with $p < 1/3$ and $2/3 \leq q$, but for no others. Then clearly, $I_{\mathrm{R}}(\omega') = [1/3, 2/3]$, but ω' is not R-random for $I_{\mathrm{R}}(\omega')$.

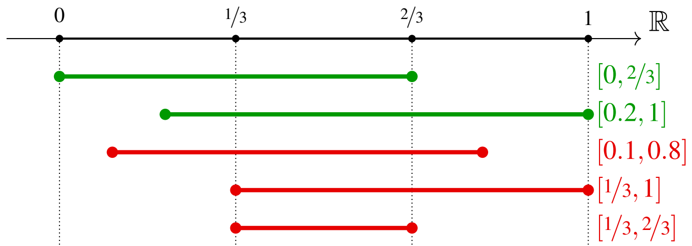

FIGURE 2. The green intervals correspond to interval forecasts for which ω' is R-random, whereas the red intervals correspond to interval forecasts for which ω' is not R-random.

In the remainder of this paper, we intend to study the smallest interval forecasts for which a path is (almost) R-random, if they exist, for several specific randomness notions R. In the next section, we start by introducing the mathematical machinery needed to introduce some of these notions, which are based on the martingale-theoretic approach to randomness.

This martingale-theoretic approach makes extensive use of the concept of betting. Generally speaking, a path $\omega \in \Omega$ will there be considered random for a forecasting system $\varphi \in \Phi$ if there is no implementable betting strategy that is allowed by φ and that, if our subject adopts it, makes him arbitrarily rich along ω. This approach will enable us to introduce the notions of Martin-Löf randomness, weak Martin-Löf randomness, computable randomness and Schnorr randomness, which differ only in what is meant by 'implementable' and in the way a subject should not be able to get arbitrarily rich [3].

3. MARTINGALE-THEORETIC APPROACH—BETTING STRATEGIES

Consider the following betting game involving an infinite sequence of binary variables $X_1, \ldots, X_n, \ldots$ There are three players: Forecaster, Sceptic and Reality.

Forecaster starts by specifying a forecasting system $\varphi \in \Phi$. For every situation $s \in \mathbb{S}$, the corresponding interval forecast $\varphi(s)$ determines for every *gamble* $f : \mathscr{X} \to \mathbb{R}$ whether or not Forecaster offers f to Sceptic, or in other words, allows Sceptic to select f. The set of all gambles is denoted by $\mathscr{L}(\mathscr{X})$. A gamble $g \in \mathscr{L}(\mathscr{X})$ is offered by Forecaster to Sceptic whenever its expectation $E_p(g) := pg(1) + (1-p)g(0)$ is non-positive for every probability $p \in I = \varphi(s)$, or equivalently, whenever $\max_{p\in I} E_p(g) \leq 0$. We will call such gambles *allowable*.

The betting game now unfolds as Reality reveals the successive elements $\omega_n \in \mathscr{X}$ of a path $\omega \in \Omega$. In particular, at every time instant $n \in \mathbb{N}_0$: Reality has revealed the situation $\omega_{1:n}$; Sceptic selects a gamble $f_{\omega_{1:n}} \in \mathscr{L}(\mathscr{X})$ from amongst the ones that are allowed to him by Forecaster, and which is specified by his betting strategy; Reality reveals the next outcome $\omega_{n+1} \in \mathscr{X}$, and Sceptic receives a (possibly negative) reward $f_{\omega_{1:n}}(\omega_{n+1})$. We furthermore assume that Sceptic starts with initial unit capital, so his running capital at every time instant $n \in \mathbb{N}_0$ equals $1 + \sum_{k=0}^{n-1} f_{\omega_{1:k}}(\omega_{k+1})$. We also do not allow Sceptic to borrow. This means that he is only allowed to adopt betting strategies that, regardless of the path that Reality reveals, will guarantee that his running capital never becomes negative.

In order to formalise Sceptic's betting strategies, we will introduce the notion of test supermartingales. We start by considering *real processes* $F\colon \mathbb{S} \to \mathbb{R}$; a process F is called *positive* if $F(s) > 0$ for all $s \in \mathbb{S}$ and *non-negative* if $F(s) \geq 0$ for all $s \in \mathbb{S}$. A real process F is called *temporal* if $F(s)$ only depends on the situation $s \in \mathbb{S}$ through its length $|s|$, meaning that $F(s) = F(t)$ for any two $s, t \in \mathbb{S}$ such that $|s| = |t|$; we then also write $F(n)$ instead of $F(s)$ for all $n \in \mathbb{N}_0$ and $s \in \mathbb{S}$ with $n = |s|$. A real process S is called a *selection process* if $S(s) \in \{0, 1\}$ for all $s \in \mathbb{S}$.

With any real process F, we can associate a *gamble process* $\Delta F\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$, defined by $\Delta F(s)(x) := F(sx) - F(s)$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, and we call it the *process difference* for F. We will use the following notation: for every $s \in \mathbb{S}$, $F(s\bullet)$ is the gamble on $\mathscr{X}$ whose value, for any $x \in \mathscr{X}$, is given by $F(sx)$. Then, clearly, $\Delta F(s) = F(s\bullet) - F(s)$ for all $s \in \mathbb{S}$.

If F is positive, then we can also consider another gamble process $D_F\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$, defined by $D_F(s) := {}^{F(s\bullet)}/_{F(s)}$ for all $s \in \mathbb{S}$, which we call the *multiplier process* for F. And vice versa, with every non-negative real gamble process $D\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$, we can associate a non-negative real process $D^{\circledcirc}\colon \mathbb{S} \to \mathbb{R}$ defined by $D^{\circledcirc}(s) := \prod_{k=0}^{n-1} D(x_{1:k})(x_{k+1})$ for all $s = (x_1, \ldots, x_n) \in \mathbb{S}$, and we then say that $D^{\circledcirc}$ is *generated by* D.

When given a forecasting system $\varphi \in \Phi$, we call a real process M a *supermartingale* for φ if for every $s \in \mathbb{S}$, $\Delta M(s)$ is an allowable gamble for the corresponding interval forecast $\varphi(s)$, meaning that $\max_{p\in\varphi(s)} E_p(\Delta M(s)) \leq 0$. Moreover, a supermartingale T is called a *test* supermartingale if it is non-negative and $T(\square) := 1$. We collect all test supermartingales for φ in the set $\overline{\mathbb{T}}(\varphi)$. It is easy to see that every test supermartingale T corresponds to an allowed betting strategy for Sceptic that starts with unit capital and avoids borrowing. Indeed, for every situation $s = (x_1, \ldots, x_n) \in \mathbb{S}$, T specifies an allowable gamble $\Delta T(s)$ for the interval forecast $\varphi(s) \in \mathcal{I}$, and Sceptic's running capital $1 + \sum_{k=0}^{n-1} \Delta T(x_{1:k})(x_{k+1})$ equals $T(s)$ and is therefore non-negative, and equals 1 in $\square$.

We recall from the discussion in Section 2 that martingale-theoretic randomness notions differ in the nature of the implementable betting strategies that are available to Sceptic. More formally, we will consider three different types of implementable test supermartingales: computable ones, lower semicomputable ones, and test supermartingales generated by lower semicomputable multiplier processes. Recall that a test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ is called *computable* if there is some algorithm that, given any $s \in \mathbb{S}$ and any $N \in \mathbb{N}_0$ as input, can compute the real number $T(s)$ within a precision of 2^{-N}. A test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ is called *lower semicomputable* if there is some algorithm that, given any $s \in \mathbb{S}$ as input, can compute a non-decreasing sequence $(q_n)_{n\in\mathbb{N}_0}$ of rational numbers that converges to the real number $T(s)$ from below—but without necessarily knowing, for any given n, how good the rational lower bound q_n is. Similarly, a real multiplier process D is called lower semicomputable if there is some algorithm that, given any $s \in \mathbb{S}$ and $x \in \mathscr{X}$ as input, can compute a non-decreasing sequence $(q_n)_{n\in\mathbb{N}_0}$ of rational numbers that converges to the real number $D(s)(x)$ from below. For all three notions of implementable betting strategies introduced above, we assume that, when the forecasting system φ at hand is non-computable, the algorithms have no access to φ by an oracle. For more details on computability theory, we refer the reader to Appendix A.

4. Several notions of (imprecise) randomness

At this point, we have introduced the mathematical machinery necessary for defining our different randomness notions R. We start by introducing four martingale-theoretic ones: they are *Martin-Löf* (R = ML) , *weak Martin-Löf* (R = wML), *computable* (R = C) and *Schnorr* (R = S) *randomness*; in the classical precise-probabilistic literature, weak Martin-Löf randomness is better known under the name of *Hitchcock randomness* [3, 14]. Generally speaking, for these notions, a path $\omega \in \Omega$ is random for a forecasting system $\varphi \in \Phi$ if Sceptic has no implementable allowed betting strategy that makes him arbitrarily rich along ω. The randomness notions above differ in how Sceptic's betting strategies are implementable, and in how he should not be able to become arbitrarily rich along a path $\omega \in \Omega$. With these types of restrictions in mind, we introduce the following sets of implementable allowed betting strategies.

$\overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$	all lower semicomputable test supermartingales for φ
$\overline{\mathbb{T}}_{\mathrm{wML}}(\varphi)$	all test supermartingales for φ generated by lower semicomputable multiplier processes
$\overline{\mathbb{T}}_{\mathrm{C}}(\varphi), \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$	all computable test supermartingales for φ

For a path ω to be Martin-Löf, weak Martin-Löf or computably random, we require that Sceptic's running capital must never be *unbounded* on ω for any implementable allowed betting strategy; that is, no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ must be *unbounded* on ω, meaning that $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$.

Definition 4 ([7]). For any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, a path $\omega \in \Omega$ is R-*random* for a forecasting system $\varphi \in \Phi$ if no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ is unbounded on ω.

For Schnorr randomness, we require instead that Sceptic's running capital must not be *computably unbounded* on ω for any implementable allowed betting strategy. More formally, we require that no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ is *computably unbounded* on ω. That T is computably unbounded on ω means that $\limsup_{n\to\infty}[T(\omega_{1:n}) - \tau(n)] \geq 0$ for some (real) *growth function* τ, that is, for some real map $\tau\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ that is

(i) computable;
(ii) non-decreasing, so $\tau(n+1) \geq \tau(n)$ for all $n \in \mathbb{N}_0$;
(iii) unbounded, so $\lim_{n\to\infty} \tau(n) = \infty$.[5]

Since any real growth function τ is unbounded, it expresses a (computable) lower bound for the 'rate' at which T increases to infinity along (a subsequence of) ω. Clearly, if $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ is computably unbounded on $\omega \in \Omega$, then it is also unbounded on ω.

Definition 5 ([7]). A path $\omega \in \Omega$ is S-*random* for a forecasting system $\varphi \in \Phi$ if no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ is computably unbounded on ω.

De Cooman and De Bock have shown that these four martingale-theoretic randomness notions satisfy Properties 1 and 2 [7, Propositions 9, 10, 17 and 18]. To describe the relationships between these martingale-theoretic imprecise-probabilistic randomness notions, we consider the sets $\Omega_{\mathrm{R}}(\varphi)$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$. They satisfy the following inclusions [7, Section 6], for every forecasting system $\varphi \in \Phi$:

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{S}}(\varphi). \tag{2}$$

[5] Since the map τ is non-decreasing, its unboundedness is equivalent to $\lim_{n\to\infty} \tau(n) = \infty$.

Thus, if a path $\omega \in \Omega$ is Martin-Löf random for a forecasting system $\varphi \in \Phi$, then it is also weakly Martin-Löf, computably and Schnorr random for φ. Consequently, for every forecasting system $\varphi \in \Phi$, there are at most as many paths that are Martin-Löf random as there are weakly Martin-Löf, computably or Schnorr random paths. We therefore call Martin-Löf randomness *stronger* than weak Martin-Löf, computable, or Schnorr randomness. And similarly, *mutatis mutandis*, for the other randomness notions.

We will also consider two other imprecise-probabilistic randomness notions that have a more frequentist flavour: *Church* ($\mathrm{R}=\mathrm{CH}$) and *weak Church* ($\mathrm{R}=\mathrm{wCH}$) *randomness*. Their definition makes use of yet another (but simpler) type of implementable real process: a selection process S is called *recursive* if there is some algorithm that, given any $s \in \mathbb{S}$ as input, outputs the binary digit $S(s) \in \{0,1\}$. It is called *adequate* along a path $\omega \in \Omega$ if $\lim_{n\to\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) = \infty$. For every path $\omega \in \Omega$, we collect the corresponding recursive adequate selection processes in the set $\mathcal{S}_{\mathrm{CH}}(\omega)$. Similarly, we collect the recursive adequate temporal selection processes in the (path-independent) set $\mathcal{S}_{\mathrm{wCH}}(\omega) = \mathcal{S}_{\mathrm{wCH}}$.

Definition 6 ([7])**.** A path $\omega \in \Omega$ is CH-*random* (respectively wCH-*random*) for a forecasting system $\varphi \in \Phi$ if for every recursive adequate (respectively recursive adequate temporal) selection process $S \in \mathcal{S}_{\mathrm{CH}}(\omega)$ (respectively $S \in \mathcal{S}_{\mathrm{wCH}}$), it holds that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})[\omega_{k+1} - \underline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0 \text{ and } \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})[\omega_{k+1} - \overline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0.$$

For a stationary forecasting system $I \in \mathcal{I}$, the conditions in this definition simplify to the perhaps more intuitive requirement that

$$\min I \leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \max I.$$

It is easy to see that these two randomness notions also satisfy Properties 1 and 2. Since wCH-randomness considers fewer selection processes than CH-randomness does, it is clear that if a path $\omega \in \Omega$ is CH-random for a forecasting system $\varphi \in \Phi$, then it is also wCH-random for φ. Hence, $\Omega_{\mathrm{CH}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$. For computable forecasting systems, we can also relate these two 'frequentist flavoured' notions to the martingale-theoretic notions considered before [7, Sections 6 and 7]: for every *computable* forecasting system $\varphi \in \Phi$,

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \begin{array}{c} \subsetneq \Omega_{\mathrm{CH}}(\varphi) \subsetneq \\ \subsetneq \Omega_{\mathrm{S}}(\varphi) \subsetneq \end{array} \Omega_{\mathrm{wCH}}(\varphi). \tag{3}$$

For what follows, it will be useful to translate the above ordering into a partial ordering $\leq$ on the set $\{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ as follows:

$$\mathrm{wCH} \begin{array}{c} \lneq \mathrm{CH} \lneq \\ \lneq \mathrm{S} \lneq \end{array} \mathrm{C} \leq \mathrm{wML} \leq \mathrm{ML}.$$

As a final remark, which we will have occasion to come back to a number of times in the sequel, we mention one of the implications of Corollary 20 in Ref. [7], namely that for any forecasting system φ, there is at least one Martin-Löf-random path, so $\Omega_{\mathrm{ML}}(\varphi) \neq \emptyset$. Equations (2) and (3) then lead to the following conclusion.

Proposition 2. *Consider any forecasting system $\varphi \in \Phi$, then $\Omega_{\mathrm{R}}(\varphi) \neq \emptyset$ for any* R *in the collection* $\{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$*, so there is at least one path that is* R*-random for φ. If φ is moreover computable, then also $\Omega_{\mathrm{R}}(\varphi) \neq \emptyset$ for any* R *in the collection* $\{\mathrm{CH}, \mathrm{wCH}\}$*,*[6] *so there is at least one path that is* R*-random for φ.*

[6] It actually holds for any forecasting system $\varphi \in \Phi$ that $\Omega_{\mathrm{R}}(\varphi) \neq \emptyset$ for any R in the collection $\{\mathrm{CH}, \mathrm{wCH}\}$; we refer to Ref. [15] for an explicit proof.

5. SMALLEST INTERVAL FORECASTS AND RANDOMNESS

From now on, we will focus on *stationary* forecasting systems and investigate the differences and similarities between the six randomness notions we have introduced above. We start by investigating whether, for any of these notions, there is a smallest interval forecast for which a path is (almost) random. To this end, we will first compare the sets $\mathcal{I}_{\mathrm{R}}(\omega)$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$. They satisfy similar relations as the sets $\Omega_{\mathrm{R}}(\varphi)$, but without the need for computability assumptions.

Proposition 3 ([7, Section 8])**.** *For every path $\omega \in \Omega$, it holds that*

$$\mathcal{I}_{\mathrm{ML}}(\omega) \subseteq \mathcal{I}_{\mathrm{wML}}(\omega) \subseteq \mathcal{I}_{\mathrm{C}}(\omega) \begin{array}{c} \subsetneq \mathcal{I}_{\mathrm{CH}}(\omega) \subsetneq \\ \subsetneq \mathcal{I}_{\mathrm{S}}(\omega) \subsetneq \end{array} \mathcal{I}_{\mathrm{wCH}}(\omega).$$

Similarly to before, if a path $\omega \in \Omega$ is Martin-Löf random for an interval forecast $I \in \mathcal{I}$, then it is also weakly Martin-Löf, computably, Schnorr and (weakly) Church random for I.

By Definition 6 [with $S = 1$], our weakest notion of randomness guarantees that all interval forecasts $I \in \mathcal{I}_{\mathrm{wCH}}(\omega)$ satisfy Property 4, and therefore, by Proposition 3, all six randomness notions that we are considering here, satisfy Property 4. Since we already know that they also satisfy Properties 1 and 2, we can infer from the discussion in Section 2 that the sets $\mathcal{I}_{\mathrm{R}}(\omega)$ are non-empty, and that the interval forecasts $I_{\mathrm{R}}(\omega)$ are well-defined and non-empty for all $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$.

Moreover, since the sets $\mathcal{I}_{\mathrm{R}}(\omega)$ satisfy the relationships in Proposition 3, their intersections $I_{\mathrm{R}}(\omega)$ satisfy the following converse relationships.

Corollary 4. *For every path $\omega \in \Omega$, it holds that*

$$I_{\mathrm{wCH}}(\omega) \begin{array}{c} \subsetneq I_{\mathrm{CH}}(\omega) \subsetneq \\ \subsetneq I_{\mathrm{S}}(\omega) \subsetneq \end{array} I_{\mathrm{C}}(\omega) \subseteq I_{\mathrm{wML}}(\omega) \subseteq I_{\mathrm{ML}}(\omega).$$

Proof. Consider any path $\omega \in \Omega$. Since all six randomness notions that we are considering satisfy Property 2 and 4, the interval forecasts $I_{\mathrm{R}}(\omega) := \bigcap \mathcal{I}_{\mathrm{R}}(\omega)$ are well-defined and non-empty for all $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$. Moreover, since the sets $\mathcal{I}_{\mathrm{R}}(\omega)$ satisfy the relations in Proposition 3, their intersections $I_{\mathrm{R}}(\omega)$ satisfy the above inverse relations. □

Let us now return to our original question: for any of our randomness notions R, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$, is there a smallest interval forecast for which a path $\omega \in \Omega$ is (almost) random? If so, then it is necessarily given by $I_{\mathrm{R}}(\omega)$, as we discussed in Section 2. For CH- and wCH-randomness, we find that every path $\omega \in \Omega$ is in fact CH- and wCH-random, respectively, for the interval forecasts $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{wCH}}(\omega)$, and that these are therefore the smallest intervals for which this is the case.

Proposition 5. *Consider any $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$ and any path $\omega \in \Omega$. Then $I_{\mathrm{R}}(\omega)$ is the smallest interval forecast for which ω is* R*-random, and*

$$I_{\mathrm{R}}(\omega) = \left[\inf_{S \in \mathcal{S}_{\mathrm{R}}(\omega)} \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})}, \sup_{S \in \mathcal{S}_{\mathrm{R}}(\omega)} \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \right].$$

Proof. Consider any $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$ and any path $\omega \in \Omega$. We need to prove that ω is R-random for the interval forecast $I_{\mathrm{R}}(\omega)$. To this end, consider the real numbers $p \leq q$ in $[0,1]$ defined by

$$p := \inf_{S \in \mathcal{S}_{\mathrm{R}}(\omega)} \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \text{ and } q := \sup_{S \in \mathcal{S}_{\mathrm{R}}(\omega)} \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})}. \quad (4)$$

It follows directly from Definition 6 that ω is R-random for $[p,q]$, whence, $I_{\mathrm{R}}(\omega) \subseteq [p,q]$. We are clearly done if we can show that $I_{\mathrm{R}}(\omega) = [p,q]$. To this end, assume towards contradiction that $I_{\mathrm{R}}(\omega) \subset [p,q]$, so there is some $I \in \mathcal{I}_{\mathrm{R}}(\omega)$ for which $p < \min I$ or $\max I <$

q. But then Equation (4) guarantees that there is some selection process $S \in \mathcal{S}_{\mathrm{R}}(\omega)$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < \min I \text{ or } \max I < \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})},$$

contradicting, via Definition 6, that ω is R-random for I, or in other words that $I \in \mathcal{I}_{\mathrm{R}}(\omega)$. We conclude that ω is indeed R-random for the interval forecast $I_{\mathrm{R}}(\omega) = [p,q]$. □

This result not only shows that there is a natural way to associate a unique smallest interval forecast with a path $\omega \in \Omega$, by adopting (w)CH-randomness, but also that our approach is very similar in spirit to earlier work by Pablo Fierens, Terrence Fine and Adrian Papamarcou [12, 13], as the interval forecast $I_{(\mathrm{w})\mathrm{CH}}(\omega)$ corresponds to the convex hull of the cluster points of the relative frequencies of ones along the path ω and a specific (countable) collection of subsequences.

A similar result need not hold for the other four types of randomness we are considering here. As an illustrative example, consider the non-stationary but temporal precise forecasting system $\varphi_{\sim 1/2}$ defined, for all $s \in \mathbb{S}$, by

$$\varphi_{\sim 1/2}(s) := \frac{1}{2} + (-1)^{|s|}\delta(|s|), \text{ with } \delta(n) := \sqrt{\frac{8}{n+33}} \text{ for all } n \in \mathbb{N}_0.$$

It has been proved [7, Section 9.2] that if a path $\omega \in \Omega$ is C-random for $\varphi_{\sim 1/2}$, then ω is also CH-random and almost C-random for the stationary precise model $1/2$, but not C-random for $1/2$; and there always is such a C-random path for $\varphi_{\sim 1/2}$ [7, Corollary 20].

While in general $I_{\mathrm{R}}(\omega)$ may not be the smallest interval forecast for which a path $\omega \in \Omega$ is R-random, De Cooman and De Bock have effectively proved that for $\mathrm{R} \in \{\mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, every path $\omega \in \Omega$ is almost R-random for $I_{\mathrm{R}}(\omega)$, essentially because in those cases the corresponding sets $\mathcal{I}_{\mathrm{R}}(\omega)$ are closed under finite intersections, that is, because these randomness notions R satisfy Property 3.

Proposition 6. *Consider any* $\mathrm{R} \in \{\mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ *and any path* $\omega \in \Omega$*. Then* $I_{\mathrm{R}}(\omega)$ *is the smallest interval forecast for which* ω *is almost* R*-random.*

Proof. This is immediate from Proposition 1, since the randomness notion R satisfies Properties 1, 2 and 4—as has already been mentioned in the text—, and since it satisfies Property 3 [7, Propositions 31–33]. □

It should be noted that there is no mention of ML-randomness in Propositions 5 and 6. Indeed, it is an open problem whether paths $\omega \in \Omega$ are generally (almost) ML-random for the interval forecast $I_{\mathrm{ML}}(\omega)$; recent results by Barmpalias et al. [16] seem to indicate the converse. We can however provide a partial answer by focusing on paths $\omega \in \Omega$ that are ML-random for a computable precise forecasting system $\varphi \in \Phi$.

Proposition 7. *If a path* $\omega \in \Omega$ *is* ML*-random for a given computable precise forecasting system* $\varphi \in \Phi$*, then* $I_{\mathrm{ML}}(\omega)$ *is the smallest interval forecast for which* ω *is almost* ML*-random.*

6. WHAT DO SMALLEST INTERVAL FORECASTS LOOK LIKE?

Having investigated for which types of randomness R the set $I_{\mathrm{R}}(\omega)$ is the smallest interval forecast for which a path ω is (almost) random, we now set out to find an alternative expression for this interval forecast $I_{\mathrm{R}}(\omega)$; note that we have already succeeded in doing so for CH- and wCH-randomness in Proposition 5 by providing alternative expressions for $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{wCH}}(\omega)$ in terms of relative frequencies. As indicated in the Introduction, we take a different approach here, and start by exploring how a (non-stationary) forecasting system for which a path ω is R-random puts bounds on the smallest interval forecast $I_{\mathrm{R}}(\omega)$ for which it is (almost) R-random.

Special dedicated interval forecasts will play a vital role in this part of the story: for every path $\omega \in \Omega$ and every forecasting system $\varphi \in \Phi$, we will consider the interval forecasts $I_{[\varphi]}(\omega)$ and $I_\varphi(\omega)$ defined by

$$I_{[\varphi]}(\omega) := \left[\lim_{n\to\infty} \inf_{s\in[\omega_{1:n}]} \underline{\varphi}(s), \lim_{n\to\infty} \sup_{s\in[\omega_{1:n}]} \overline{\varphi}(s)\right] \text{ and}$$

$$I_\varphi(\omega) := \left[\liminf_{n\to\infty} \underline{\varphi}(\omega_{1:n}), \limsup_{n\to\infty} \overline{\varphi}(\omega_{1:n})\right].$$

To make the definitions of the interval forecasts $I_{[\varphi]}(\omega)$ and $I_\varphi(\omega)$ more intuitive, we have provided a graphical representation in Figure 3.

In general, clearly, $I_\varphi(\omega) \subseteq I_{[\varphi]}(\omega)$. However, for *temporal* forecasting systems $\varphi \in \Phi$, these interval forecasts are easily seen to coincide with the path-independent interval forecast

$$I_\varphi := \left[\liminf_{n\to\infty} \underline{\varphi}(n), \limsup_{n\to\infty} \overline{\varphi}(n)\right] = I_{[\varphi]}(\omega) = I_\varphi(\omega). \tag{5}$$

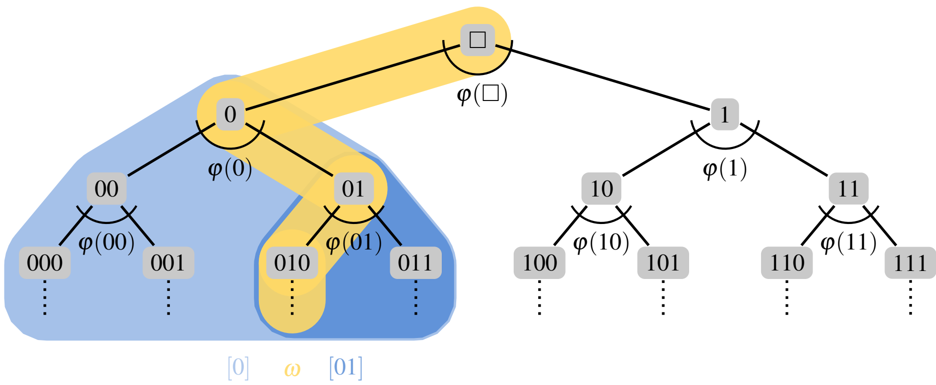

FIGURE 3. In yellow, light blue and dark blue, we have respectively denoted the path $\omega = (0,1,0\dots)$, and the sets of situations $[0]$ and $[01]$. For $I_\varphi(\omega)$, we only need to consider the values of φ in the situations along the yellow path ω, whereas for $I_{[\varphi]}(\omega)$, we need to take into account the values of φ in the (increasingly smaller) nested blue cones $[0]$, $[01]$, ... along the yellow path ω.

So, what does the smallest interval forecast $I_{\mathrm{R}}(\omega)$ for which a path ω is (almost) R-random look like? We start by considering arbitrary forecasting systems $\varphi \in \Phi$ and by assuming that ω is R-random for such a forecasting system φ, for any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$. The forecasting system φ then imposes *outer bounds* on the interval forecast $I_{\mathrm{R}}(\omega)$ in the following sense.

Proposition 8. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given forecasting system* $\varphi \in \Phi$*, it holds that* $I_{\mathrm{R}}(\omega) \subseteq I_\varphi(\omega)$ *if* $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$*, and* $I_{\mathrm{R}}(\omega) \subseteq I_{[\varphi]}(\omega)$ *if* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$.

When we restrict our attention to *computable* forecasting systems $\varphi \in \Phi$, the bounds can be made (at least as tight or) tighter for $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ as well.

Proposition 9. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable forecasting system* $\varphi \in \Phi$*, it holds that* $I_{\mathrm{R}}(\omega) \subseteq I_\varphi(\omega)$.

To see that the outer bounds on $I_{\mathrm{R}}(\omega)$ in Proposition 9, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, can indeed be tighter than the ones given in Proposition 8, consider the following example, with a path $\omega \in \Omega$ that is R-random for a computable forecasting system $\varphi \in \Phi$ such that $I_{\mathrm{R}}(\omega) = I_\varphi(\omega) \subset I_{[\varphi]}(\omega)$.

Example 1. Define the computable precise forecasting system $\varphi_{0,1}$ by letting $\varphi_{0,1}(\Box) := 1$ and $\varphi_{0,1}(x_{1:n}) := x_n$ for all $x_{1:n} \in \mathbb{S}$ and $n \in \mathbb{N}$. Fix any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ and consider the path $\omega \in \Omega$ defined by $\omega_n := 1$ for all $n \in \mathbb{N}$. This ω is R-random for $\varphi_{0,1}$. Indeed, consider any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{0,1})$. Since $\varphi_{0,1}(\omega_{1:n}) = 1$ for all $n \in \mathbb{N}_0$, it holds for all $n \in \mathbb{N}_0$ that

$$0 \geq \overline{E}_{\varphi_{0,1}(\omega_{1:n})}(\Delta T(\omega_{1:n})) = E_1(\Delta T(\omega_{1:n})) = \Delta T(\omega_{1:n})(1) = \Delta T(\omega_{1:n})(\omega_{n+1}),$$

so $T(\omega_{1:n}) = T(\Box) + \sum_{k=0}^{n-1} \Delta T(\omega_{1:k})(\omega_{k+1}) \leq T(\Box) = 1$. All test supermartingales $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{0,1})$ are therefore bounded above by 1 on ω, which guarantees that ω is indeed R-random for $\varphi_{0,1}$. It therefore follows from Proposition 9 that $I_{\mathrm{R}}(\omega) \subseteq I_{\varphi_{0,1}}(\omega)$. Furthermore, clearly, $I_{\varphi_{0,1}}(\omega) = 1$ and $I_{[\varphi_{0,1}]}(\omega) = [0,1]$. Since the only non-empty interval contained in $[1,1]$ is $[1,1]$ itself, it follows that $I_{\mathrm{R}}(\omega) = I_{\varphi_{0,1}}(\omega) \subset I_{[\varphi_{0,1}]}(\omega)$. ◊

If ω is R-random for a computable forecasting system $\varphi \in \Phi$, now with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$, then the forecasting system φ also imposes *inner bounds* on the interval forecast $I_{\mathrm{R}}(\omega)$.

Proposition 10. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable forecasting system* $\varphi \in \Phi$*, it holds that*

$$\min I_{\mathrm{R}}(\omega) \leq \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n}) \text{ and } \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n}) \leq \max I_{\mathrm{R}}(\omega).$$

When the forecasting system $\varphi \in \Phi$ is also *temporal*, we can extend this to $\mathrm{R} \in \{\mathrm{S}, \mathrm{wCH}\}$.

Proposition 11. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable temporal forecasting system* $\varphi \in \Phi$*, it holds that*

$$\min I_{\mathrm{R}}(\omega) \leq \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n}) \text{ and } \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n}) \leq \max I_{\mathrm{R}}(\omega).$$

Interestingly, if ω is R-random for a computable *precise* forecasting systems $\varphi \in \Phi$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$, then the inner bounds on $I_{\mathrm{R}}(\omega)$ in Proposition 10 simplify to $I_{\varphi}(\omega) \subseteq I_{\mathrm{R}}(\omega)$. If we combine this with the outer bounds on $I_{\mathrm{R}}(\omega)$ in Proposition 9, we see that in this case the forecasting system φ characterises the interval forecast $I_{\mathrm{R}}(\omega)$ completely in the following sense.

Theorem 12. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable precise forecasting system* $\varphi \in \Phi$*, it holds that* $I_{\mathrm{R}}(\omega) = I_{\varphi}(\omega)$.

Proof. Since the forecasting system $\varphi \in \Phi$ is precise by assumption, it holds that $\underline{\varphi} = \overline{\varphi}$. Since ω is R-random for the computable precise forecasting system φ, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$, it therefore follows from Proposition 10 that

$$\min I_{\mathrm{R}}(\omega) \leq \liminf_{n\to\infty} \varphi(\omega_{1:n}) \leq \limsup_{n\to\infty} \varphi(\omega_{1:n}) \leq \max I_{\mathrm{R}}(\omega),$$

so, on the one hand, $I_{\varphi}(\omega) \subseteq I_{\mathrm{R}}(\omega)$. Since, on the other hand, it follows from Proposition 9 that $I_{\mathrm{R}}(\omega) \subseteq I_{\varphi}(\omega)$, we may conclude that, indeed, $I_{\mathrm{R}}(\omega) = I_{\varphi}(\omega)$. □

And of course, when the computable precise forecasting system $\varphi \in \Phi$ is also *temporal*, this result applies to Schnorr and weak Church randomness as well.

Theorem 13. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable temporal precise forecasting system* $\varphi \in \Phi$*, it holds that* $I_{\mathrm{R}}(\omega) = I_{\varphi}$.

Proof. Since the temporal forecasting system $\varphi \in \Phi$ is precise by assumption, it holds that $\underline{\varphi} = \overline{\varphi}$. Since ω is R-random for the computable precise temporal forecasting system φ, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$, it therefore follows from Proposition 11 that

$$\min I_{\mathrm{R}}(\omega) \leq \liminf_{n\to\infty} \varphi(\omega_{1:n}) \leq \limsup_{n\to\infty} \varphi(\omega_{1:n}) \leq \max I_{\mathrm{R}}(\omega),$$

so, on the one hand, $I_{\varphi}(\omega) \subseteq I_{\mathrm{R}}(\omega)$. Since, on the other hand, it follows from Proposition 9 that $I_{\mathrm{R}}(\omega) \subseteq I_{\varphi}(\omega)$, we may conclude that, indeed, $I_{\mathrm{R}}(\omega) = I_{\varphi}(\omega) = I_{\varphi}$. □

It is now natural to wonder whether this result can be extended to general imprecise computable forecasting systems. The following example contradicts that.

Example 2. Fix any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ and consider any path $\omega \in \Omega$ that is R-random for $1/2$, which is always possible by Proposition 2. Obviously, $I_{\mathrm{R}}(\omega) = 1/2$. Since R-randomness satisfies Property 1, we know that ω is also R-random for the rational—and therefore computable—interval forecast $\varphi = [1/4, 3/4]$. Clearly, $I_{\varphi} = [1/4, 3/4] \supset 1/2$, so $I_{\mathrm{R}}(\omega) \neq I_{\varphi}$. ◊

Perhaps the results in Theorems 12 or 13 can be extended to general precise (but non-computable) forecasting systems? Again, we provide a counterexample.

Example 3. As in the previous example, fix any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ and consider any path $\omega \in \Omega$ that is R-random for $1/2$, which is always possible by Proposition 2. Obviously, as before, $I_{\mathrm{R}}(\omega) = 1/2$.

Observe that the path ω must have an infinite number of zeroes and ones, because otherwise it would follow that $\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \omega_k = 0$ or $\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \omega_k = 1$, contradicting Property 4 as ω is R-random for $1/2$. This implies that ω cannot be computable, because computable paths with infinitely many zeroes and ones are only R-random for $[0,1]$ [7, Proposition 34].

Meanwhile, the path ω is also R-random for the—clearly non-computable because ω is non-computable—temporal precise forecasting system $\varphi^{\omega}_{0,1}$, defined by $\varphi^{\omega}_{0,1}(n) := \omega_{n+1}$ for all $n \in \mathbb{N}_0$.

To see why, first assume that $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, and consider any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\omega}_{0,1})$. Since T is a supermartingale for $\varphi^{\omega}_{0,1}$, it holds for any $n \in \mathbb{N}_0$ that

$$0 \geq E_{\varphi^{\omega}_{0,1}(\omega_{1:n})}(\Delta T(\omega_{1:n})) = E_{\omega_{n+1}}(\Delta T(\omega_{1:n})) = \Delta T(\omega_{1:n})(\omega_{n+1}),$$

and therefore,

$$T(\omega_{1:n}) = T(\square) + \sum_{k=0}^{n-1} \Delta T(\omega_{1:k})(\omega_{k+1}) \leq T(\square) = 1.$$

Consequently, all test supermartingales $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\omega}_{0,1})$ are bounded above by 1 on ω, so ω is indeed R-random for $\varphi^{\omega}_{0,1}$.

Next, since $\overline{\varphi}^{\omega}_{0,1}(\omega_{1:n}) = \underline{\varphi}^{\omega}_{0,1}(\omega_{1:n}) = \omega_{n+1}$, we see that the conditions in Definition 6 are trivially satisfied, so ω is indeed also R-random for $\varphi^{\omega}_{0,1}$ for $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$.

However, since we know that ω contains an infinite number of zeroes and ones,

$$\begin{aligned} I_{[\varphi^{\omega}_{0,1}]}(\omega) \overset{(5)}{=} I_{\varphi^{\omega}_{0,1}}(\omega) \overset{(5)}{=} I_{\varphi^{\omega}_{0,1}} &\overset{(5)}{=} \Big[\liminf_{n\to\infty} \underline{\varphi}^{\omega}_{0,1}(n), \limsup_{n\to\infty} \overline{\varphi}^{\omega}_{0,1}(n)\Big] \\ &= \Big[\liminf_{n\to\infty} \omega_{n+1}, \limsup_{n\to\infty} \omega_{n+1}\Big] = [0,1], \end{aligned}$$

and therefore $I_{\mathrm{R}}(\omega) \neq I_{\varphi^{\omega}_{0,1}}$. ◊

7. When do smallest interval forecasts (not) coincide?

We mentioned in the Introduction that we restrict our attention in this paper to associating interval forecasts with idealised infinite amounts of data, and therefore will not be concerned with trying to elicit such interval forecasts from finite amounts of data. It will nevertheless be instructive to start our discussion in this section by considering a situation $s \in \mathbb{S}$ and assuming that it is an initial segment of some idealised path ω.

What does it mean to learn an uncertainty model from a finite amount of data, that is, from a situation s? First of all, it implies that we assume the data-generating process to be erratic in some way, and that, therefore, putting forth some uncertainty model is the most accurate description we can provide.

What type of uncertainty model is to be used, then? The answer depends on the assumptions we (want to) make about the data generating process—that is, about the process that governs the path ω. Alonzo Church [17], for example, suggested associating a probability $p \in [0,1]$ with a path ω when the relative frequencies of ones along ω and along all computably selectable infinite subsequences converge to p.

How to elicit this uncertainty model from a finite amount of data, then? When adopting Church's above-mentioned assumption on the data-generating process, and when given a finite amount of data $s \sqsubseteq \omega$, the relative frequency of ones along s seems a reasonable estimator for the uncertainty model p that is associated with the path ω—for one thing, this estimator can then be expected to converge to p as more data comes in, in the sense that, if our assumption is correct, $(\forall \varepsilon > 0)(\exists t \sqsubseteq \omega)(\forall s \in \mathbb{S} \colon t \sqsubseteq s \sqsubseteq \omega)\left|\frac{1}{|s|}\sum_{k=1}^{|s|} s_k - p\right| < \varepsilon$.

However, no matter how straightforward matters seem to be in the above classical (so called precise-probabilistic) case, we ought to keep an open mind and remain wary of the justifiability of our assumptions. In [18], for instance, Igor Gorban has argued that quite some phenomena display behaviour where relative frequencies do not tend to converge with increasing amounts of data.

As one possible solution, our approach allows for more general types of uncertainty models: we have allowed for (more general) interval forecasts while still considering a variety of randomness assumptions $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ that can be associated with the data generating process. We can thus associate with a path ω several smallest interval forecasts $I_{\mathrm{R}}(\omega)$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$, and we have shown for all randomness notions R but ML-randomness that $I_{\mathrm{R}}(\omega)$ is the smallest interval forecast for which a path ω is (almost) random. As a possible (baby) step towards developing a statistical theory for such interval forecasts one day, we may now ask ourselves the important question: how robust is the notion of a smallest interval for which a path is (almost) random, with respect to randomness assumptions about the data generating process: how strong are the assumptions we want to make and what is their impact?

If, for instance, we impose ML-randomness on a path ω, how does the corresponding smallest interval forecast $I_{\mathrm{ML}}(\omega)$ for which a path is (almost) ML-random—if it exists—then compare to that for weaker randomness notions? In this section, we investigate conditions for the choice of randomness assumption to have no effect on the smallest interval forecast for which a path is (almost) random, in the sense that it coincides with the smallest interval forecasts for which the path is (almost) random, for weaker randomness assumptions. On the other hand, we will provide examples where the choice does matter, and where the stronger the randomness assumption is, the wider the corresponding smallest interval forecast will be.

Let us start by considering a path $\omega \in \Omega$ that is ML-random for some computable precise forecasting system $\varphi \in \Phi$; similar results hold when focusing on weaker notions of randomness. We know from Equation (3) that ω is then also wML-, C- and CH-random for φ. By invoking Propositions 5–7, we infer that $I_{\mathrm{R}}(\omega)$ is the smallest interval forecast for which ω is (almost) R-random, for any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$. Moreover, by Theorem 12, these smallest interval forecasts all equal $I_{\varphi}(\omega)$ and therefore coincide, i.e., $I_{\mathrm{ML}}(\omega) = I_{\mathrm{wML}}(\omega) = I_{\mathrm{C}}(\omega) = I_{\mathrm{CH}}(\omega) = I_{\varphi}(\omega)$; the previous exposition is formalised in the following statement, with the partial ordering $\leq$ on the set $\{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ as defined on page 9.

Corollary 14. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$ *and any path* $\omega \in \Omega$ *that is* R*-random for a given computable precise forecasting system* $\varphi \in \Phi$ *it holds that* $I_{\varphi}(\omega) = I_{\mathrm{Q}}(\omega)$ *for all* $\mathrm{Q} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$ *such that* $\mathrm{Q} \leq \mathrm{R}$.

Proof. Since ω is R-random for the computable precise forecasting system φ, Equation (3) tells us that ω is also Q-random for φ, with $\mathrm{Q} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$ such that $\mathrm{Q} \leq \mathrm{R}$, and Theorem 12 then guarantees that $I_{\mathrm{Q}}(\omega) = I_\varphi(\omega)$ for all these Q. □

By only looking at temporal computable precise forecasting systems $\varphi \in \Phi$, we can even strengthen these conclusions. For example, using a similar argument as before—but using Theorem 13 instead of 12—we see that if ω is ML-random for such a forecasting system φ, then the smallest interval forecasts $I_{\mathrm{R}}(\omega)$ for which ω is (almost) R-random, coincide for all six randomness notions that we consider; this is formalised in the following statement.

Corollary 15. *For any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any path* $\omega \in \Omega$ *that is* R-*random for a computable temporal precise forecasting system* $\varphi \in \Phi$ *it holds that* $I_{\mathrm{Q}}(\omega) = I_\varphi$ *for all* $\mathrm{Q} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *such that* $\mathrm{Q} \leq \mathrm{R}$.

Proof. Since ω is R-random for the computable precise temporal forecasting system φ, Equation (3) tells us that ω is also Q-random for φ, with $\mathrm{Q} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ such that $\mathrm{Q} \leq \mathrm{R}$, and Theorem 13 then guarantees that $I_{\mathrm{Q}}(\omega) = I_\varphi$ for all these Q. □

Looking at these results, the question arises whether there are paths $\omega \in \Omega$ for which the various interval forecasts $I_{\mathrm{R}}(\omega)$ do not coincide. It turns out that such paths do exist; notice that this approach to comparing randomness notions differs from what is classically done [3, 4], since there is for example, as we discussed in Section 5, a path $\omega \in \Omega$ that is CH-random but not C-random for $1/2$, whilst $I_{\mathrm{CH}}(\omega) = I_{\mathrm{C}}(\omega) = 1/2$. We start by showing that the smallest interval forecasts $I_{\mathrm{C}}(\omega)$ and $I_{\mathrm{S}}(\omega)$ for which a path $\omega \in \Omega$ is respectively almost C- and almost S-random, do not always coincide.

Proposition 16. *There is a path* $\omega \in \Omega$ *such that* $I_{\mathrm{S}}(\omega) = 1/2 \in [1/2, 1] \subseteq I_{\mathrm{C}}(\omega)$.

The following proof is based on ideas in Ref. [7], which are in their turn based on a construction by Wang [4] that shows that S-randomness does not entail CH-randomness; a more recent and perhaps more simple construction that leads to this result can be found in Ref. [19, Theorem 2.2.21 and Corollary 2.2.23].

Proof. Yongge Wang [4, Theorem 3.3.5] has proved the existence of a path $\omega \in \Omega$ and a recursive selection process $S \in \mathcal{S}_{\mathrm{CH}}(\omega)$ such that

(i) $1/2 \in \mathcal{I}_{\mathrm{S}}(\omega)$;
(ii) for all $n \in \mathbb{N}_0$, if $S(\omega_{1:n}) = 1$, then $\omega_{n+1} = 1$.

We will prove that ω is exactly the path we are after.

To show that $I_{\mathrm{S}}(\omega) = 1/2$, we simply observe that, since $1/2 \in \mathcal{I}_{\mathrm{S}}(\omega)$ and since $I_{\mathrm{S}}(\omega)$ is non-empty because Properties 2 and 4 hold for R = S, it follows that $I_{\mathrm{S}}(\omega) = \bigcap \mathcal{I}_{\mathrm{S}}(\omega) = 1/2$.

We continue by showing that $[1/2, 1] \subseteq I_{\mathrm{C}}(\omega)$. By Corollary 4, it suffices to show that $[1/2, 1] \subseteq I_{\mathrm{CH}}(\omega)$, which is what we will now do. Since $I_{\mathrm{S}}(\omega) = 1/2$ and since $I_{\mathrm{wCH}}(\omega)$ is non-empty by Properties 2 and 4, it follows from Corollary 4 that $I_{\mathrm{wCH}}(\omega) = 1/2 \neq \emptyset$, and therefore, again by Corollary 4, we already find that $\min I_{\mathrm{CH}}(\omega) \leq 1/2$. It now only remains to show that $1 \leq \max I_{\mathrm{CH}}(\omega)$. Since the path ω is CH-random for the interval forecast $I_{\mathrm{CH}}(\omega)$ by Proposition 5, we gather using Definition 6 and (ii) that, indeed,

$$1 = \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \max I_{\mathrm{CH}}(\omega).$$ □

We are also able to show that there is a path $\omega \in \Omega$ such that $I_{\mathrm{C}}(\omega) = 1/2$ is the smallest interval forecast for which it is almost C-random, whereas ω is not almost ML-random for $1/2$. It suffices to apply the following result with $p < 1/2 < q$; for this result, we have drawn inspiration from Ref. [20].

Proposition 17. *For every two real numbers $p,q \in \mathbb{R}$ such that $0 < p \leq q < 1$, there is a path $\omega \in \Omega$ such that $I_{\mathrm{C}}(\omega) = 1/2$ and $[p,q] \notin \mathcal{I}_{\mathrm{ML}}(\omega)$.*

Clearly, for $p < 1/2 < q$, the path $\omega \in \Omega$ in Proposition 17 cannot be Martin-Löf random for a precise computable forecasting system $\varphi \in \Phi$, because otherwise, the interval forecasts $I_{\mathrm{C}}(\omega)$ and $I_{\mathrm{ML}}(\omega)$ would coincide by Corollary 14, and ω would therefore be almost Martin-Löf random for $1/2$ by Proposition 7, contradicting the result. So, for $p < 1/2 < q$, the path ω in this result is an example of a path for which we do not know whether there is a smallest interval forecast for which ω is almost Martin-Löf random. However, if there is such a smallest interval forecast, then Definition 3 and Proposition 17 show it is definitely not equal to $1/2$; due to Corollary 4, it must then strictly include $1/2$.

We continue by showing that there is at least one path $\omega \in \Omega$ such that the smallest interval forecasts $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{wCH}}(\omega)$ for which it is CH- and wCH-random, respectively, do not coincide. To do so, we will make use of the forecasting system $\varphi_{p,q} \in \Phi$, defined for all $p,q \in \mathbb{R}$ by

$$\varphi_{p,q}(\square) = 0 \text{ and } \varphi_{p,q}(x_{1:n}) := \begin{cases} p & \text{if } x_n = 1 \\ q & \text{if } x_n = 0 \end{cases} = q + (p-q)x_n \qquad \text{for all } n \in \mathbb{N} \text{ and } x_{1:n} \in \mathbb{S}.$$

When considering any two computable real numbers $p,q \in \mathbb{R}$ such that $0 < p < q < 1$, it turns out that $[p,q] = I_{\mathrm{CH}}(\omega) \supset I_{\mathrm{wCH}}(\omega)$ for any path $\omega \in \Omega_{\mathrm{CH}}(\varphi_{p,q})$; since the forecasting system $\varphi_{p,q}$ is clearly computable for any two computable reals p and q, there always is such a path, by Proposition 2.

Proposition 18. *Consider any two computable real numbers $p,q \in \mathbb{R}$ such that $0 < p < q < 1$ and any path $\omega \in \Omega_{\mathrm{CH}}(\varphi_{p,q})$. Then $I_{\mathrm{CH}}(\omega) = [p,q]$ and*

$$p < q - (q-p)q \leq \min I_{\mathrm{wCH}}(\omega) \text{ and } \max I_{\mathrm{wCH}}(\omega) \leq q - (q-p)p < q.$$

We will make use of the following lemma for proving this proposition.

Lemma 19. *Consider any two real numbers $p,q \in \mathbb{R}$ such that $0 \leq p \leq q \leq 1$, any path $\omega \in \Omega_{\mathrm{CH}}([p,q])$ and any recursive temporal selection process S for which $\lim_{n\to\infty} \sum_{k=0}^{n-1} S(k) = \infty$. Then*

$$p \leq \liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} \leq \limsup_{n\to\infty} \frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} \leq q.$$

Proof. Consider the recursive temporal selection process S' defined by $S'(n) := S(n+1)$ for all $n \in \mathbb{N}_0$ and observe that

$$\frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} = \frac{\sum_{k=0}^{n-2} S'(k)\omega_{k+1}}{S(0) + \sum_{k=0}^{n-2} S'(k)} = \frac{\sum_{k=0}^{n-2} S'(k)\omega_{k+1}}{\sum_{k=0}^{n-2} S'(k)} \frac{\sum_{k=0}^{n-2} S'(k)}{S(0) + \sum_{k=0}^{n-2} S'(k)}.$$

Since $\lim_{n\to\infty} \sum_{k=0}^{n-1} S(k) = \infty$ implies that also $\lim_{n\to\infty} \sum_{k=1}^{n-1} S'(k) = \infty$, this guarantees that

$$\liminf_{n\to\infty} \frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} = \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S'(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S'(k)} \geq p,$$

where the last inequality follows from Definition 6, and in a completely similar vein,

$$\limsup_{n\to\infty} \frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} = \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S'(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S'(k)} \leq q. \qquad \square$$

Proof of Proposition 18. Since ω is CH-random for $\varphi_{p,q}$, and Property 1 holds for $\mathrm{R} = \mathrm{CH}$, we find that $\omega \in \Omega_{\mathrm{CH}}([p,q])$.

As a first step, we show that ω then must have an infinite number of zeroes and an infinite number of ones. Indeed, assume towards contradiction that ω contains only a finite number of ones; the other case can be proved by a similar line of reasoning. Since $\omega \in \Omega_{\mathrm{CH}}([p,q])$, we infer from Definition 6 [with $S=1$] that

$$p \leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} \omega_{k+1}}{n} = 0,$$

contradicting the assumption that $0 < p$.

Since $\varphi_{p,q}$ is clearly a computable precise forecasting system, it follows from Theorem 12 that

$$\begin{aligned} I_{\mathrm{CH}}(\omega) = I_{\varphi_{p,q}}(\omega) &= \Big[\liminf_{n\to\infty} \varphi_{p,q}(\omega_{1:n}), \limsup_{n\to\infty} \varphi_{p,q}(\omega_{1:n})\Big] \\ &= \Big[\liminf_{n\to\infty}\big(q+(p-q)\omega_n\big), \limsup_{n\to\infty}\big(q+(p-q)\omega_n\big)\Big] = [p,q], \end{aligned} \tag{6}$$

where the last equality holds because ω consists of an infinite number of zeroes and an infinite number of ones.

It is a matter of straightforward verification to show that $p < q-(q-p)q$ and that $q-(q-p)p < q$. It therefore remains to prove that $I_{\mathrm{wCH}}(\omega) \subseteq [q-(q-p)q, q-(q-p)p]$. To this end, fix any recursive temporal selection process S such that $\lim_{n\to\infty}\sum_{k=0}^{n-1} S(k) = \infty$. Since $\omega \in \Omega_{\mathrm{CH}}(\varphi_{p,q})$ and since $\varphi_{p,q}$ is a computable forecasting system, it follows from Equation (3) that $\omega \in \Omega_{\mathrm{wCH}}(\varphi_{p,q})$, and therefore Definition 6 guarantees that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)[\omega_{k+1} - \varphi_{p,q}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(k)} \geq 0 \text{ and } \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)[\omega_{k+1} - \varphi_{p,q}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(k)} \leq 0. \tag{7}$$

Now,

$$\begin{aligned} \frac{\sum_{k=0}^{n-1} S(k)[\omega_{k+1} - \varphi_{p,q}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(k)} &= \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - \frac{\sum_{k=0}^{n-1} S(k)\varphi_{p,q}(\omega_{1:k})}{\sum_{k=0}^{n-1} S(k)} \\ &= \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - \frac{\sum_{k=1}^{n-1} S(k)\big(q+(p-q)\omega_k\big)}{\sum_{k=0}^{n-1} S(k)} \\ &= \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - q\frac{\sum_{k=1}^{n-1} S(k)}{\sum_{k=0}^{n-1} S(k)} + (q-p)\frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)}. \end{aligned} \tag{8}$$

Since we assumed that $\lim_{n\to\infty}\sum_{k=0}^{n-1} S(k) = \infty$, taking the $\liminf$ on both sides in Equation (8) then leads to

$$\begin{aligned} &\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)[\omega_{k+1} - \varphi_{p,q}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(k)} \\ &\quad= \liminf_{n\to\infty}\left(\frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - q\frac{\sum_{k=1}^{n-1} S(k)}{\sum_{k=0}^{n-1} S(k)} + (q-p)\frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)}\right) \\ &\quad\leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} + \limsup_{n\to\infty}\left(-q\frac{\sum_{k=1}^{n-1} S(k)}{\sum_{k=0}^{n-1} S(k)} + (q-p)\frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)}\right) \\ &\quad= \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - q + (q-p)\limsup_{n\to\infty} \frac{\sum_{k=1}^{n-1} S(k)\omega_k}{\sum_{k=0}^{n-1} S(k)} \\ &\quad\leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(k)\omega_{k+1}}{\sum_{k=0}^{n-1} S(k)} - q + (q-p)q, \end{aligned}$$

where the (in)equalities follow from the properties of the $\liminf$ operator, $q > p$ and Lemma 19. Similarly, taking the $\limsup$ on both sides in Equation (8) leads to

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)[\omega_{k+1}-\varphi_{p,q}(\omega_{1:k})]}{\sum_{k=0}^{n-1}S(k)}\\
&\qquad=\limsup_{n\to\infty}\left(\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}-q\frac{\sum_{k=1}^{n-1}S(k)}{\sum_{k=0}^{n-1}S(k)}+(q-p)\frac{\sum_{k=1}^{n-1}S(k)\omega_k}{\sum_{k=0}^{n-1}S(k)}\right)\\
&\qquad\geq\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}+\liminf_{n\to\infty}\left(-q\frac{\sum_{k=1}^{n-1}S(k)}{\sum_{k=0}^{n-1}S(k)}+(q-p)\frac{\sum_{k=1}^{n-1}S(k)\omega_k}{\sum_{k=0}^{n-1}S(k)}\right)\\
&\qquad=\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}-q+(q-p)\liminf_{n\to\infty}\frac{\sum_{k=1}^{n-1}S(k)\omega_k}{\sum_{k=0}^{n-1}S(k)}\\
&\qquad\geq\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}-q+(q-p)p.
\end{aligned}$$

If we combine these inequalities with Equation (7), then a few algebraic manipulations lead us to

$$q-(q-p)q\leq\liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}\leq\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(k)\omega_{k+1}}{\sum_{k=0}^{n-1}S(k)}\leq q-(q-p)p.$$

We now infer via Definition 6 that ω is wCH-random for $[q-(q-p)q,q-(q-p)p]$, and therefore, indeed, $I_{\mathrm{wCH}}(\omega)\subseteq[q-(q-p)q,q-(q-p)p]$. □

Proposition 18 also shows that R can only equal wCH in Theorem 13 and Corollary 15, and not in Theorem 12 and Corollary 14. Indeed, consider any computable precise forecasting system $\varphi_{p,q}$ as in Proposition 18 and any path $\omega\in\Omega$ that is CH-random for $\varphi_{p,q}$, for which then $I_{\mathrm{CH}}(\omega)=[p,q]$. If Theorem 12 or Corollary 14 allowed for R $=$ wCH, then $I_{\mathrm{wCH}}(\omega)$ would equal $I_{\mathrm{CH}}(\omega)=[p,q]$, which contradicts Proposition 18 because that says that $p<\min I_{\mathrm{wCH}}(\omega)$ and $\max I_{\mathrm{wCH}}(\omega)<q$.

When considering a forecasting system $\varphi_{p,q}$ as in Proposition 18, but a path $\omega\in\Omega$ that is now C-random for $\varphi_{p,q}$ [which is always possible by Proposition 2], then we can prove that the smallest interval forecasts $I_{\mathrm{S}}(\omega)$ and $I_{\mathrm{wCH}}(\omega)$ for which ω is respectively almost S- and wCH-random, do not coincide.

Proposition 20. *Consider any two computable real numbers $p,q\in\mathbb{R}$ such that $0<p<q<1$ and any path $\omega\in\Omega_{\mathrm{C}}(\varphi_{p,q})$. Then $I_{\mathrm{S}}(\omega)=I_{\mathrm{C}}(\omega)=I_{\mathrm{CH}}(\omega)=[p,q]$ and*

$$p<q-(q-p)q\leq\min I_{\mathrm{wCH}}(\omega)\text{ and }\max I_{\mathrm{wCH}}(\omega)\leq q-(q-p)p<q.$$

So, Propositions 16–20 show that the smallest interval forecasts $I_{\mathrm{R}}(\omega)$, with R $\in\{$ML, wML, C, S, CH, wCH$\}$ and $\omega\in\Omega$, do not always coincide. In order to keep track of those randomness notions for which we have been able to prove this, we provide an overview in Figure 4.

We believe that allowing for interval forecasts in the study of algorithmic randomness allows for a more detailed differentiation between the more common randomness notions in the literature. For example, instead of only being able to say that some path $\omega\in\Omega$ is Church random but not computably nor Martin-Löf random for some probability $p\in[0,1]$, we can now also compare the smallest interval forecasts $I_{\mathrm{CH}}(\omega)$, $I_{\mathrm{C}}(\omega)$ and $I_{\mathrm{ML}}(\omega)$ for which ω is (almost) CH-, C- or ML-random, and for example find that they nevertheless coincide. This is for example the case for any path $\omega\in\Omega$ that is ML-random for $\varphi_{\sim 1/2}$ [which always exists by Proposition 2]. Indeed, by Corollary 15 and Propositions 5–7, ω is almost ML-, almost C- and CH-random for $I_{\varphi}={}^1\!/_2$, while it is not C-random for ${}^1\!/_2$ [7, Section 9.2], and therefore also not ML-random for ${}^1\!/_2$, by Equation (3).

Alternatively, as is indicated by a green arrow in Figure 4, there are also paths ω for which $I_{\mathrm{C}}(\omega)\subset I_{\mathrm{ML}}(\omega)$, which means that C- and ML-randomness are quite different for such ω. As already mentioned above, we have been able to reveal such potentially different behaviour for the randomness notions that are connected by the full green arrows

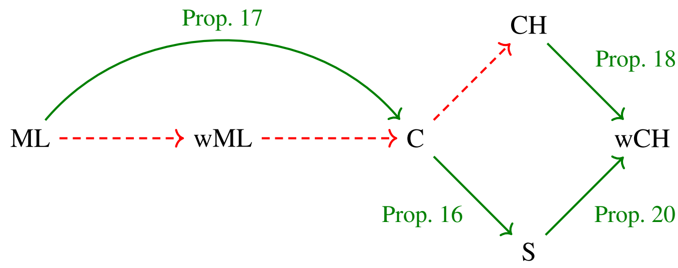

FIGURE 4. Arrows represent the known hierarchy between the notions of randomness we consider, and the green, full arrows denote that we have shown that these relations are strict, in the sense that the corresponding smallest interval forecasts $I_{\mathrm{R}}(\omega)$ do not coincide for every path $\omega \in \Omega$. The red, dashed arrows indicate that more work is required to find out whether the relations are strict in that sense.

in Figure 4, and we need a closer investigation for those randomness notions connected by dashed red arrows.

Consider for example the dashed red arrow connecting C- and CH-randomness. We know that there are paths $\omega \in \Omega$ such that $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{C}}(\omega)$ are the smallest interval forecasts for which they are CH- and C-random, respectively. Consider, for instance, any path $\omega \in \Omega$ that is C-random for $1/2$ [which is always possible by Proposition 2]. By Equation (3), any such path ω is then also CH-random for $1/2$. From the discussion after Corollary 15, we also know that there are paths $\omega \in \Omega$ such that $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{C}}(\omega)$ are the smallest interval forecasts for which they are CH- and almost C-random, respectively, while not being C-random for $I_{\mathrm{C}}(\omega)$. Meanwhile, it is still an open question whether there are paths $\omega \in \Omega$ for which the smallest interval forecasts $I_{\mathrm{CH}}(\omega)$ and $I_{\mathrm{C}}(\omega)$ for which they are CH- and (almost) C-random, respectively, do not coincide, implying that then $I_{\mathrm{CH}}(\omega) \subset I_{\mathrm{C}}(\omega)$.

8. CONCLUSIONS AND FUTURE WORK

We have come to the conclusion that various (non-stationary) precise-probabilistic randomness notions in the literature are, in some respects, not all that different: if a path is random for a computable precise (temporal) forecasting system, then the smallest interval forecasts for which it is (almost) random will coincide for several randomness notions. The computability condition on the precise forecasting system is important for this result, but we do not think it is that big a restriction from a practical point of view. After all, computable forecasting systems are those that can be computed by an algorithm up to any desired precision, and therefore, they are arguably the only ones that are of practical relevance.

An important concept that made several of our results possible was that of almost randomness, a notion that is closely related to randomness but (slightly) easier to satisfy. In future work, we would like to take a closer look at the difference between these two notions. In particular, the present discussion, together with our discussion in Section 7 of Ref. [21], makes us wonder to what extent the distinction between them is relevant in more practical contexts.

Furthermore, building upon the work by Vladimir Vovk [22, 23], we would like to find out if there is some path-dependent 'distance' between any two computable (imprecise) forecasting systems $\varphi, \varphi' \in \Phi$ such that, if a path $\omega \in \Omega$ is random for φ, then it will be random for φ' if (and only if) the distance between both forecasting systems remains bounded on ω; for this to be possible, we may need to impose additional properties on the forecasting systems, such as computability and non-degeneracy. When assuming that the path ω is R-random for a computable precise (temporal) forecasting system φ, as we do in Theorems 12 and 13, we expect such distances to tell us whether ω is random for its smallest interval forecast $\varphi' = I_{\mathrm{R}}(\omega)$, instead of merely almost random.

We also plan to continue investigating the open question whether there is for every path some smallest interval forecast for which it is (almost) Martin-Löf random. Moreover, there is still quite some work to be done in finding out whether the randomness notions considered here are all different from a stationary imprecise-probabilistic perspective: are there paths for which the smallest interval forecasts for which they are (almost) random, do not coincide? This corresponds to having a closer look at those pairs of randomness notions connected by red, dashed arrows in Figure 4; a recent separation result by Barmpalias et al. [16, Theorem 1.2] seems to indicate the existence of a path $\omega \in \Omega$ such that $I_{\mathrm{wML}}(\omega) \subset I_{\mathrm{ML}}(\omega)$.

Finally, we wonder whether the results in this paper can be generalised from binary to arbitrary finite sample spaces: does every non-binary infinite sequence have a smallest *credal set*—that is, a smallest closed convex set of probabilities—for which it is (almost) random? And if so, what does it look like?

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APPENDIX A. COMPUTABILITY THEORY

Computability theory studies what it means for a mathematical object to be 'implementable'. As its basic building blocks, it has *recursive* natural maps $\phi\colon \mathbb{N}_0 \to \mathbb{N}_0$, which are maps that can be computed by a Turing machine [25]. By the Church–Turing thesis, the natural map ϕ being recursive is equivalent to the existence of an algorithm that, given any input $n \in \mathbb{N}_0$, outputs the non-negative integer $\phi(n) \in \mathbb{N}_0$; in what follows, we will often use this equivalence without explicitly mentioning it. Note that the domain of ϕ can be replaced by any (countably infinite) set $\mathscr{D}$ that can be effectively encoded as a subset of $\mathbb{N}_0$ [3, 25, 26]; examples of such sets $\mathscr{D}$ are given by $\mathbb{N}$, $\mathbb{S}$, $\mathscr{X}$, $\mathbb{S}\times\mathscr{X}$, $\mathbb{S}\times\mathbb{N}_0$ and $\mathbb{S}\times\mathscr{X}\times\mathbb{N}_0$.

The notion of a recursive natural map ϕ can also be extended to maps whose codomain is the set of rational numbers $\mathbb{Q}$. We call a rational map $q\colon \mathscr{D}\to\mathbb{Q}$ *recursive* if there are three recursive natural maps $a,b,c\colon \mathscr{D}\to\mathbb{N}_0$ such that

$$b(d)\neq 0 \text{ and } q(d)=(-1)^{c(d)}\frac{a(d)}{b(d)} \text{ for all } d\in\mathscr{D}.$$

Since a finite number of algorithms can always be combined into one algorithm, a rational map q is recursive if and only if there is some algorithm that, given any input $d\in\mathscr{D}$, outputs the rational number $q(d)\in\mathbb{Q}$ [27].

We can use recursive rational maps to provide several interpretations of what it means for a real map $r\colon \mathscr{D}\to\mathbb{R}$ to be 'implementable'.

We call a real map $r\colon \mathscr{D}\to\mathbb{R}$ *lower semicomputable* if there is some recursive rational map $q\colon \mathscr{D}\times\mathbb{N}_0\to\mathbb{Q}$ such that

$$q(d,n+1)\geq q(d,n) \text{ and } r(d)=\lim_{m\to\infty} q(d,m) \text{ for all } d\in\mathscr{D} \text{ and } n\in\mathbb{N}_0.$$

This means that there is some algorithm that, given any $d\in\mathscr{D}$ as input, allows us to approach the real number $r(d)$ from below—but without knowing, for any given $n\in\mathbb{N}_0$, how good the lower bound $q(d,n)$ is. Correspondingly, we call a real map r *upper semicomputable* if the real map $-r$ is lower semicomputable. A real multiplier process $D\colon \mathbb{S}\to\mathscr{L}(\mathscr{X})$ is called *lower* (*upper*) *semicomputable* if it is lower (upper) semicomputable as a real map on $\mathbb{S}\times\mathscr{X}$ that maps any $(s,x)\in\mathbb{S}\times\mathscr{X}$ to $D(s)(x)$.

A real map r is called *computable* if it is both lower and upper semicomputable. Equivalently [5], a real map $r\colon \mathscr{D}\to\mathbb{R}$ is computable if and only if there is some recursive rational map $q\colon \mathscr{D}\times\mathbb{N}_0\to\mathbb{Q}$ such that

$$|r(d)-q(d,n)|<2^{-n} \text{ for all } d\in\mathscr{D} \text{ and } n\in\mathbb{N}_0.$$

By the Church-Turing thesis, this means that there is some algorithm that, given any $d\in\mathscr{D}$ and $N\in\mathbb{N}_0$ as input, allows us to approximate the real number $r(d)$ to within a precision of 2^{-N}. A real number $x\in\mathbb{R}$ is called *computable* if there is some recursive rational map $q\colon \mathbb{N}_0\to\mathbb{Q}$ such that $|x-q(n)|<2^{-n}$ for all $n\in\mathbb{N}_0$; this corresponds to the general definition with $\mathscr{D}$ a singleton. An interval forecast $I\in\mathcal{I}$ is called *computable* if the reals $\min I$ and $\max I$ are both computable; since finitely many algorithms can be combined into one, this corresponds to the general definition with $\mathscr{D}$ a doubleton—such as $\mathscr{X}$. A forecasting system $\varphi\in\Phi$ is called *computable* if the real processes $\underline{\varphi}$ and $\overline{\varphi}$ are both computable; since finitely many algorithms can be combined into one, this corresponds to the general definition with $\mathscr{D}$ equal to $\mathbb{S}\times\mathscr{X}$. A real multiplier process $D\colon \mathbb{S}\to\mathscr{L}(\mathscr{X})$ is called *computable* if it is computable as a real map on $\mathbb{S}\times\mathscr{X}$ that maps any $(s,x)\in\mathbb{S}\times\mathscr{X}$ to $D(s)(x)$.

APPENDIX B. PROOFS AND ADDITIONAL MATERIAL

In this part of the Appendix, we have gathered all of the more lengthy proofs that make use of computability theory, and some additional material necessary for understanding the argumentation in these proofs. It is divided into three sections: in Section B.1 we provide some additional material that is used in our proofs, and in Sections B.2–B.3 we have collected most of the proofs for Sections 6 and 7, respectively. Moreover, Section B.2 also contains the proof of Proposition 7 (which appears at the end of Section 5); we postpone this proof until the end of Section B.2.3 because it will follow immediately from Proposition 31—whose proof can be found in Section B.2.2—and Theorem 12—whose proof makes use of a result that is proved in Section B.2.3.

B.1. **Some additional material about allowable gambles, test supermartingales and multiplier processes.**

In our proofs, we will use an operator to characterise whether a gamble $f \in \mathscr{L}(\mathscr{X})$ is allowed by an interval forecast $I \in \mathcal{I}$ or not. We associate with every interval forecast $I \in \mathcal{I}$ the so-called *upper expectation* $\overline{E}_I \colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$, defined by

$$\overline{E}_I(f) := \max_{p \in I} E_p(f) = \max_{p \in I}\{pf(1) + (1-p)f(0)\} \text{ for all } f \in \mathscr{L}(\mathscr{X}). \tag{9}$$

Clearly, a gamble $f \in \mathscr{L}(\mathscr{X})$ is allowable for an interval forecast $I \in \mathcal{I}$ if and only if its upper expectation $\overline{E}_I(f)$ is non-positive, i.e., if and only if $\overline{E}_I(f) \leq 0$.

It will be convenient to have the following properties at our disposal. For all $f \in \mathscr{L}(\mathscr{X})$, it readily follows from Equation (9) that

$$\overline{E}_I(f) = \max\{\min I f(1) + (1-\min I)f(0), \max I f(1) + (1-\max I)f(0)\} \tag{10}$$

$$= \begin{cases} \min I f(1) + (1-\min I)f(0) & \text{if } f(1) \leq f(0) \\ \max I f(1) + (1-\max I)f(0) & \text{if } f(1) > f(0). \end{cases} \tag{11}$$

The upper expectation operator $\overline{E}_I$ also satisfies the following properties [28, 29].[7]

Proposition 21. *Consider any interval forecast $I \in \mathcal{I}$. Then for all gambles $f, g \in \mathscr{L}(\mathscr{X})$, all sequences of gambles $(f_n)_{n\in\mathbb{N}_0} \in \mathscr{L}(\mathscr{X})^{\mathbb{N}_0}$, and all $\mu \in \mathbb{R}$ and $\lambda \geq 0$:*

C1. $\min f \leq \overline{E}_I(f) \leq \max f$; [boundedness]
C2. $\overline{E}_I(\lambda f) = \lambda \overline{E}_I(f)$; [non-negative homogeneity]
C3. $\overline{E}_I(f+g) \leq \overline{E}_I(f) + \overline{E}_I(g)$; [subadditivity]
C4. $\overline{E}_I(f+\mu) = \overline{E}_I(f) + \mu$; [constant additivity]
C5. *if $f \leq g$ then $\overline{E}_I(f) \leq \overline{E}_I(g)$*; [increasingness]
C6. if $\lim_{n\to\infty} f_n = f$ then $\lim_{n\to\infty} \overline{E}_I(f_n) = \overline{E}_I(f)$. [continuity]

In our proofs, we will also use the fact that, for every forecasting system $\varphi \in \Phi$, there is a close connection between test supermartingales $T \in \overline{\mathbb{T}}(\varphi)$ and a specific type of real multiplier processes D. To reveal this connection, we introduce the following terminology: a real multiplier process D is called a real supermartingale multiplier for the forecasting system φ if $\overline{E}_{\varphi(s)}(D(s)) \leq 1$ for all $s \in \mathbb{S}$. Moreover, it is called positive (respectively non-negative) if $D(s)(x) > 0$ (respectively $D(s)(x) \geq 0$) for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$.

Lemma 22. *Consider a forecasting system $\varphi \in \Phi$ and a computable positive real supermartingale multiplier D for φ. Then $D^{\circledcirc}$ is a computable positive test supermartingale for φ.*

Proof. The computability of $D^{\circledcirc}$ follows immediately from Proposition 23 in Ref. [30]. To prove its positivity, recall that $D^{\circledcirc}(s) := \prod_{k=0}^{n-1} D(x_{1:k})(x_{k+1})$ for all $s = (x_1, \dots, x_n) \in \mathbb{S}$. Since $D(s)(x) > 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, clearly also $D^{\circledcirc}(s) > 0$ for all $s \in \mathbb{S}$. It now only remains to prove that $D^{\circledcirc}$ is a supermartingale for φ. To this end, note that

$$\begin{aligned}\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) &= \overline{E}_{\varphi(s)}(D^{\circledcirc}(s\,\bullet) - D^{\circledcirc}(s)) \\ &= \overline{E}_{\varphi(s)}(D^{\circledcirc}(s)D(s) - D^{\circledcirc}(s)) \overset{\text{C2}}{=} D^{\circledcirc}(s)\overline{E}_{\varphi(s)}(D(s)-1),\end{aligned}$$

and therefore we have the following chain of equivalences:

$$\begin{aligned}\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) \leq 0 &\Leftrightarrow D^{\circledcirc}(s)\overline{E}_{\varphi(s)}(D(s)-1) \leq 0 \\ &\overset{D^{\circledcirc}(s)>0}{\Leftrightarrow} \overline{E}_{\varphi(s)}(D(s)-1) \leq 0 \\ &\overset{\text{C4}}{\Leftrightarrow} \overline{E}_{\varphi(s)}(D(s)) \leq 1.\end{aligned}$$

□

[7]We note that C6 is usually presented as a property that requires uniform instead of pointwise convergence. However, since $\mathscr{X}$ is a finite sample space, uniform convergence is equivalent to pointwise convergence in this context.

B.2. **Proofs for Section 6 and of Proposition 7.**

B.2.1. *Outer bounds on $I_{\mathrm{R}}(\omega)$ for general forecasting systems.*

Lemma 23. *Consider any interval forecast $I \in \mathcal{I}$ such that $0 < \max I$ and $\min I < 1$, and any test supermartingale $T \in \overline{\mathbb{T}}(I)$. Then*

$$T(s\bullet) \leq T(s)\max\left\{\frac{1}{\max I}, \frac{1}{1-\min I}\right\} \text{ for all } s \in \mathbb{S}.$$

Proof. Since $0 < \max I$ and $\min I < 1$, both $1/\max I$ and $1/1-\min I$ are positive real numbers. Fix any $s \in \mathbb{S}$. T being a supermartingale for I implies that

$$\begin{aligned}0 \geq \overline{E}_I(\Delta T(s)) &\geq (\max I)\Delta T(s)(1) + (1-\max I)\Delta T(s)(0)\\ &= (\max I)T(s1) + (1-\max I)T(s0) - T(s)\\ &\geq (\max I)T(s1) - T(s),\end{aligned}$$

where the second inequality follows from Equation (10) and the last one from the fact that $\max I \leq 1$ and $T(s0) \geq 0$. Hence, $T(s1) \leq T(s)/\max I$.

Similarly, T being a supermartingale for I implies that

$$\begin{aligned}0 \geq \overline{E}_I(\Delta T(s)) &\geq (\min I)\Delta T(s)(1) + (1-\min I)\Delta T(s)(0)\\ &= (\min I)T(s1) + (1-\min I)T(s0) - T(s)\\ &\geq (1-\min I)T(s0) - T(s),\end{aligned}$$

where the second inequality follows from Equation (10) and the last one form the fact that $\min I \geq 0$ and $T(s1) \geq 0$. Hence, $T(s0) \leq T(s)/(1-\min I)$. □

Lemma 24. *Consider any lower semicomputable real process $F_1 \colon \mathbb{S} \to \mathbb{R}$ and any recursive non-negative rational process $F_2 \colon \mathbb{S} \to \mathbb{Q}$. Then the real process $F \colon \mathbb{S} \to \mathbb{R}$, defined by $F(s) := F_1(s)F_2(s)$ for all $s \in \mathbb{S}$, is lower semicomputable.*

Proof. F_1 is lower semicomputable, so there is some recursive rational map $q \colon \mathbb{S} \times \mathbb{N}_0 \to \mathbb{Q}$ such that

(i) $q(s,n+1) \geq q(s,n)$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$;
(ii) $F_1(s) = \lim_{n\to\infty} q(s,n)$ for all $s \in \mathbb{S}$.

Consider the map $q' \colon \mathbb{S} \times \mathbb{N}_0 \to \mathbb{Q}$ defined by

$$q'(s,n) := q(s,n)F_2(s) \text{ for all } s \in \mathbb{S} \text{ and } n \in \mathbb{N}_0.$$

Since q is a recursive rational map and since the process F_2 is recursive and rational, the map q' is recursive and rational as well. Due to (i) and the non-negativity of F_2, we find that

$$q'(s,n+1) = q(s,n+1)F_2(s) \geq q(s,n)F_2(s) = q'(s,n) \text{ for all } s \in \mathbb{S} \text{ and } n \in \mathbb{N}_0.$$

Moreover,

$$\lim_{n\to\infty} q'(s,n) = \lim_{n\to\infty} q(s,n)F_2(s) = F_1(s)F_2(s) = F(s) \text{ for all } s \in \mathbb{S},$$

taking into account (ii), so F is indeed lower semicomputable. □

Lemma 25. *Consider any non-negative real process $D_1^{\circledcirc} \colon \mathbb{S} \to \mathbb{R}$ generated by some lower semicomputable non-negative real multiplier process $D_1 \colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ and any recursive positive rational process $F_2 \colon \mathbb{S} \to \mathbb{Q}$ such that $F_2(\Box) = 1$. Then the real process $F \colon \mathbb{S} \to \mathbb{R}$, defined by $F(s) := D_1^{\circledcirc}(s)F_2(s)$ for all $s \in \mathbb{S}$, is generated by some lower semicomputable multiplier process.*

Proof. D_1 is lower semicomputable, so there is some recursive rational map $q \colon \mathbb{S} \times \mathscr{X} \times \mathbb{N}_0 \to \mathbb{Q}$ such that

(i) $q(s,x,n+1) \geq q(s,x,n)$ for all $s \in \mathbb{S}$, $x \in \mathscr{X}$ and $n \in \mathbb{N}_0$;
(ii) $D_1(s)(x) = \lim_{n\to\infty} q(s,x,n)$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$.

To prove that F is generated by a lower semicomputable multiplier process, consider the map $q'\colon \mathbb{S}\times\mathscr{X}\times\mathbb{N}_0\to\mathbb{Q}$ defined by

$$q'(s,x,n) := q(s,x,n)\frac{F_2(sx)}{F_2(s)} \text{ for all } s\in\mathbb{S},\, x\in\mathscr{X} \text{ and } n\in\mathbb{N}_0.$$

Since q is a recursive rational map and since the process F_2 is recursive, positive and rational, the map q' is well-defined, recursive and rational. Due to (i) and the positivity of F_2, we find that

$$q'(s,x,n+1) = q(s,x,n+1)\frac{F_2(sx)}{F_2(s)} \geq q(s,x,n)\frac{F_2(sx)}{F_2(s)} = q'(s,x,n) \\ \text{for all } s\in\mathbb{S},\, x\in\mathscr{X} \text{ and } n\in\mathbb{N}_0.$$

Moreover,

$$\lim_{n\to\infty} q'(s,x,n) = \lim_{n\to\infty} q(s,x,n)\frac{F_2(sx)}{F_2(s)} = D_1(s)(x)\frac{F_2(sx)}{F_2(s)} \text{ for all } s\in\mathbb{S} \text{ and } x\in\mathscr{X}.$$

This tells us that the non-negative multiplier process $D\colon \mathbb{S}\to\mathscr{L}(\mathscr{X})$, defined by $D(s) := D_1(s)\frac{F_2(s\bullet)}{F_2(s)}$ for all $s\in\mathbb{S}$, is lower semicomputable. We complete the proof by showing that F is generated by D. Indeed, since F_2 is positive and $F_2(\square)=1$, we find that

$$F(s\bullet) = D_1^{\circledcirc}(s\bullet)F_2(s\bullet) = D_1^{\circledcirc}(s)D_1(s)F_2(s\bullet) = D_1^{\circledcirc}(s)F_2(s)D(s) = F(s)D(s) \text{ for all } s\in\mathbb{S}.$$

□

Lemma 26. *Consider any computable real process $F_1\colon \mathbb{S}\to\mathbb{R}$ and any recursive rational process $F_2\colon \mathbb{S}\to\mathbb{Q}$ such that $0\leq F_2\leq 1$. Then the process $F\colon \mathbb{S}\to\mathbb{R}$, defined by $F(s) := F_1(s)F_2(s)$ for all $s\in\mathbb{S}$, is computable.*

Proof. Since F_1 is computable, there is some recursive rational map $q\colon \mathbb{S}\times\mathbb{N}_0\to\mathbb{Q}$ such that $|F_1(s)-q(s,n)|<2^{-n}$ for all $s\in\mathbb{S}$ and $n\in\mathbb{N}_0$. Consider the map $q'\colon \mathbb{S}\times\mathbb{N}_0\to\mathbb{Q}$ defined by

$$q'(s,n) := q(s,n)F_2(s) \text{ for all } s\in\mathbb{S} \text{ and } n\in\mathbb{N}_0.$$

Since q is a recursive rational map and since the process F_2 is recursive and rational, the map q' is recursive and rational as well. Since $0\leq F_2\leq 1$, it now follows that

$$|F(s)-q'(s,n)| = |F_1(s)F_2(s)-q(s,n)F_2(s)| = F_2(s)|F_1(s)-q(s,n)| \\ \leq |F_1(s)-q(s,n)| < 2^{-n} \text{ for all } s\in\mathbb{S} \text{ and } n\in\mathbb{N}_0,$$

so F is indeed computable. □

Lemma 27. *Consider any path $\omega\in\Omega$, any $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$, any forecasting system $\varphi\in\Phi$, any interval forecast $I\in\mathcal{I}$ such that $\min I<\max I$, and any recursive selection process S such that, for all $s\in\mathbb{S}$, $\varphi(s)\subseteq I$ if $S(s)=0$, and such that $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})<\infty$. If ω is R-random for φ, then it is also R-random for I.*

Proof. We assume that ω is R-random for φ and set out to prove that it is then also R-random for I. In line with Definitions 4 and 5, we fix any test supermartingale $T\in\overline{\mathbb{T}}_{\mathrm{R}}(I)$ and show that it remains (computably) bounded on ω.

To this end, we use the selection process S to introduce the process S' defined by

$$S'(s) := \sum_{k=0}^{n-1} S(x_{1:k}) \text{ for all } s=(x_1,\dots,x_n)\in\mathbb{S},$$

which counts for every situation $s\in\mathbb{S}$ how many times $S(t)=1$ for all strictly preceding situations $t\sqsubset s$. Since S is recursive and only takes values in $\{0,1\}$, and since there is clearly a single algorithm that, for every $s\in\mathbb{S}$, can enumerate the finite number of situations $t\in\mathbb{S}$ for which $t\sqsubset s$, the process S' is recursive and assumes values in the non-negative integers.

Since $0 \leq \min I < \max I \leq 1$, it follows that $0 < \max I$ and $\min I < 1$, so we can fix some rational $K > 1$ such that $K > \max\{1/\max I, 1/1-\min I\}$. We introduce a new process F' defined by

$$F'(s) := K^{-S'(s)} \text{ for all } s \in \mathbb{S}.$$

What are the relevant properties of this process F'? Since K is rational and since the process S' is recursive and takes on non-negative integers, the process F' is recursive and rational. Furthermore, since K is positive and since the process S' is non-negative, the process F' is positive as well, that is, $F'(s) > 0$ for all $s \in \mathbb{S}$. Also, since $K > 1$ and $S'(s) \in \mathbb{N}_0$ for all $s \in \mathbb{S}$, $F'(s) = K^{-S'(s)} \leq 1$ for all $s \in \mathbb{S}$. Finally, note that $F'(\square) = K^{-S'(\square)} = K^{-0} = 1$.

Next, consider the real process T', defined by

$$T'(s) := T(s)F'(s) \text{ for all } s \in \mathbb{S}. \tag{12}$$

Let us now show that T' is a test supermartingale for φ. First, since T and F' are non-negative, so is T'. Second, $T'(\square) = 1$, since $T(\square) = 1$ and $F'(\square) = 1$. Last, to show that it is a supermartingale for φ, we fix any $s \in \mathbb{S}$ and prove that $\overline{E}_{\varphi(s)}(\Delta T'(s)) \leq 0$. Note that

$$\begin{aligned}\Delta T'(s) &= T'(s\bullet) - T'(s) = T(s\bullet)F'(s\bullet) - T(s)F'(s)\\ &= T(s\bullet)K^{-S'(s\bullet)} - T(s)K^{-S'(s)} = T(s\bullet)K^{-S'(s)-S(s)} - T(s)K^{-S'(s)}\\ &= K^{-S'(s)}\Big[T(s\bullet)K^{-S(s)} - T(s)\Big].\end{aligned} \tag{13}$$

We now consider two cases. If $S(s) = 1$, then

$$\Delta T'(s) \overset{(13)}{=} K^{-S'(s)}\Big[T(s\bullet)K^{-1} - T(s)\Big]. \tag{14}$$

Since $0 < \max I$ and $\min I < 1$, $K > \max\{1/\max I, 1/1-\min I\}$ and $T \in \overline{\mathbb{T}}_{\mathrm{R}}(I) \subseteq \overline{\mathbb{T}}(I)$, it follows from Lemma 23 that

$$T(s\bullet) \leq T(s)\max\left\{\frac{1}{\max I}, \frac{1}{1-\min I}\right\} \leq T(s)K,$$

so $T(s\bullet)K^{-1} - T(s) \leq 0$. Hence also, by Equation (14), $\Delta T'(s) \leq 0$, and therefore, by C1, indeed $\overline{E}_{\varphi(s)}(\Delta T'(s)) \leq 0$.

Otherwise, that is, if $S(s) = 0$, then

$$\overline{E}_{\varphi(s)}(\Delta T'(s)) \overset{(13)}{=} \overline{E}_{\varphi(s)}\big(K^{-S'(s)}\Delta T(s)\big) \overset{\mathrm{C2}}{=} K^{-S'(s)}\overline{E}_{\varphi(s)}(\Delta T(s)). \tag{15}$$

By assumption, $\varphi(s) \subseteq I$ since $S(s) = 0$. Equation (9) therefore leads to the conclusion that $\overline{E}_{\varphi(s)}(\Delta T(s)) \leq \overline{E}_I(\Delta T(s)) \leq 0$, where the last equality holds because T is a supermartingale for I. If we plug this into Equation (15), we find that, in this case too, $\overline{E}_{\varphi(s)}(\Delta T'(s)) \leq 0$. Hence, T' is indeed a supermartingale for φ.

To also show that $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$, in addition to T' being a test supermartingale for φ, we need to check whether it has the appropriate implementability properties.

If $\mathrm{R} = \mathrm{ML}$, then T is a lower semicomputable real process. Since F' is a recursive non-negative rational process, it follows from Equation (12) and Lemma 24 that T' is lower semicomputable.

If $\mathrm{R} = \mathrm{wML}$, then T is generated by a lower semicomputable non-negative real multiplier process. Since F' is a recursive positive rational process with $F'(\square) = 1$, it follows from Equation (12) and Lemma 25 that T' is generated by a lower semicomputable multiplier process.

If $\mathrm{R} \in \{\mathrm{C}, \mathrm{S}\}$, then T is a computable real process. Since F' is a recursive rational process with $0 < F' \leq 1$, it follows from Equation (12) and Lemma 26 that T' is computable.

We may therefore conclude that, in all cases, $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$.

To conclude the proof, it only remains to show that T' is (computably) bounded on ω. By recalling that $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})<\infty$ by assumption, we can fix some $N\in\mathbb{N}_0$ such that $\lim_{n\to\infty}S'(\omega_{1:n})=N$, so

$$\limsup_{n\to\infty}T'(\omega_{1:n})\overset{(12)}{=}\limsup_{n\to\infty}T(\omega_{1:n})F'(\omega_{1:n})=K^{-N}\limsup_{n\to\infty}T(\omega_{1:n}). \tag{16}$$

We now consider two cases.

If $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C}\}$, then since ω is R-random for φ by assumption and since $T'\in\overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$, it follows from Definition 4 that $\limsup_{n\to\infty}T'(\omega_{1:n})<\infty$, and therefore also, by Equation (16), that $\limsup_{n\to\infty}T(\omega_{1:n})<\infty$. ω is therefore indeed R-random for I.

If $\mathrm{R}=\mathrm{S}$, then since ω was assumed to be S-random for φ, no computable test supermartingale for φ is computably unbounded on ω; see Definition 5. Assume towards contradiction that $T\in\overline{\mathbb{T}}_{\mathrm{S}}(I)$ is computably unbounded on ω, meaning that there is some real growth function τ such that $\limsup_{n\to\infty}[T(\omega_{1:n})-\tau(n)]\geq 0$. Consider the real growth function τ' defined by $\tau'(n):=K^{-N}\tau(n)$ for all $n\in\mathbb{N}_0$. A similar argument as in Equation (16) then leads to

$$\limsup_{n\to\infty}[T'(\omega_{1:n})-\tau'(n)]=K^{-N}\limsup_{n\to\infty}[T(\omega_{1:n})-\tau(n)]\geq 0,$$

so the test supermartingale T' for φ, which is computable [see the argumentation above], is computably unbounded on ω, a contradiction. ω is therefore indeed R-random for I. □

Proposition 28. *For any* $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$ *and any path* $\omega\in\Omega$ *that is* R*-random for a given forecasting system* $\varphi\in\Phi$ *it holds that* $I_{\mathrm{R}}(\omega)\subseteq I_{[\varphi]}(\omega)$.

Proof. Consider any $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$ and any path $\omega\in\Omega$ that is R-random for a given forecasting system φ. Consider any $N\in\mathbb{N}_0$, and let $I_N:=[p_N,q_N]$, with

$$p_N:=\max\left\{0,\inf_{s\in[\omega_{1:N}]}\underline{\varphi}(s)-2^{-N}\right\}<q_N:=\min\left\{1,\sup_{s\in[\omega_{1:N}]}\overline{\varphi}(s)+2^{-N}\right\}.$$

Observe that $p_{N+1}\geq p_N$ and $q_{N+1}\leq q_N$, and therefore $I_{N+1}\subseteq I_N$. Since obviously also $\lim_{N\to\infty}p_N=\lim_{N\to\infty}\inf_{s\in[\omega_{1:N}]}\underline{\varphi}(s)$ and $\lim_{N\to\infty}q_N=\lim_{N\to\infty}\sup_{s\in[\omega_{1:N}]}\overline{\varphi}(s)$, we find that $I_{[\varphi]}(\omega)=\lim_{N\to\infty}I_N=\bigcap_{N\in\mathbb{N}_0}I_N$. This tells us that if we can show that the path ω is R-random for the interval forecast I_N, we will be essentially done, because in that case we will have that $I_{\mathrm{R}}(\omega)\subseteq I_N$, and therefore also $I_{\mathrm{R}}(\omega)\subseteq\bigcap_{N\in\mathbb{N}_0}I_N=I_{[\varphi]}(\omega)$.

So let us now fix any $N\in\mathbb{N}_0$ and prove that path ω is indeed R-random for the interval forecast I_N.

To this end, consider the selection process S defined by

$$S(s):=\begin{cases}0 & \text{if } \omega_{1:N}\sqsubseteq s\\ 1 & \text{otherwise}\end{cases}\quad\text{for all } s\in\mathbb{S}.$$

Since $\omega_{1:N}$ is a fixed situation, the selection process S is recursive. By construction, $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})=\sum_{k=0}^{N-1}S(\omega_{1:k})=N<\infty$. Moreover, for all $s\in\mathbb{S}$, $\varphi(s)\subseteq I_N$ if $S(s)=0$ (or equivalently, if $\omega_{1:N}\sqsubseteq s$), because $p_N\leq\inf_{s\in[\omega_{1:N}]}\underline{\varphi}(s)$ and $\sup_{s\in[\omega_{1:N}]}\overline{\varphi}(s)\leq q_N$ by definition. Consequently, since ω is assumed to be R-random for φ, it follows from Lemma 27 [with $I=I_N$] that ω is R-random for I_N. □

Lemma 29. *For any path* $\omega\in\Omega$*, any real sequence* $F\colon\mathbb{N}_0\to\mathbb{R}$*, any* $M\in\mathbb{N}_0$*, and any selection process* S *such that* $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})=\infty$*, it holds that*

$$\liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}=\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=M}^{n-1}S(\omega_{1:k})}$$

and

$$\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}=\limsup_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=M}^{n-1}S(\omega_{1:k})}.$$

Proof. We will give the proof for the first equality; the proof for the second equality is very similar. Since $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})=\infty$, it holds that

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{M-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}=0 \text{ and } \lim_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})}{\sum_{k=0}^{n-1}S(\omega_{1:k})}=1,$$

and therefore

$$\begin{aligned}\liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}&=\liminf_{n\to\infty}\left(\frac{\sum_{k=0}^{M-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}+\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\right)\\&=\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\\&=\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\\&=\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})F(k)}{\sum_{k=M}^{n-1}S(\omega_{1:k})}.\end{aligned}$$

□

Proposition 30. *For any* $\mathrm{R}\in\{\mathrm{CH},\mathrm{wCH}\}$ *and any path* $\omega\in\Omega$ *that is* R*-random for a given forecasting system* $\varphi\in\Phi$ *it holds that* $I_{\mathrm{R}}(\omega)\subseteq I_{\varphi}(\omega)$.

Proof. We will give the proof for $\mathrm{R}=\mathrm{CH}$. The proof for $\mathrm{R}=\mathrm{wCH}$ is very similar.[8] Consider any path $\omega\in\Omega$ that is CH-random for a given forecasting system φ. Consider any $N\in\mathbb{N}_0$, and let $I_N:=[p_N,q_N]$, with

$$p_N:=\max\left\{0,\liminf_{n\to\infty}\underline{\varphi}(\omega_{1:n})-2^{-N}\right\}<q_N:=\min\left\{1,\limsup_{n\to\infty}\overline{\varphi}(\omega_{1:n})+2^{-N}\right\}.$$

Note that $p_{N+1}\geq p_N$ and $q_{N+1}\leq q_N$, so $I_{N+1}\subseteq I_N$. Also $\lim_{N\to\infty}p_N=\liminf_{n\to\infty}\underline{\varphi}(\omega_{1:n})$ and $\lim_{N\to\infty}q_N=\limsup_{n\to\infty}\overline{\varphi}(\omega_{1:n})$, so we find that $I_{\varphi}(\omega)=\lim_{N\to\infty}I_N=\bigcap_{N\in\mathbb{N}_0}I_N$. This tells us that if we can show that the path ω is CH-random for the interval forecast I_N, we will be essentially done, because in that case we will have that $I_{\mathrm{CH}}(\omega)\subseteq I_N$, and therefore also $I_{\mathrm{CH}}(\omega)\subseteq\bigcap_{N\in\mathbb{N}_0}I_N=I_{\varphi}(\omega)$.

So let us now fix an arbitrary $N\in\mathbb{N}_0$ and show that the path ω is indeed CH-random for the interval forecast I_N. Consider any recursive selection process S that is adequate along ω. Since ω is CH-random for φ by assumption, it follows from Definition 6 that

$$\liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\underline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\geq 0 \text{ and } \limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\overline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\leq 0. \tag{17}$$

The definition of the p_N and q_N makes sure we can fix some $M\in\mathbb{N}_0$ such that

$$p_N\leq\underline{\varphi}(\omega_{1:n})\leq\overline{\varphi}(\omega_{1:n})\leq q_N \text{ for all } n\geq M. \tag{18}$$

Since $S\in\mathcal{S}_{\mathrm{CH}}(\omega)$, and therefore $\lim_{n\to\infty}\sum_{k=0}^{n-1}S(\omega_{1:k})=\infty$, it follows from Equation (17) and Lemma 29 [with $F(k)=\omega_{k+1}-\underline{\varphi}(\omega_{1:k})$ and $F(k)=\omega_{k+1}-\overline{\varphi}(\omega_{1:k})$, respectively, for all $k\in\mathbb{N}_0$] that

$$\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\underline{\varphi}(\omega_{1:k})]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\geq 0 \text{ and } \limsup_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\overline{\varphi}(\omega_{1:k})]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\leq 0,$$

[8] Simply replace CH by wCH, and consider recursive *adequate* temporal selection processes $S\in\mathcal{S}_{\mathrm{wCH}}$ instead of the more general set $\mathcal{S}_{\mathrm{CH}}(\omega)$ of recursive selection processes that are adequate along ω.

so we can infer from Equation (18) that

$$\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-p_N]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\geq\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\underline{\varphi}(\omega_{1:k})]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\geq 0$$

and similarly

$$\limsup_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-q_N)]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\leq\limsup_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})[\omega_{k+1}-\overline{\varphi}(\omega_{1:k})]}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\leq 0.$$

Consequently,

$$p_N\leq\liminf_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})\omega_{k+1}}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\leq\limsup_{n\to\infty}\frac{\sum_{k=M}^{n-1}S(\omega_{1:k})\omega_{k+1}}{\sum_{k=M}^{n-1}S(\omega_{1:k})}\leq q_N,$$

and therefore also by Lemma 29 [with $F(k)=\omega_{k+1}$ for all $k\in\mathbb{N}_0$], again since S is adequate along ω,

$$p_N\leq\liminf_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\leq\limsup_{n\to\infty}\frac{\sum_{k=0}^{n-1}S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1}S(\omega_{1:k})}\leq q_N. \qquad \square$$

Proof of Proposition 8. This proposition is an immediate consequence of Propositions 28 and 30. □

B.2.2. *Outer bounds on $\mathrm{I}_{\mathrm{R}}(\omega)$ for computable forecasting systems.*

Proposition 31. *For any* $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$*, any path* $\omega\in\Omega$ *that is* R*-random for a computable forecasting system* $\varphi\in\Phi$ *and any* $\varepsilon_1,\varepsilon_2>0$*, it holds that*

$$\Big[\liminf_{n\to\infty}\underline{\varphi}(\omega_{1:n})-\varepsilon_1,\limsup_{n\to\infty}\overline{\varphi}(\omega_{1:n})+\varepsilon_2\Big]\cap[0,1]\in\mathcal{I}_{\mathrm{R}}(\omega).$$

Proof. Consider any $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$, any path $\omega\in\Omega$ that is R-random for a computable forecasting system φ and any $\varepsilon_1,\varepsilon_2>0$. If we let $p:=\liminf_{n\to\infty}\underline{\varphi}(\omega_{1:n})$ and $q:=\limsup_{n\to\infty}\overline{\varphi}(\omega_{1:n})$, then we need to show that the path ω is R-random for the interval forecast $I:=[p-\varepsilon_1,q+\varepsilon_2]\cap[0,1]$; observe that $\min I<\max I$. To this end, fix any rational numbers $\underline{r}$ and $\overline{r}$ and any natural number N such that

$$p-\frac{3}{4}\varepsilon_1<\underline{r}<p-\frac{1}{2}\varepsilon_1\text{ and }q+\frac{1}{2}\varepsilon_2<\overline{r}<q+\frac{3}{4}\varepsilon_2\text{ and }2^{-N}<\frac{1}{4}\min\{\varepsilon_1,\varepsilon_2\}. \tag{19}$$

Since φ is a computable forecasting system, there are two recursive rational maps $\underline{q},\overline{q}\colon\mathbb{S}\times\mathbb{N}_0\to\mathbb{Q}$ such that $|\underline{\varphi}(s)-\underline{q}(s,n)|<2^{-n}$ and $|\overline{\varphi}(s)-\overline{q}(s,n)|<2^{-n}$ for all $s\in\mathbb{S}$ and $n\in\mathbb{N}_0$. Consequently,

$$\big|\underline{\varphi}(s)-\underline{q}(s,N)\big|<2^{-N}<\frac{1}{4}\varepsilon_1\text{ and }\big|\overline{\varphi}(s)-\overline{q}(s,N)\big|<2^{-N}<\frac{1}{4}\varepsilon_2\text{ for all }s\in\mathbb{S}. \tag{20}$$

To show that ω is R-random for I, consider the selection process S defined by

$$S(s):=\begin{cases}1 & \text{if }\underline{q}(s,N)<\underline{r}\text{ or }\overline{r}<\overline{q}(s,N)\\ 0 & \text{otherwise}\end{cases}\quad\text{for all }s\in\mathbb{S}.$$

Since $\underline{r}$ and $\overline{r}$ are rational numbers, N is a natural number, and $\underline{q}$ and $\overline{q}$ are recursive rational maps, the inequalities in the above expression are decidable for every $s\in\mathbb{S}$, so the selection process S is recursive. We continue by proving that the recursive selection process S satisfies the conditions in Lemma 27.

Fix any $s\in\mathbb{S}$. If $S(s)=0$, or equivalently, if $\underline{r}\leq\underline{q}(s,N)$ and $\overline{q}(s,N)\leq\overline{r}$, then also

$$p-\frac{3}{4}\varepsilon_1\overset{(19)}{<}\underline{r}\leq\underline{q}(s,N)\overset{(20)}{<}\underline{\varphi}(s)+\frac{1}{4}\varepsilon_1\text{ and }q+\frac{3}{4}\varepsilon_2\overset{(19)}{>}\overline{r}\geq\overline{q}(s,N)\overset{(20)}{>}\overline{\varphi}(s)-\frac{1}{4}\varepsilon_2,$$

so $p-\varepsilon_1<\underline{\varphi}(s)$ and $\overline{\varphi}(s)<q+\varepsilon_2$, and therefore $\varphi(s)\subseteq I$.

Moreover, for any $n \in \mathbb{N}_0$, if $\underline{q}(\omega_{1:n}, N) < \underline{r}$, then

$$\underline{\varphi}(\omega_{1:n}) \overset{(20)}{<} \underline{q}(\omega_{1:n}, N) + \frac{1}{4}\varepsilon_1 < \underline{r} + \frac{1}{4}\varepsilon_1 \overset{(19)}{<} p - \frac{1}{2}\varepsilon_1 + \frac{1}{4}\varepsilon_1 = p - \frac{1}{4}\varepsilon_1.$$

Similarly, if $\overline{r} < \overline{q}(\omega_{1:n}, N)$, then

$$\overline{\varphi}(\omega_{1:n}) \overset{(20)}{>} \overline{q}(\omega_{1:n}, N) - \frac{1}{4}\varepsilon_2 > \overline{r} - \frac{1}{4}\varepsilon_2 \overset{(19)}{>} q + \frac{1}{2}\varepsilon_2 - \frac{1}{4}\varepsilon_2 = q + \frac{1}{4}\varepsilon_2.$$

If we recall the definition of p and q, it becomes clear that there is only a finite number of $n \in \mathbb{N}_0$ for which $\underline{\varphi}(\omega_{1:n}) < p - \frac{1}{4}\varepsilon_1$ or $q + \frac{1}{4}\varepsilon_2 < \overline{\varphi}(\omega_{1:n})$, and as a result, there is only a finite number of natural numbers $n \in \mathbb{N}_0$ for which $\underline{q}(\omega_{1:n}, N) < \underline{r}$ or $\overline{r} < \overline{q}(\omega_{1:n}, N)$, or, equivalently, for which $S(\omega_{1:n}) = 1$. Consequently, $\lim_{n\to\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) < \infty$.

By invoking Lemma 27, since ω is assumed to be R-random for φ, we then infer that ω is R-random for I. □

Proof of Proposition 9. Consider any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ and any path $\omega \in \Omega$ that is R-random for a given computable forecasting system $\varphi \in \Phi$. In the interest of notational brevity, we let $p := \liminf_{n\to\infty} \underline{\varphi}(\omega_{1:n})$ and $q := \limsup_{n\to\infty} \overline{\varphi}(\omega_{1:n})$, so $I_\varphi(\omega) = [p, q]$. We consider two possible cases.

If $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$, the stated result follows immediately from Proposition 30.

If $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, then it follows from Proposition 31 that

$$[p - \varepsilon_1, q + \varepsilon_2] \cap [0,1] \in \mathcal{I}_\mathrm{R}(\omega) \text{ for all } \varepsilon_1, \varepsilon_2 > 0,$$

and therefore that

$$\begin{aligned} I_\mathrm{R}(\omega) = \bigcap \mathcal{I}_\mathrm{R}(\omega) &\subseteq \bigcap_{\varepsilon_1, \varepsilon_2 > 0} \big([p - \varepsilon_1, q + \varepsilon_2] \cap [0,1]\big) \\ &= [0,1] \cap \bigcap_{\varepsilon_1, \varepsilon_2 > 0} [p - \varepsilon_1, q + \varepsilon_2] = [p, q] = I_\varphi(\omega). \end{aligned}$$ □

B.2.3. *Inner bounds on $I_\mathrm{R}(\omega)$ for computable forecasting systems.*

Proposition 32. *If a path $\omega \in \Omega$ is* (w)CH*-random for a computable* (*temporal*) *forecasting system $\varphi \in \Phi$, then*

$$\min I_{(\mathrm{w})\mathrm{CH}}(\omega) \leq \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n}) \text{ and } \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n}) \leq \max I_{(\mathrm{w})\mathrm{CH}}(\omega).$$

Proof. We will give the proof for R = CH. The proof for R = wCH is very similar.[9] Consider any path $\omega \in \Omega_\mathrm{CH}(\varphi)$ and any $\varepsilon > 0$. If, for ease of notation, we let $p := \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n})$ and $q := \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n})$, then it is clearly enough to show that $\min I_\mathrm{CH}(\omega) \leq p + \varepsilon$ and $q - \varepsilon \leq \max I_\mathrm{CH}(\omega)$. We will prove the first inequality; the proof of the second one is very similar. Fix any rational number r and any natural number N such that

$$p + \frac{1}{2}\varepsilon < r < p + \frac{3}{4}\varepsilon \text{ and } 2^{-N} < \frac{1}{4}\varepsilon. \tag{21}$$

Since φ is a computable forecasting system, there is some recursive rational map $\overline{q} \colon \mathbb{S} \times \mathbb{N}_0 \to \mathbb{Q}$ such that $|\overline{\varphi}(s) - \overline{q}(s, n)| \leq 2^{-n}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}_0$. Consequently,

$$\big|\overline{\varphi}(s) - \overline{q}(s, N)\big| < 2^{-N} < \frac{1}{4}\varepsilon \text{ for all } s \in \mathbb{S}. \tag{22}$$

[9] Simply replace CH by wCH, and consider the forecasting system φ to be a map from the non-negative integers to the set of interval forecasts $\mathcal{I}$. Following the line of reasoning of the proof, the computability of $\overline{\varphi}$ then results in the existence of a recursive rational map $\overline{q} \colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ such that $|\overline{\varphi}(m) - \overline{q}(m, n)| \leq 2^{-n}$ for all $m, n \in \mathbb{N}_0$. Correspondingly, since the first argument of the map $\overline{q}$ takes non-negative integers instead of situations, the recursive selection process S becomes temporal, as is needed for the notion of wCH-randomness.

Consider the selection process S, defined by

$$S(s) := \begin{cases} 1 & \text{if } \overline{q}(s,N) < r \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{S}.$$

Since r is a rational number, N is a natural number, and $\overline{q}$ is a recursive rational map, the inequality in the above expression is decidable for every $s \in \mathbb{S}$, and the selection process S is recursive. If we recall the definition of p, we see that the subset $A := \{n \in \mathbb{N}_0 \colon \overline{\varphi}(\omega_{1:n}) < p + \frac{1}{4}\varepsilon\}$ of $\mathbb{N}_0$ is infinite. Observe that

$$\overline{q}(\omega_{1:n},N) \overset{(22)}{<} \overline{\varphi}(\omega_{1:n}) + \frac{1}{4}\varepsilon < p + \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = p + \frac{1}{2}\varepsilon \overset{(21)}{<} r \text{ for all } n \in A,$$

and therefore also $S(\omega_{1:n}) = 1$ for all $n \in A$, so $\lim_{n\to\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) = \infty$. Consequently, the CH-randomness of ω for φ guarantees through Definition 6 that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})[\omega_{k+1} - \overline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0.$$

Moreover, for any $s \in \mathbb{S}$, if $S(s) = 1$ or, equivalently, if $\overline{q}(s,N) < r$, then

$$\overline{\varphi}(s) \overset{(22)}{<} \overline{q}(s,N) + \frac{1}{4}\varepsilon < r + \frac{1}{4}\varepsilon \overset{(21)}{<} p + \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = p + \varepsilon.$$

Hence,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[\omega_{k+1} - (p+\varepsilon)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})[\omega_{k+1} - \overline{\varphi}(\omega_{1:k})]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0,$$

so, on the one hand,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq p + \varepsilon.$$

Since ω is CH-random for $I_{\mathrm{CH}}(\omega)$ by Proposition 5, we can also infer from Definition 6 that, on the other hand,

$$\min I_{\mathrm{CH}}(\omega) \leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\omega_{k+1}}{\sum_{k=0}^{n-1} S(\omega_{1:k})}.$$

Hence, indeed, $\min I_{\mathrm{CH}}(\omega) \leq p + \varepsilon$. □

Proof of Proposition 10. By Corollary 4, it is enough to show that

$$\min I_{\mathrm{CH}}(\omega) \leq \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n}) \text{ and } \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n}) \leq \max I_{\mathrm{CH}}(\omega).$$

Since ω is assumed to be R-random for φ [with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{CH}\}$] and since φ is assumed to be computable, it follows from Equation (3) that ω is CH-random for φ. The above then immediately follows from Proposition 32. □

Proof of Proposition 11. By Corollary 4, it is enough to show that

$$\min I_{\mathrm{wCH}}(\omega) \leq \liminf_{n\to\infty} \overline{\varphi}(\omega_{1:n}) \text{ and } \limsup_{n\to\infty} \underline{\varphi}(\omega_{1:n}) \leq \max I_{\mathrm{wCH}}(\omega).$$

Since ω is assumed to be R-random for φ [with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$] and since φ is assumed to be computable (and temporal), it follows from Equation (3) that ω is wCH-random for φ. The inequalities above then immediately follow from Proposition 32. □

We are now, as promised at the end of Section 5, in a better position to furnish a short proof for Proposition 7.

Proof of Proposition 7. Assume that the path $\omega \in \Omega$ is ML-random for a given computable precise forecasting system $\varphi \in \Phi$. We let, for ease of notation, $p := \liminf_{n\to\infty} \varphi(\omega_{1:n})$ and $q := \limsup_{n\to\infty} \varphi(\omega_{1:n})$, and then infer from Theorem 12 that $I_{\mathrm{ML}}(\omega) = [p,q]$.

First of all, consider any interval forecast $I \in \mathcal{I}$ for which ω is almost ML-random. Then it follows from Definition 3 that

$$[\min I - \varepsilon_1, \max I + \varepsilon_2] \cap [0,1] \in \mathcal{I}_{\mathrm{ML}}(\omega) \text{ for all } \varepsilon_1, \varepsilon_2 > 0,$$

and therefore also that

$$\begin{aligned} I_{\mathrm{ML}}(\omega) = \cap \mathcal{I}_{\mathrm{ML}}(\omega) &\subseteq \bigcap_{\varepsilon_1,\varepsilon_2>0} [\min I - \varepsilon_1, \max I + \varepsilon_2] \cap [0,1] \\ &= [0,1] \cap \bigcap_{\varepsilon_1,\varepsilon_2>0} [\min I - \varepsilon_1, \max I + \varepsilon_2] = I. \end{aligned}$$

It is therefore enough to show that ω is almost ML-random for the interval forecast $I_{\mathrm{ML}}(\omega)$.

We infer from Proposition 31 that ω is ML-random for any interval forecast of the form $[p - \varepsilon_1, q + \varepsilon_2] \cap [0,1]$ with $\varepsilon_1, \varepsilon_2 > 0$, which tells us that ω is indeed almost ML-random for the interval forecast $I_{\mathrm{ML}}(\omega) = [p,q]$; see Definition 3. □

B.3. **Proofs and additional material for Section 7.**

B.3.1. C-*randomness versus* ML-*randomness.*

Lemma 33. *For every computable interval forecast $I \in \mathcal{I}$ such that $0 < \max I$ and $\min I < 1$, there is a so-called* universal *lower semicomputable test supermartingale U for I, with the property that any path $\omega \in \Omega$ is not* ML*-random for I if and only if* $\lim_{n\to\infty} U(\omega_{1:n}) = \infty$.

Proof. This follows immediately from Corollary 32 in Ref. [8] since a computable interval forecast $I \in \mathcal{I}$ such that $0 < \max I$ and $\min I < 1$ is a special case of a so-called non-degenerate computable forecasting system; a forecasting system φ is called *non-degenerate* if $\underline{\varphi}(s) < 1$ and $\overline{\varphi}(s) > 0$ for all $s \in \mathbb{S}$. □

Lemma 34 ([31, Proposition 21]). *Consider any path $\omega \in \Omega$ and any interval forecast $I \subset (0,1)$. If ω is recursive, then ω is not* C*-random for I.*

Proposition 35. *A path $\omega \in \Omega$ is* C*-random for a forecasting system $\varphi \in \Phi$ if and only if no recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ is unbounded on ω.*

Proof. By Definition 4, a path $\omega \in \Omega$ is C-random for a forecasting system $\varphi \in \Phi$ if no computable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ is unbounded on ω. By Proposition 6 in Ref. [24], the path ω is C-random for φ if and only if there is no recursive positive rational strict test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ such that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$, where a real process $M \colon \mathbb{S} \to \mathbb{R}$ is called a strict supermartingale for φ if $\overline{E}_{\varphi(s)}(\Delta M(s)) < 0$ for every situation $s \in \mathbb{S}$. As a consequence, whenever we restrict the betting strategies in Definition 4 to a set that is smaller than the one in Definition 4, but larger than the one in Proposition 6 in Ref. [24], we obtain a definition for computably random sequences that is equivalent to Definition 4. □

For the following lemma and propositions, we draw inspiration from Schnorr's discussion of the difference between Martin-Löf randomness and its conjugate notion in terms of *upper* semicomputable test supermartingales [20].

Lemma 36. *For every recursive positive rational supermartingale M for $1/2$, every positive real $y \in \mathbb{R}_{>0}$ and every situation $s \in \mathbb{S}$ for which $M(t) \le y$ for all $t \sqsubseteq s$, there is a recursive path $\omega \in [\![s]\!]$ such that $M(\omega_{1:n}) \le y$ for all $n \in \mathbb{N}_0$.*

Proof. The path $\omega \in \Omega$ will be constructed by an induction argument. We begin by letting $\omega_{1:|s|} := s$. Now, assume that $\omega_{1:n}$ has already been constructed in such a way that $M(\omega_{1:k}) \leq y$ for all $k \leq n$; note that this condition is satisfied trivially for $n \leq |s|$. Since M is a supermartingale for $1/2$, there is always some $x_n \in \mathscr{X}$ for which $M(\omega_{1:n}x_n) \leq M(\omega_{1:n})$. Indeed, assume towards contradiction that $\Delta M(\omega_{1:n})(x) > 0$ for all $x \in \mathscr{X}$, then also $E_{1/2}(\Delta M(\omega_{1:n})) = \frac{1}{2}\Delta M(\omega_{1:n})(1) + \frac{1}{2}\Delta M(\omega_{1:n})(0) > 0$, which is impossible. Since M is a recursive rational process, we can determine such an $x_n \in \mathscr{X}$ recursively: if $M(\omega_{1:n}1) \leq M(\omega_{1:n})$, let $x_n = 1$, and otherwise, if $M(\omega_{1:n}1) > M(\omega_{1:n})$ and then $M(\omega_{1:n}0) \leq M(\omega_{1:n})$, let $x_n = 0$. Put $\omega_{1:n+1} := \omega_{1:n}x_n$, thus guaranteeing that $M(\omega_{1:n+1}) = M(\omega_{1:n}x_n) \leq M(\omega_{1:n}) \leq y$, and therefore $M(\omega_{1:k}) \leq y$ for all $k \leq n+1$.

To show that ω is recursive, it suffices to prove that there is an algorithm that, given any $n \in \mathbb{N}$, outputs the binary digit ω_n. By construction, if $n \leq |s|$, then $\omega_n = s_n$. Otherwise, that is, if $n > |s|$, then there is by construction a unique recursive way to extend $\omega_{1:|s|} = s$ up to any arbitrary length $N \in \mathbb{N}_0$ in a finite number of steps, which then allows to output ω_n, and thus concludes the proof. □

Proposition 37. *For any computable interval forecast $I \subseteq (0,1)$, there is a path $\omega \in \Omega$ that is* C*-random for $1/2$, but not* ML*-random for I.*

Proof. Fix any computable interval forecast $I \subseteq (0,1)$. By Lemma 33, there is a lower semicomputable test supermartingale $U \in \overline{\mathbb{T}}_{\mathrm{ML}}(I)$ that is unbounded on every path $\omega \in \Omega$ that is not ML-random for I. Let $(T_\ell)_{\ell \in \mathbb{N}_0}$ be an enumeration (not necessarily recursive) of all recursive positive rational test supermartingales for $1/2$; this is always possible because the set of all recursive processes is countable [26, 32].

Taking into account Proposition 35, it is enough to find a path $\omega \in \Omega$ for which it holds that $\limsup_{n\to\infty} U(\omega_{1:n}) = \infty$ and for which at the same time $\limsup_{n\to\infty} T_\ell(\omega_{1:n}) < \infty$ for all $\ell \in \mathbb{N}_0$. We will 'construct' such a path using an induction method on $k \in \mathbb{N}_0$. To kickstart the induction, for $k = 0$, we let $n_k := 0$ and $\omega_{1:n_k} := \square$. Next, we consider any $k \in \mathbb{N}_0$, and we will assume that $\omega_{1:n_k}$, with $0 = n_0 < n_1 < \cdots < n_k \in \mathbb{N}_0$, has already been defined in such a way that

$$U(\omega_{1:n_k}) \geq k \text{ and } \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \leq \sum_{\ell=0}^{k} 2^{-\ell} \text{ for all } n \leq n_k; \tag{23}$$

we will call this the *induction condition* for k. Note that the induction condition is trivially satisfied for $k = 0$, since $U(\square) = 1 \geq 0$ and $T_0(\square) = 1 \leq 1$. We are now going to 'construct' an n_{k+1} and $\omega_{1:n_{k+1}}$ in such a way that the induction condition will also be satisfied for $k+1$, and in this way 'construct' the path ω inductively.

We begin by showing that the process T'_k, defined by

$$T'_k(s) := \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(s) \text{ for all } s \in \mathbb{S}, \tag{24}$$

is a recursive positive rational supermartingale for $1/2$. Since this T'_k is a finite weighted sum of recursive positive rational test supermartingales for $1/2$ with positive rational coefficients, it is a recursive positive rational process. We are therefore left with proving the supermartingale property:

$$E_{1/2}(\Delta T'_k(s)) = E_{1/2}\left(\sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} \Delta T_\ell(s)\right) = \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} E_{1/2}(\Delta T_\ell(s)) \leq 0 \text{ for all } s \in \mathbb{S},$$

where the last inequality holds because all T_ℓ, with $\ell \in \mathbb{N}_0$, are supermartingales for $1/2$. We conclude that T'_k is indeed a recursive positive rational supermartingale for $1/2$.

Now for the actual 'construction' of n_{k+1} and $\omega_{1:n_{k+1}}$. We already know from Equation (23), the induction condition for k, that $T'_k(\omega_{1:n}) \leq \sum_{\ell=0}^{k} 2^{-\ell}$ for all $n \leq n_k$. Now invoke Lemma 36 to find that there is a recursive path ω' such that

$$\omega'_{1:n_k} = \omega_{1:n_k} \text{ and } T'_k(\omega'_{1:n}) \leq \sum_{\ell=0}^{k} 2^{-\ell} \text{ for all } n \in \mathbb{N}_0. \tag{25}$$

Since ω' is recursive, it follows from Lemma 34 that it is not C-random for I, and hence, by Proposition 3, it is also not ML-random for I. Consequently, $\limsup_{n\to\infty} U(\omega'_{1:n}) = \infty$, and therefore, there is some natural number $n_{k+1} > n_k$ such that $U(\omega'_{1:n_{k+1}}) \geq k+1$. Now let $\omega_{1:n_{k+1}} := \omega'_{1:n_{k+1}}$, and observe that this leaves $\omega_{1:n_k}$ unchanged. It now holds by construction that

$$\sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \overset{(25)}{=} \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega'_{1:n}) \overset{(24)}{=} T'_k(\omega'_{1:n}) \overset{(25)}{\leq} \sum_{\ell=0}^{k} 2^{-\ell} \text{ for all } n \leq n_{k+1}. \tag{26}$$

Consider the recursive positive rational test supermartingale $T_{k+1} \in \overline{\mathbb{T}}_{\mathrm{C}}(1/2)$. By repeatedly applying Lemma 23, we obtain that $T_{k+1}(\omega_{1:n}) \leq 2T_{k+1}(\omega_{1:n-1}) \leq \dots \leq 2^n T(\square) = 2^n$ for all $n \in \mathbb{N}_0$, and therefore

$$2^{-(k+1)} \frac{T_{k+1}(\omega_{1:n})}{2^{n_{k+1}}} \leq 2^{-(k+1)} \text{ for all } n \leq n_{k+1}.$$

These inequalities, together with Equation (26), now lead to

$$\sum_{\ell=0}^{k+1} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \leq \sum_{\ell=0}^{k+1} 2^{-\ell} \text{ for all } n \leq n_{k+1}.$$

If we recall that also $U(\omega_{1:n_{k+1}}) = U(\omega'_{1:n_{k+1}}) \geq k+1$, we see that the induction condition is satisfied for $k+1$ as well.

In our construction, we have made sure that $n_{k+1} > n_k$ for all $k \in \mathbb{N}_0$, and this makes sure that the consecutive situations $\omega_{1:n_k}$, $k \in \mathbb{N}_0$ define a unique path ω. It now only remains to show that this path does satisfy all the requirements.

If we recall that $n_0 < n_1 < \dots < n_k < n_{k+1} < \dots$, then we see that, on the one hand,

$$\limsup_{n\to\infty} U(\omega_{1:n}) \geq \limsup_{k\to\infty} U(\omega_{1:n_k}) \overset{(23)}{\geq} \limsup_{k\to\infty} k = \infty,$$

which tells us that, indeed, ω is not ML-random for the interval forecast I.

On the other hand, that $n_0 < n_1 < \dots < n_k < n_{k+1} < \dots$, implies that for all $n \in \mathbb{N}_0$ there is some $k \in \mathbb{N}_0$ such that $n \leq n_k < n_{k+1} < \dots$, and it therefore follows from Equation (23) that

$$\lim_{k\to\infty} \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \leq \lim_{k\to\infty} \sum_{\ell=0}^{k} 2^{-\ell} = 2 \text{ for all } n \in \mathbb{N}_0.$$

Consequently, for all $k, n \in \mathbb{N}_0$,

$$\begin{aligned} T_k(\omega_{1:n}) = 2^{n_k+k} 2^{-(n_k+k)} T_k(\omega_{1:n}) &\leq 2^{n_k+k} \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \\ &\leq 2^{n_k+k} \lim_{k\to\infty} \sum_{\ell=0}^{k} 2^{-(n_\ell+\ell)} T_\ell(\omega_{1:n}) \leq 2^{n_k+k+1}. \end{aligned}$$

This shows that, for every $k \in \mathbb{N}_0$, the recursive positive rational test supermartingale T_k for $1/2$ is bounded above on ω by 2^{n_k+k+1}. Hence, ω is indeed C-random for $1/2$. □

Proof of Proposition 17. Fix any two real numbers $p, q \in \mathbb{R}$ such that $0 < p \leq q < 1$, then there is a computable interval forecast $I' \subseteq (0,1)$ such that $[p,q] \subseteq I'$. By Proposition 37, there is a path $\omega \in \Omega$ that is C-random for $1/2$ but not ML-random for I'.

To show that $I_{\mathrm{C}}(\omega) = 1/2$, simply observe that since $1/2 \in \mathcal{I}_{\mathrm{C}}(\omega)$ and since $I_{\mathrm{C}}(\omega)$ is non-empty because Properties 2 and 4 hold for R = C, it follows that $I_{\mathrm{C}}(\omega) = \bigcap \mathcal{I}_{\mathrm{C}}(\omega) = 1/2$.

To conclude, assume towards contradiction that ω is ML-random for $[p,q]$. Since $[p,q] \subseteq I'$ and Property 1 holds for $\mathrm{R} = \mathrm{C}$, we find that $I' \in \mathcal{I}_{\mathrm{ML}}(\omega)$, a contradiction. □

B.3.2. S-*randomness versus* wCH-*randomness.*

For any $\alpha, \beta \in (0,1)$, we define the gamble $f_{\alpha,\beta}$ on $\mathscr{X}$ by

$$f_{\alpha,\beta}(1) := \frac{\sqrt{\beta/\alpha}}{\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}} \text{ and } f_{\alpha,\beta}(0) := \frac{\sqrt{(1-\beta)/(1-\alpha)}}{\sqrt{\alpha\beta} + \sqrt{(1-\alpha)(1-\beta)}}. \tag{27}$$

Lemma 38 ([7, Lemma 35])**.** *For any $\alpha, \beta \in (0,1)$, we consider the gamble $f_{\alpha,\beta}$ on $\mathscr{X}$ defined in Equation* (27)*. Then for any $\alpha, \beta \in (0,1)$ and $I \in \mathcal{I}$, the following statements hold:*

(i) $E_\alpha(f_{\alpha,\beta}) = 1$*,* $f_{\alpha,\beta}(0) > 0$ *and* $f_{\alpha,\beta}(1) > 0$*;*
(ii) $f_{\alpha,\beta}(1) > f_{\alpha,\beta}(0)$ *if and only if* $\alpha < \beta$*;*
(iii) *if* $\max I \leq \alpha < \beta$*, then* $\overline{E}_I(f_{\alpha,\beta}) \leq 1$*;*
(iv) *if* $\alpha < \beta \leq \min I$*, then* $\overline{E}_I(f_{\beta,\alpha}) \leq 1$*;*
(v) $f_{\alpha,\beta}(0) f_{\beta,\alpha}(0) = f_{\alpha,\beta}(1) f_{\beta,\alpha}(1) \geq \left(1 - \frac{1}{4}(\alpha-\beta)^2\right)^{-1}$.

The following proof uses ideas from Ref. [7], which are in their turn inspired by Ref. [22].

Proof of Proposition 20. Since $\varphi_{p,q}$ is clearly a computable precise forecasting system and since ω is C-random for $\varphi_{p,q}$, Equation (3) implies that $\omega \in \Omega_{\mathrm{CH}}(\varphi_{p,q})$, so it already follows from Proposition 18 that $I_{\mathrm{CH}}(\omega) = [p,q]$, $p < q - (q-p)q \leq \min I_{\mathrm{wCH}}(\omega)$ and $\max I_{\mathrm{wCH}}(\omega) \leq q - (q-p)p < q$. That ω is C-random for the computable precise forecasting system $\varphi_{p,q}$ also implies, taking into account Corollary 14, that $I_{\mathrm{C}}(\omega) = I_{\mathrm{CH}}(\omega)$, so $I_{\mathrm{C}}(\omega) = I_{\mathrm{CH}}(\omega) = [p,q]$.

Since $I_{\mathrm{C}}(\omega) = [p,q]$, Corollary 4 implies that $I_{\mathrm{S}}(\omega) \subseteq [p,q]$. It now only remains to prove that $I_{\mathrm{S}}(\omega) = [p,q]$. This is what we now set out to do.

First, since we already know that $\min I_{\mathrm{S}}(\omega) \geq p$, we assume towards contradiction that $\min I_{\mathrm{S}}(\omega) > p$. Since $p > 0$, there is then some positive rational number $r \in \mathbb{Q}$ such that $0 < p < r < \min I_{\mathrm{S}}(\omega) \leq 1$. By Proposition 6 and Definition 3, ω must be S-random for the interval forecast $[r,1]$. Let us now prove that this is impossible, by constructing a computable test supermartingale for $[r,1]$ that is computably unbounded on ω.

Let $0 < \varepsilon < 1$ be any rational number [which there always is] such that $p + 3\varepsilon < r$. We define a multiplier process D by $D(\square) := 1$ and

$$D(x_{1:n}) := \begin{cases} f_{r,r-\varepsilon} & \text{if } x_n = 1 \\ 1 & \text{if } x_n = 0 \end{cases} \text{ for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N},$$

where, for any $\alpha, \beta \in (0,1)$, the gamble $f_{\alpha,\beta} \in \mathscr{L}(\mathscr{X})$ is defined by Equation (27). The multiplier process D is computable because r and ε are rational. D is furthermore positive by Lemma 38(i), since $1 > r > 0$ and $1 > r - \varepsilon > p + 2\varepsilon > 0$.

To prove that D is a supermartingale multiplier for $[r,1]$, we show that $\overline{E}_{[r,1]}(D(s)) \leq 1$ for all $s = x_{1:n} \in \mathbb{S}$ with $n \in \mathbb{N}_0$. There are three possible cases.

If $n = 0$ and therefore $s = \square$, we find that $\overline{E}_{[r,1]}(D(\square)) = \overline{E}_{[r,1]}(1) \overset{\mathrm{C1}}{=} 1$.

If $n > 0$ and $x_n = 0$, then $\overline{E}_{[r,1]}(D(s)) = \overline{E}_{[r,1]}(1) \overset{\mathrm{C1}}{=} 1$.

If $n > 0$ and $x_n = 1$, it follows from Lemma 38(iv) that $\overline{E}_{[r,1]}(D(s)) = \overline{E}_{[r,1]}(f_{r,r-\varepsilon}) \leq 1$.

So D is indeed a supermartingale multiplier for $[r,1]$. Since we had already established that D is computable and positive, it follows from Lemma 22 that $D^{\circledcirc}$ is a positive computable test supermartingale for $[r,1]$.

We will now show that this $D^{\circledcirc}$ is computably unbounded on ω. Consider the multiplier process D' defined by $D'(\square) := 1$ and

$$D'(x_{1:n}) := \begin{cases} f_{r-\varepsilon,r} & \text{if } x_n = 1 \\ 1 & \text{if } x_n = 0 \end{cases} \text{ for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N}.$$

We first prove that D' is a positive computable supermartingale multiplier for the forecasting system $\varphi_{p,q}$. The argumentation is fairly similar to the one given above for D and $[r,1]$. Computability follows from the rationality of r and ε. Positivity follows from Lemma 38(i) since $1 > r > 0$ and $1 > r-\varepsilon > 0$. To prove that D' is a supermartingale multiplier for $\varphi_{p,q}$, we need to show that $E_{\varphi_{p,q}(s)}(D'(s)) \leq 1$ for all $s = x_{1:n} \in \mathbb{S}$ with $n \in \mathbb{N}_0$.

The cases where $s = \square$, or where $n > 0$ and $x_n = 0$, are again trivial.

If $n > 0$ and $x_n = 1$, then $D'(x_{1:n}) = f_{r-\varepsilon,r}$ and $\varphi_{p,q}(x_{1:n}) = p$, so, since $p < r-\varepsilon < r$, it follows from Lemma 38(iii) that $E_{\varphi_{p,q}(x_{1:n})}(D'(x_{1:n})) = E_p(f_{r-\varepsilon,r}) \leq 1$.

So, D' is indeed a positive computable supermartingale multiplier for $\varphi_{p,q}$. Taking into account Lemma 22, we can conclude that $D'^{\circledcirc}$ is a positive computable test supermartingale for $\varphi_{p,q}$. This computable test supermartingale $D'^{\circledcirc}$ must furthermore be bounded above on ω, because of the assumed C-randomness of the path ω for $\varphi_{p,q}$.

Let us now show that the product process $D^{\circledcirc}D'^{\circledcirc}$ is computably unbounded on ω, which will then immediately imply that $D^{\circledcirc}$ must be computably unbounded too.

Let B be any positive rational upper bound on $D'^{\circledcirc}$ along ω. Consider the rational number $\delta := \left(1-\frac{1}{4}\varepsilon^2\right)^{-1}$, then $\delta > 1$ because $0 < \varepsilon < 1$, and Lemma 38(v) guarantees that

$$f_{r-\varepsilon,r}(1)f_{r,r-\varepsilon}(1) = f_{r-\varepsilon,r}(0)f_{r,r-\varepsilon}(0) \geq \delta.$$

Hence,

$$D(x_{1:n})D'(x_{1:n}) \begin{cases} = 1 & \text{if } n = 0 \\ = 1 & \text{if } n > 0 \text{ and } x_n = 0 \\ \geq \delta & \text{if } n > 0 \text{ and } x_n = 1 \end{cases} \quad \text{for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N}_0. \tag{28}$$

Now recall that $I_{\mathrm{CH}}(\omega) = [p,q]$, so we infer from Proposition 5 [with $S = 1$] that $p \leq \liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1}$. Consider any rational α such that $0 < \alpha < p$ [which there always is], then there is some $M \in \mathbb{N}$ such that $\alpha < \frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1}$ and therefore also $\alpha n < \sum_{k=1}^{n}\omega_k$ for all $n \geq M$. Then, for any $n \geq M$,

$$\begin{aligned} D^{\circledcirc}(\omega_{1:n+1})D'^{\circledcirc}(\omega_{1:n+1}) &= \prod_{k=0}^{n} D(\omega_{1:k})(\omega_{k+1}) \prod_{k=0}^{n} D'(\omega_{1:k})(\omega_{k+1}) \\ &= \prod_{k=0}^{n}\left[D(\omega_{1:k})(\omega_{k+1})D'(\omega_{1:k})(\omega_{k+1})\right] \\ &\overset{(28)}{\geq} \prod_{k=1}^{n}\delta^{\omega_k} = \delta^{\sum_{k=1}^{n}\omega_k} \overset{\delta>1}{>} \delta^{\alpha n}, \end{aligned}$$

and therefore, since $D'^{\circledcirc}$ is positive and bounded above by B along ω, also

$$D^{\circledcirc}(\omega_{1:n+1}) \geq \frac{\delta^{\alpha n}}{D'^{\circledcirc}(\omega_{1:n+1})} \geq B^{-1}\delta^{\alpha n} \text{ for all } n \geq M.$$

Now, let $\tau\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ be defined by $\tau(0) = 0$ and $\tau(n+1) := B^{-1}\delta^{\alpha n}$ for all $n \in \mathbb{N}_0$. Then τ is computable because α, δ and B are rational, and τ is non-decreasing and unbounded because $B > 0$, $\delta > 1$ and $\alpha > 0$. So, τ is a real growth function for which $D^{\circledcirc}(\omega_{1:n}) \geq \tau(n)$ for all $n \geq M+1$, and therefore also $\limsup_{n\to\infty}[D^{\circledcirc}(\omega_{1:n}) - \tau(n)] \geq 0$. We conclude that the computable test supermartingale $D^{\circledcirc}$ for $[r,1]$ is indeed computably unbounded on ω, so ω cannot be S-random for $[r,1]$.

This tells us that, necessarily, $\min I_{\mathrm{S}}(\omega) = p$. In a similar way, we will now prove that $q = \max I_{\mathrm{S}}(\omega)$, and therefore indeed $I_{\mathrm{S}}(\omega) = [p,q]$.

First, since we already know that $\max I_{\mathrm{S}}(\omega) \leq q$, we assume towards contradiction that $\max I_{\mathrm{S}}(\omega) < q$. Since $q < 1$, there is then some positive rational number $r \in \mathbb{Q}$ such that $0 \leq \max I_{\mathrm{S}}(\omega) < r < q < 1$. By Proposition 6 and Definition 3, ω must be S-random for the interval forecast $[0,r]$. Let us now prove that this is impossible, by constructing a computable test supermartingale for $[0,r]$ that is computably unbounded on ω.

Let $0 < \varepsilon < 1$ be any rational number [which there always is] such that $r < q - 3\varepsilon$. We define a multiplier process D by $D(\square) := 1$ and

$$D(x_{1:n}) := \begin{cases} 1 & \text{if } x_n = 1 \\ f_{r,r+\varepsilon} & \text{if } x_n = 0 \end{cases} \text{ for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N},$$

where, for any $\alpha, \beta \in (0,1)$, the gamble $f_{\alpha,\beta} \in \mathscr{L}(\mathscr{X})$ is again defined by Equation (27). The multiplier process D is computable because r and ε are rational. D is furthermore positive by Lemma 38(i), since $0 < r < 1$ and $0 < r + \varepsilon < q - 2\varepsilon < 1$. To prove that D is a supermartingale multiplier for $[0,r]$, we show that $\overline{E}_{[0,r]}(D(s)) \leq 1$ for all $s = x_{1:n} \in \mathbb{S}$ with $n \in \mathbb{N}_0$. There are three possible cases.

If $n = 0$ and therefore $s = \square$, we find that $\overline{E}_{[0,r]}(D(\square)) = \overline{E}_{[0,r]}(1) \overset{\text{C1}}{=} 1$.

If $n > 0$ and $x_n = 1$, then $\overline{E}_{[0,r]}(D(s)) = \overline{E}_{[0,r]}(1) \overset{\text{C1}}{=} 1$.

If $n > 0$ and $x_n = 0$, it follows from Lemma 38(iii) that $\overline{E}_{[0,r]}(D(s)) = \overline{E}_{[0,r]}(f_{r,r+\varepsilon}) \leq 1$.

So D is indeed a supermartingale multiplier for $[0,r]$. Since we had already established that D is computable and positive, it follows from Lemma 22 that $D^{\circledcirc}$ is a positive computable test supermartingale for $[0,r]$.

We will now show that this $D^{\circledcirc}$ is computably unbounded on ω. Consider the multiplier process D' defined by $D'(\square) := 1$ and

$$D'(x_{1:n}) := \begin{cases} 1 & \text{if } x_n = 1 \\ f_{r+\varepsilon,r} & \text{if } x_n = 0 \end{cases} \text{ for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N}.$$

We first prove that D' is a positive computable supermartingale multiplier for the forecasting system $\varphi_{p,q}$. The argumentation is fairly similar to the one given above for D and $[0,r]$. Computability follows from the rationality of r and ε. Positivity follows from Lemma 38(i) since $0 < r < 1$ and $0 < r + \varepsilon < 1$. To prove that D' is a supermartingale multiplier for $\varphi_{p,q}$, we need to show that $E_{\varphi_{p,q}(s)}(D'(s)) \leq 1$ for all $s = x_{1:n} \in \mathbb{S}$ with $n \in \mathbb{N}_0$.

The cases where $s = \square$, or where $n > 0$ and $x_n = 1$, are again trivial.

If $n > 0$ and $x_n = 0$, then $D'(x_{1:n}) = f_{r+\varepsilon,r}$ and $\varphi_{p,q}(x_{1:n}) = q$, so, since $r < r + \varepsilon < q$, it follows from Lemma 38(iv) that $E_{\varphi_{p,q}(x_{1:n})}(D'(x_{1:n})) = E_q(f_{r+\varepsilon,r}) \leq 1$.

So, D' is indeed a positive computable supermartingale multiplier for $\varphi_{p,q}$. Taking into account Lemma 22, we can conclude that $D'^{\circledcirc}$ is a positive computable test supermartingale for $\varphi_{p,q}$. This computable test supermartingale $D'^{\circledcirc}$ must furthermore be bounded above on ω, because of the assumed C-randomness of the path ω for $\varphi_{p,q}$.

Let us now show that the product process $D^{\circledcirc}D'^{\circledcirc}$ is computably unbounded on ω, which will then immediately imply that $D^{\circledcirc}$ must be computably unbounded too.

Let B be any positive rational upper bound on $D'^{\circledcirc}$ along ω. Consider the rational number $\delta := \left(1 - \frac{1}{4}\varepsilon^2\right)^{-1}$, then $\delta > 1$ because $0 < \varepsilon < 1$, and Lemma 38(v) guarantees that

$$f_{r+\varepsilon,r}(1) f_{r,r+\varepsilon}(1) = f_{r+\varepsilon,r}(0) f_{r,r+\varepsilon}(0) \geq \delta.$$

Hence,

$$D(x_{1:n})D'(x_{1:n}) \begin{cases} = 1 & \text{if } n = 0 \\ \geq \delta & \text{if } n > 0 \text{ and } x_n = 0 \\ = 1 & \text{if } n > 0 \text{ and } x_n = 1 \end{cases} \quad \text{for all } x_{1:n} \in \mathbb{S} \text{ with } n \in \mathbb{N}_0. \tag{29}$$

Now recall that $I_{\mathrm{CH}}(\omega) = [p,q]$, so we infer from Proposition 5 [with $S = 1$] that $q \geq \limsup_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \omega_{k+1}$. Consider any rational α such that $1 > \alpha > q$ [which there always is], then there is some $M \in \mathbb{N}$ such that $\alpha > \frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1}$ and therefore also $\alpha n > \sum_{k=1}^{n}\omega_k$ for all $n \geq M$. Then, for any $n \geq M$,

$$\begin{aligned} D^{\circledcirc}(\omega_{1:n+1})D'^{\circledcirc}(\omega_{1:n+1}) &= \prod_{k=0}^{n} D(\omega_{1:k})(\omega_{k+1}) \prod_{k=0}^{n} D'(\omega_{1:k})(\omega_{k+1}) \\ &= \prod_{k=0}^{n} \big[D(\omega_{1:k})(\omega_{k+1})D'(\omega_{1:k})(\omega_{k+1})\big] \\ &\overset{(29)}{\geq} \prod_{k=1}^{n} \delta^{1-\omega_k} = \delta^{\sum_{k=1}^{n}(1-\omega_k)} = \delta^{n-\sum_{k=1}^{n}(\omega_k)} \overset{\delta>1}{>} \delta^{(1-\alpha)n}, \end{aligned}$$

and therefore, since $D'^{\circledcirc}$ is positive and bounded above by B along ω, also

$$D^{\circledcirc}(\omega_{1:n+1}) \geq \frac{\delta^{(1-\alpha)n}}{D'^{\circledcirc}(\omega_{1:n+1})} \geq B^{-1}\delta^{(1-\alpha)n} \text{ for all } n \geq M.$$

Now, let $\tau\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ be defined by $\tau(0) = 0$ and $\tau(n+1) := B^{-1}\delta^{(1-\alpha)n}$ for all $n \in \mathbb{N}_0$. Then τ is computable because α, δ and B are rational, and τ is non-decreasing and unbounded because $B > 0$, $\delta > 1$ and $1-\alpha > 0$. So, τ is a real growth function for which $D^{\circledcirc}(\omega_{1:n}) \geq \tau(n)$ for all $n \geq M+1$, and therefore also $\limsup_{n\to\infty}[D^{\circledcirc}(\omega_{1:n}) - \tau(n)] \geq 0$. We conclude that the computable test supermartingale $D^{\circledcirc}$ for $[0,r]$ is indeed computably unbounded on ω, so ω cannot be S-random for $[0,r]$.

This tells us that, indeed, $\max I_{\mathrm{S}}(\omega) = q$.

□

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