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Imprecise Probabilities in Algorithmic Randomness

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Preface

In the past months, I put a great deal of time in thinking about what to put into this preface. I for example came up with the idea of comparing my act of writing this dissertation with me being a whale that would be happy to be pulled ashore, after having spent too much time at the dark bottom of the sea; I probably lent this picture from Wim Opbrouck's performance 'Ik Ben De Walvis'. Bruno Bauwens provided me with a different picture: writing a dissertation is like eating a whole cake. At first it seems undoable, but if you cut it into pieces, and start eating it piece by piece, you will eventually finish it; for your information, I really like to eat tarte maison. On another more humoristic note, I thought about including a before and after image of my face that would reveal how much I aged while writing this dissertation. It's only recently that it became clear to me what I would do with this preface. In order for it to be(come) meaningful, I would try to be precise, and in order to be precise, I have to be elaborate. In what follows, I will try thanking a number of people for a number of specific reasons, and I will sometimes do so in my mother's tongue, since that is the language that binds some of us.

Aan mijn ouders Ronny Albert Josephine Persiau en Martine Rooman geef ik graag de volgende gedachte(n) mee. Enkele jaren geleden schreef Janne op de achterkant van een postkaart de 5 basisbehoeften van een mens volgens Al Pesso: plaats, voeding, steun, bescherming en begrenzing. Hoewel ik de afgelopen jaren beschouw als mijn laattijdige puberteit—waarin mijn kijk op en mening over vele zaken verschilt van die van jullie—, doet deze kaart me al geruime tijd nadenken over wat ons verbindt en wat jullie meegegeven hebben. Een paar jaar geleden las ik 'De ondraaglijke lichtheid van het bestaan' van Milan Kundera, en beluisterde ik een bespreking waarin uitgelegd wordt dat dit boek (ondermeer) gaat over hoezeer een mensenleven anders kan zijn/verlopen door enkele kleine ingrepen/wijzigingen. Het voelt echter prettig om stil te staan bij wat er (al) is en mogelijk is, en niet bij wat er niet is of (geweest) kon zijn. Het is enorm dankbaar om terug te kijken op hoe ik onder jullie hoede en veilige, warme vleugels deels tot wasdom ben gekomen, vertrekkende van een weliswaar nieuwsgierig kind dat moeite had met verandering en het onbekende. Bedankt om Charlotte en mij vijf jaar geleden op dergelijke wijze af te zetten bij wat jullie 'de start van het

(volwassen) leven' noemen. In het bijzonder wil ik jullie bedanken om een manier te tonen waarop een mens zorg kan dragen voor zichzelf, voor de herhaaldelijke hulp bij verhuizen, om me uit te leggen hoe je een strijkijzer hanteert (onderbroeken en keukenhanddoeken dienen wel écht niet gestreken te worden), voor het ontvangen van mijn tranen na een relatiebreuk, en om zo goed voor elkaar te blijven zorgen terwijl ik in Gent zat. Het is fijn om te weten dat er iemand mee over je schouder kijkt, om te weten dat je niet alleen bent, om terug te kunnen keren naar een nest.

Dear Jasper and Gert, thanks for having given me a chance and the trust to continue your research on Imprecise Probabilities in Algorithmic Randomness. I remember Gert using two metaphors for describing this line of research: he considered it both a playground he liked playing in, and his newborn baby. I can only say that I had a great deal of fun in the past 5 years, and that I hope you're happy about the way I took care of the baby. The past 5 years have actually been about way more than just fun, it was also about finding a passion, which has felt as a great gift in life, and which I'm extremely grateful for. It's been nice to be surrounded by people I look up to, and who I want to/can learn from. I consider you both—in your very own way—very generous people. In particular, I thank Jasper for his support and involvement—especially at the start and at the end of my PhD—, for trusting me with co-organising the SIPTA School 2024, and for sharing (and applying) his impressive mathematical (clockwork) precision. In particular, I thank Gert for letting me be me—especially in those cases when I was not having the best mood—, for letting me feel that you believe in me, and for your clever statements and sentences that stuck in my head and provided me with reflection and understanding.

For my research stay in the beautiful city of Montpellier, I want to thank Alexander Shen for sharing his house and for providing me with many relevant pointers to the literature that allowed me to better position my research. For my research stay at Carnegie Mellon University, I thank Francesca Zaffora Blando for the time she was willing to spend on discussing research. Your—what I call feminine—way of lucidly and calmly explaining and discussing things is really inspiring. I really enjoy(ed) working together.

To my non-professorial colleagues of FLip—Alexander, Natan, Arne, Keano and Adrian—I write that it's a privilege to be surrounded by so many young, bright and rather funny mathematicians. In the past year, it's been pleasant to work on this dissertation in the morning, and to look forward to sharing lunch with you at noon (which consists of eating my daily cheese sandwich). In particular, I thank Alexander and Natan for so generously sharing the hurdles they faced and the mistakes they made while writing their dissertations; you definitely made writing this one a lot easier. In addition, I want to thank Alexander for sharing an—what he long considered his—office with me. It has been nice to get to know you a bit better. Moreover, I immensely enjoyed working together on our take on the SIPTA School 2024,

and I'm immensely proud about both the process and the outcome.

To my beloved friends, doing a PhD in mathematics has only been pleasant by the counterbalance of your sheer existence. I am extremely grateful for receiving your love, for your openness to discuss ideas and your thoughts, for sharing, analysing and exploring (y)our emotions, for going on new exciting adventures (with old exciting people—Kae Tempest), for you being you, and of course for just having a dance and fun. It moves me how much I learned from you.

Liefste Jannevrouw. Zoals zo vaak geschreven wordt over een doctoraat: 'what a ride'. Hoewel de bovenstaande oplijsting een verzameling is van dankbaarheden voor verscheidene vrienden, beschouw ik ze allen als van toepassing op jou. In het bijzonder dank ik je om me te laten zien en voelen dat er vele mogelijkheden zijn in het leven, dat er heel veel kan en niet moet. Het afgelopen jaar was mede dankzij jou(w duwtjes en sleeptouw) de max.

Floris Persiau
September 2024

Summary

This dissertation is concerned with the study of 'Imprecise Probabilities in Algorithmic Randomness'. To explain what it's all about, we'll start by unpacking the title.

The field of Imprecise Probabilities questions whether precise-probabilistic uncertainty models are always sufficient to capture one's uncertainty, and puts forward alternative and (even) more general uncertainty models that allow for reasoning in an informative and conservative way, even in those situations where it's infeasible or inappropriate to specify a single probability (measure). In particular, in this thesis, we consider a sequence of discrete-time outcome variables whose states assume values in an arbitrary but finite state space, and we model our uncertainty about these values by adopting so-called *forecasting systems*, which associate with every possible finite outcome sequence a possibly different set of probability mass functions—which is called a credal set—to express our uncertainty about the next unknown outcome. If a forecasting system only specifies a single probability mass function for every finite outcome sequence, then it's called precise; every such precise forecasting system defines a unique probability measure on the elements of the standard Borel (sigma) algebra, and for every probability measure on this algebra there's at least one forecasting system that generates it.

The field of Algorithmic Randomness, on the other hand, studies what it means for an infinite outcome sequence to be random. Consider for example infinite binary sequences that are generated by flipping a fair coin—which corresponds to probability 1/2: the infinite binary sequence 01010101... doesn't seem random at all, whereas the sequence 10001011... seems more random. Algorithmic randomness notions try to formalise our intuition behind random sequences, by defining what it means for an infinite sequence to be random for an uncertainty model. Classically, these uncertainty models are probability measures (or precise forecasting systems).

It's the endeavour of this dissertation to allow for imprecise-probabilistic forecasting systems in several algorithmic randomness notions, and to see and study what happens when doing so, building on initial work that has

been done in the intersection of both fields.

After introducing the necessary tools from imprecise probability theory and computability theory in a first part, we adopt three approaches to algorithmic randomness: a *martingale-theoretic*, a *frequentist* and a *test-theoretic* one. Under the first *martingale-theoretic* approach to algorithmic randomness, a sequence is random for a forecasting system if there's no implementable betting strategy for getting arbitrarily rich along this sequence without borrowing, where the bets that are allowed, are determined by the forecasting system. By changing the type of implementability that's imposed on the betting strategies and the way of getting arbitrarily rich, we obtain 4 different randomness notions: Martin-Löf randomness, weak Martin-Löf randomness, computable randomness and Schnorr randomness. We explain that they are natural imprecise-probabilistic generalisations of several classical precise-probabilistic randomness notions, and show that they satisfy the same relations as their precise-probabilistic counterparts. Moreover, as yet another argument in favour of our approach, we prove several other properties, which are again reminiscent of the classical ones: for every computable forecasting system there's a so-called universal betting strategy such that a sequence is Martin-Löf random for the forecasting system if and only if this particular betting strategy doesn't allow you to get arbitrarily rich along the sequence, the randomness of a sequence with respect to a computable forecasting system only depends on the forecasts that are specified along the sequence, these randomness notions are reasonably robust with respect to changes to both the forecasting systems and the betting strategies, etc.

Under a more *frequentist* approach to algorithmic randomness, a sequence is random for a forecasting system if the sequence and its computably selectable subsequences satisfy an imprecise-probabilistic version of the law of large numbers. By changing what it means for a subsequence to be computably selectable, we obtain 2 different randomness notions: Church randomness and weak Church randomness. We explain that they are imprecise-probabilistic generalisations of classical precise-probabilistic randomness notions, show how they relate to each other and to the martingale-theoretic randomness notions—which relation is analogous to the ones for their precise-probabilistic counterparts—, show that they have an equivalent alternative frequentist and martingale-theoretic characterisation, and show that they satisfy similar properties as the martingale-theoretic ones.

As should be clear by now, there are many ways to come up with a notion of randomness. What makes a randomness notion interesting then? Of course, its definition should have an intuitive interpretation and should come with a number of interesting properties. Furthermore, an interesting notion of randomness typically carries several equivalent characterisations; we show that this holds for some of our martingale-theoretic randomness notions. Under a *test-theoretic* approach to algorithmic randomness, a sequence is random for a forecasting system if it passes all implementable statistical tests that are associated with the forecasting system. By changing what it means for a statistical test to be implementable, we obtain 2 randomness notions: Martin-Löf test randomness and Schnorr test randomness. Under the restriction of computable forecasting systems, we show that these test-theoretic definitions coincide with the corresponding martingale-theoretic ones—thereby generalising classical equivalence results by Schnorr and Levin [1, 2, 3]—and that Martin-Löf test randomness coincides with Levin's notion of uniform randomness [4, 5, 6]—which considers effectively compact classes of probability measures.

Next, we move away from the classical/standard approach to algorithmic randomness by questioning whether it's always possible/opportune to define the randomness of a sequence with respect to a forecasting system, which specifies credal sets for all finite outcome sequences that could have been observed. We answer this question in the negative, and address it by adopting a so-called *prequential* approach to randomness, which is based on the work by Dawid and Vovk [7, 8, 9] and which allows to define the randomness of an infinite sequence only with respect to the credal sets that are actually forecast along the sequence. In particular, we develop a prequential version of both the martingale- and test-theoretic approach to Martin-Löf randomness—which we call game- and test-randomness, respectively—, show that both randomness notions coincide, and prove that they also coincide in a specific sense with the standard version of Martin-Löf randomness when imposing some mild (computability) conditions on the forecasting systems.

In a final part, we zoom out and question whether imprecise forecasting systems are really needed to capture a sequence's randomness. We answer this question both in the positive and in the negative. On the one hand, for our martingale-theoretic and frequentist notions of randomness, we show the existence of sequences that are random for an imprecise forecasting system, but that aren't random for any *computable* precise forecasting system. On the other hand, we show that a sequence is martingale-theoretically random for a forecasting system if and only if it's random for some compatible (typically non-computable) precise forecasting system. These answers lay bare the importance of the computability assumption on the forecasting systems. We finish our exposition by explaining why computable forecasting systems are to be favoured from the point of view of statistics whose aim it is to learn an uncertainty model from data.

Samenvatting

Dit proefschrift richt zich op de studie van 'Imprecieze waarschijnlijkheden in algoritmische toevalligheid'. Om uit te leggen waarover het gaat, starten we met het ontleden van de titel.

Het veld van imprecieze waarschijnlijkheden betwist of precieze onzekerheidsmodellen altijd voldoende zijn om iemands onzekerheid te beschrijven, en schuift alternatieve en (zelfs) algemenere onzekerheidsmodellen naar voren die het mogelijk maken om op een informatieve en conservatieve manier te redeneren, zelfs in situaties waarin het niet haalbaar of zelfs ongepast is om een enkele waarschijnlijkheid(smaat) op te geven. In het bijzonder kijken we naar discrete-tijdsvariabelen waarvan de toestanden waarden aannemen in een eindige toestandsruimte, en modelleren we onze onzekerheid over deze waarden door zogenaamde voorspellingssystemen te adopteren, die met elke mogelijke eindige uitkomstenrij een mogelijks andere verzameling van massafuncties associëren—wat een credale verzameling wordt genoemd—om onze onzekerheid over de volgende onbekende uitkomst te beschrijven. Indien een voorspellingssysteem maar een enkele massafunctie specificeert voor elke eindige uitkomstenrij, dan wordt het precies genoemd; elk precies voorspellingssysteem definieert een unieke waarschijnlijkheidsmaat op de elementen van de gebruikelijke borelalgebra, en voor elke waarschijnlijkheidsmaat op deze algebra is er minstens één voorspellingssysteem dat ze genereert.

Het veld van algoritmische toevalligheid bestudeert daarentegen wat het betekent voor een oneindige uitkomstenrij om toevallig te zijn. Beschouw bijvoorbeeld oneindige binaire rijen die worden gegenereerd door het opgooien van een eerlijk muntstuk—wat overeenkomt met de kans $1/2$: de oneindige binaire rij 01010101... lijkt helemaal niet toevallig te zijn, terwijl de rij 10001011... toevalliger lijkt. Noties van algoritmische toevalligheid proberen onze intuïtie achter toevallige rijen te formaliseren door te definiëren wat het betekent voor een oneindige rij om toevallig te zijn voor een onzekerheidsmodel. Klassiek gezien zijn deze onzekerheidsmodellen waarschijnlijkheidsmaten (of precieze voorspellingssystemen).

Het is de betrachting van deze dissertatie om imprecieze voorspellingssystemen toe te staan in diverse noties van algoritmische toevalligheid, en om te zien en te bestuderen wat er gebeurt wanneer we daarin slagen.

Na in een eerste deel het nodige materiaal uit de theorie van imprecieze waarschijnlijktheorie en berekenbaarheidstheorie te introduceren, volgen we drie aanpakken van algoritmische toevalligheid: een martingaaltheoretische, een frequentistische en een testtheoretische aanpak. Onder de eerste, martingaaltheoretische, aanpak van algoritmische toevalligheid wordt een rij als toevallig beschouwd voor een voorspellingssysteem als er geen implementeerbare gokstrategie bestaat om willekeurig rijk te worden langsheen deze rij zonder te lenen, waarbij de toegestane gokken worden bepaald door het voorspellingssysteem. Door het soort implementeerbaarheid dat wordt opgelegd aan de gokstrategieën en de manier waarop ze je willekeurig rijk moeten laten worden, te veranderen, krijgen we vier verschillende toevalligheidsnoties: Martin-Löf-toevalligheid, zwakke Martin-Löf-toevalligheid, berekenbare toevalligheid en Schnorr-toevalligheid. We leggen uit dat deze noties natuurlijke imprecieze veralgemeningen zijn van verscheidene klassieke precieze toevalligheidsnoties, en tonen aan dat ze voldoen aan dezelfde relaties als hun precieze tegenhangers. Bovendien, als extra argument ten gunste van onze aanpak, bewijzen we verschillende andere eigenschappen, die weer doen denken aan de klassieke eigenschappen: voor elk berekenbaar voorspellingssysteem is er een zogenaamde universele gokstrategie zodanig dat een rij Martin-Löf-toevallig is voor het voorspellingssysteem als en slechts als deze specifieke gokstrategie je niet in staat stelt om willekeurig rijk te worden langsheen de rij; de toevalligheid van een rij ten opzichte van een berekenbaar voorspellingssysteem hangt alleen af van de voorspellingen die langsheen de rij worden gespecificeerd; deze toevalligheidsnoties zijn tamelijk robuust met betrekking tot veranderingen in zowel de voorspellingssystemen als de gokstrategieën; enzovoort.

Onder een frequentistische aanpak van algoritmische toevalligheid wordt een rij als toevallig beschouwd voor een voorspellingssysteem indien de rij en haar deelrijen die op berekbare wijze selecteerbaar zijn, voldoen aan een 'imprecieze' versie van de wet van de grote aantallen. Door te veranderen wat het betekent dat een deelrij op berekenbare wijze selecteerbaar is, verkrijgen we twee verschillende toevalligheidsnoties: Church-toevalligheid en zwakke Church-toevalligheid. We leggen uit dat deze noties 'imprecieze' veralgemeningen zijn van klassieke precieze toevalligheidsnoties, tonen aan hoe ze zich tot elkaar en tot de martingaaltheoretische toevalligheidsnoties verhouden—wat analoog is met de relaties voor hun precieze tegenhangers—, tonen aan dat ze een alternatieve equivalente frequentistische en martingaaltheoretische karakterisering hebben, en tonen aan dat ze vergelijkbare eigenschappen hebben als de martingaaltheoretische toevalligheidsnoties.

Zoals ondertussen duidelijk zou moeten zijn, zijn er veel manieren om een notie van toevalligheid vast te leggen. Wat maakt een toevalligheidsnotie dan interessant? Natuurlijk moet de definitie een intuïtieve interpretatie hebben en moet ze gepaard gaan met een aantal interessante eigenschappen. Bovendien heeft een interessante notie van toevalligheid doorgaans verschillende equivalente karakteriseringen; we tonen aan dat dit geldt voor enkele van onze martingaaltheoretische toevalligheidsnoties. Onder een testtheoretische aanpak van algoritmische toevalligheid wordt een rij als toevallig beschouwd voor een voorspellingssysteem als ze alle implementeerbare statistische testen doorstaat die geassocieerd zijn met het voorspellingssysteem. Door te veranderen wat het betekent dat een statistische test implementeerbaar is, krijgen we twee toevalligheidsnoties: Martin-Löf-test-toevalligheid en Schnorr-test-toevalligheid. We tonen aan dat, onder de beperking van berekenbare voorspellingssystemen, deze testtheoretische definities samenvallen met de overeenkomstige martingaaltheoretische definities—wat een veralgemening vormt van klassieke equivalentieresultaten van Schnorr en Levin—en dat Martin-Löf-test-toevalligheid samenvalt met Levins notie van uniforme toevalligheid—die effectief compacte klassen van waarschijnlijkheidsmaten beschouwt.

Vervolgens stappen we af van de klassieke aanpak van algoritmische toevalligheid door ons af te vragen of het altijd mogelijk/gepast is om de toevalligheid van een rij te definiëren met betrekking tot een voorspellingssysteem dat credale verzamelingen specificeert voor alle eindige uitkomstenrijen die in principe waargenomen kunnen worden. We moeten een negatief antwoord op deze vraag geven, en volgen daarom een zogenaamde prequentiële aanpak van toevalligheid, die gestoeld is op het werk van Dawid en Vovk en die het mogelijk maakt om de toevalligheid van een oneindige rij te definiëren met betrekking tot enkel de credale verzamelingen die daadwerkelijk langsheen de rij worden gespecificeerd. In het bijzonder ontwikkelen we een prequentiële versie van zowel de martingaaltheoretische als de testtheoretische aanpak van Martin-Löf-toevalligheid—die we respectievelijk speltoevalligheid en testtoevalligheid noemen—, tonen we aan dat beide toevalligheidsnoties samenvallen, en bewijzen we dat ze ook samenvallen met de standaardversie van Martin-Löf-toevalligheid wanneer enkele milde (berekenbaarheids)voorwaarden worden opgelegd aan de voorspellingssystemen.

In een laatste deel nemen we een stap terug en stellen we de vraag of imprecieze voorspellingssystemen weldegelijk nodig zijn om de toevalligheid van een rij te vatten. We geven zowel een positief als een negatief antwoord op deze vraag. Enerzijds tonen we voor onze martingaaltheoretische en frequentistische noties van toevalligheid aan dat er rijen zijn die toevallig zijn voor een imprecies voorspellingssysteem, maar die niet toevallig zijn voor enig berekenbaar precies voorspellingssysteem. Anderzijds tonen we aan dat een rij martingaaltheoretisch toevallig is voor een voorspellingssysteem als en slechts als ze toevallig is voor een compatibel (typisch onberekenbaar) precies voorspellingssysteem. Deze antwoorden onthullen het belang van de berekenbaarheidsvoorwaarde op de voorspellingssystemen. We eindigen onze uiteenzetting met een argument waarom berekenbare voorspellingssystemen de voorkeur verdienen vanuit het oogpunt van statistiek, die erop gericht is om een onzekerheidsmodel te leren uit data.

Introduction

In line with so many expositions on and introductions to algorithmic randomness [10, 11, 12, 13], we get things moving by putting forward an infinite binary sequence

$$1000100000011010001101110011000110100100100101001 0\dots$$

and by asking a question: (when) do you consider this sequence to be generated by flipping a fair coin? Or put differently, (when) would you say that the sequence agrees with probability $1/2$, where $1/2$ is the probability for the coin landing heads? If you were given the infinite binary sequence $0101010101\dots$, then you'd probably be inclined to answer this question in the negative. Meanwhile, the above infinite binary sequence looks at least more random. So, what sequences do (and don't) you deem random? The field of *algorithmic randomness* tries to answer this question by putting forward formal definitions of what it means for an infinite sequence to be random for an uncertainty model. Classically, these uncertainty models are probability measures, including the special case of those that describe the process of flipping a (possibly un)fair coin.

In recent decades, however, there has been a scholarly push towards the development of alternative and (even) more general uncertainty models, the study of which belongs to the field called *imprecise probabilities* [14, 15, 16, 17]. Such models typically allow for reasoning in an informative and conservative way, even in those situations where it's infeasible or inappropriate to specify a single probability (measure); such situations may arise because of having available only a limited amount of data [18], dealing with incomplete datasets [19, 20], conflicting expert judgements [21], time constraints, realism in expert judgements, etc. In this dissertation, we'll embrace such uncertainty models and push the field of algorithmic randomness beyond precise probabilities: what happens when we allow for such imprecise probability models in the field of algorithmic randomness? In particular, how do we allow for imprecise uncertainty models in several classical randomness definitions, and how do the corresponding generalisations shine new light on our understanding of random sequences?

This introductory chapter consists of 4 parts. In Section 1, we provide a historical overview of some of the key developments in the field of algorithmic randomness, and explain and substantiate how and why we want to allow for imprecise uncertainty models in several randomness notions. Section 2_7 gives an overview of the six chapters this dissertation consists of, and Section 3_8 lists my publications that led to this dissertation. We conclude this chapter in Section 4_{10} by providing some information on how we manage the internal and external references in this work.

1 History, context and motivation

We'll start in Section 1.1 by describing the origin of and some developments in the field of algorithmic randomness. Afterwards, in Section 1.2_4, we explain why precise uncertainty models aren't always sufficient to describe one's uncertainty, and motivate the usage of imprecise uncertainty models in algorithmic randomness notions. We end with an overview of the pioneering works that allow for 'Imprecise Probabilities in Algorithmic Randomness', which this dissertation builds upon.

1.1 Algorithmic randomness: some history

As early as 1919, Richard von Mises wondered about how to give a mathematical account of the notion of an individual random sequence $(x_1, x_2, \dots, x_n, ...) \in \mathscr{X}^{\mathbb{N}}$ [22, 23], with $\mathscr{X}$ some arbitrary but finite non-empty set; $\mathscr{X}$ may for example substitute for the binary outcome set corresponding with the flipping of a (possibly un)fair coin—as we considered above—, or for the senary set that corresponds to the possible outcomes of rolling a die. This question was widely discussed during the following 20 years. In von Mises' opinion, random sequences should be considered infinite in order to construct a simple and elegant mathematical theory, and he formulated the following two conditions such sequences need to satisfy in order to be random for a probability mass function $m \colon \mathscr{X} \to [0,1]$ [24]. First, the relative frequency of every outcome $x \in \mathscr{X}$ along the sequence should converge to the probability $m(x)$ of x. Second, these limiting values must remain the same in all subsequences that can be obtained from the sequence by so-called *place selection rules*; infinite sequences that satisfy both requirements are called *collectives*. Consider for example again the infinite binary sequence 0101010101... Clearly, the limiting relative frequency of ones and zeros along the whole sequence equals $1/2$, whereas along the subsequence obtained by only selecting the outcomes at odd positions, that is, $(x_1, x_3, x_5, \dots) = 00000\dots$, the limiting relative frequency of ones equals 0. Hence, according to von Mises' notion, this sequence isn't random. This is also clear from an intuitive point of view. However, when considering the displayed infinite binary sequence at the beginning of this Introduction again, intuition is no longer so readily available. Moreover, the question remains what set of place selection rules must be adopted. Obviously, we mustn't allow for all selection rules, since this would always allow for the selection of a subsequence that consists only of zeros or ones. Consequently, no infinite binary sequence would then be random for a probability mass function.

In 1937, Abraham Wald proved the existence of random infinite sequences when the set of place selection rules is countable [23, 25, 26]. Still, it hadn't been decided yet what countable set of place selection rules should be adopted. It was Alonzo Church who suggested in 1940—based on Wald's work—to adopt the countable set of computable selection rules [23, 25, 27]; an infinite sequence is then random for a probability mass function m if the relative frequency of every outcome $x \in \mathscr{X}$ along all computably selectable infinite subsequences converges to $m(x)$, where 'computably selectable' essentially means that there's some finite algorithm that decides which outcomes to keep and which to discard. This notion of randomness is currently known as *Church stochasticity*.

Such *frequentist* notions of randomness weren't free from debate though. To test the randomness of an infinite sequence, as is obvious from the above definition, von Mises highly prioritises the law of large numbers by requiring convergence of relative frequencies along a number of its subsequences. Meanwhile, such random sequences don't necessarily comply with other statistical laws. As proven by Jean Ville [28, 29], there are infinite binary sequences that satisfy the above requirements for probability $1/2$, but for which the running frequency of ones along the sequence converges to $1/2$ from below. Such random sequences seem to posses a clear pattern, and disobey another statistical law, which is known as the law of the iterated algorithm. For this reason, Jean Ville criticised this type of randomness definitions, and argued that besides the law of large numbers, a random sequence also ought to satisfy other statistical laws [25]. Nowadays, Church stochasticity is generally considered to be too weak a randomness notion, and is therefore called a stochasticity notion instead of a randomness notion.

Objections of this kind against Church stochasticity led to the development of many other randomness notions. Some of the most well-known and well-studied amongst these are Martin-Löf randomness, computable randomness and Schnorr randomness [2, 30]. The reasons for this are twofold: they have an intuitive interpretation and they can be defined in several equivalent ways [31, 32]. From a *test-theoretic* point of view, for example, an infinite sequence is random for a probability mass function m if it passes all implementable statistical tests that are associated with m. On the other hand, if we adopt a *martingale-theoretic* approach, then a sequence is random for a probability mass function m if there's no implementable betting strategy for getting arbitrarily rich along this sequence without borrowing, where the bets that are allowed are determined by m. The randomness notions mentioned above will, amongst other things, differ in what type of implementability is imposed on the betting strategies and the statistical tests.

The above frequentist, martingale- and test-theoretic randomness notions don't only allow for defining the randomness of an infinite sequence with respect to a *stationary* probability mass function m. In general, the classical approach consists in allowing for *non-stationary* probability mass functions that describe one's uncertainty about the next unknown outcome $X_n \in \mathscr{X}$, which may depend on the already observed outcomes $(x_1, \dots, x_{n-1})$ or only on n. Such (conditional) uncertainty models that associate with every observed finite outcome sequence $(x_1, \dots, x_{n-1})$ a possibly different probability mass function m are called (precise) *forecasting systems*. For those readers who are used to working with probability measures, it's worth mentioning that every such forecasting system defines a unique probability measure on the elements of the standard Borel (sigma) algebra over the set $\mathscr{X}^{\mathbb{N}}$. Vice versa, for every probability measure on this algebra there's at least one forecasting system that generates it; the connection is one-to-one when restricting attention to positive measures and to forecasting systems that never assign a probability zero. Consequently, forecasting systems are slightly more expressive in this sense, since they provide/contain full conditional information.

1.2 Imprecise probabilities in algorithmic randomness: context and motivation

Algorithmic randomness notions basically try to capture our intuition behind random sequences. Meanwhile, as we'll illustrate below, precise probability models aren't always sufficiently expressive to capture that intuition. In particular, we'll support that claim by explaining what imprecise uncertainty models we'll adopt and how they remedy issues that originate from only allowing for precise (or fully specified) probabilities.

In addition to probability mass functions, we consider closed and convex sets of probability mass functions, which are called *credal sets*. In what follows, we explain why we deem them natural and necessary in this context, and we'll do so by adopting a frequentist, a subjective and a martingale-theoretic approach.

Let's start by assuming a simple stationary description of a sequence's randomness, that is, by defining the randomness of a sequence for a single probability mass function. Since all randomness notions typically impose or imply adherence to the law of large numbers, this assumption implies convergence of the relative frequencies of every outcome along the sequence. However, such behaviour isn't always satisfied in our material world. In fact, such violations aren't even rare; consider for example air temperature fluctuations over long time intervals [33], or the relative occurrence of vowels in messages drawn from Internet job postings [34]. When insisting on a simple stationary description, this problem is easily addressed by defining the randomness of a sequence with respect to a credal set, because then the credal set only imposes imprecise bounds on the relative frequencies of the outcomes along the sequence, which allows for the fluctuation of relative frequencies.

As another argument in favour of allowing for credal sets, imagine you have to estimate the probability for heads of a possibly unfair coin. Are you willing to specify a probability based on the observation of 5 tosses? In this case, I'd be more inclined to put forward an interval of probabilities—which corresponds to a credal set. This example reveals that probabilities aren't always sufficiently expressive, because a subject may be incapable of having precise credences, which can result from dealing with a small amount of data.

On the other hand, we can also do away with the stationarity assumption, and consider a possibly non-stationary precise forecasting system. As touched upon above, a martingale-theoretic approach to randomness bases the randomness of a sequence with respect to a forecasting system on the impossibility of getting arbitrarily rich by adopting an implementable betting strategy, where the bets that are allowed are determined by the forecasting system. When a forecasting system outputs a probability mass function associated with a finite outcome sequence $(x_1,\ldots,x_{n-1})$, this functions as an inclination to accept or reject bets on the following unknown outcome $X_n \in \mathscr{X}$; in particular, it provides fair prices for the *a priori* unknown rewards associated with these bets, and regards bets with positive fair prices as acceptable. However, it's for example clear from the behaviour of the stock market that fair prices are in general not sufficiently expressive, because maximum buying prices (bid) and minimum selling prices (ask) typically don't coincide. Here as well, we can remedy this issue by adopting credal sets, because they allow for associating maximum acceptable buying and minimum acceptable selling prices with every bet, whose values don't necessarily have to coincide.

In general, because of the aforementioned reasons, we'll consider so-called *imprecise* forecasting systems, which associate with every finite outcome sequence $(x_1,\ldots,x_{n-1})$ a possibly different credal set. In 2017, Gert de Cooman and Jasper De Bock succeeded in allowing for such imprecise forecasting systems in a frequentist and martingale-theoretic approach to algorithmic randomness [35]. That is, they defined what it means for an infinite binary sequence to be Church stochastic and computably random for a forecasting system. As we've found out later thanks to Alexander Shen, while De Cooman and De Bock were the first to introduce imprecision in frequentist and martingale-theoretic approaches to algorithmic randomness [35, 36], they weren't the first to move beyond probabilities in a test-theoretic approach to algorithmic randomness [37]. In 1966, Martin-Löf constructed so-called Bernoulli tests, which test the randomness of a sequence with respect to the set of all Bernoulli measures, that is, the set of measures generated by single probability mass functions m [4, 30]. He showed that if a sequence withstands these tests, then it's a von Mises' collective relative to some probability mass function m [4]. Consequently, in the above sense, Bernoulli tests allow to test whether a sequence is random for some Bernoulli measure. In 1973, Levin generalised this result by putting forward a test-theoretic notion of randomness—which is nowadays known as *uniform randomness*—that allows for so-called 'effectively compact classes of probability measures' [4, 5, 6];[1] the set of all Bernoulli measures is a specific example of such an effectively compact class of probability measures. In particular, there are class tests such that a sequence passes this test if and only if it's random with respect to some probability measure in the considered class. While De Cooman and De Bock thus weren't the first to move beyond probabilities in the field of algorithmic randomness, we find it important to emphasize that their endeavour to allow for imprecise probabilities in algorithmic randomness has a different motivation: whereas previous approaches test whether a sequence is random for some member in a class of probability measures, De Cooman and De Bock take forecasting systems as their central and elementary object to work with, without necessarily considering it to be composed of precise forecasting systems.

In this dissertation, in which we build upon the work by Levin, Martin-Löf, Schnorr, Church, Wald, De Cooman and De Bock, we'll allow for imprecise uncertainty models in various frequentist, test- and martingale-theoretic notions of randomness. We'll argue that these definitions are natural since (i) they coincide with the classical definitions when considering precise (computable) forecasting systems, and since (ii) they have similar properties as the classical precise-probabilistic definitions. In particular, we'll study how all definitions relate to each other, and these relationships will be reminiscent of the classical (precise-probabilistic) relations. Given the state of the art in algorithmic randomness, some of the properties and relations will seem unsurprising. That's not to say, however, that proving them is a straightforward matter, especially since a number of the techniques used for precise (and therefore additive) probabilities and their (linear) expectations become unworkable, or require a fundamentally different approach, when dealing with imprecise or game-theoretic probabilities and expectations, which are typically non-additive and non-linear. The fact that we can identify *new* ways of establishing these properties and relations in a more general and arguably more abstract setting would argue in favour of our method of approach. Moreover, we're able to ask and address some questions for which imprecise probabilities are pivotal. For instance, should or could the randomness of a sequence always be defined with respect to a precise uncertainty model? And how do imprecise probability models change our understanding of random sequences?

[1] As is nowadays standard in the precise-probabilistic randomness setting [38, 39, 40], this imprecise-probabilistic notion of uniform randomness has also been defined and studied in the (even) more general setting of computable Polish spaces, which are also called constructive metric spaces [6, Sections 7 and 8].

2 Overview of the chapters

Besides this introductory chapter and the conclusions, this dissertation consists of six chapters that are numbered by the six sides of a die—as you have already noticed or will duly notice. This section contains a brief description of every such chapter.

Chapter ⚀$_{13}$ formally introduces most of the mathematical objects and concepts that are used in the other five chapters, and therefore forms the central pillar of this dissertation. In particular, we introduce *credal sets*, equip them with both a frequentist and a betting interpretation, extend them to *forecasting systems*, explain how they function as an inclination to accept or reject bets, formalise betting strategies and the corresponding capital processes in so-called *sub- and supermartingales*, and use them to introduce (global) upper and lower expectations. We end the chapter by explaining what it means for some of the above objects to be implementable; that is, we resort to computability theory and explain the concepts of recursiveness, lower semicomputability and computability. Throughout this chapter, which thus contains a collection of results that we borrow from imprecise probability theory and computability theory, we've interwoven pointers to the algorithmic randomness literature both to draft a compelling story and to justify why we introduce and consider something.

At the start of Chapter ⚁$_{49}$, we have everything ready to allow for imprecise uncertainty models in algorithmic randomness notions a first time in this dissertation. We introduce imprecise-probabilistic *martingale-theoretic* versions of (weak) Martin-Löf, computable and Schnorr randomness, study some of their properties, and examine how robust these notions are with respect to changes to the forecasting systems and the betting strategies at hand.

In Chapter ⚂$_{85}$, we allow for imprecise forecasting systems in algorithmic randomness notions a second time, but now adopt a *frequentist* approach. We define a rather general frequentist notion of randomness that requires a sequence and some of its subsequences to satisfy an imprecise-probabilistic version of the law of large numbers. In particular, when we consider all *(totally) computably selectable* infinite subsequences, we obtain an imprecise-probabilistic version of (weak) Church stochasticity. We explain how these frequentist randomness notions relate to the other martingale-theoretic ones, and show that they have an equivalent alternative frequentist and martingale-theoretic characterisation.

We adopt a third, *test-theoretic*, approach to randomness in Chapter ⚃$_{111}$, where we allow for imprecise forecasting systems in test-theoretic versions of Martin-Löf and Schnorr randomness. We show that these test-theoretic definitions coincide with the corresponding martingale-theoretic ones when restricting attention to (non-degenerate) computable forecasting systems, thereby generalising classical equivalence results by Schnorr and Levin. Moreover, while still restricting our attention to computable forecasting systems, we prove that our imprecise-probabilistic test-theoretic version of Martin-Löf randomness—which considers forecasting systems—coincides with Levin's notion of uniform randomness—which considers effectively compact classes of probability measures—; in particular, every given computable forecasting system turns out to correspond with a specific effectively compact class of probability measures.

In Chapter ⚄$_{143}$, we question whether it's always desirable to define the randomness of an infinite sequence with respect to a *forecasting system*, which associates a credal set with every finite outcome sequence that could have been observed instead of merely those that actually have been observed (namely the finite precursors of the infinite sequence whose randomness we are actually considering). Why should the randomness of a sequence depend on forecasts for situations that have never been observed? We address this question by developing a so-called *prequential* version of both the martingale- and test-theoretic approach to Martin-Löf randomness, which we'll call game- and test-randomness respectively. We show that both notions coincide, and explain how they relate to the standard version of Martin-Löf randomness.

After having allowed for imprecise probabilities in several approaches to and notions of algorithmic randomness, we reach a finale in Chapter ⚅$_{179}$ by questioning the need for *imprecise probabilities* in algorithmic randomness. In a first part, we show that for every non-precise stationary forecasting system—that is, a credal set that doesn't correspond to a single probability mass function—there's a sequence that is random for this stationary imprecise forecasting system but not for any *computable* (possibly non-stationary) precise forecasting system. In this sense, imprecision is needed to capture some sequences' randomness. In a second part, we show that things change drastically when allowing for non-computable forecasting systems: for every imprecise forecasting system there's a *non-computable* compatible precise forecasting system that has the exact same set of random sequences. In this sense, you could naively say that imprecise probabilities aren't needed in algorithmic randomness. We however stand our ground by explaining why non-computable uncertainty models are rather awkward, and we do so by resorting to the practical grounds of statistics.

3 List of publications

This dissertation gathers (part of) the research I—in cooperation with fellow researchers—have developed during my past five years as a PhD student. Many of the presented results can be found elsewhere, although they're often only stated and proved in the specific and more simple case of binary state spaces. This thesis aims to generalise and unite these results in a single place. Specifically, this dissertation encompasses results from the following publications.

(i) Floris Persiau, Jasper De Bock & Gert de Cooman. Computable randomness is about more than probabilities. In: *Lecture Notes in Computer Science* (2020). See [42] for an extended version

(ii) Floris Persiau, Jasper De Bock & Gert de Cooman. A remarkable equivalence between non-stationary precise and stationary imprecise uncertainty models in computable randomness. In: *Proceedings of the Twelfth International Symposium on Imprecise Probability: Theories and Applications*. 2021

(iii) Floris Persiau, Jasper De Bock & Gert de Cooman. On the (dis)similarities between stationary imprecise and non-stationary precise uncertainty models in algorithmic randomness. In: *International Journal of Approximate Reasoning* (2022)

(iv) Floris Persiau & Gert de Cooman. Imprecision in martingale-theoretic prequential randomness. In: *Proceedings of the Thirteenth International Symposium on Imprecise Probability: Theories and Applications*. 2023

(v) Floris Persiau & Gert de Cooman. Imprecision in martingale- and test-theoretic prequential randomness. In: *International Journal of Approximate Reasoning* (2024)

(vi) Gert de Cooman, Floris Persiau & Jasper De Bock. Randomness and imprecision: from supermartingales to randomness tests. Submitted for publication. 2024

Chapter ⚁$_{49}$ is largely based on (i), Chapter ⚂$_{85}$ on (i) and (iii), Chapter ⚃$_{111}$ on (vi), Chapter ⚄$_{143}$ on (iv) and (v), and Chapter ⚅$_{179}$ on (ii) and (iii).

The present work also includes some novel ideas, proofs and results that haven't been published yet. The three most significant amongst these, we believe, are listed below.

- We equip our imprecise-probabilistic frequentist notion of Church stochasticity in Section 11.2$_{92}$ with an interesting equivalent frequentist characterisation which differs substantially from our original frequentist definition.
- In [36, Theorem 37], De Cooman and De Bock came up with a beautiful result that leads them to say that randomness is 'inherently imprecise', because there are infinite binary sequences that are random for a probability interval but not for any computable (possibly non-stationary) precise forecasting system. In Section 19$_{181}$, we generalise this result from binary to arbitrary finite state spaces, and thereby allow for more general credal sets instead of probability intervals; to do so, we use a drastically different proof strategy.
- In (ii) and (iii), we showed that an infinite binary sequence is martingale-theoretically random for an imprecise forecasting system

if and only if it's random for a compatible precise forecasting system. Hence, an imprecise forecasting system can (also) be seen as a set of precise compatible forecasting systems from the vantage point of algorithmic randomness. We generalise this statement to arbitrary state spaces in Section 20$_{192}$, and we do so by adopting a way more simple proof strategy that was suggested to us by Alexander Shen.

More modest, but still noteworthy contributions are listed below.

- We provide a lucid explanation for the implementability of several mathematical objects in Section 7$_{40}$, and in particular for the computability of forecasting systems in Section 7.4$_{46}$.
- In Section 10$_{66}$, we study the robustness of several martingale-theoretic randomness notions with respect to changes in the forecasting systems and betting strategies at hand.
- When restricting our attention to stationary forecasting systems—that is, to credal sets—we're able to provide the frequentist notion of Church stochasticity with an equivalent martingale-theoretic characterisation [see Section 12.3$_{106}$].

Furthermore, there's some published work that we decided not to include in this dissertation. The reasons for not doing so are twofold. First of all, we wanted to keep the length of this dissertation within reasonable (and readable) bounds. Secondly, and perhaps more importantly, the work presented here is all stated in the setting of arbitrary but finite state spaces. The results in the publications below, on the other hand, haven't yet been generalised from binary to arbitrary finite state spaces. Moreover, the last contribution only adopts precise-probabilistic uncertainty models. Hence, to bring a compact, coherent and cohesive story, we decided to omit the material in the below publications from this dissertation.

(vii) Floris Persiau, Jasper De Bock & Gert de Cooman. The smallest probability interval a sequence is random for: a study for six types of randomness. In: *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*. See [48] for an extended version. 2021

(viii) Floris Persiau, Gert de Cooman & Jasper De Bock. A comparative study of the smallest probability intervals a binary sequence is random for. Submitted for publication. 2024

(ix) Floris Persiau & Francesca Zaffora Blando. Randomness and invariance. Accepted for publication in the Journal of Logic and Computation. 2024

4 (Internal) references

As you have probably (pun intended) already noticed by now, external references are denoted by a number between square brackets; the full bibliographical details of these references are available at the corresponding numbers in the Bibliography, which starts on p. 217. So, for instance, [51] points to a book on the Game-Theoretic Foundations for Probability and Finance written by Glenn Shafer and Vladimir Vovk.

This dissertation also contains many internal references to chapters, (sub)sections, equations, figures, theorems, propositions, lemmas, corollaries and examples. We'll first spend some words on the numbering system that we've adopted. As mentioned before, this dissertation consists of six chapters—excluding the Introduction and Conclusions—which are numbered from 1 to 6 by the sides of a die. The sections are continuously numbered from 1_{2} to 21_{199}; for instance, Section 8_{50} is the first section of Chapter ⚁$_{49}$. This numbering continues up to subsection level; for instance, Section 14.3_{126} is the third subsection of Section 14_{119}. In every section, a single counter is used to number equations, figures, theorems, propositions, lemmas, corollaries and examples, which results for each one of them in a unique identification label consisting of two numbers; for instance, Eq. $(17.10)_{155}$ can be found after Lemma 17.8_{154} in Section 17_{150}.

Still, locating the referenced material can be a rather cumbersome affair. For this reason, we resort to Quaeghebeur's [52] system of locational clues, as has become a tradition for doctoral dissertations and monographs written by members of FLip[2] [16, 52, 53, 54, 55, 56, 57, 58]. We accompany internal references with a subscript number that indicates the page on which the referred content can be found; for instance, Theorem 14.1_{120} can be found on p. 120. If the content we refer to is on the previous or subsequent page, then the subscript number is replaced by the symbols ↶ or ↷, respectively, and if it's on the same page, then no number or symbol is added; for instance, this text is part of Section 4$_{↶}$.

[2]The Foundations Lab for imprecise probabilities; a research group at Ghent University formerly known as SYSTeMS.

Computable uncertainty modelling

In a nutshell, this dissertation contains a study of a specific type of phenomena that are prone to uncertainty; in this chapter we'll introduce some (basic) terminology and mathematical notation that will allow us to do so. Examples of such phenomena are the flipping of a coin, rolling a die, your favourite newspaper's rating of the new album by Stromae, or one of my supervisors' mood after proofreading my dissertation. For every such phenomenon, the set of possible realisations is called the *state space* and is generically denoted by $\mathscr{X}$. Throughout this dissertation, we'll always assume the state space $\mathscr{X}$ to be non-empty and finite ($1 \leq |\mathscr{X}| < \infty$) and exhaustive; possible state spaces that correspond to the previous examples are $\mathscr{X} = \{\text{heads}, \text{tails}\}$, $\mathscr{X} = \{$⚀, ⚁, ⚂, ⚃, ⚄, ⚅$\}$, $\mathscr{X} = \{$★, ★★, ★★★, ★★★★, ★★★★★$\}$ and $\mathscr{X} = \{$☺, ☹$\}$, respectively. How then to describe a subject's uncertainty about the uncertain outcomes of such phenomena?

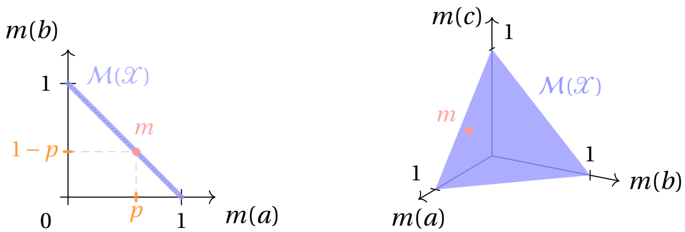
5 Uncertainty about a single variable

We start this endeavour by discussing a number of ways to describe the/your uncertainty related to the unknown outcome of a single phenomenon. To this end, consider a (single) *variable* X that assumes some (yet) unknown *outcome* x in the state space $\mathscr{X}$.[3]

[3] In line with de Finetti's approach [59], and in contradistinction with what the measure-theoretic approach to probability theory would have us do, we don't define or see a 'variable' as a map, but rather as a primitive notion: something that assumes a value in some set. This is also the reason why we don't use the term 'random variable', because that would definitely confuse some readers into using unnecessary measure-theoretic associations.

5.1 Probability mass functions

Classically, a subject describes his uncertainty about the unknown outcome of such a variable X by providing a *probability mass function* $m\colon \mathscr{X} \to [0,1]$ that associates a *probability* $m(x) \in [0,1]$ with every possible outcome $x \in \mathscr{X}$ for X in such a way that $\sum_{x\in\mathscr{X}} m(x) = 1$. We denote the collection of all probability mass functions on $\mathscr{X}$ by $\mathcal{M}(\mathscr{X})$. A probability mass function $m \in \mathcal{M}(\mathscr{X})$ is called *positive* if $m(x) > 0$ for all $x \in \mathscr{X}$. m is called *rational* if $m(x) \in \mathbb{Q}$[4] for all $x \in \mathscr{X}$; we'll then typically denote it by m_{rat}. The collection of all rational probability mass functions is denoted by $\mathcal{M}_{\text{rat}}(\mathscr{X})$. In the case of a binary state space such as $\mathscr{X} = \{a, b\}$, a probability mass function $m \in \mathcal{M}(\mathscr{X})$ is fully specified by the specification of the single probability $p = m(a)$ that's associated with the outcome $X = a$, since then automatically $m(b) = 1 - p$. Note that $\mathcal{M}(\mathscr{X})$ can be identified with the so-called unit simplex, and that a probability mass function m can then be identified with a point in the simplex; this is visualised in Figures 5.1(a) and 5.1(b) for a binary and ternary state space, respectively.



(a) $\mathscr{X} = \{a, b\}$ (b) $\mathscr{X} = \{a, b, c\}$

Figure 5.1. The purple regions depict the set $\mathcal{M}(\mathscr{X})$ of all probability mass functions, the pink dot depicts a probability mass function $m \in \mathcal{M}(\mathscr{X})$, and the orange markings represent the probabilities $p = m(a)$ and $1 - p = m(b)$.

Probability mass functions can be given a frequentist interpretation. They are then considered to be physical properties of a phenomenon, not depending on a subject. More precisely, the frequentist probability of an outcome is the relative frequency of the times this outcome occurs in a long series of observations of a phenomenon [14, Section 2.1]; this interpretation will shimmer through in Section 11_{87}, where we introduce a very general frequentist notion of randomness. In our present context, however, we'll mainly adopt a

[4] $\mathbb{Q}$ denotes the set of rational numbers, $\mathbb{Q}_{\geq 0}$ that of the non-negative[5] rational numbers and $\mathbb{Q}_{>0}$ that of the positive rational numbers.

[5] $\mathbb{R}$ denotes the set of real numbers. A real number $x \in \mathbb{R}$ is called *non-negative* if $x \geq 0$ and *positive* if $x > 0$. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ denote the set of non-negative and positive real numbers, respectively.

subjective interpretation, which we'll introduce in the following subsection by associating so-called *linear expectations* with probability mass functions.

5.2 Linear expectations

We can equip a subject's (pronouns he/him)[6] specification of a probability mass function m with a so-called betting interpretation; it characterises the gambles he's willing to offer to an opponent (pronouns she/her). To this end, any map $f\colon \mathscr{X} \to \mathbb{R}$ from the state space to the real numbers will be called a *gamble*; note that, in particular, every probability mass function can be interpreted as a non-negative gamble whose entries sum to one. $f(X)$ is then an unknown reward that's associated with the unknown outcome of a variable X; we'll assume that gambles are expressed in units of some linear utility scale. We denote the set of all gambles by $\mathscr{L}(\mathscr{X})$. Any constant gamble $f \in \mathscr{L}(\mathscr{X})$ will also be identified with the real number $c \in \mathbb{R}$ for which $f(x) = c$ for all $x \in \mathscr{X}$. A gamble $f \in \mathscr{L}(\mathscr{X})$ is called *rational* if $f(x) \in \mathbb{Q}$ for all $x \in \mathscr{X}$. The set of all rational gambles will be denoted by $\mathscr{L}_{\mathrm{rat}}(\mathscr{X})$. A gamble $f \in \mathscr{L}(\mathscr{X})$ is called *positive* if $f(x) > 0$ for all $x \in \mathscr{X}$—and we then also write $f > 0$; it is called *non-negative* if $f(x) \geq 0$ for all $x \in \mathscr{X}$—and we then also write $f \geq 0$. From time to time we'll also make use of the set $\mathscr{L}_1(\mathscr{X}) := \{f \in \mathscr{L}(\mathscr{X})\colon 0 \leq f \leq 1\}$, which includes the set $\mathcal{M}(\mathscr{X})$ of all probability mass functions. With every outcome $x \in \mathscr{X}$, we associate the gamble $\mathbb{I}_x \in \mathscr{L}(\mathscr{X})$ that assumes the value 1 on x and 0 elsewhere, and call it the *indicator* of x; note that the indicators $\mathbb{I}_x$, with $x \in \mathscr{X}$, correspond to the extreme elements of the closed convex set $\mathcal{M}(\mathscr{X})$. With every probability mass function $m \in \mathcal{M}(\mathscr{X})$, we associate a *linear expectation* $E_m\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ that's defined by

$$E_m(f) := \sum_{x \in \mathscr{X}} m(x) f(x) \text{ for all } f \in \mathscr{L}(\mathscr{X}). \tag{5.2}$$

Under the frequentist interpretation, the linear expectation $E_m(f)$ of a gamble $f \in \mathscr{L}(\mathscr{X})$ corresponds to the average gain associated with f in a long series of observations of a phenomenon. For our present purposes, however, it's inconvenient that the interpretation of the uncertainty models we work with depends on a long series of observations; this will become clear in Section 6.224 when introducing so-called forecasting systems that associate a possibly different local uncertainty model with every finite series of observations. Therefore, we also introduce a betting interpretation, which only considers a single observation. To this end, we start by observing that it's a matter of straightforward verification that the linear expectation E_m satisfies the following properties [60, Section 2.2.2].

[6] These pronouns help us lighten our exposition, by having several options at our disposal to refer to a subject.

Proposition 5.3. *Consider a probability mass function $m \in \mathcal{M}(\mathcal{X})$. Then for all $f, g \in \mathcal{L}(\mathcal{X})$ and $\mu, \lambda \in \mathbb{R}$:*

m1. $\min f \leq E_m(f) \leq \max f$; [boundedness]

m2. $E_m(\mu f + \lambda g) = \mu E_m(f) + \lambda E_m(g)$. [linearity]

More generally, any functional $E\colon \mathcal{L}(\mathcal{X}) \to \mathbb{R}$ that satisfies m1 and m2 is called a *linear expectation*; this is compatible with our earlier definition because it can actually be proven that for every such linear expectation E there's a probability mass function m such that $E(f) = E_m(f)$ for all $f \in \mathcal{L}(\mathcal{X})$ [61, Section 1.6.1]. Our subject's probability mass function m can then be interpreted as providing a fair price $E_m(f)$ for every gamble $f \in \mathcal{L}(\mathcal{X})$ [60, Section 2.2.2], expressed in units of the linear utility scale for gambles.[7] This means that he's willing to accept the uncertain pay-off $f(X) - r$ for any buying price $r \leq E_m(f)$, and to accept the uncertain pay-off $q - f(X)$ for any selling price $q \geq E_m(f)$.[8]

From the perspective of an opponent who bets against our subject, and who thus can adopt a gamble $f \in \mathcal{L}(\mathcal{X})$ if our subject is willing to accept the gamble $-f \in \mathcal{L}(\mathcal{X})$, this implies that our subject is willing to *offer* her any uncertain reward $f(X)$ for which $E_m(f) \leq 0$ [to see so, let $q = 0$ in the previous paragraph], and we'll call such rewards—or gambles—*allowable* for an opponent with respect to the subject's specification of the probability mass function m.

So, by his specification of a probability mass function $m \in \mathcal{M}(\mathcal{X})$, a subject only allows his opponent to pick those gambles that he expects to have non-positive gain, that is, he offers those gambles $f \in \mathcal{L}(\mathcal{X})$ for which $E_m(f) \leq 0$. This collection of gambles corresponds to a closed half-space; for the binary case, this is a half-plane, as is depicted in Figure 5.4↷.

With this betting interpretation at our disposal, we introduce a betting game on a single binary variable X. There are three players involved: *Forecaster* (who will take up our subject's part), *Sceptic* (who is his opponent) and *Reality*; we borrow this terminology from the field of game-theoretic probabilities [51]. Forecaster initiates the game by providing a probability mass

[7] Although we mostly adopt this subjective interpretation for probability mass functions, thus focusing more on linear expectations than on the probability mass functions they are derived from, we nevertheless choose to take probability mass functions as our primitive objects instead of linear expectations. Our reasons for doing so are twofold: we believe that researchers are in general less acquainted with the latter, and we find it more intuitive to introduce the implementability of uncertainty models starting from probability mass functions (and credal sets) in Section 7.4₄₆. A similar choice is made further on in Sections 5.3↷ and 5.4₁₉, where we first introduce credal sets—which are sets of probability mass functions— and then continue by introducing upper and lower expectations.

[8] In the (im)precise probabilities literature, a linear expectation is usually interpreted as providing a subject's coinciding *supremum* acceptable buying prices and *infimum* acceptable selling prices. However, in our context, as will for example be clear from Propositions 10.9₇₃, 10.14₇₈, 10.16₇₈ and 10.22₈₂, this subtlety isn't important, and we choose to work with *maximum* acceptable buying prices and *minimum* acceptable selling prices.

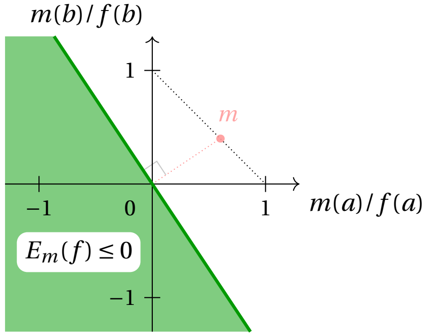


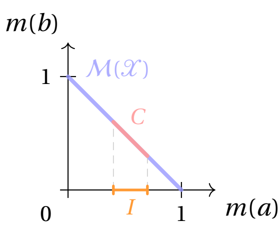
Figure 5.4. Let $\mathscr{X} = \{a, b\}$ and $(m(a), m(b)) = (0.6, 0.4)$. The pink dot depicts the probability mass function m and the green halfplane depicts all gambles that are allowable by a subject's specification of m.

function $m \in \mathcal{M}(\mathscr{X})$, which describes, as we explained, his beliefs about—and betting commitments related to—the uncertain outcome $X \in \mathscr{X}$. Next, Sceptic, being Forecaster's opponent, is allowed to pick any gamble $f \in \mathscr{L}(\mathscr{X})$ that Forecaster is willing to offer, in the specific sense that $E_m(f) \leq 0$. This leads to an uncertain (possibly negative) gain $f(X)$ for Sceptic and $-f(X)$ for Forecaster. Finally, Reality reveals the outcome $x \in \mathscr{X}$, which leads to an actual (possibly negative) gain $f(x)$ for Sceptic and $-f(x)$ for Forecaster.

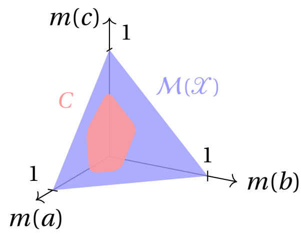
5.3 Credal sets

We won't restrict Forecaster to only adopting probability mass functions to express his beliefs about the uncertain outcome $X \in \mathscr{X}$. We'll also allow him to adopt so-called *credal sets* $C \subseteq \mathcal{M}(\mathscr{X})$ to do so, which are non-empty closed convex sets of probability mass functions; we provide graphical examples for a binary and ternary state space in Figures 5.5(a) and 5.5(b)$_{\curvearrowright}$, respectively. We denote the set of all credal sets by $\mathscr{C}(\mathscr{X})$. In the case of a binary state space such as $\mathscr{X} = \{a, b\}$, a credal set $C \in \mathscr{C}(\mathscr{X})$ is fully specified by the specification of the set of probabilities $I = \{m(a) \in [0,1] \colon m \in C\}$ that's associated with the outcome a, since then automatically $C = \big\{m \in \mathcal{M}(\mathscr{X}) \colon m(a) \in I\big\}$. Since the credal set C is closed and convex, I can then be identified with the closed probability interval $[\min_{m \in C} m(a), \max_{m \in C} m(a)] \subseteq [0,1]$, and is called an *interval forecast*; all such closed probability intervals are collected in the set $\mathscr{I}$.

A credal set $C \in \mathscr{C}(\mathscr{X})$ is called *non-degenerate* if no outcome $x \in \mathscr{X}$ has probability zero with respect to every probability mass function $m \in C$, that is, if $\max_{m \in C} m(x) = \max_{m \in C} E_m(\mathbb{I}_x) > 0$ for all $x \in \mathscr{X}$. A credal set $C \in \mathscr{C}(\mathscr{X})$ is called *rational*, and is then typically denoted by C_{rat}, if there's a finite number



(a) $\mathscr{X} = \{a, b\}$



(b) $\mathscr{X} = \{a, b, c\}$

Figure 5.5. The purple regions depict the set $\mathcal{M}(\mathscr{X})$ of all probability mass functions, the pink subset depicts a credal set $C \subseteq \mathcal{M}(\mathscr{X})$, and the orange interval represents the interval forecast $I = [\min_{m \in C} m(a), \max_{m \in C} m(a)]$.

of rational probability mass functions $\{m_1, \dots, m_n\} \subseteq \mathcal{M}_{\text{rat}}(\mathscr{X})$, with $n \in \mathbb{N}$[9], such that the (closed) convex hull of $\{m_1, \dots, m_n\}$ equals C, which implies that $\max_{m \in C} E_m(f) = \max_{1 \leq i \leq n} E_{m_i}(f)$ and $\min_{m \in C} E_m(f) = \min_{1 \leq i \leq n} E_{m_i}(f)$ for all $f \in \mathscr{L}(\mathscr{X})$. The set of all rational credal sets is denoted by $\mathscr{C}_{\text{rat}}(\mathscr{X})$, and the set of all finite non-empty subsets of rational probability mass functions is denoted by $\mathcal{P}_{\text{fin}}(\mathcal{M}_{\text{rat}}(\mathscr{X}))$; note that both sets are countable. In what follows, for ease of notation, we'll make use of the convex hull operator $\text{CH} \colon \mathcal{P}_{\text{fin}}(\mathcal{M}(\mathscr{X})) \to \mathscr{C}(\mathscr{X})$, with $\mathcal{P}_{\text{fin}}(\mathcal{M}(\mathscr{X}))$ the set of all finite non-empty subsets of probability mass functions; a credal set $C \in \mathscr{C}(\mathscr{X})$ is then rational if and only if there's some finite non-empty set $\{m_1, \dots, m_n\} \in \mathcal{P}_{\text{fin}}(\mathcal{M}_{\text{rat}}(\mathscr{X}))$ such that $\text{CH}(\{m_1, \dots, m_n\}) = C$. As is guaranteed by Hadwiger's Lemma [62], the set $\mathscr{C}_{\text{rat}}(\mathscr{X})$ of all rational credal sets is dense in $\mathscr{C}(\mathscr{X})$ under the *Hausdorff distance*, where the *Hausdorff distance* between credal sets $d_{\text{H}} \colon \mathscr{C}(\mathscr{X}) \times \mathscr{C}(\mathscr{X}) \to \mathbb{R}_{\geq 0}$ is defined by

$$d_{\text{H}}\big(C, C'\big) \coloneqq \max\left\{\max_{m \in C} d(m, C'), \max_{m' \in C'} d(m', C)\right\} \text{ for all } C, C' \in \mathscr{C}(\mathscr{X}),$$

with $d(m, C') \coloneqq \min_{m' \in C'} \|m - m'\|_{\text{tv}}$ for all $m \in \mathcal{M}(\mathscr{X})$ and $C' \in \mathscr{C}(\mathscr{X})$, and the *total variation norm* $\|\bullet\|_{\text{tv}} \colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}_{\geq 0}$ defined by $\|f\|_{\text{tv}} \coloneqq \frac{1}{2} \sum_{x \in \mathscr{X}} |f(x)|$ for all $f \in \mathscr{L}(\mathscr{X})$.

Lemma 5.6. *$\mathscr{C}_{\text{rat}}(\mathscr{X})$ is dense in $\mathscr{C}(\mathscr{X})$ under the Hausdorff distance.*

Proof. Consider any credal set $C \in \mathscr{C}(\mathscr{X})$ and any real $\epsilon > 0$. By Hadwiger's lemma [62, Chapter 13, Satz 3], there's a finite set of probability mass functions $\{m_1, \dots, m_n\} \subseteq \mathcal{M}(\mathscr{X})$ such that $d_{\text{H}}(C, \text{CH}(\{m_1, \dots, m_n\})) \leq \epsilon/2$. Since the rational numbers are a dense subset of the real numbers, there's a finite set of rational probability mass

[9] $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$ the set of non-negative integers; we denote the set of integers by $\mathbb{Z}$

functions $\{m'_1,\dots,m'_n\} \subseteq \mathcal{M}_{\mathrm{rat}}(\mathscr{X})$ such that $\|m_i - m'_i\|_{\mathrm{tv}} \leq \epsilon/2$ for all $1 \leq i \leq n$. It's then immediate from the triangle inequality for the Hausdorff distance and [63, Proposition 4] that

$$\begin{aligned}
&d_{\mathrm{H}}\big(C, \mathrm{CH}\big(\{m'_1,\dots,m'_n\}\big)\big)\\
&\quad\leq d_{\mathrm{H}}(C, \mathrm{CH}(\{m_1,\dots,m_n\})) + d_{\mathrm{H}}\big(\mathrm{CH}(\{m_1,\dots,m_n\}), \mathrm{CH}\big(\{m'_1,\dots,m'_n\}\big)\big)\\
&\quad\leq \frac{\epsilon}{2} + d_{\mathrm{H}}\big(\mathrm{CH}(\{m_1,\dots,m_n\}), \mathrm{CH}\big(\{m'_1,\dots,m'_n\}\big)\big)\\
&\quad\overset{[63]}{=} \frac{\epsilon}{2} + \max\left\{\max_{1\leq i\leq n} d\big(m_i, \mathrm{CH}\big(\{m'_1,\dots,m'_n\}\big)\big), \max_{1\leq j\leq n} d\Big(m'_j, \mathrm{CH}(\{m_1,\dots,m_n\})\Big)\right\}\\
&\quad\leq \frac{\epsilon}{2} + \max\left\{\max_{1\leq i\leq n} d(m_i, \{m'_i\}), \max_{1\leq j\leq n} d(m'_j, \{m_j\})\right\}\\
&\quad= \frac{\epsilon}{2} + \max_{1\leq i\leq n} \|m_i - m'_i\|_{\mathrm{tv}} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Credal sets can be given various interpretations. Under a frequentist interpretation, they can be seen as bounds on the relative frequencies of the outcomes in a long series of observations of a phenomenon; these relative frequencies then don't necessarily have to converge. More precisely, the relative frequencies of the outcomes in a long series of observations of a phenomenon should converge to the credal set under d; this interpretation will shimmer through in Section 11_{87}, where we introduce a very general frequentist notion of randomness. Note that this frequentist interpretation for a credal set C coincides with the one for a probability mass function m if $C = \{m\}$. Again, for our purposes, it's inconvenient that the above interpretation depends on a long series of observations; as mentioned earlier, this will become clear in Section 6.2_{24} when introducing forecasting systems. Therefore, we'll make use of a betting interpretation, which only depends on a single observation. To this end, we associate so-called *upper* and *lower expectations* with credal sets.

5.4 (Local) upper and lower expectations

When C consists of a single probability mass function $m \in \mathcal{M}(\mathscr{X})$, that is, when $C = \{m\}$, we considered the linear expectation E_m. As a generalisation, we associate with every credal set C the *upper expectation* $\overline{E}_C \colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ defined by

$$\overline{E}_C(f) := \max_{m\in C} E_m(f) \text{ for all } f \in \mathscr{L}(\mathscr{X}). \tag{5.7}$$

As a closely related operator, we consider the *lower expectation* $\underline{E}_C \colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ defined by

$$\underline{E}_C(f) := \min_{m\in C} E_m(f) \text{ for all } f \in \mathscr{L}(\mathscr{X}). \tag{5.8}$$

It's clear that upper and lower expectations are related to each other through the following conjugacy relationship: $\overline{E}_C(f) = -\underline{E}_C(-f)$ for all $f \in \mathscr{L}(\mathscr{X})$.

Moreover, with this notation at our disposal, a credal set $C \in \mathscr{C}(\mathscr{X})$ is non-degenerate if and only if $\overline{E}_C(\mathbb{I}_x) > 0$ for all $x \in \mathscr{X}$.

Under a frequentist interpretation, the upper expectation $\overline{E}_C(f)$ and lower expectation $\underline{E}_C(f)$ of a gamble $f \in \mathscr{L}(\mathscr{X})$ can respectively be seen as an upper and lower bound for the average gain associated with f in a long series of observations of a phenomenon; this average gain then doesn't necessarily have to converge. Similar to before, we now work towards a betting interpretation, which only requires a single observation. To this end, we start by observing that it's a matter of straightforward verification that the upper and lower expectation $\overline{E}_C$ and $\underline{E}_C$ satisfy the following so-called *coherence* properties [17, Section 2.6.1].

Proposition 5.9. *Consider a credal set $C \in \mathscr{C}(\mathscr{X})$. Then for all $f, g \in \mathscr{L}(\mathscr{X})$, $(f_n)_{n\in\mathbb{N}_0} \in \mathscr{L}(\mathscr{X})^{\mathbb{N}_0}$*[10]*, $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}_{\geq 0}$:*

C1. $\min f \leq \underline{E}_C(f) \leq \overline{E}_C(f) \leq \max f$; [boundedness]

C2. $\overline{E}_C(\lambda f) = \lambda \overline{E}_C(f)$ *and* $\underline{E}_C(\lambda f) = \lambda \underline{E}_C(f)$; [non-negative homogeneity]

C3. $\underline{E}_C(f) + \underline{E}_C(g) \leq \underline{E}_C(f+g) \leq \overline{E}_C(f) + \underline{E}_C(g) \leq \overline{E}_C(f+g) \leq \overline{E}_C(f) + \overline{E}_C(g)$; [(mixed) sub- and superadditivity]

C4. $\overline{E}_C(f+\mu) = \overline{E}_C(f) + \mu$ *and* $\underline{E}_C(f+\mu) = \underline{E}_C(f) + \mu$; [constant additivity]

C5. *if* $f \leq g$, *then* $\overline{E}_C(f) \leq \overline{E}_C(g)$ *and* $\underline{E}_C(f) \leq \underline{E}_C(g)$. [monotonicity]

C6. *if* $\lim_{n\to\infty} \max|f_n - f| = 0$ *then* $\lim_{n\to\infty} \overline{E}_C(f_n) = \overline{E}_C(f)$ *and* $\lim_{n\to\infty} \underline{E}_C(f_n) = \underline{E}_C(f)$. [uniform continuity]

Any functional $\overline{E}\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$—with corresponding functional $\underline{E}\colon \mathscr{L}(\mathscr{X}) \to \mathbb{R}$ defined by $\underline{E}(f) := -\overline{E}(-f)$ for all $f \in \mathscr{L}(\mathscr{X})$—that satisfies $\overline{E}_C(f) \leq \max f$ [C1], $\overline{E}_C(\lambda f) = \lambda \overline{E}_C(f)$ [C2] and $\overline{E}_C(f+g) \leq \overline{E}_C(f) + \overline{E}_C(g)$ [C3] for all $f, g \in \mathscr{L}(\mathscr{X})$ and $\lambda \in \mathbb{R}_{\geq 0}$ is called an *upper expectation* [60, Section 2.2.1]; it can be proven that for every upper expectation $\overline{E}$ there's a unique credal set $C \in \mathscr{C}(\mathscr{X})$ such that $\overline{E}(f) = \overline{E}_C(f)$ for all $f \in \mathscr{L}(\mathscr{X})$ [60, Section 2.2.2]. A subject's specification of a credal set C can then be interpreted as providing a maximum acceptable buying price $\underline{E}_C(f)$ and a minimum acceptable selling price $\overline{E}_C(f)$ for every gamble $f \in \mathscr{L}(\mathscr{X})$. This means that he's willing to accept the uncertain pay-off $f(X) - r$ for any buying price $r \leq \min_{m\in C} E_m(f) = \underline{E}_C(f)$, and to accept the uncertain pay-off $q - f(X)$ for any selling price $q \geq \max_{m\in C} E_m(f) = \overline{E}_C(f)$. Under a sensitivity analysis interpretation [60, Section 2.2], a subject's specification of a credal set C can then be seen as an inclination to buy and sell gambles, which he'll do in a very conservative way: he'll only buy (or sell) those gambles for prices that are in agreement with every probability mass function m in the set C.

From the perspective of an opponent who bets against our subject, and who thus can adopt a gamble $f \in \mathscr{L}(\mathscr{X})$ if our subject is willing to accept the

[10] For any two sets A and B, we denote by A^B the collection of all total maps from B to A.

gamble $-f \in \mathscr{L}(\mathscr{X})$, this implies that our subject is willing to *offer* her any uncertain reward $f(X)$ for which $\max_{m \in C} E_m(f) = \overline{E}_C(f) \leq 0$ [to see so, let $q = 0$ in the previous paragraph], and we'll call such rewards—or gambles—*allowable* for an opponent with respect to a subject's specification of the credal set C. Under a sensitivity analysis interpretation [60, Section 2.2], a subject's specification of a credal set C can then be seen as an inclination to offer gambles, which he'll do in a very conservative way: he'll only offer those gambles that he would offer with respect to every probability mass function m in the set C.

If for two credal sets $C, C' \in \mathscr{C}(\mathscr{X})$ it holds that $C \subseteq C'$, then we say that C' is *less informative*—or *more conservative*—than C'; equivalently, C' is less informative than C if and only if $\overline{E}_C(f) \leq \overline{E}_{C'}(f)$ for all $f \in \mathscr{L}(\mathscr{X})$. If the credal set C consists of a single probability mass function $m \in \mathcal{M}(\mathscr{X})$, then a gamble $f \in \mathscr{L}(\mathscr{X})$ is allowable if $E_m(f) \leq 0$, which is in agreement with our discussion in Section 5.215. Under the betting interpretation, this is a most informative—or least conservative—model for a subject's uncertainty, since buying and selling prices coincide. If the credal set C coincides with the set of all probability mass functions $\mathcal{M}(\mathscr{X})$, then we call C the *vacuous* credal set and often write C_{v} instead, and then $\overline{E}_C(f) = \max f$ and $\underline{E}_C(f) = \min f$ for all $f \in \mathscr{L}(\mathscr{X})$, because

$$\max f \overset{\mathrm{C1}}{\geq} \overline{E}_{C_{\mathrm{v}}}(f) \geq \max_{x \in \mathscr{X}} E_{\mathbb{I}_x}(f) = \max_{x \in \mathscr{X}} f(x)$$

and

$$\min f \overset{\mathrm{C1}}{\leq} \underline{E}_{C_{\mathrm{v}}}(f) \leq \min_{x \in \mathscr{X}} E_{\mathbb{I}_x}(f) = \min_{x \in \mathscr{X}} f(x).$$

Under the betting interpretation, this is a least informative—or most conservative—model for Forecaster's uncertainty because the buying and selling prices are maximally apart, and this corresponds to a guaranteed non-positive profit for Sceptic, i.e., $\max f = \overline{E}_C(f) \leq 0$ for all allowable gambles $f \in \mathscr{L}(\mathscr{X})$.

With the betting interpretation at our disposal, we can now easily generalise the previous betting game on a single variable X as introduced at the end of Section 5.215. There are again three players involved: Forecaster, Sceptic and Reality. This time, Forecaster initiates the game by providing a credal set $C \in \mathscr{C}(\mathscr{X})$. Next, Sceptic is allowed to pick any gamble $f \in \mathscr{L}(\mathscr{X})$ that's allowed by Forecaster, in the specific sense that $\overline{E}_C(f) \leq 0$; this collection of gambles is depicted in Figure 5.10 for a binary state space, and corresponds to the intersection of the half-spaces of gambles that are associated with every probability mass function $m \in C$. This leads to an uncertain (possibly negative) gain $f(X)$ for Sceptic and $-f(X)$ for Forecaster. Finally, Reality reveals the outcome $x \in \mathscr{X}$, which leads to an actual (possibly negative) gain $f(x)$ for Sceptic and $-f(x)$ for Forecaster.

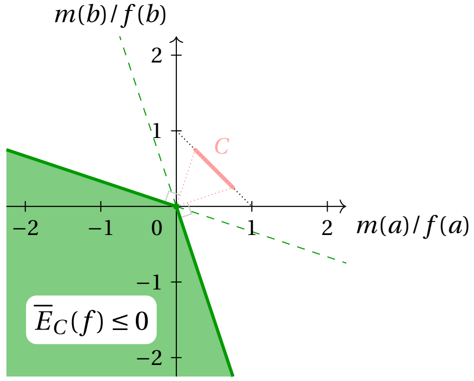


Figure 5.10. Let $\mathscr{X} = \{a, b\}$, $I = [1/4, 3/4]$ and $C = \{m \in \mathcal{M}(\mathscr{X})\colon m(a) \in I\}$. The pink region depicts the credal set C, and the green region depicts the gambles $f \in \mathscr{L}(\mathscr{X})$ that are allowable by C.

Having equipped credal sets with a betting interpretation, we touch upon the seemingly trivial but definitely non-obvious and central question in the field of Algorithmic Randomness a first time: how to test/define the agreement between a subject's specification of an (imprecise) uncertainty model and an uncertain phenomenon. Usually, and confusingly, the very same question is formulated as follows: when do we consider a phenomenon to be *random* for an uncertainty model? To answer that question, we'll consider an infinite sequence of uncertain phenomena (that all take values in some finite state space). This approach finds its origins in the work by von Mises, who's considered one of the founding fathers of the field of algorithmic randomness, and had the (different) aim of providing probabilities with a proper frequentist interpretation [24].

> *"The rational concept of probability, which is the only basis of probability calculus, applies only to problems in which either the same event repeats itself again and again, or a great number of uniform elements are involved at the same time. Using the language of physics, we may say that in order to apply the theory of probability we must have a practically unlimited sequence of uniform observations." [24, p. 11]*

Whereas von Mised considered an infinite repetition of some uncertain phenomenon, and thus assumes a fixed probability mass function m (or credal set C), the field of algorithmic randomness allows to consider non-stationary probability mass functions (or credal sets).

6 Uncertainty about a sequence of variables

As a first step to answering the just mentioned randomness question, we consider in this section an infinite sequence of uncertain phenomena (that have a common finite state space); and we explain how to describe the uncertainty about it. That is, we consider an infinite sequence of variables $X_1, \dots, X_n, \dots$, where every variable X_n takes values in a fixed but arbitrary finite state space $\mathscr{X}$; we generically denote such values by x_n.

6.1 Paths and situations

We're interested in the corresponding infinite outcome sequences $(x_1, \dots, x_n, \dots)$, and, in particular, in their possible randomness. We denote such a sequence generically by ω and call it a *path*. All such paths are collected in the set $\Omega := \mathscr{X}^{\mathbb{N}}$, which we'll also call the *sample space*. For any path $\omega = (x_1, \dots, x_n, \dots) \in \Omega$, we let $\omega_{1:n} := (x_1, \dots, x_n)$ and $\omega_n := x_n$ for all $n \in \mathbb{N}$. For $n = 0$, the empty sequence $\omega_{1:0} := \omega_0 := ()$ is called the *initial situation*, and we also denote it by $\square$. For any $n \in \mathbb{N}_0$, a finite outcome sequence $(x_1, \dots, x_n) \in \mathscr{X}^n$ is called a *situation*, also generically denoted by s, and its length is then denoted by $|s| := n$. All situations are collected in the set $\mathbb{S} := \bigcup_{n \in \mathbb{N}_0} \mathscr{X}^n$. For any situation $s = (x_1, \dots, x_n) \in \mathbb{S}$, we let $s_{1:k} = (x_1, \dots, x_k)$ and $s_k = x_k$ for all $1 \leq k \leq n$, and $x_{1:0} = x_0 = \square$. Also, for any $s = (x_1, \dots, x_n) \in \mathbb{S}$ and $x \in \mathscr{X}$, we use sx to denote the concatenation $(x_1, \dots, x_n, x)$. Figure 6.1 depicts situations and paths in the so-called *event tree* for a binary state space.

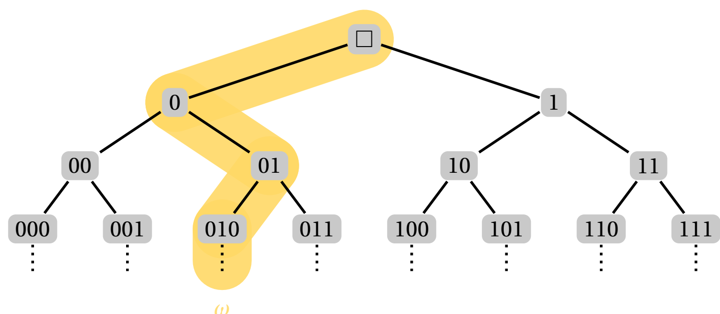


Figure 6.1. Let $\mathscr{X} = \{0, 1\}$. Each grey node in the event tree corresponds to a situation $s \in \mathbb{S}$, and the path $(0, 1, 0, \dots) \in \Omega$ is depicted in yellow.

For any situation $s \in \mathbb{S}$ and any path $\omega \in \Omega$, we say that ω *goes through* s if there's some $n \in \mathbb{N}_0$ such that $\omega_{1:n} = s$, and we then also write $s \sqsubseteq \omega$. We denote by $[\![s]\!]$ the so-called *cylinder set* of all paths $\omega \in \Omega$ that go through s. More generally, if $S \subseteq \mathbb{S}$ is some set of situations, then we denote by $[\![S]\!] := \bigcup_{s \in S} [\![s]\!]$ the set of all paths that go through (some situation in) S.

For any two situations $s, t \in \mathbb{S}$, we write $s \sqsubseteq t$ when every path that goes through t also goes through s, and we then say that the situation s *precedes* the situation t; so s is a precursor of t. An equivalent condition is of course that $[\![t]\!] \subseteq [\![s]\!]$. We may then also write $t \sqsupseteq s$ and say that t *follows* s. For any situation $s \in \mathbb{S}$, we denote the set $\{t \in \mathbb{S} : s \sqsubseteq t\}$ of all situations that follow s by $[s]$. We say that s *strictly precedes* t, and write $s \sqsubset t$, when $s \sqsubseteq t$ and $s \neq t$. If neither $s \sqsubseteq t$ nor $t \sqsubseteq s$, then we say that the situations s and t are *incomparable*, and write $s \parallel t$. Equivalently, s and t are incomparable if and only if $[\![s]\!] \cap [\![t]\!] = \emptyset$, so there's no path that goes through both s and t.

A subset S of $\mathbb{S}$ is called a *partial cut*—the term '*prefix free set*' is also commonly used in the algorithmic randomness literature—, and is then also denoted by K, if all its elements are mutually incomparable, or in other words constitute an anti-chain for the partial order $\sqsubseteq$, meaning that $s \parallel t$, or equivalently, $[\![s]\!] \cap [\![t]\!] = \emptyset$, for all $s, t \in S$ with $s \neq t$; the corresponding collection of cylinder sets $\{[\![s]\!] : s \in K\}$ constitutes a partition of $[\![K]\!]$. For any situation $s \in \mathbb{S}$ and any subset $S \subseteq \mathbb{S}$, there are a number of possibilities. We say that s *precedes* S, and write $s \sqsubseteq S$, if s precedes some situation in S: so $(\exists t \in S)\, s \sqsubseteq t$. Similarly, we say that s *strictly precedes* S, and write $s \sqsubset S$, if s strictly precedes some situation in S and doesn't follow any situation in S: $(\exists t \in S)\, s \sqsubset t$ and $(\nexists t \in S)\, t \sqsubseteq s$; for a partial cut K it holds that $s \sqsubset K \Leftrightarrow (\exists t \in K)\, s \sqsubset t$. We say that s *follows* S, and write $s \sqsupseteq S$, if s follows some situation in S: $(\exists t \in S)\, s \sqsupseteq t$; if S is a partial cut, then the situation t is necessarily unique. Similarly for s *strictly follows* S, written as $s \sqsupset S$. Of course, the situations in a partial cut K are the only ones that both precede and follow K. And, finally, we say that s is *incomparable* with S, and write $s \parallel S$, if s neither follows nor precedes (any situation in) S: $(\forall t \in S)\, s \parallel t$.

As is quite often done, we provide the set of all paths Ω with the *Cantor topology*, whose base is the collection $\{[\![s]\!] : s \in \mathbb{S}\}$ of all cylinder sets; see for instance Ref. [32, Sec. 1.2]. The corresponding Borel algebra $\mathcal{B}(\Omega)$ is the σ-algebra generated by this Cantor topology. A subset $A \subseteq \Omega$, which we call a *(global) event*, is then *open* if there's a subset $S \subseteq \mathbb{S}$ such that $A = [\![S]\!]$; it's *closed* if there's a subset $S \subseteq \mathbb{S}$ such that $\Omega \setminus A = [\![S]\!]$. If an event $A \subseteq \Omega$ is both open and closed, then it's *clopen*; this is equivalent to the existence of a finite set of situations $S \subseteq \mathbb{S}$ such that $A = [\![S]\!]$ [32, p. 4]. In particular, all cylinder sets $[\![s]\!]$ are clopen in this topology.

6.2 Forecasting systems

The randomness of a path $\omega \in \Omega$ is always defined with respect to an uncertainty model. Classically, this uncertainty model is often simply a probability mass function $m \in \mathcal{M}(\mathscr{X})$, which, for every $x \in \mathscr{X}$ and any $n \in \mathbb{N}$, specifies a probability $m(x)$ for the outcome $X_n = x$, and this independent of the outcome(s) at any other time instance(s). As explained in Section 1_2, we can generalise this by considering a credal set $C \in \mathscr{C}(\mathscr{X})$ instead. Another generalisation of the classical approach consists in allowing for non-stationary probability mass functions for describing the uncertainty related to the variables X_n, which may depend on the already observed outcomes $s = (x_1, \dots, x_{n-1})$ or on their length $|s| = n$; in associating a probability mass function with every situation, the event tree is turned into a so-called *precise probability tree*. Each of these generalisations can themselves be seen as a special case of an even more general approach, which consists in providing every situation $s \in \mathbb{S}$ with a (possibly different) credal set in $\mathscr{C}(\mathscr{X})$, denoted by $\varphi(s)$. This credal set $\varphi(s) \in \mathscr{C}(\mathscr{X})$ then describes the uncertainty about the *a priori* unknown outcome of $X_{|s|+1}$, given that the situation s has been observed. We call such general uncertainty models *forecasting systems*, and provide a graphical representation in Figure 6.3.

Definition 6.2. A *forecasting system* is a map $\varphi\colon \mathbb{S} \to \mathscr{C}(\mathscr{X})$ that associates with every situation $s \in \mathbb{S}$ a credal set $\varphi(s) \in \mathscr{C}(\mathscr{X})$, and, in doing so, turns the event tree into a so-called *imprecise probability tree*. We denote by $\Phi(\mathscr{X})$ the set of all such forecasting systems.

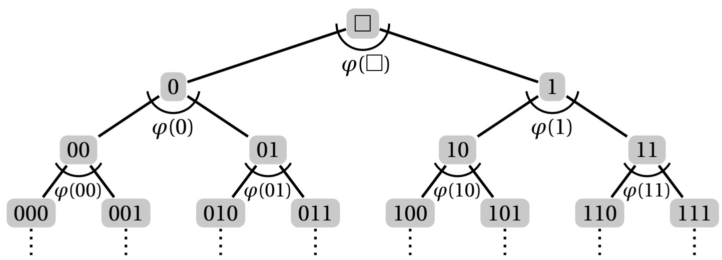


Figure 6.3. Let $\mathscr{X} = \{0, 1\}$. The arc around every situation $s \in \mathbb{S}$ in the imprecise probability tree corresponds to a credal set $\varphi(s) \in C$ that is specified by the forecasting system φ.

A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *precise* if $\varphi(s)$ consists of a single probability mass function for every $s \in \mathbb{S}$, and we then also denote it by φ_{pr}; the set of all precise forecasting systems is denoted by $\Phi_{\mathrm{pr}}(\mathscr{X})$. A precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ is called *positive* if $\varphi_{\mathrm{pr}}(s)$ is a positive probability mass function for all $s \in \mathbb{S}$. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *non-degenerate* if $\varphi(s)$ is non-degenerate for all $s \in \mathbb{S}$, and it's called *degenerate* otherwise. So, a forecasting system φ is degenerate as soon as there's some situation s and some outcome x for which $\overline{E}_{\varphi(s)}(\mathbb{I}_x) = 0$, meaning that according to Forecaster, after observing s, the next outcome can't be x. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *rational* if $\varphi(s) \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ for all $s \in \mathbb{S}$, and we then also denote it by φ_{rat}; the set of all rational forecasting systems is denoted by $\Phi_{\mathrm{rat}}(\mathscr{X})$. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *stationary* if there's some credal set $C \in \mathscr{C}(\mathscr{X})$ such that $\varphi(s) = C$ for all $s \in \mathbb{S}$; for ease of notation, we'll then denote this forecasting system simply by C; the case of a single probability mass function m corresponds to a stationary forecasting system with $C = \{m\}$. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *temporal* if its credal sets $\varphi(s)$ only depend on the situations $s \in \mathbb{S}$ through their length $|s|$, meaning that $\varphi(s) = \varphi(t)$ for any two situations $s, t \in \mathbb{S}$ that have the same length $|s| = |t|$. Allowing ourselves a slight abuse of notation, we'll also consider a temporal forecasting system $\varphi\colon \mathbb{S} \to \mathscr{C}(\mathscr{X})$ to be a map from the non-negative integers to the set of credal sets $\mathscr{C}(\mathscr{X})$, thus enabling us to write $\varphi(n)$ instead of $\varphi(s)$ for all $n \in \mathbb{N}_0$ and $s \in \mathbb{S}$ with $|s| = n$.

In the case of a binary state space such as $\mathscr{X} = \{a, b\}$, we recall from Section 5.3_{17} that every credal set $C \in \mathscr{C}(\mathscr{X})$ can be fully specified by the specification of an interval forecast $I \in \mathscr{I}$ that's associated with the outcome $X = a$. Accordingly, a forecasting system $\varphi\colon \mathbb{S} \to \mathscr{C}(\mathscr{X})$ can then be fully specified by associating an interval forecast $I_s \in \mathscr{I}$ with every situation $s \in \mathbb{S}$. In Section 20.1_{196}, where we'll restrict our attention to binary state spaces, we'll therefore consider forecasting systems to be maps from situations to interval forecasts, and will allow ourselves to write $\varphi\colon \mathbb{S} \to \mathscr{I}$; if the forecasting system φ is moreover precise, then we'll allow ourselves to write $\varphi\colon \mathbb{S} \to [0,1]$. If the forecasting system $\varphi\colon \mathbb{S} \to \mathscr{I}$ is stationary, that is, if there is some $I \in \mathscr{I}$ such that $\varphi(s) = I$ for all $s \in \mathbb{S}$, then we'll also denote this forecasting system simply by I, and *mutatis mutandis* for stationary precise forecasting systems $\varphi\colon \mathbb{S} \to [0,1]$ for which there is—by definition—a probability $p \in [0,1]$ such that $\varphi(s) = p$ for all $s \in \mathbb{S}$; we'll also call $1/2$ the *fair-coin forecasting system* and denote it by $\varphi_{1/2}$. In this binary setting, we also associate with every forecasting system $\varphi \in \Phi(\mathscr{X})$ two precise forecasting systems $\underline{\varphi}, \overline{\varphi} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ defined by $\underline{\varphi}(s) := \min \varphi(s)$ and $\overline{\varphi}(s) := \max \varphi(s)$ for all $s \in \mathbb{S}$. Clearly, a forecasting system $\varphi \in \Phi(\mathscr{X})$ is then precise if and only if $\underline{\varphi}(s) = \overline{\varphi}(s)$ for all $s \in \mathbb{S}$.

If for two forecasting systems $\varphi, \varphi^* \in \Phi(\mathscr{X})$ it holds that $\varphi(s) \subseteq \varphi^*(s)$ for all $s \in \mathbb{S}$, then we say that φ^* is *less informative*—or *more conservative*—than φ, and denote this by $\varphi \subseteq \varphi^*$. In this case, if φ is precise, then we also say that φ is *compatible* with φ^*, and denote this by $\varphi \in \varphi^*$; every forecasting system $\varphi \in \Phi(\mathscr{X})$ can then be seen as a specification of a set $\{\varphi_{\mathrm{pr}}\colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X}) \text{ and } \varphi_{\mathrm{pr}} \in \varphi\}$ of (compatible) precise forecasting systems.

For those readers only familiar with *probability measures* $\mu\colon \Omega \to [0,1]$, it's also worth noting that, through Ionescu Tulcea's extension theorem [64, Thm. II.9.2], every precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$—which corresponds to a conditional specification of probabilities—uniquely determines a *probability measure* $\mu^{\varphi_{\mathrm{pr}}}$ on the Borel algebra $\mathscr{B}(\Omega)$ (of sets of paths) generated by the cylinder sets, and that this probability measure is completely defined by its values on the cylinder sets. In particular, for these cylinder sets $[\![s]\!]$ themselves, with $s \in \mathbb{S}$, it assumes the values $\mu^{\varphi_{\mathrm{pr}}}([\![s]\!]) = \prod_{k=0}^{|s|-1} \varphi_{\mathrm{pr}}(s_{1:k})(s_{k+1})$; if $\varphi_{\mathrm{pr}}(s)(x) \neq 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, then the probability measure $\mu^{\varphi_{\mathrm{pr}}}$ only assumes positive values on the cylinder sets. Vice versa, any probability measure μ that associates positive probability with every cylinder set $[\![s]\!] \subseteq \Omega$, with $s \in \mathbb{S}$, determines a unique precise forecasting system φ_μ defined as $\varphi_\mu(s)(x) = \mu([\![sx]\!])/\mu([\![s]\!])$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$. If all probabilities that make up a precise forecasting system φ_{pr} stay away from zero, that is, if $\varphi_{\mathrm{pr}}(s)(x) \neq 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, then specifying φ_{pr} is equivalent to specifying the probability measure $\mu^{\varphi_{\mathrm{pr}}}$ on the Borel algebra $\mathscr{B}(\Omega)$, in the sense that, as discussed above, φ_{pr} uniquely determines $\mu^{\varphi_{\mathrm{pr}}}$ on $\mathscr{B}(\Omega)$, and $\mu^{\varphi_{\mathrm{pr}}}$—which then assumes positive values on the cylinder sets—determines the unique precise forecasting system $\varphi_{\mu^{\varphi_{\mathrm{pr}}}}$ for which $\varphi_{\mu^{\varphi_{\mathrm{pr}}}}(s)(x) = \mu^{\varphi_{\mathrm{pr}}}([\![sx]\!])/\mu^{\varphi_{\mathrm{pr}}}([\![s]\!]) = \varphi_{\mathrm{pr}}(s)(x)$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$. In classical (measure-theoretic) notions of randomness, one will typically focus on measures instead of forecasting systems. In our (game-theoretic) approach, however, forecasting systems take centre stage.

Forecasting systems can also be given a betting interpretation, which is the topic of the next section.

6.3 Betting strategies: sub- and supermartingales

In order to equip a forecasting system with a betting interpretation, which we'll do by extending the betting game in Section 5.4$_{19}$ to an infinite betting game involving a sequence of successively revealed variables $X_1, \dots, X_n, \dots$, we require a bit more terminology.

It will be useful to be able to deal with objects that depend on the situations. Formally, we define a *process* F as a map on the set $\mathbb{S}$ of all situations. In particular, a *real* process $F\colon \mathbb{S} \to \mathbb{R}$ is a map from situations to real numbers; the set of all real processes is denoted by $\mathscr{F}$. A real process F is called *bounded below* if there's some natural number $N \in \mathbb{N}_0$ such that $F \geq -N$ for all $s \in \mathbb{S}$. In particular, a real process F is called *non-negative* if $F(s) \geq 0$ for all $s \in \mathbb{S}$; it's called *positive* if $F(s) > 0$ for all $s \in \mathbb{S}$. We call a non-negative real process F a *test process* if additionally $F(\square) = 1$. A zero-one valued process S—with $S(s) \in \{0,1\}$ for all $s \in \mathbb{S}$—is called a *selection process*; the set of all selection processes is denoted by $\mathcal{S}$. If a process F depends only on the situations $s \in \mathbb{S}$ through their length $|s|$, we call it *temporal,* and then also write $F(n)$ instead of $F(s)$ for all $n \in \mathbb{N}_0$ and $s \in \mathbb{S}$ with $n = |s|$. Note that this is compatible with our terminology for forecasting systems; these are in fact also processes that map situations to credal sets.

We're now ready to consider a sequential version of the betting game in Section 5.4$_{19}$. We again consider three players: Forecaster, Sceptic and Reality. Forecaster's part in the game now consists in providing a forecasting system $\varphi \in \Phi(\mathscr{X})$. Subsequently, Sceptic is allowed to adopt any *betting strategy* that, for every situation $s \in \mathbb{S}$, selects an allowable gamble $f_s \in \mathscr{L}(\mathscr{X})$ that Forecaster is bound to offer by his specification of the credal set $\varphi(s) \in \mathscr{C}(\mathscr{X})$; that is, she selects a *gamble process* $\sigma\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ for which $\overline{E}_{\varphi(s)}(\sigma(s)) \leq 0$ for all $s \in \mathbb{S}$. Afterwards, Reality reveals the successive outcomes $X_n = x_n$ at each successive *time instant* $n \in \mathbb{N}$, leading to the sequence $\omega = (x_1, \ldots, x_n, \ldots)$. At every time instant n, after Reality has revealed the outcome x_n, Sceptic uses the gamble $\sigma(x_{1:n})$ that corresponds to her betting strategy. Next, Reality reveals the subsequent outcome $X_{n+1} = x_{n+1} \in \mathscr{X}$ and the reward $\sigma(x_{1:n})(x_{n+1})$ goes to Sceptic. If we assume that she starts with capital $c \in \mathbb{R}$, then her total *capital* at that stage in the game becomes $c + \sum_{k=0}^{n} \sigma(x_{1:k})(x_{k+1})$. Moreover, we'll prohibit Sceptic from borrowing.[11] To do so, it's useful to look at a gamble process's corresponding *capital process*; it's these processes that will take centre stage throughout this dissertation.

In order to formalise these capital processes for Sceptic, we associate with every real process F a *process difference* $\Delta F\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$, which is the gamble process that maps any $s \in \mathbb{S}$ to the gamble $\Delta F(s) := F(s\,\cdot) - F(s)$, where we use $F(s\,\cdot)$ to denote the gamble on $\mathscr{X}$ whose value, for any $x \in \mathscr{X}$, is given by $F(sx)$. Note that $F(x_{1:n}) = F(\square) + \sum_{k=0}^{n-1} \Delta F(x_{1:k})(x_{k+1})$ for all $x_{1:n} \in \mathbb{S}$, with $n \in \mathbb{N}_0$.

Given a forecasting system $\varphi \in \Phi(\mathscr{X})$, we call a real process M a *supermartingale* for φ if $\overline{E}_{\varphi(s)}(\Delta M(s)) \leq 0$ for all $s \in \mathbb{S}$—or equivalently, if $\overline{E}_{\varphi(s)}(M(s\,\cdot)) \leq M(s)$ for all $s \in \mathbb{S}$ [use C4$_{20}$]; it's called a *strict* supermartingale for φ if $\overline{E}_{\varphi(s)}(\Delta M(s)) < 0$ for all $s \in \mathbb{S}$—or equivalently, if $\overline{E}_{\varphi(s)}(M(s\,\cdot)) < M(s)$ for all $s \in \mathbb{S}$. A real process M is called a (*strict*) *submartingale* for φ if $-M$ is a (strict) supermartingale for φ, meaning by conjugacy that $\underline{E}_{\varphi(s)}(\Delta M(s)) \geq 0$ ($\underline{E}_{\varphi(s)}(\Delta M(s)) > 0$) for all $s \in \mathbb{S}$. A real process M is called a *martingale* for φ it it's both a super- and a submartingale for φ. All supermartingales and submartingales for φ are respectively collected in the sets $\overline{\mathbb{M}}(\varphi)$ and $\underline{\mathbb{M}}(\varphi)$.

Supermartingales correspond to Sceptic's allowed betting strategies; for that reason, in what follows, we'll also refer to supermartingales as betting strategies. Indeed, assume that Forecaster adopts the forecasting system $\varphi \in \Phi(\mathscr{X})$, consider a time instant $n \in \mathbb{N}_0$, and consider the situation where Reality has revealed a finite outcome sequence $x_{1:n} \in \mathbb{S}$. A supermartingale M for φ then specifies a gamble $\Delta M(x_{1:n}) \in \mathscr{L}(\mathscr{X})$ that Sceptic is allowed to pick. If she does, and Reality reveals the outcome $x_{n+1} \in \mathscr{X}$, the (possibly negative) amount $\Delta M(x_{1:n})(x_{n+1})$ goes to Sceptic and her total capital becomes

$$M(x_{1:n+1}) = M(x_{1:n}) + \Delta M(x_{1:n})(x_{n+1}) = M(\square) + \sum_{k=0}^{n} \Delta M(x_{1:k})(x_{k+1}),$$

with $M(\square)$ her initial capital. So, supermartingales can be seen as the possible evolutions of Sceptic's capital; submartingales, on the other hand, correspond to the possible evolutions of Forecaster's capital. As the following proposition reveals, the more conservative the forecasting system $\varphi \in \Phi(\mathscr{X})$ that Forecaster puts forwards, the less betting strategies Sceptic has at her disposal.

[11] We need the Axiom of dependent choice for doing so. We'll make use of it several other times throughout this dissertation, although without always providing an explicit mention.

Proposition 6.4. *Consider any two forecasting systems $\varphi, \psi \in \Phi(\mathscr{X})$ such that $\varphi \subseteq \psi$. Then any supermartingale for ψ is also a supermartingale for φ, so $\overline{\mathbb{M}}(\psi) \subseteq \overline{\mathbb{M}}(\varphi)$.*

Proof. Consider any supermartingale M for ψ, which means that $\overline{E}_{\psi(s)}(\Delta M(s\cdot)) \leq 0$ for all $s \in \mathbb{S}$, or equivalently [use C4$_{20}$], that $\overline{E}_{\psi(s)}(M(s\cdot)) \leq M(s)$ for all $s \in \mathbb{S}$. Now simply observe that also

$$\overline{E}_{\varphi(s)}(M(s\cdot)) = \sup_{m\in\varphi(s)} E_m(M(s\cdot)) \leq \sup_{m\in\psi(s)} E_m(M(s\cdot)) = \overline{E}_{\psi(s)}(M(s\cdot)) \leq M(s),$$

where the first inequality holds because $\varphi(s) \subseteq \psi(s)$ for all $s \in \mathbb{S}$. □

In the context of randomness, we'll also want to prevent Sceptic from borrowing, leading us to focus on non-negative supermartingales. Furthermore, as long as it's positive, the exact value of the initial capital will not matter. For these reasons, as an important special case, we consider *test supermartingales* $T : \mathbb{S} \to \mathbb{R}$ for φ. These are non-negative supermartingales for φ for which $T(\square) := 1$. We collect all test supermartingales for φ in the set $\overline{\mathbb{T}}(\varphi)$. It's exactly these capital processes that will be used in Chapter ⚁$_{49}$ to define several (martingale-theoretic) notions of randomness.

In one of these randomness notions, we'll adopt a particular way of defining such test supermartingales by focusing on 'multiplicative' rather than 'additive' betting strategies. For this reason, in addition to process differences—which we regard as additive betting strategies—, we introduce the notion of a *multiplier process*, which is a non-negative gamble process—and which we('ll) regard as a multiplicative betting strategy. With every such multiplier process $D : \mathbb{S} \to \mathscr{L}(\mathscr{X})$, we associate a test process $D^{\circledcirc} : \mathbb{S} \to \mathbb{R}$, defined by the initial condition $D^{\circledcirc}(\square) := 1$ and, for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, by the recursion equation $D^{\circledcirc}(sx) := D^{\circledcirc}(s)D(s)(x)$, and we say that $D^{\circledcirc}$ is *generated by D*; in this setting, it's also immediate that $D^{\circledcirc}(x_{1:n}) = \prod_{k=0}^{n-1} D(x_{1:k})(x_{k+1})$ for all $x_{1:n} \in \mathbb{S}$, with $n \in \mathbb{N}$. In particular, every positive test process F is generated by a (positive) multiplier process D_F.

Proposition 6.5. *Consider any positive test process F. Then there's a unique positive multiplier process $D_F : \mathbb{S} \to \mathscr{L}(\mathscr{X}) : s \mapsto \frac{F(s\cdot)}{F(s)}$ such that $F(s) = D_F^{\circledcirc}(s)$ for all $s \in \mathbb{S}$.*

Proof. Fix any positive test process F. Obviously, D_F is well-defined and positive because F is. Moreover, $F(\square) = 1 = D_F^{\circledcirc}(\square)$ and

$$F(x_{1:n}) = \prod_{k=0}^{n-1} \frac{F(x_{1:k+1})}{F(x_{1:k})} = \prod_{k=0}^{n-1} D_F(x_{1:k})(x_{k+1}) = D_F^{\circledcirc}(x_{1:n})$$

for all $x_{1:n} \in \mathbb{S}$ and $n \in \mathbb{N}$. From the positivity of F it's also immediate that D_F is the unique positive multiplier process for which $F(s) = D_F^{\circledcirc}(s)$ for all $s \in \mathbb{S}$. □

So, we say that a test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ is *generated by a multiplier process* if there's some non-negative gamble process D such that $T(x_{1:n}) = D^{\circledcirc}(x_{1:n})$ for all $x_{1:n} \in \mathbb{S}$, with $n \in \mathbb{N}$. Any multiplier process D that satisfies the condition that $\overline{E}_{\varphi(s)}(D(s)) \leq 1$ for all $s \in \mathbb{S}$, is called a *supermartingale multiplier* for the forecasting system φ; it's called a *strict supermartingale multiplier* for φ if $\overline{E}_{\varphi(s)}(D(s)) < 1$ for all $s \in \mathbb{S}$. As the following Proposition shows, a positive multiplier process D is a supermartingale multiplier for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if $D^{\circledcirc}$ is a positive test supermartingale for φ.

Proposition 6.6. *Consider a multiplier process D and a forecasting system $\varphi \in \Phi(\mathscr{X})$. If D is a (positive (strict)) supermartingale multiplier for φ, then the test process $D^{\circledcirc}$ is a (positive (strict)) test supermartingale for φ. And if $D^{\circledcirc}$ is a positive (strict) test supermartingale for φ, then D is a positive (strict) supermartingale multiplier for φ.*

Proof. Let's start by assuming that D is a supermartingale multiplier for φ. For every $s \in \mathbb{S}$, since

$$\Delta D^{\circledcirc}(s)(x) = D^{\circledcirc}(sx) - D^{\circledcirc}(s) = D^{\circledcirc}(s)D(s)(x) - D^{\circledcirc}(s) = D^{\circledcirc}(s)[D(s)(x) - 1]$$

for all $x \in \mathscr{X}$, we see that $\Delta D^{\circledcirc}(s) = D^{\circledcirc}(s)[D(s) - 1]$ and therefore, that

$$\begin{aligned}\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) = \overline{E}_{\varphi(s)}(D^{\circledcirc}(s)[D(s)-1]) &\overset{\text{C2}_{20}}{=} D^{\circledcirc}(s)\overline{E}_{\varphi(s)}(D(s)-1)\\ &\overset{\text{C4}_{20}}{=} D^{\circledcirc}(s)\Big[\overline{E}_{\varphi(s)}(D(s)) - 1\Big] \leq 0,\end{aligned}$$

where the inequality holds since $\overline{E}_{\varphi(s)}(D(s)) \leq 1$ for all $s \in \mathbb{S}$. Since $D^{\circledcirc}$ is non-negative and $D^{\circledcirc}(\square) = 1$ by definition, we conclude that $D^{\circledcirc}$ is a test supermartingale for φ. Furthermore, if $D(s)(x) > 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$ by assumption, then it immediately follows from the definition that the test supermartingale $D^{\circledcirc}$ is positive as well. Moreover, if additionally $\overline{E}_{\varphi(s)}(D(s)) < 1$ for all $s \in \mathbb{S}$, then

$$\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) = D^{\circledcirc}(s)\Big[\overline{E}_{\varphi(s)}(D(s)) - 1\Big] \overset{D^{\circledcirc}>0}{<} 0,$$

and hence, $D^{\circledcirc}$ is a positive strict test supermartingale for φ.

Let's continue by assuming that $D^{\circledcirc}$ is a positive test supermartingale for φ. It's immediate from Proposition 6.5↶ that D is positive. For every $s \in \mathbb{S}$, since $D(s) = 1 + \frac{\Delta D^{\circledcirc}(s)}{D^{\circledcirc}(s)}$, we see that

$$\overline{E}_{\varphi(s)}(D(s)) = \overline{E}_{\varphi(s)}\left(1 + \frac{\Delta D^{\circledcirc}(s)}{D^{\circledcirc}(s)}\right) \overset{\text{C2}_{20},\text{C4}_{20}}{=} 1 + \frac{1}{D^{\circledcirc}(s)}\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) \leq 1,$$

where the inequality holds because $\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) \leq 0$ for all $s \in \mathbb{S}$. We conclude that D is a positive supermartingale multiplier for φ. Moreover, if additionally $\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) < 0$ for all $s \in \mathbb{S}$, then

$$\overline{E}_{\varphi(s)}(D(s)) = 1 + \frac{1}{D^{\circledcirc}(s)}\overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) < 1,$$

and hence, D is a positive strict supermartingale multiplier for φ. □

It's test supermartingales and (positive) supermartingale multipliers that we'll use to introduce several imprecise-probabilistic martingale-theoretic notions of randomness in Chapter □49. On this martingale-theoretic account, a path $\omega \in \Omega$ will be considered random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if Sceptic won't be able to adopt any betting strategy (chosen from a well-chosen proper subset of all betting strategies) that makes her rich without bounds along ω. As indicated before, it's the capital processes $T \in \overline{\mathbb{T}}(\varphi)$ that correspond to/result from betting strategies that will take centre stage in our martingale-theoretic notions of randomness. For such notions, if we want for every forecasting system φ at least one path to be random, then we need to sufficiently restrict these betting strategies; we'll explain in Section 850 what restricted sets we'll allow Sceptic to choose from. In the following two examples, we'll explore what minimal conditions we have to impose on $T \in \overline{\mathbb{T}}(\varphi)$ to end up with a non-vacuous randomness notion. As the following example shows, we should indeed at least impose non-negativity, because no binary path is random for the prototypical stationary precise forecasting system $\varphi_{1/2}$ if we don't impose a lower bound on the capital processes that correspond to Sceptic's allowed betting strategies.

Example 6.7. Consider the binary state space $\mathscr{X} = \{0, 1\}$, any path $\omega \in \Omega$ and the fair-coin forecasting system $\varphi_{1/2}$. Then the so-called 'martingale betting system' will guarantee the existence of a betting strategy that gives an unbounded profit on ω. Assume that ω contains an infinite number of ones; if it contains only an infinite number of zeros, then reverse the bets in the following betting strategy. Let the betting strategy $\sigma\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ be recursively defined as $\sigma(\square)(0) = -1$, $\sigma(\square)(1) = 1$ and

$$\sigma(sx) := \begin{cases} 2\sigma(s) & \text{if } x = 0 \\ \sigma(\square) & \text{if } x = 1 \end{cases} \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}.$$

This betting strategy doubles the stakes if loss occurs—that is, if 0 is the last observed entry—and resets the stakes to the initial bet if gain occurs—that is, if 1 is the last observed outcome. By construction, Sceptic's initial capital has increased by one unit by the time she observes outcome 1 for the first time, has increased by two units by the time she observes outcome 1 for the second time, and so on. So, Sceptic has increased her capital by n units by the time she observes outcome 1 for the n-th time. Since ω is assumed to contain an infinite number of ones, this provides an unbounded gain for Sceptic on ω. ◇

Imposing non-negativity does however not suffice to obtain a non-trivial notion of randomness, since, as the following example reveals, no binary path is random for the fair-coin forecasting system $\varphi_{1/2}$ if we consider the class of all test supermartingales $\overline{\mathbb{T}}(\varphi_{1/2})$.

Example 6.8. Consider the binary state space $\mathscr{X} = \{0,1\}$, any path $\omega \in \Omega$ and the fair-coin forecasting system $\varphi_{1/2}$. Let the temporal multiplier process $D\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ be defined as

$$D(n)(x) := \begin{cases} \frac{1}{2} & \text{if } x = 1 - \omega_{1:n+1} \\ \frac{3}{2} & \text{if } x = \omega_{1:n+1} \end{cases} \text{ for all } n \in \mathbb{N}_0 \text{ and } x \in \mathscr{X}.$$

By construction, $D(s)(x) > 0$ and $E_{1/2}(D(s)) = \frac{1}{2}\big(\frac{1}{2} + \frac{3}{2}\big) = 1$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, and hence, by Proposition 6.6$_{30}$, $D^{\circledcirc}$ is a positive test supermartingale for $\varphi_{1/2}$. Also, by construction, $\lim_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \lim_{n\to\infty}\big(\frac{3}{2}\big)^n = \infty$. ⋄

So, since no binary path is random for the fair-coin forecasting system $\varphi_{1/2}$ if we allow for all test supermartingales, to what set should we restrict Sceptic's betting strategies? Interesting martingale-theoretic notions of randomness are typically obtained by additionally imposing some computability constraints, as we'll explain at the beginning of Section 7$_{40}$.

But before we get to that, let us move on with the topic of this section, which is how to model uncertainty about a sequence of variables. In particular, we'll use super- and submartingales in Section 6.4 to associate upper and lower expectations–and corresponding upper and lower probabilities—with forecasting systems. These uncertainty models will especially come in handy in Chapter ⚃$_{111}$ to equip some of our martingale-theoretic randomness notions with a measure-theoretic characterisation in Chapter ⚃$_{111}$. Classically, in a precise-probabilistic measure-theoretic setting, the randomness of a path $\omega \in \Omega$ with respect to a precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ is defined by means of so-called *null covers*: a path ω isn't random for φ_{pr} if, for every positive threshold $\delta > 0$, there's some computable/effective way to specify a set of paths that contains ω and whose probability is smaller than δ. To do this in our (imprecise-probabilistic) setting, we'll make use of upper probabilities instead of probabilities to express what it means for a set of paths to be small with respect to a forecasting system.

6.4 (Global) upper and lower expectations

Recall that we associated (local) upper and lower expectations with credal sets in Section 5.4$_{19}$. In our current (global) setting, we can also associate (global) upper and lower expectations with forecasting systems, which then take global gambles $u \in \mathscr{L}(\Omega)$ on the sample space Ω as their argument, instead of local gambles $f \in \mathscr{L}(\mathscr{X})$ on the state space $\mathscr{X}$.

Global gambles. A *(global) gamble* $u\colon \Omega \to \mathbb{R}$ is a bounded map from the set Ω of all paths to the real numbers. All such gambles are collected in the set $\mathscr{L}(\Omega)$. If a global gamble $u \in \mathscr{L}(\Omega)$ is constant, that is, if there's some $c \in \mathbb{R}$ such that $u(\omega) = c$ for all $\omega \in \Omega$, then we also allow ourselves to write c instead of u. With every (global) event $A \subseteq \Omega$, we associate the gamble $\mathbb{I}_A \in \mathscr{L}(\Omega)$

that assumes the value 1 on A and 0 elsewhere, and call it the *indicator* of A; observe that, since $0 \leq \mathbb{I}_A(\omega) \leq 1$ for any $\omega \in \Omega$, $\mathbb{I}_A$ is bounded and therefore indeed a gamble. The *complement* $\Omega \setminus A$ of an event $A \subseteq \Omega$ is denoted by A^c and its indicator $\mathbb{I}_{A^c}$ satisfies $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$. Also, with every so-called *local gamble* $f \in \mathscr{L}(\mathscr{X})$ and any situation $s \in \mathbb{S}$, we associate the global gamble $f_s \in \mathscr{L}(\Omega)$, defined by

$$f_s(\omega) := \begin{cases} f(x) & \text{if } \omega \in [\![sx]\!] \text{ with } x \in \mathscr{X} \\ 0 & \text{otherwise, so if } \omega \notin [\![s]\!] \end{cases} \text{ for all } \omega \in \Omega, \tag{6.9}$$

which is equivalent to writing that $f_s = \sum_{x \in \mathscr{X}} f(x) \mathbb{I}_{[\![sx]\!]}$.

Global upper and lower expectations. We're now ready to associate a (global) upper and lower expectation with every forecasting system $\varphi \in \Phi(\mathscr{X})$. We do so by adopting the so-called *game-theoretic conditional upper expectation* $\overline{E}^{\varphi}(\bullet|\bullet) \colon \mathscr{L}(\Omega) \times \mathbb{S} \to \mathbb{R}$ and *conditional lower expectation* $\underline{E}^{\varphi}(\bullet|\bullet) \colon \mathscr{L}(\Omega) \times \mathbb{S} \to \mathbb{R}$, which are defined by[12]

$$\overline{E}^{\varphi}(u|s) := \inf\left\{M(s) \colon M \in \overline{\mathbb{M}}(\varphi) \text{ and } \liminf_{n\to\infty} M(\omega_{1:n}) \geq u(\omega) \text{ for all } \omega \in [\![s]\!]\right\} \tag{6.10}$$

and

$$\underline{E}^{\varphi}(u|s) := \sup\left\{M(s) \colon M \in \underline{\mathbb{M}}(\varphi) \text{ and } \limsup_{n\to\infty} M(\omega_{1:n}) \leq u(\omega) \text{ for all } \omega \in [\![s]\!]\right\} \tag{6.11}$$

for all $u \in \mathscr{L}(\Omega)$ and $s \in \mathbb{S}$. These global conditional upper and lower expectations are related to each other through the following conjugacy relationship [66, Eq. (5)]: $\overline{E}^{\varphi}(u|s) = -\underline{E}^{\varphi}(-u|s)$ for all $u \in \mathscr{L}(\Omega)$ and $s \in \mathbb{S}$. For any gamble $u \in \mathscr{L}(\Omega)$, we'll also refer to the conditional upper expectation $\overline{E}^{\varphi}(u|\square)$ and the conditional lower expectation $\underline{E}^{\varphi}(u|\square)$ as simply the (global) *upper* and *lower expectation* of u, respectively, and we then denote them by $\overline{E}^{\varphi}(u)$ and $\underline{E}^{\varphi}(u)$.

Extensive discussions in related contexts about why these expressions are relevant can be found in Refs. [14, 51, 65, 66, 67, 68, 69, 70, 71, 72]. For our present purposes, it suffices to know that the global conditional upper expectation $\overline{E}^{\varphi}(u|s)$ of a gamble $u \in \mathscr{L}(\Omega)$ is the infimum capital Sceptic has to start with in s such that there's an allowed betting strategy [supermartingale]

[12] Several versions of these definitions exist, which differ only in the type of supermartingales that are used (real-valued, extended real-valued, unbounded, bounded, bounded below); see for example De Cooman & De Bock [36], Shafer & Vovk [51], T'Joens, De Bock & De Cooman [65], De Cooman, De Bock & Lopatatzidis [66], and T'Joens, De Bock & De Cooman [67]. For gambles, however, all these definitions are equivalent; see De Cooman, De Bock & Lopatatzidis [66, Proposition 10] and T'Joens, De Bock & De Cooman [65, Proposition 36]. This allows us to apply properties that were proved for these alternative expressions in our context as well.

that guarantees her ending up with a higher capital than the reward that's associated with u, along all paths that go through s. So, in particular, $\overline{E}^{\varphi}(u)$ is Forecaster's infimum selling price for the gamble u. Indeed, consider any supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ such that $\liminf_{n\to\infty} M(\omega_{1:n}) \geq u(\omega)$ for all $\omega \in \Omega$. By his specification of the forecasting system φ, Forecaster allows Sceptic to play the betting strategy that corresponds to $M - M(\square)$, which is worth more than the gamble $u - M(\square)$, implying that he's willing to offer her this gamble, and thus to sell u for $M(\square)$. The global conditional lower expectation $\underline{E}^{\varphi}(u|s)$ of a gamble $u \in \mathscr{L}(\Omega)$, on the other hand, is the supremum capital Forecaster has to start with in s such that he has a betting strategy [submartingale] that guarantees him ending up with a lower capital than the reward that's associated with u, along all paths that go through s. In particular, $\underline{E}^{\varphi}(u)$ is Forecaster's supremum buying price for the gamble u. Indeed, consider any submartingale $M \in \underline{\mathbb{M}}(\varphi)$ such that $\limsup_{n\to\infty} M(\omega_{1:n}) \leq u(\omega)$ for all $\omega \in \Omega$. By his specification of the forecasting system φ, Forecaster accepts the betting strategy that corresponds to $M - M(\square)$ [and he allows Sceptic to play the betting strategy that corresponds to the supermartingale $M(\square) - M$], which is worth less than the gamble $u - M(\square)$, implying that he's willing to accept this gamble, and thus to buy u for $M(\square)$.

Conveniently, $\overline{E}^{\varphi}(\bullet|\bullet)$ and $\underline{E}^{\varphi}(\bullet|\bullet)$ satisfy a number of properties, the first six of which resemble C1 to C6_{20}.

Proposition 6.12. *Consider a forecasting system $\varphi \in \Phi(\mathscr{X})$. Then for all $u, v \in \mathscr{L}(\Omega)$, $(u_n)_{n\in\mathbb{N}_0} \in \mathscr{L}(\Omega)^{\mathbb{N}_0}$, $f \in \mathscr{L}(\mathscr{X})$, $s \in \mathbb{S}$ and $\lambda \in \mathbb{R}_{\geq 0}$:*

E1. $\inf_{\omega\in[\![s]\!]} u(\omega) \leq \underline{E}^{\varphi}(u|s) \leq \overline{E}^{\varphi}(u|s) \leq \sup_{\omega\in[\![s]\!]} u(\omega)$; [boundedness]

E2. $\overline{E}^{\varphi}(\lambda u|s) = \lambda\overline{E}^{\varphi}(u|s)$ *and* $\underline{E}^{\varphi}(\lambda u|s) = \lambda\underline{E}^{\varphi}(u|s)$; [non-negative homogeneity]

E3. $\overline{E}^{\varphi}(u+v|s) \leq \overline{E}^{\varphi}(u|s) + \overline{E}^{\varphi}(v|s)$ *and* $\underline{E}^{\varphi}(u|s) + \underline{E}^{\varphi}(v|s) \leq \underline{E}^{\varphi}(u+v|s)$; [sub- and superadditivity]

E4. $\overline{E}^{\varphi}(u+v|s) = \overline{E}^{\varphi}(u|s) + v_s$ *and* $\underline{E}^{\varphi}(u+v|s) = \underline{E}^{\varphi}(u|s) + v_s$ *if v assumes the constant value v_s on $[\![s]\!]$*; [constant additivity]

E5. *if $u(\omega) \leq v(\omega)$ for all $\omega \in [\![s]\!]$, then* $\overline{E}^{\varphi}(u|s) \leq \overline{E}^{\varphi}(v|s)$ *and* $\underline{E}^{\varphi}(u|s) \leq \underline{E}^{\varphi}(v|s)$; [monotonicity]

E6. $\overline{E}^{\varphi}(u|s) = \overline{E}^{\varphi}(u\mathbb{I}_{[\![s]\!]}|s)$ *and* $\underline{E}^{\varphi}(u|s) = \underline{E}^{\varphi}(u\mathbb{I}_{[\![s]\!]}|s)$; [restriction]

E7. if $u_n \nearrow u$ point-wise on $[\![s]\!]$, then $\overline{E}^{\varphi}(u|s) = \lim_{n\to\infty} \overline{E}^{\varphi}(u_n|s)$; [convergence]

E8. $\overline{E}^{\varphi}(f_s|s) = \overline{E}_{\varphi(s)}(f)$ and $\underline{E}^{\varphi}(f_s|s) = \underline{E}_{\varphi(s)}(f)$; [locality]

E9. $\overline{E}_{\varphi(s)}(\overline{E}^{\varphi}(u|s\,\cdot)) = \overline{E}^{\varphi}(u|s)$ and $\underline{E}_{\varphi(s)}(\underline{E}^{\varphi}(u|s\,\cdot)) = \underline{E}^{\varphi}(u|s)$. [super- and submartingale]

Proof. From De Cooman, De Bock & Lopatatzidis [66, Equation (5) and Proposition 10], it follows that the conditional upper expectation $\overline{E}^{\varphi}(u|s)$, with $u \in \mathscr{L}(\Omega)$ and $s \in \mathbb{S}$, can be equivalently defined in terms of bounded below supermartingales. By

T'Joens, De Bock & De Cooman [65, Proposition 36], this equivalence continues to hold when considering extended real-valued bounded below supermartingales. Consequently, E1 to E3 follow from conjugacy and Proposition 14 in Ref. [66]. E4 follows from conjugacy and Proposition 4.4.3 in Ref. [14]: $\overline{E}^{\varphi}(u+v|s) = \overline{E}^{\varphi}(u\mathbb{I}_{[\![s]\!]} + v\mathbb{I}_{[\![s]\!]}|s) = \overline{E}^{\varphi}(u\mathbb{I}_{[\![s]\!]} + v_s\mathbb{I}_{[\![s]\!]}|s) = \overline{E}^{\varphi}(u+v_s|s) = \overline{E}^{\varphi}(u|s) + v_s$, where the first and third equality follow from EC6 in Ref. [14], and where the fourth inequality follows from EC5 in Ref. [14]. E5 follows from conjugacy and Proposition 13 in Ref. [65]. E6 follows from conjugacy and Proposition 4.4.3 in Ref. [14]. E7 follows from Theorem 23 in Ref. [65] and E6. E8 follows from conjugacy and Proposition 14 in Ref. [65]. And E9 follows from conjugacy, E6, E8, and Theorem 15 in Ref. [65]. □

As yet another property, we see that more conservative forecasting systems lead to more conservative (larger) upper expectations.

Proposition 6.13. *Consider any two forecasting systems $\varphi, \psi \in \Phi(\mathscr{X})$ such that $\varphi \subseteq \psi$. Then $\overline{E}^{\varphi}(u|s) \leq \overline{E}^{\psi}(u|s)$ for all global gambles $u \in \mathscr{L}(\Omega)$ and all situations $s \in \mathbb{S}$.*

Proof. Use Eq. (6.10)33:

$$\overline{E}^{\varphi}(u|s) = \inf\Big\{M(s)\colon M \in \overline{\mathbb{M}}(\varphi) \text{ and } \liminf_{n\to\infty} M(\omega_{1:n}) \geq u(\omega) \text{ for all } \omega \in [\![s]\!]\Big\}$$
$$\leq \inf\Big\{M(s)\colon M \in \overline{\mathbb{M}}(\psi) \text{ and } \liminf_{n\to\infty} M(\omega_{1:n}) \geq u(\omega) \text{ for all } \omega \in [\![s]\!]\Big\} = \overline{E}^{\psi}(u|s),$$

where the inequality holds because $\overline{\mathbb{M}}(\psi) \subseteq \overline{\mathbb{M}}(\varphi)$ by Proposition 6.429. □

Global upper and lower probabilities. Global conditional upper and lower expectations allow us to also define their corresponding *conditional upper and lower probabilities*: for any event $A \subseteq \Omega$, any forecasting system φ, and any situation $s \in \mathbb{S}$, $\overline{P}^{\varphi}(A|s) := \overline{E}^{\varphi}(\mathbb{I}_A|s)$ and $\underline{P}^{\varphi}(A|s) := \underline{E}^{\varphi}(\mathbb{I}_A|s)$. For any event $A \subseteq \Omega$, we'll also refer to the conditional upper probability $\overline{P}^{\varphi}(A|\square)$ and the conditional lower probability $\underline{P}^{\varphi}(A|\square)$ as simply the (global) *upper* and *lower probability* of A, respectively, and we then denote them by $\overline{P}^{\varphi}(A)$ and $\underline{P}^{\varphi}(A)$.

Conveniently, $\overline{P}^{\varphi}(\bullet|\bullet)$ and $\underline{P}^{\varphi}(\bullet|\bullet)$ satisfy the following properties, with the first five being special instantiations of properties E1, E3, E5, E7 and E9 in Proposition 6.12.

Corollary 6.14. *Consider a forecasting system $\varphi \in \Phi(\mathscr{X})$. Then for all $A, A_n, B \subseteq \Omega$, with $n \in \mathbb{N}_0$, and $s \in \mathbb{S}$:*

P1. $\inf_{\omega\in[\![s]\!]} \mathbb{I}_A(\omega) \leq \underline{P}^{\varphi}(A|s) \leq \overline{P}^{\varphi}(A|s) \leq \sup_{\omega\in[\![s]\!]} \mathbb{I}_A(\omega)$; [boundedness]

P2. $\overline{P}^{\varphi}(A\cup B|s) \leq \overline{P}^{\varphi}(A|s) + \overline{P}^{\varphi}(B|s)$ *and* $\underline{P}^{\varphi}(A|s) + \underline{P}^{\varphi}(B|s) \leq \underline{P}^{\varphi}(A\cup B|s)$; [subadditivity]

P3. *if* $A \cap [\![s]\!] \subseteq B \cap [\![s]\!]$, *then* $\overline{P}^{\varphi}(A|s) \leq \overline{P}^{\varphi}(B|s)$ *and* $\underline{P}^{\varphi}(A|s) \leq \underline{P}^{\varphi}(B|s)$; [monotonicity]

P4. *if* $\mathbb{I}_{A_n} \nearrow \mathbb{I}_A$ *point-wise on* $[\![s]\!]$, *then* $\overline{P}^{\varphi}(A|s) = \lim_{n\to\infty} \overline{P}^{\varphi}(A_n|s)$; [convergence]

P5. $\overline{E}_{\varphi(s)}(\overline{P}^{\varphi}(A|s\,\cdot)) = \overline{P}^{\varphi}(A|s)$ *and* $\underline{E}_{\varphi(s)}(\underline{P}^{\varphi}(A|s\,\cdot)) = \underline{P}^{\varphi}(A|s)$; [super- and submartingale]

P6. $\overline{P}^{\varphi}(A^c) = 1 - \underline{P}^{\varphi}(A)$. [complement]

Proof. To prove P6, observe that by E4$_{34}$ and conjugacy,

$$\overline{P}^{\varphi}(A^c) = \overline{E}^{\varphi}(\mathbb{I}_{A^c}) = \overline{E}^{\varphi}(1-\mathbb{I}_A) = 1 + \overline{E}^{\varphi}(-\mathbb{I}_A) = 1 - \underline{E}^{\varphi}(\mathbb{I}_A) = 1 - \underline{P}^{\varphi}(A)$$

for all $A \subseteq \Omega$. □

When considering events that are open in the Cantor topology, that is, events $A \subseteq \Omega$ for which there's some subset $S \subseteq \mathbb{S}$ such that $A = [\![S]\!]$, we'll make use of the following additional properties of $\overline{P}^{\varphi}(\bullet|\bullet)$.

Corollary 6.15. *Consider any forecasting system* $\varphi \in \Phi(\mathscr{X})$, *any subset* $S \subseteq \mathbb{S}$, *and any* $s \in \mathbb{S}$. *Then the following statements hold for the real process* $\overline{P}^{\varphi}([\![S]\!]|\bullet)\colon \mathbb{S} \to \mathbb{R}$:

(i) $s \sqsupseteq S \Rightarrow \overline{P}^{\varphi}([\![S]\!]|s) = 1$ *and* $s \parallel S \Rightarrow \overline{P}^{\varphi}([\![S]\!]|s) = 0$;

(ii) $\liminf_{n\to\infty} \overline{P}^{\varphi}([\![S]\!]|\omega_{1:n}) \geq \mathbb{I}_{[\![S]\!]}(\omega)$ *for all* $\omega \in \Omega$.

Proof. For (i), observe on the one hand that $s \sqsupseteq S$ implies that the global gamble $\mathbb{I}_{[\![S]\!]}$ assumes the constant value 1 on $[\![s]\!]$, and use P1$_{\curvearrowleft}$. If, on the other hand, $s \parallel S$, then $\mathbb{I}_{[\![S]\!]}$ assumes the constant value 0 on $[\![s]\!]$, and the desired result again follows from P1$_{\curvearrowleft}$.

For (ii), observe that it follows from P1$_{\curvearrowleft}$ that $\overline{P}^{\varphi}([\![S]\!]|\bullet) \geq 0$. It therefore suffices to consider any $\omega \in [\![S]\!]$ and to prove that then $\liminf_{n\to\infty} \overline{P}^{\varphi}([\![S]\!]|\omega_{1:n}) = 1$. But if $\omega \in [\![S]\!]$, then there must be some $s \in S$ such that $\omega \in [\![s]\!]$. Hence, for all $n \geq |s|$, $\omega_{1:n} \sqsupseteq S$ and therefore, by (i), also $\overline{P}^{\varphi}([\![S]\!]|\omega_{1:n}) = 1$. □

It will also prove useful to have expressions for the upper and lower probabilities of cylinder sets. Unlike those for more general global events, they turn out to be particularly simple and elegant.

Proposition 6.16. *Consider any forecasting system* $\varphi \in \Phi(\mathscr{X})$ *and any situation* $s \in \mathbb{S}$, *then*

$$\overline{P}^{\varphi}([\![s]\!]) = \prod_{k=0}^{|s|-1} \overline{E}_{\varphi(s_{1:k})}(\mathbb{I}_{s_{k+1}}) \text{ and } \underline{P}^{\varphi}([\![s]\!]) = \prod_{k=0}^{|s|-1} \underline{E}_{\varphi(s_{1:k})}(\mathbb{I}_{s_{k+1}}).$$

Proof. We give the proof for the upper probability. The proof for the lower probability is completely similar.

First of all, fix any $\ell \in \{0,1,\dots,|s|-1\}$. For any $x \in \mathscr{X}$,

$$\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell}\,x\big) = \overline{E}^{\varphi}(\mathbb{I}_{[\![s]\!]}|s_{1:\ell}\,x) \overset{\text{E6}_{34}}{=} \overline{E}^{\varphi}(\mathbb{I}_{[\![s]\!]}\mathbb{I}_{[\![s_{1:\ell}\,x]\!]}|s_{1:\ell}\,x) = \overline{E}^{\varphi}(\mathbb{I}_{[\![s]\!]}\mathbb{I}_{s_{\ell+1}}(x)|s_{1:\ell}\,x)$$
$$\overset{\text{E2}_{34}}{=} \overline{E}^{\varphi}(\mathbb{I}_{[\![s]\!]}|s_{1:\ell}\,x)\mathbb{I}_{s_{\ell+1}}(x) = \overline{E}^{\varphi}(\mathbb{I}_{[\![s]\!]}|s_{1:\ell+1})\mathbb{I}_{s_{\ell+1}}(x)$$

$$= \overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell+1}\big)\mathbb{I}_{s_{\ell+1}}(x).$$

Hence,

$$\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell}\,\cdot\big) = \overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell+1}\big)\mathbb{I}_{s_{\ell+1}}, \tag{6.17}$$

so we can infer from the recursion equation in P5 that

$$\begin{aligned}\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell}\big) &= \overline{E}_{\varphi(s_{1:\ell})}(\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell}\,\cdot\big)) \overset{\text{Eq. (6.17)}}{=} \overline{E}_{\varphi(s_{1:\ell})}(\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell+1}\big)\mathbb{I}_{s_{\ell+1}})\\ &= \overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell+1}\big)\overline{E}_{\varphi(s_{1:\ell})}(\mathbb{I}_{s_{\ell+1}}),\end{aligned}$$

where the third equality follows from $\mathrm{C2}_{20}$ and the fact that $\overline{P}^{\varphi}\big([\![s]\!]|s_{1:\ell+1}\big) \geq 0$ [use $\mathrm{P1}_{35}$]. A simple iteration on ℓ now shows that, indeed,

$$\begin{aligned}\overline{P}^{\varphi}([\![s]\!]) = \overline{P}^{\varphi}([\![s]\!]|\square) = \overline{P}^{\varphi}([\![s]\!]|s)\prod_{k=0}^{|s|-1}\overline{E}_{\varphi(s_{1:k})}(\mathbb{I}_{s_{k+1}})\\ = \prod_{k=0}^{|s|-1}\overline{E}_{\varphi(s_{1:k})}(\mathbb{I}_{s_{k+1}}),\end{aligned}$$

where the last equality follows from $\overline{P}^{\varphi}([\![s]\!]|s) = 1$, as is guaranteed by $\mathrm{P1}_{35}$, or alternatively, by Corollary 6.15(i). □

We'll also make use of the following elegant and powerful inequality, the idea for which in its simplest form is due to Ville [28]; its proof is based on Shafer and Vovk's work on game-theoretic probabilities [51, 69].

Proposition 6.18 (Ville's inequality)**.** *Consider any forecasting system φ, any non-negative supermartingale T for φ, and any $C > 0$, then*

$$\overline{P}^{\varphi}\left(\left\{\omega \in \Omega\colon \sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) \geq C\right\}\right) \leq \frac{1}{C}T(\square).$$

Proof. Let $G_C := \{\omega \in \Omega\colon \sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) \geq C\}$. Consider any $0 < \epsilon < C$, and let T_ϵ be the real process given for all $s \in \mathbb{S}$ by

$$T_\epsilon(s) := \begin{cases} T(t) & \text{if there's some first } t \sqsubseteq s \text{ such that } T(t) \geq C-\epsilon\\ T(s) & \text{if } T(t) < C-\epsilon \text{ for all } t \sqsubseteq s,\end{cases}$$

so T_ϵ is the version of T that mimics the behaviour of T but is stopped—kept constant—as soon as it reaches a value of at least $C-\epsilon$. Observe that $T_\epsilon(\square) = T(\square)$, and that $\frac{1}{C-\epsilon}T_\epsilon$ is still a non-negative supermartingale for φ. For any $\omega \in G_C$, we have that $\sup_{n\in\mathbb{N}_0} T(\omega_{1:n}) \geq C > C-\epsilon$, so there's some $n \in \mathbb{N}_0$ such that $T(\omega_{1:n}) > C-\epsilon$, implying that $T_\epsilon(\omega_{1:m}) = T_\epsilon(\omega_{1:n}) \geq C-\epsilon$ for all $m \geq n$, and therefore $\liminf_{n\to\infty}\frac{1}{C-\epsilon}T_\epsilon(\omega_{1:n}) \geq 1$. Hence

$$\liminf_{n\to\infty}\frac{1}{C-\epsilon}T_\epsilon(\omega_{1:n}) \geq \mathbb{I}_{G_C}(\omega) \text{ for all } \omega \in \Omega,$$

and therefore Eq. $(6.10)_{33}$ tells us that $\overline{P}^{\varphi}(G_C) \leq \frac{1}{C-\epsilon}T_\epsilon(\square) = \frac{1}{C-\epsilon}T(\square)$. Since this holds for all $0 < \epsilon < C$, we're done. □

Almost sure events. (Global) upper and lower probabilities allow for an imprecise-probabilistic generalisation of 'almost sure events'; in the precise-probabilistic setting, an event $A \subseteq \Omega$ is said to happen almost surely for a probability measure μ if it has probability 1, that is, if $\mu(A) = 1$. We say that an event A is *almost sure* for a forecasting system φ if $\underline{P}^{\varphi}(A) = 1$; if the forecasting system φ isn't important, or clear from the context, we simply say that A is almost sure. Since we know by P6$_{36}$ that $\overline{P}^{\varphi}(A^c) = 1 - \underline{P}^{\varphi}(A)$ for all $A \subseteq \Omega$, an event A is almost sure if and only if $\overline{P}^{\varphi}(A^c) = 0$. This alternative characterisation is often more convenient in proofs, and we'll use it implicitly.

There are three features of almost sure events that will be useful to us. The first is that they are never empty.

Lemma 6.19. *Any almost sure event $A \subseteq \Omega$ is non-empty.*

Proof. Assume *ex absurdo* that A is empty. This would imply that $A^c = \Omega$ and therefore, since A is almost sure, that $\overline{P}^{\varphi}(\Omega) = 0$. But it follows from P1$_{35}$ that, actually, $\overline{P}^{\varphi}(\Omega) = 1$. □

The second feature is that countable intersections of almost sure events are still almost sure. We start with finite intersections.

Lemma 6.20. *Consider two almost sure events $A, B \subseteq \Omega$, then their intersection $A \cap B$ is almost sure as well.*

Proof. Since A and B are almost sure events, we know that $\overline{P}^{\varphi}(A^c) = 0$ and $\overline{P}^{\varphi}(B^c) = 0$. By invoking P1 and P2$_{35}$, it follows that

$$0 \overset{\text{P1}_{35}}{\leq} \overline{P}^{\varphi}\big((A \cap B)^c\big) = \overline{P}^{\varphi}\big(A^c \cup B^c\big) \overset{\text{P2}_{35}}{\leq} \overline{P}^{\varphi}\big(A^c\big) + \overline{P}^{\varphi}\big(B^c\big) = 0.$$

So $\overline{P}^{\varphi}((A \cap B)^c) = 0$ and, therefore, $A \cap B$ is almost sure. □

By combining this result with P4$_{36}$, we obtain the version for countable intersections.

Corollary 6.21. *For any sequence $(A_n)_{n \in \mathbb{N}_0}$ of almost sure events, their intersection $\bigcap_{n \in \mathbb{N}_0} A_n$ is almost sure as well.*

Proof. For any $n \in \mathbb{N}_0$, we know from Lemma 6.20 that the event $\bigcap_{k=0}^{n} A_k$ is almost sure and, therefore, that $0 = \overline{P}^{\varphi}((\bigcap_{k=0}^{n} A_k)^c) = \overline{P}^{\varphi}(\bigcup_{k=0}^{n} A_k^c)$. Since the sequence $(\mathbb{I}_{\bigcup_{k=0}^{n} A_k^c})_{n \in \mathbb{N}_0}$ in $\mathscr{L}(\Omega)$ is non-decreasing and converges pointwise to the gamble $\mathbb{I}_{\bigcup_{n \in \mathbb{N}_0} A_n^c} \in \mathscr{L}(\Omega)$, it follows from P4$_{36}$ that

$$\overline{P}^{\varphi}\left(\left(\bigcap_{n \in \mathbb{N}_0} A_n\right)^c\right) = \overline{P}^{\varphi}\left(\bigcup_{n \in \mathbb{N}_0} A_n^c\right) = \lim_{n \to \infty} \overline{P}^{\varphi}\left(\bigcup_{k=0}^{n} A_k^c\right) = 0,$$

so $\bigcap_{n \in \mathbb{N}_0} A_n$ is indeed almost sure. □

The third feature is that, for any countable collection of betting strategies, it's almost sure that none of them allow Sceptic to get rich without bounds.

Lemma 6.22. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$ and any sequence $(T_k)_{k\in\mathbb{N}_0}$ of test supermartingales for φ. Then the event*

$$\bigcap_{k\in\mathbb{N}_0}\left\{\omega\in\Omega\colon \limsup_{n\to\infty} T_k(\omega_{1:n})<\infty\right\}\subseteq\Omega$$

is almost sure for φ.

Proof. Fix any $k\in\mathbb{N}_0$ and consider the event

$$A_k := \left\{\omega\in\Omega\colon \limsup_{n\to\infty} T_k(\omega_{1:n})<\infty\right\}.$$

We'll be done if we can show that $\underline{P}^{\varphi}(A_k)=1$, or equivalently, that $\overline{P}^{\varphi}(A_k^c)=0$, since it will then be immediate from Corollary 6.21↶ that the event $\bigcap_{n\in\mathbb{N}_0} A_n$ is almost sure.

To that end, we now construct, for any real $\alpha>1$, a test supermartingale T_k^{α} for φ. Let

$$T_k^{\alpha}(s) := \begin{cases}\alpha & \text{if } T_k(t)\geq\alpha \text{ for some precursor } t\sqsubseteq s \text{ of } s\\ T_k(s) & \text{if } T_k(t)<\alpha \text{ for all precursors } t\sqsubseteq s \text{ of } s\end{cases} \quad \text{for all } s\in\mathbb{S}.$$

It's a matter of direct verification to show that T_k^{α} is indeed a test supermartingale for φ. It's clear that, for any $\omega\in A_k^c$, T_k will eventually exceed α on ω, and therefore $\lim_{n\to\infty} T_k^{\alpha}(\omega_{1:n})=\alpha$ for all $\omega\in A_k^c$. This implies that $\liminf_{n\to\infty} T_k^{\alpha}(\omega_{1:n})\geq\alpha\mathbb{I}_{A^c}(\omega)$ for all $\omega\in\Omega$, and therefore

$$0\overset{\text{P1}_{35}}{\leq}\overline{P}^{\varphi}(A_k^c)=\overline{E}^{\varphi}\left(\mathbb{I}_{A_k^c}\right)\overset{\text{E2}_{34}}{=}\frac{1}{\alpha}\overline{E}^{\varphi}\left(\alpha\mathbb{I}_{A_k^c}\right)\leq\frac{1}{\alpha}T_k^{\alpha}(\square)=\frac{1}{\alpha},$$

where the last inequality follows from Eq. (6.10)₃₃, and the last equality from the fact that T_k^{α} is a test supermartingale. Since this statement holds for all real $\alpha>1$, this implies that, indeed, $\overline{P}^{\varphi}(A_k^c)=0$. □

Connection with measures. Although we've already highlighted the close connection between precise forecasting systems and measures at the end of Section 6.2$_{24}$, it will be enlightening to elaborate on this connection. First recall from Section 6.2$_{24}$ that every precise forecasting system φ_{pr} has a corresponding countably additive probability measure $\mu^{\varphi_{\mathrm{pr}}}$ on the Borel algebra $\mathscr{B}(\Omega)$. Our global upper and lower expectation $\overline{E}^{\varphi_{\mathrm{pr}}}(\bullet)$ and $\underline{E}^{\varphi_{\mathrm{pr}}}(\bullet)$ then both coincide with the usual measure-theoretic *global expectation* that's associated with $\mu^{\varphi_{\mathrm{pr}}}$, at least on all global gambles $u\in\mathscr{L}(\Omega)$ that are measurable with respect to $\mathscr{B}(\Omega)$ [58, Section 4.8, Theorem 5.3.1 and Corollaries 5.2.5 and 5.3.4], and we then write $E^{\varphi_{\mathrm{pr}}}(u):=\overline{E}^{\varphi_{\mathrm{pr}}}(u)=\underline{E}^{\varphi_{\mathrm{pr}}}(u)$. Similarly, our global upper and lower probability $\overline{P}^{\varphi_{\mathrm{pr}}}(\bullet)$ and $\underline{P}^{\varphi_{\mathrm{pr}}}(\bullet)$ then both coincide with $\mu^{\varphi_{\mathrm{pr}}}$, at least on all events $A\in\mathscr{B}(\Omega)$, and we then write $P^{\varphi_{\mathrm{pr}}}(A) := \overline{P}^{\varphi_{\mathrm{pr}}}(A) = \underline{P}^{\varphi_{\mathrm{pr}}}(A)$. In particular, for any partial cut $K \subseteq \mathbb{S}$, since the global gamble $\mathbb{I}_{[\![K]\!]} \in \mathscr{L}(\Omega)$ is obviously measurable with respect to $\mathscr{B}(\Omega)$, we then have that

$$E^{\varphi_{\mathrm{pr}}}(\mathbb{I}_{[\![K]\!]}) = \overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = P^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \mu^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \sum_{s \in K} \prod_{k=0}^{|s|-1} \varphi_{\mathrm{pr}}(s_{1:k})(s_{k+1}). \tag{6.23}$$

This connection will be particularly useful in Chapter ⚃[111], where we not only equip some of the martingale-theoretic notions of randomness that are introduced in Chapter ⚁[49] with a measure-theoretic characterisation, but also explain that the measure-theoretic randomness tests that we introduce to do so, coincide with the ones found in the classical precise-probabilistic literature when restricting our attention to precise forecasting systems.

The above close connection between precise forecasting systems and measures is also a particular instantiation of a more general result. First of all, recall from our discussion in Section 6.2[24] that an arbitrary forecasting system $\varphi \in \Phi(\mathscr{X})$ can be seen as a set of compatible precise forecasting systems. Now, for every forecasting system φ, the corresponding upper expectation $\overline{E}^{\varphi}(\bullet)$ coincides with the upper envelope $\sup_{\varphi_{\mathrm{pr}} \in \varphi} E^{\varphi_{\mathrm{pr}}}(\bullet)$ of the global expectations $E^{\varphi_{\mathrm{pr}}}$ that correspond to a compatible precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$, and this on all global gambles that are measurable with respect to $\mathscr{B}(\Omega)$ [70, Theorem 13]. In particular, for any event $A \in \mathscr{B}(\Omega)$, we then have that

$$\overline{P}^{\varphi}(A) = \sup_{\varphi_{\mathrm{pr}} \in \varphi} \mu^{\varphi_{\mathrm{pr}}}(A). \tag{6.24}$$

Via the conjugacy relationship for global upper and lower expectations, we then immediately have that $\underline{E}^{\varphi}(u) = \inf_{\varphi_{\mathrm{pr}} \in \varphi} E^{\varphi_{\mathrm{pr}}}(u)$ for all global gambles $u \in \mathscr{L}(\Omega)$ that are measurable with respect to $\mathscr{B}(\Omega)$, and $\underline{P}^{\varphi}(A) = \inf_{\varphi_{\mathrm{pr}} \in \varphi} P^{\varphi_{\mathrm{pr}}}(A)$ for all events $A \in \mathscr{B}(\Omega)$. So, in this sense, for every forecasting system $\varphi \in \Phi(\mathscr{X})$, the global lower and upper expectations $\underline{E}^{\varphi}(\bullet)$ and $\overline{E}^{\varphi}(\bullet)$ provide tight lower and upper bounds on the global expectations $E^{\varphi_{\mathrm{pr}}}(\bullet)$ determined by the precise forecasting systems φ_{pr} that are compatible with φ, and similarly for $\underline{P}^{\varphi_{\mathrm{pr}}}(\bullet)$, $\overline{P}^{\varphi_{\mathrm{pr}}}(\bullet)$ and $\mu^{\varphi_{\mathrm{pr}}}(\bullet)$. This for example implies that an event $A \in \mathscr{B}(\Omega)$ is almost sure for a forecasting system φ [$\underline{P}^{\varphi}(A) = 1$] if and only if A is almost sure for all measures $\mu^{\varphi_{\mathrm{pr}}}$ with $\varphi_{\mathrm{pr}} \in \varphi$ [$\mu^{\varphi_{\mathrm{pr}}}(A) = 1$ for all $\varphi_{\mathrm{pr}} \in \varphi$].

7 Computability theory

After introducing global upper and lower expectations in the previous section, which will be especially useful for generalising measure-theoretic notions of randomness in Chapter ⚃[111], let's continue with introducing the remaining necessary mathematical tools for defining non-trivial martingale-theoretic (and measure-theoretic) notions of randomness. As is clear from Examples 6.7$_{31}$ and 6.8$_{32}$, Sceptic can't adopt just any betting strategy to test a path's randomness: for instance, as we've seen in Example 6.8$_{32}$, if we allow for all test supermartingales, then no binary path is random for the fair-coin forecasting system $\varphi_{1/2}$. Consequently, imposing non-negativity and requiring unit initial capital doesn't suffice to obtain a non-trivial martingale-theoretic notion of randomness. Again, to what set should we restrict Sceptic's betting strategies then?

As is common knowledge nowadays, random paths do exist when restricting Sceptic's betting strategies to a countable class [32, p. 235]. The question then still remains what countable class to adopt. According to common sense, when making statements, Sceptic makes use of an alphabet that has a finite number of letters, and uses these to formulate finite sentences, of which there are only a countably infinite number. Sensible martingale-theoretic algorithmic randomness notions typically adopt this idea by imposing that Sceptic should be able to describe her betting strategies in a finite way. This idea originates from a seminal paper by Alonzo Church, where he advocated that a path's randomness should be tested by 'effectively calculable' functions; the *Spielsystem* below refers to the betting strategies as considered and put forward by von Mises [22].

> *"It may be held that the representation of a* Spielsystem *by an arbitrary function ϕ is too broad. To a player who would beat the wheel at roulette a system is unusable which corresponds to a mathematical function known to exist but not given by explicit definition; and even the explicit definition is of no use unless it provides a means of calculating the particular values of the function. As a less frivolous example, the scientist concerned with making predictions or probable predictions of some phenomenon must employ an effectively calculable function: if the law of the phenomenon isn't approximable by such a function, prediction is impossible. Thus a* Spielsystem *should be represented mathematically, not as a function, or even as a definition of a function, but as an effective algorithm for the calculation of the values of a function." [27, p. 133]*

7.1 (Partial) recursive maps

To understand what it means for a mathematical object to be 'effectively calculable', we introduce some basic notions/definitions and results from computability theory. It considers as basic building blocks *partial recursive* natural maps $\phi\colon \mathbb{N} \to \mathbb{N}$, which are maps that can be computed by a

Turing machine.[13] This means that there's some Turing machine that halts on the input $n \in \mathbb{N}$—which we denote by $\phi(n)\downarrow$—and outputs the natural number $\phi(n) \in \mathbb{N}$ if $\phi(n)$ is defined, and doesn't halt otherwise—which we denote by $\phi(n)\uparrow$. By the Church-Turing (hypo)thesis, this is equivalent to the existence of a finite algorithm that, given any input $n \in \mathbb{N}$, outputs the natural number $\phi(n) \in \mathbb{N}$ if $\phi(n)$ is defined, and never finishes otherwise; in what follows, we'll often use this equivalence without mentioning it explicitly. If the Turing machine halts for all inputs $n \in \mathbb{N}$, that is, if the Turing machine computes the natural number $\phi(n)$ in a finite number of steps for every $n \in \mathbb{N}$, then the map ϕ is defined for all arguments and we call it *total recursive*, or simply *recursive* [32, Chapter 2].

Instead of maps from $\mathbb{N}$ to $\mathbb{N}$, we'll also consider maps with domain or codomain $\mathbb{N}$, $\mathbb{S}$, $\mathbb{S} \times \mathbb{N}$, $\mathbb{Q}$, $\mathcal{M}_{\mathrm{rat}}(\mathscr{X})$, $\mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, $\mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$, or any other (countably in)finite set whose elements can be encoded by the natural numbers; we'll denote such a generic set by $\mathcal{D}$, and also call it an *encodable set*.[14] The choice of encoding isn't important, provided we can algorithmically decide whether a natural number is an encoding of an object and, if this is the case, we can find an encoding of the same object with respect to any other encoding [74, p. *xvi*]. A function $\phi\colon \mathcal{D} \to \mathcal{D}'$ is then called *partial recursive* if there's a Turing machine that, when given a natural-valued encoding of any $d \in \mathcal{D}$, outputs a natural-valued encoding of $\phi(d) \in \mathcal{D}'$ if $\phi(d)$ is defined, and never halts otherwise. By the Church-Turing thesis, this is again equivalent to the existence of a finite algorithm that, when given the input $d \in \mathcal{D}$, outputs the object $\phi(d) \in \mathcal{D}'$ if $\phi(d)$ is defined, and never finishes otherwise. If the Turing machine halts on all natural numbers that encode some element $d \in \mathcal{D}$, or equivalently, if the finite algorithm outputs an element $\phi(d) \in \mathcal{D}'$ for every $d \in \mathcal{D}$, then we call ϕ *total recursive*, or simply *recursive*. In practice, in line with the approach of Pour-El & Richards [75], we'll provide or describe an algorithm whenever we want to establish a map's recursive character.

When $\mathcal{D}' = \mathbb{Q}$, then for any rational number $\alpha \in \mathbb{Q}$ and any two recursive

[13] A Turing machine is an accurate mathematical model of a general purpose computer, and can do everything that a real computer based on classical physics can do [73]. If a Turing machine is provided with an additional infinite read-only oracle tape, which it can access one bit at a time while performing its computation, then we call it an *oracle machine* [32]. Oracle machines thus allow for accessing information that cannot be described in a finitary manner. In what follows, when we want to stress that a Turing machine has no access to such information, we'll say that such information is *not accessible by an oracle*.

[14] The (partial) map $\mathbb{N} \to \mathcal{D}$ that corresponds to this encoding doesn't have to be one-to-one, it merely has to be surjective. We thus require that every element in the (countably in)finite set $\mathcal{D}$ has to be associated with at least one natural number. Several natural numbers may thus be associated with the same element, and not every natural number has to correspond to an element in the set. Moreover, we remark that our choice to use natural numbers for our encoding is a bit arbitrary. Instead of the natural numbers, we could for example as well have chosen the countably infinite set of binary strings; this is however of no real importance since there's an obvious (encodable) bijection between both sets.

rational maps $q_1, q_2\colon \mathcal{D} \to \mathbb{Q}$, the following rational maps are clearly recursive as well: $q_1 + q_2$, $q_1 \cdot q_2$, q_1 / q_2 with $q_2(d) \neq 0$ for all $d \in \mathcal{D}$, $\max\{q_1, q_2\}$, αq_1 and $\lceil q_1 \rceil$.[15] Since a finite number of algorithms can always be combined into one, it follows from the foregoing that the rational maps $\min\{q_1, q_2\}$ and $\lfloor q_1 \rfloor$ are also recursive.

For $\mathcal{D} = \mathbb{S}$ and $\mathcal{D}' = \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, a rational forecasting system $\varphi_{\mathrm{rat}}\colon \mathbb{S} \to \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ is called *recursive* if there's a recursive map $q\colon \mathbb{S} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $\mathrm{CH}\big(q(s)\big) = \varphi_{\mathrm{rat}}(s)$ for all $s \in \mathbb{S}$. Any finite description of such a map q that establishes the recursiveness of the forecasting system φ in the above sense, will also be called a *code* for φ. In the specific case of a stationary rational forecasting system C_{rat}, C_{rat} is *recursive* by definition because there's a finite set of rational probability mass functions $\{m_1, \dots, m_n\} \in \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $\mathrm{CH}(\{m_1, \dots, m_n\}) = C_{\mathrm{rat}}$; any such finite set of rational probability mass functions that establishes the recursive character of a credal set C_{rat} in the above sense will then also be called a *code* for C_{rat}.

If we consider a gamble that's rational as well, then we can check the supermartingale (multiplier) property in a recursive manner.

Lemma 7.1. *There's a single algorithm that, when provided with a code for a recursive rational credal set $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and a rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, outputs a rational $q \in \mathbb{Q}$ such that $q = \overline{E}_{C_{\mathrm{rat}}}(f)$.*

Proof. Let $\{m_1, \dots, m_n\} \in \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ be a code for C_{rat}. Then, $\overline{E}_{C_{\mathrm{rat}}}(f) = \max_{m_{\mathrm{rat}} \in \{m_1,\dots,m_n\}} E_{m_{\mathrm{rat}}}(f) = \max_{m_{\mathrm{rat}} \in \{m_1,\dots,m_n\}} \sum_{x \in \mathscr{X}} m_{\mathrm{rat}}(x) f(x)$, which is clearly rational. Let $q := \max_{m_{\mathrm{rat}} \in \{m_1,\dots,m_n\}} \sum_{x \in \mathscr{X}} m_{\mathrm{rat}}(x) f(x)$. The single algorithm that performs this operation then consists of taking the maximum of the inproduct between m_{rat} and f, where m_{rat} ranges over the finite set $\{m_1, \dots, m_n\}$. □

We'll consider one other particular case that will be of interest to us. Observe that the set of all recursive rational maps $q\colon \mathcal{D} \to \mathbb{Q}$ is countably infinite and can substitute for $\mathcal{D}$ and $\mathcal{D}'$, which implies that the recursiveness of maps that take rational recursive maps as their domain and/or image is well-defined. By the Church–Turing thesis, a map $(q\colon \mathcal{D} \to \mathbb{Q}) \to (q'\colon \mathcal{D}' \to \mathbb{Q})$ is then recursive if there's some finite algorithm that outputs a finite description of the rational recursive map q' in a finite number of steps, when it's given a finite description of the rational recursive map q as an input.

We'll also come across the implementability of (countably in)finite families of recursive maps: for any indexed family $(\phi_{d''})_{d'' \in \mathcal{D}''}$, with $\phi_{d''}\colon \mathcal{D} \to \mathcal{D}'$ for all $d'' \in \mathcal{D}''$, we say that $\phi_{d''}$ is *(partial) recursive uniformly in $d'' \in \mathcal{D}''$* if there's a (partial) recursive function $\phi\colon \mathcal{D}'' \times \mathcal{D} \to \mathcal{D}'$ such that $\phi_{d''}(\bullet) = \phi(d'', \bullet)$ for all $d'' \in \mathcal{D}''$. In particular, there's a sequence $(\phi_n)_{n \in \mathbb{N}}$, with

[15] The floor function $\lfloor \bullet \rfloor\colon \mathbb{R} \to \mathbb{Z}$ is the function that outputs for every real number $x \in \mathbb{R}$ the greatest integer less than or equal to x. Similarly, the ceiling function $\lceil \bullet \rceil\colon \mathbb{R} \to \mathbb{Z}$ maps every real number $x \in \mathbb{R}$ to the smallest integer greater than or equal to x.

$\phi_n : \mathcal{D} \to \mathcal{D}'$ partial recursive uniformly in $n \in \mathbb{N}$, that contains all partial recursive functions $\phi : \mathcal{D} \to \mathcal{D}'$.

Lemma 7.2 ([32, Proposition 2.1.2])**.** *For any two encodable sets $\mathcal{D}$ and $\mathcal{D}'$ there's a partial recursive function $\phi : \mathbb{N} \times \mathcal{D} \to \mathcal{D}'$ such that for any partial recursive function $\phi' : \mathcal{D} \to \mathcal{D}'$ there's some $n \in \mathbb{N}$ for which $\phi' = \phi(n, \bullet)$.*

7.2 Recursive(ly enumerable) and effectively open sets

(Partial) recursive functions allow for defining notions of implementability for subsets of countable sets and for subsets of effectively second countable spaces, as we'll now explain.

Countable sets

A subset $D \subseteq \mathcal{D}$ is called *recursively enumerable* if there's some Turing machine that halts on every natural number that encodes an element $d \in D$, but never halts on any natural number that encodes an element $d \in \mathcal{D} \setminus D$ [32, Definition 2.2.1]. For any non-empty $D \subseteq \mathcal{D}$, this is equivalent to the existence of a finite algorithm that enumerates (finite descriptions of) the elements of the set D, meaning that there's some total recursive map $\phi : \mathbb{N} \to \mathcal{D}$ such that $D = \phi(\mathbb{N})$, with $\phi(\mathbb{N}) := \{\phi(n) : n \in \mathbb{N}\}$ [32, Proposition 2.2.2]. If both the set D and its complement $\mathcal{D} \setminus D$ are recursively enumerable, then we call D *recursive*. This is equivalent to the existence of a recursive indicator $\mathbb{I}_D : \mathcal{D} \to \{0, 1\}$ that outputs 1 for all $d \in D$, and outputs 0 otherwise [32, p. 11].

For any indexed family $(D_{d'})_{d' \in \mathcal{D}'}$, with $D_{d'} \subseteq \mathcal{D}$ for all $d' \in \mathcal{D}'$ and $\mathcal{D}'$ a countable set whose elements can be encoded by the natural numbers, we say that $D_{d'}$ is *recursive(ly enumerable) uniformly in $d' \in \mathcal{D}'$* if there's a recursive(ly enumerable) set $\mathfrak{D} \subseteq \mathcal{D}' \times \mathcal{D}$ such that $D_{d'} = \{d \in \mathcal{D} : (d', d) \in \mathfrak{D}\}$ for all $d' \in \mathcal{D}'$; if moreover $D_{d'} \neq \emptyset$ for all $d' \in \mathcal{D}'$, then this implies the existence of some recursive map $q : \mathbb{N} \times \mathcal{D}' \to \mathcal{D}$ such that $D_{d'} = \{q(n, d') : n \in \mathbb{N}\}$ for all $d' \in \mathcal{D}'$. In particular, there's a sequence $(D_n)_{n \in \mathbb{N}}$, with $D_n \subseteq \mathcal{D}$ recursively enumerable uniformly in $n \in \mathbb{N}$, that contains all recursively enumerable subsets $D \subseteq \mathcal{D}$.

Corollary 7.3. *For any encodable set $\mathcal{D}$, there's a sequence $(D_n)_{n \in \mathbb{N}}$, with $D_n \subseteq \mathcal{D}$ recursively enumerable uniformly in $n \in \mathbb{N}$, such that for any recursively enumerable subset $D \subseteq \mathcal{D}$ there's some $n \in \mathbb{N}$ for which $D = D_n$.*

Proof. Consider the partial recursive function $\phi : \mathbb{N} \times \mathcal{D} \to \mathbb{N}$ from Lemma 7.2 [thus with $\mathcal{D}' = \mathbb{N}$], and let $\mathfrak{D} := \{(n, d) \in \mathbb{N} \times \mathcal{D} : \phi(n, d)\downarrow\}$ and $D_n := \{d \in \mathcal{D} : (n, d) \in \mathfrak{D}\}$ for all $n \in \mathbb{N}$; $\mathfrak{D}$ is obviously recursively enumerable, and hence, $D_n \subseteq \mathcal{D}$ is recursively enumerable uniformly in $n \in \mathbb{N}$. Consider any recursively enumerable subset $D \subseteq \mathcal{D}$. Then there's a Turing machine that halts on every natural number that encodes an element $d \in D$, but never halts on any natural number that encodes an element $d \in$

$\mathcal{D} \setminus D$. This is equivalent to the existence of a partial recursive function $\phi' \colon \mathcal{D} \to \mathbb{N}$ such that $D = \{d \in \mathcal{D} \colon \phi'(d)\downarrow\}$. Then there's some $n \in \mathbb{N}$ such that $\phi' = \phi(n, \bullet)$, and, for this same n, $D = \{d \in \mathcal{D} \colon \phi(n, d)\downarrow\} = D_n$. □

Effectively second countable spaces

A topological space (X, τ) is called *effectively second countable* if its topology τ has a countable base $\mathcal{D}$ that can be encoded by the natural numbers. A subset $G \subseteq X$ is called *effectively open* if there's a recursively enumerable subset $D \subseteq \mathcal{D}$ such that $G = \bigcup_{d \in D} d$. A subset $G \subseteq X$ is called *effectively closed* if $X \setminus G$ is effectively open. Consider, as an example, the Cantor topology on Ω generated by the countable base $\mathcal{D} = \{[\![s]\!] \subseteq \Omega \colon s \in \mathbb{S}\}$; for ease of notation and manipulation, we write and consider $\mathcal{D} = \mathbb{S}$ instead. A set of paths $G \subseteq \Omega$ is then *effectively open* if there's some recursively enumerable subset $S \subseteq \mathbb{S}$ such that $G = [\![S]\!]$.

For any indexed family $(G_{d'})_{d' \in \mathcal{D}'}$, with $G_{d'} \subseteq X$ for all $d' \in \mathcal{D}'$, we say that $G_{d'}$ is *effectively open uniformly in* $d' \in \mathcal{D}'$ if there's some recursively enumerable set $\mathfrak{D} \subseteq \mathcal{D}' \times \mathcal{D}$ such that $G_{d'} = \bigcup\{d \in \mathcal{D} \colon (d', d) \in \mathfrak{D}\}$ for all $d' \in \mathcal{D}'$. In particular, if $\mathcal{D}' = \mathbb{N}$ (or $\mathcal{D}' = \mathbb{N}_0$), then we also say that the $(G_n)_{n \in \mathbb{N}}$ (or $(G_n)_{n \in \mathbb{N}_0}$) constitute a *computable sequence of effectively open sets*. Interestingly, there's a computable sequence of effectively open sets that contains all effectively open sets, in the following sense.

Corollary 7.4. *For any effectively second countable space (X, τ) with encodable base $\mathcal{D}$, there's a computable sequence of effectively open sets $(G_n)_{n \in \mathbb{N}}$, with $G_n \subseteq X$ for all $n \in \mathbb{N}$, such that for any effectively open set $G \subseteq X$ there's some $n \in \mathbb{N}$ for which $G = G_n$.*

Proof. This follows immediately from Corollary 7.3. □

7.3 Lower (and upper) semicomputable extended real maps

We'll also use recursive maps and recursively enumerable sets to provide notions of implementability for (extended) real maps of the form $r \colon \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, the co-domain of which isn't countably infinite. Such a map r is called *lower semicomputable* if there's a recursive rational map $q \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ such that $q(d, n+1) \geq q(d, n)$ and $r(d) = \lim_{m \to \infty} q(d, m)$ for all $d \in \mathcal{D}$ and $n \in \mathbb{N}$; as a shorthand notation, we'll then write $q(d, \bullet) \nearrow r(d)$. Any finite description of such a map q that establishes the lower semicomputability of the map r in the above sense, will also be called a *code* for r. We may always assume that this approximation from below is strictly increasing.

Lemma 7.5. *There's a single algorithm that, upon the input of a code for a lower semicomputable extended real map $r \colon \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, outputs a (finite description of a) recursive rational map $q \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ such that $\lim_{m \to \infty} q(d, m) = r(d)$ and $q(d, n) < q(d, n+1)$ for all $d \in \mathcal{D}$ and $n \in \mathbb{N}$.*

Proof. Start from a code $q' \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ for the map r that is lower semicomputable, which implies that $q'(d, \bullet) \nearrow r(d)$ for all $d \in \mathcal{D}$, and output (a finite description of) the recursive rational map $q \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ defined by $q(d, n) := q'(d, n) - 2^{-n}$ for all $d \in \mathcal{D}$ and $n \in \mathbb{N}$. Then $\lim_{m \to \infty} q(d, m) = \lim_{m \to \infty} q'(d, m) = r(d)$ and $q(d, n) < q'(d, n) - 2^{-(n+1)} \leq q'(d, n+1) - 2^{-(n+1)} = q(d, n+1)$ for all $d \in \mathcal{D}$ and $n \in \mathbb{N}$. □

Equivalently, a real map $r \colon \mathcal{D} \to \mathbb{R}$ is lower semicomputable if and only if the set $\{(d, q) \in \mathcal{D} \times \mathbb{Q} \colon r(d) > q\}$ is recursively enumerable [32, Section 5.2]; in this case, it's equivalent to say that the set $\{(d, x) \in \mathcal{D} \times \mathbb{R} \colon r(d) > x\}$ is effectively open [with $X = \mathcal{D} \times \mathbb{R}$ and encodable countable base $\{\{d\} \times (-\infty, \alpha) \subseteq \mathcal{D} \times \mathbb{R} \colon d \in \mathcal{D} \text{ and } \alpha \in \mathbb{Q}\}$]. An (extended) real map $r \colon \mathcal{D} \to \mathbb{R} \cup \{-\infty\}$ is called *upper semicomputable* if $-r$ is lower semicomputable.

For any indexed family $(r_{d'})_{d' \in \mathcal{D}'}$, with $r_{d'} \colon \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ for all $d' \in \mathcal{D}'$, we say that $r_{d'}$ is *lower semicomputable uniformly in* $d' \in \mathcal{D}'$ if there's a lower semicomputable map $r \colon \mathcal{D}' \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ such that $r_{d'}(\bullet) = r(d', \bullet)$ for all $d' \in \mathcal{D}'$. In particular, when we restrict ourselves to non-negative extended real maps, there's a sequence $(r_n)_{n \in \mathbb{N}}$, with $r_n \colon \mathcal{D} \to [0, +\infty]$ lower semicomputable uniformly in $n \in \mathbb{N}$, that contains all lower semicomputable non-negative extended real maps $r \colon \mathcal{D} \to [0, +\infty]$.

Lemma 7.6 ([9, Lemma 13])**.** *For any encodable set $\mathcal{D}$ there's a lower semicomputable function $r \colon \mathbb{N} \times \mathcal{D} \to [0, +\infty]$ such that for any lower semicomputable function $r' \colon \mathcal{D} \to [0, +\infty]$ there's some $n \in \mathbb{N}$ for which $r'(d) = r(n, d)$ for all $d \in \mathcal{D}$.*

7.4 Computable maps

If a real map $r \colon \mathcal{D} \to \mathbb{R}$ is both lower and upper semicomputable, then we call it *computable*; obviously then, every computable real map is lower semicomputable, and every recursive rational map is computable. Computability is equivalent to the existence of a recursive rational map $q \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ such that $|r(d) - q(d, N)| \leq 2^{-N}$ for all $d \in \mathcal{D}$ and $N \in \mathbb{N}$ [36, Propositions 3 and 4]. Any finite description of such a map q that establishes the computability of the map r in the above sense, will also be called a *code* for r. This is also equivalent to the existence of two recursive maps $q \colon \mathcal{D} \times \mathbb{N} \to \mathbb{Q}$ and $e \colon \mathcal{D} \times \mathbb{N} \to \mathbb{N}$ such that $|r(d) - q(d, \ell)| \leq 2^{-N}$ for all $d \in \mathcal{D}$, $N \in \mathbb{N}$ and $\ell \geq e(d, N)$ [36, Proposition 3]. A real number $\alpha \in \mathbb{R}$ is then called computable if it's computable as a real map on a singleton, or equivalently, if there's a recursive rational map $q \colon \mathbb{N} \to \mathbb{Q}$ such that $|\alpha - q(N)| \leq 2^{-N}$ for all $N \in \mathbb{N}$. For any computable real number $\alpha \in \mathbb{R}$ and any two computable real maps $r_1, r_2 \colon \mathcal{D} \to \mathbb{R}$, the following real maps are computable as well: $r_1 + r_2$, $r_1 \cdot r_2$, r_1 / r_2 with $r_2(d) \neq 0$ for all $d \in \mathcal{D}$, $\max\{r_1, r_2\}$, αr_1, $\exp(r_1)$, and $\log_2(r_1)$ with $r_1(d) > 0$ for all $d \in \mathcal{D}$ [75, Section 0.2].

To show that a real map $r \colon \mathcal{D} \to \mathbb{R}$ is computable or lower semicomputable, we can also use computable real maps rather than recursive rational maps. If there's some computable real map $q\colon \mathcal{D}\times\mathbb{N}\to\mathbb{R}$ such that $|r(d)-q(d,N)|\le 2^{-N}$ for all $d\in\mathcal{D}$ and $N\in\mathbb{N}$, then the real map r is computable and we say that q *converges effectively* to r [75, Section 0.2]; the opposite direction is trivially true as well: if the real map r is computable, then there's a recursive rational map $q\colon \mathcal{D}\times\mathbb{N}\to\mathbb{Q}$—which is obviously computable and real-valued—such that $|r(d)-q(d,N)|\le 2^{-N}$ for all $d\in\mathcal{D}$ and $N\in\mathbb{N}$. Equivalently, the real map r is computable if and only if there's a computable real map $q\colon \mathcal{D}\times\mathbb{N}\to\mathbb{R}$ and a recursive map $e\colon \mathcal{D}\times\mathbb{N}\to\mathbb{N}$ such that $|r(d)-q(d,\ell)|\le 2^{-N}$ for all $d\in\mathcal{D}$, $N\in\mathbb{N}$ and $\ell\ge e(d,N)$ [75, Section 0.2], and we then also say that q *converges effectively* to r.[16] Finally, if there's some computable real map $q\colon \mathcal{D}\times\mathbb{N}\to\mathbb{R}$ such that $q(d,n+1)\ge q(d,n)$ and $r(d)=\lim_{m\to\infty}q(d,m)$ for all $d\in\mathcal{D}$ and $n\in\mathbb{N}$, then the real map r is lower semicomputable [37].

For any indexed family $(r_{d'})_{d'\in\mathcal{D}'}$, with $r_{d'}\colon \mathcal{D}\to\mathbb{R}$ for all $d'\in\mathcal{D}'$, we say that $r_{d'}$ is *computable uniformly in* $d'\in\mathcal{D}'$ if there's a computable map $r\colon \mathcal{D}'\times\mathcal{D}\to\mathbb{R}$ such that $r_{d'}(\bullet)=r(d',\bullet)$ for all $d'\in\mathcal{D}'$. In particular, if $\mathcal{D}'=\mathbb{N}$ (or $\mathcal{D}'=\mathbb{N}_0$), then we also say that the $(r_n)_{n\in\mathbb{N}}$ (or $(r_n)_{n\in\mathbb{N}_0}$) constitute a *computable sequence of computable real maps*; if, moreover, $\mathcal{D}$ is a singleton, then we also say that $(r_n)_{n\in\mathbb{N}}$ (or $(r_n)_{n\in\mathbb{N}_0}$) is a *computable sequence of real numbers.*

Forecasting systems

So far, we've only defined computability for real maps, but, instead of the reals, we could have considered any uncountable codomain that has some dense encodable subset $\mathcal{D}$. In particular, we'll consider *computable* forecasting systems $\varphi\in\Phi(\mathcal{X})$. To this end, let $\mathcal{D}:=\mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathcal{X}))$; recall that the corresponding set $\mathscr{C}_{\mathrm{rat}}(\mathcal{X})$ of closed convex hulls is dense in $\mathscr{C}(\mathcal{X})$ under the Hausdorff distance [Lemma 5.6_{18}]. A forecasting system $\varphi\in\Phi(\mathcal{X})$ is then called *computable* if there's a recursive map $q\colon \mathbb{S}\times\mathbb{N}\to\mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathcal{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s),\mathrm{CH}\big(q(s,N)\big)\big)\le 2^{-N}$ for all $s\in\mathbb{S}$ and $N\in\mathbb{N}$. Any finite description of such a map q that establishes the computability of the forecasting system φ in the above sense, will also be called a *code* for φ. In the specific case of a stationary forecasting system C, C is *computable* if there's a recursive map $q\colon \mathbb{N}\to\mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathcal{X}))$ such that $d_{\mathrm{H}}\big(C,\mathrm{CH}\big(q(N)\big)\big)\le 2^{-N}$ for all $N\in\mathbb{N}$; any finite description of such a map q that establishes the computability of the credal set C in the above sense will then also be called a *code* for C.

If we consider a gamble $f\in\mathscr{L}(\mathcal{X})$ that's computable as well, then the upper expectation $\overline{E}_C(f)$ is computable when given a code for C and f.

[16] This second criterion for effective convergence is weaker than the first one: for any real map $r\colon \mathcal{D}\to\mathbb{R}$ and recursive rational map $q\colon \mathcal{D}\times\mathbb{N}\to\mathbb{Q}$ such that $|r(d)-q(d,N)|\le 2^{-N}$ for all $d\in\mathcal{D}$ and $N\in\mathbb{N}$, it holds that $|r(d)-q(d,\ell)|\le 2^{-N}$ for all $d\in\mathcal{D}$, $N\in\mathbb{N}$ and $\ell\ge N=: e(d,N)$.

Lemma 7.7. *There's a single algorithm that, upon the input of a code for a computable credal set $C \in \mathscr{C}(\mathscr{X})$ and a code for a computable gamble $f \in \mathscr{L}(\mathscr{X})$, outputs a code for $\overline{E}_C(f)$; that is, a finite description of a recursive rational map $q\colon \mathbb{N} \to \mathbb{Q}$ such that $|\overline{E}_C(f) - q(N)| \leq 2^{-N}$ for all $N \in \mathbb{N}$.*

Proof. Let $q_C\colon \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathcal{M}_{\text{rat}}(\mathscr{X}))$ and $q_f\colon \mathscr{X} \times \mathbb{N} \to \mathbb{Q}$ be recursive maps that establish the computability of C and f, respectively. Let

$$N_f := 1 + \max_{x \in \mathscr{X}} \lceil |q_f(x,1)| \rceil \in \mathbb{N}$$

and

$$q\colon \mathbb{N} \to \mathbb{Q}\colon n \mapsto \overline{E}_{\text{CH}(q_C(n+2+N_f))}(q_f(\bullet, n+1)).$$

By Lemma 7.1[43] and by the definition of N_f, q is clearly recursive when given the codes q_C and q_f. Since $|f(x) - q_f(x,1)| \leq 2^{-1}$ for all $x \in \mathscr{X}$, it holds that $|f(x)| - |q_f(x,1)| < 2^{-1}$ for all $x \in \mathscr{X}$, and hence, $0 \leq \max_{x \in \mathscr{X}} |f(x)| < N_f$, which implies that $0 \leq (f+N_f)/(2N_f) \leq 1$. For all $N \in \mathbb{N}$,

$$\begin{aligned}
\overline{E}_C(f) &\overset{\text{C2}_{20},\text{C4}_{20}}{=} 2N_f \overline{E}_C\left(\frac{f+N_f}{2N_f}\right) - N_f \\
&\leq 2N_f\left(\overline{E}_{\text{CH}(q_C(N+2+N_f))}\left(\frac{f+N_f}{2N_f}\right) + 2^{-(N+2+N_f)}\right) - N_f \\
&\overset{\text{C2}_{20},\text{C4}_{20}}{=} \overline{E}_{\text{CH}(q_C(N+2+N_f))}(f) + N_f 2^{-(N+1+N_f)} \\
&\overset{\text{C5}_{20}}{\leq} \overline{E}_{\text{CH}(q_C(N+2+N_f))}\left(q_f(\bullet, N+1) + 2^{-(N+1)}\right) + N_f 2^{-(N+1+N_f)} \\
&\leq \overline{E}_{\text{CH}(q_C(N+2+N_f))}\left(q_f(\bullet, N+1) + 2^{-(N+1)}\right) + 2^{-(N+1)} \\
&\overset{\text{C4}_{20}}{=} \overline{E}_{\text{CH}(q_C(N+2+N_f))}\left(q_f(\bullet, N+1)\right) + 2^{-N} \\
&= q(N) + 2^{-N},
\end{aligned}$$

where the first inequality holds by Corollary 7.9 because $d_{\text{H}}(C, \text{CH}(q_C(N))) \leq 2^{-N}$ for all $N \in \mathbb{N}$, and where the last inequality holds because $N < 2^N$ for all $N \in \mathbb{N}$. In a completely analogous way, we can prove that $\overline{E}_C(f) \geq q(N) - 2^{-N}$ for all $N \in \mathbb{N}$, and therefore $|\overline{E}_C(f) - q(N)| \leq 2^{-N}$ for all $N \in \mathbb{N}$. □

Lemma 7.8 ([76, Section 4.1])**.** *Consider any credal sets $C, C' \in \mathscr{C}(\mathscr{X})$. Then $d_H(C, C') = \max_{f \in \mathscr{L}_1(\mathscr{X})} |\overline{E}_C(f) - \overline{E}_{C'}(f)|$.*

Corollary 7.9. *Consider any credal sets $C, C' \in \mathscr{C}(\mathscr{X})$ and any gamble $f \in \mathscr{L}_1(\mathscr{X})$. Then, $|\overline{E}_C(f) - \overline{E}_{C'}(f)| \leq d_H(C, C')$.*

⚁

Martingale-theoretic notions of randomness

What sequences do we consider to be random for a forecasting system $\varphi \in \Phi(\mathscr{X})$? Or put differently, when do we say that a sequence agrees with a forecasting system $\varphi \in \Phi(\mathscr{X})$. In this chapter, we formally address and answer this question a first time in this dissertation by introducing several (imprecise-probabilistic) martingale-theoretic randomness notions. After reading Chapter ⚀13, which contains all the mathematical machinery that will allow us to define several such randomness notions, it should already be clear that it's Sceptic who will test the compatibility between a path $\omega \in \Omega$ and Forecaster's forecasting system $\varphi \in \Phi(\mathscr{X})$. Generally speaking, for all of these martingale-theoretic randomness notions, a path $\omega \in \Omega$ is considered to be random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if Sceptic has no implementable allowed betting strategy that makes her arbitrarily rich along ω, without borrowing. All these randomness notions are thus based on the following intuition: a path shouldn't be called random for φ if Sceptic can get arbitrarily rich in a betting game by exploiting a pattern/structure in the outcomes along ω. These randomness notions will differ in *how* Sceptic's betting strategies are implementable, and in *how* she shouldn't be able to become arbitrarily rich along a path $\omega \in \Omega$.

In Section 8↷, we'll explain in which ways we'll require Sceptic's betting strategies to be implementable, and in which ways she shouldn't be able to get arbitrarily rich. This will allow us to introduce four (different) martingale-theoretic randomness notions: *Martin-Löf* (ML) , *weak Martin-Löf* (wML), *computable* (C) and *Schnorr* (S) *randomness*. All these randomness notions are imprecise-probabilistic generalisations of classical precise-probabilistic ones, and we'll provide pointers to the relevant literature to highlight this feature.

Afterwards, in Section 9_{54}, we show how these randomness notions relate to each other, and discuss a number of their properties. Section 10_{66}, which is the last section of this chapter, contains a collection of properties that are all related to the robustness of the considered randomness notions: to what extent can we change a set of implementable betting strategies and the way they should allow Sceptic to get infinitely rich, without changing the randomness notion at hand? Some of the results in Sections 9_{54} and 10_{66} will spark insight and raise questions that form the starting point for later chapters, where these questions will be addressed.

8 Martingale-theoretic randomness definitions

Recall that, loosely speaking, a path $\omega \in \Omega$ is considered to be martingale-theoretically random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if Sceptic can adopt no allowable implementable betting strategy that makes her arbitrarily rich along ω, without borrowing. The four martingale-theoretic randomness notions that we'll introduce in this section will differ only in how they formalise what it means to be implementable, and what it means to become arbitrarily rich. To formalise the implementability constraints, we introduce a number of sets of implementable allowed betting strategies; we'll do so by considering implementable real processes first. A real process $F\colon \mathbb{S} \to \mathbb{R}$ can be implementable by being recursive, lower semicomputable, upper semicomputable or computable, but also, if it's generated by a multiplier process D, by D being of one of these four types. In what follows, it will be useful to consider the following sets of implementable real processes:

$\mathscr{F}_{\mathrm{ML}}$	all lower semicomputable test processes;
$\mathscr{F}_{\mathrm{wML}}$	all positive test processes generated by lower semicomputable multiplier processes;
$\mathscr{F}_{\mathrm{C}} = \mathscr{F}_{\mathrm{S}}$	all computable positive test processes.

From the discussion in Section 7_{40} of Chapter ⚀$_{13}$, it's clear that if F is computable, then it is lower semicomputable as well; so $\mathscr{F}_{\mathrm{S}} = \mathscr{F}_{\mathrm{C}} \subseteq \mathscr{F}_{\mathrm{ML}}$. What is less immediate, is that all four sets are in fact nested.

Proposition 8.1. $\mathscr{F}_{\mathrm{S}} = \mathscr{F}_{\mathrm{C}} \subseteq \mathscr{F}_{\mathrm{wML}} \subseteq \mathscr{F}_{\mathrm{ML}}$.

Proof. It follows immediately from Lemma 8.2 below that $\mathscr{F}_{\mathrm{wML}} \subseteq \mathscr{F}_{\mathrm{ML}}$, and from Proposition 6.5_{29} above and Lemma 8.3↷ below that $\mathscr{F}_{\mathrm{C}} \subseteq \mathscr{F}_{\mathrm{wML}}$. □

Lemma 8.2. *Consider any multiplier process D. If D is lower semicomputable, then so is $D^{\circledcirc}$.*

Proof. Assume that the multiplier process D is lower semicomputable. This implies that there's a recursive rational map $q\colon \mathbb{S}\times\mathscr{X}\times\mathbb{N}\to\mathbb{Q}$ such that $q(s,x,\bullet)\nearrow D(s)(x)$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$. Since $D(s)(x)\geq 0$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$, we may assume without loss of generality that $q(s,x,n)\geq 0$ too for all $s\in\mathbb{S}$, $x\in\mathscr{X}$ and $n\in\mathbb{N}$ [otherwise replace $q(s,x,n)$ by $\max\{0,q(s,x,n)\}$ for all $s\in\mathbb{S}$, $x\in\mathscr{X}$ and $n\in\mathbb{N}$]. We now construct a recursive rational map $q'\colon \mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ as follows: for any $s\in\mathbb{S}$ and any $n\in\mathbb{N}$, we let $q'(s,n):=\prod_{k=0}^{|s|-1} q(s_{1:k},s_{k+1},n)$. Then, since also $D^{\circledcirc}(s)=\prod_{k=0}^{|s|-1} D(s_{1:k})(s_{k+1})$ and $q(s_{1:k},s_{k+1},\bullet)\nearrow D(s_{1:k})(s_{k+1})$ for all $k\in\{0,1,\dots,|s|-1\}$, we find that $q'(s,\bullet)\nearrow D^{\circledcirc}(s)$ for all $s\in\mathbb{S}$, so $D^{\circledcirc}$ is indeed lower semicomputable. □

Lemma 8.3. *Consider any positive test process $F\colon \mathbb{S}\to\mathbb{R}$. If F is computable, then so is the positive multiplier process $D_F\colon \mathbb{S}\to\mathscr{L}(\mathscr{X})$ defined by $D_F(s)(x):={}^{F(sx)}/_{F(s)}$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$.*

Proof. Since F is positive, it follows trivially that D_F is positive as well. Assume that the positive test process F is computable. The positive real maps $\mathbb{S}\times\mathscr{X}\to\mathbb{R}_{>0}\colon (s,x)\mapsto F(s)$ and $\mathbb{S}\times\mathscr{X}\to\mathbb{R}_{>0}\colon (s,x)\mapsto F(sx)$ are then obviously computable, and so, therefore, is D_F as the quotient of both. □

As a result, for any $\mathrm{R}\in\{\mathrm{ML},\mathrm{wML},\mathrm{C},\mathrm{S}\}$, the set $\mathscr{F}_{\mathrm{R}}$ is countably infinite, because the lower semicomputable non-negative test processes are countable in number by Lemma 7.646.

With these sets of implementable real processes at our notational disposal, we gather Sceptic's implementable allowed betting strategies for the notions of Martin-Löf (ML-), weak Martin-Löf (wML-), computable (C-) and Schnorr (S-)randomness—for every forecasting system $\varphi\in\Phi(\mathscr{X})$—in the following sets:

$\overline{\mathbb{T}}_{\mathrm{ML}}(\varphi):=\mathscr{F}_{\mathrm{ML}}\cap\overline{\mathbb{T}}(\varphi)$	all lower semicomputable test supermartingales for φ;
$\overline{\mathbb{T}}_{\mathrm{wML}}(\varphi):=\mathscr{F}_{\mathrm{wML}}\cap\overline{\mathbb{T}}(\varphi)$	all positive test supermartingales for φ generated by lower semicomputable positive supermartingale multipliers for φ;
$\overline{\mathbb{T}}_{\mathrm{C}}(\varphi):=\mathscr{F}_{\mathrm{C}}\cap\overline{\mathbb{T}}(\varphi)$	all computable positive test supermartingales for φ;
$\overline{\mathbb{T}}_{\mathrm{S}}(\varphi):=\mathscr{F}_{\mathrm{S}}\cap\overline{\mathbb{T}}(\varphi)$	all computable positive test supermartingales for φ.

By recalling that $\mathscr{F}_{\mathrm{S}}=\mathscr{F}_{\mathrm{C}}\subseteq\mathscr{F}_{\mathrm{wML}}\subseteq\mathscr{F}_{\mathrm{ML}}$, it readily follows that these sets of betting strategies satisfy the following relations for any forecasting system $\varphi\in\Phi(\mathscr{X})$.

Corollary 8.4. *Consider any forecasting system $\varphi\in\Phi(\mathscr{X})$, then $\overline{\mathbb{T}}_{\mathrm{S}}(\varphi)=\overline{\mathbb{T}}_{\mathrm{C}}(\varphi)\subseteq\overline{\mathbb{T}}_{\mathrm{wML}}(\varphi)\subseteq\overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$.*

Proof. This is an immediate corollary of Proposition 8.1. □

Since $\overline{\mathbb{T}}_{\mathrm{R}}(\varphi) \subseteq \mathscr{F}_{\mathrm{R}}$ for any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, it's immediate that the sets $\overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ are countably infinite as well.

Now, for a path $\omega \in \Omega$ to be R-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, we require that Sceptic's running capital must never be *unbounded* on ω for any implementable allowed betting strategy; that is, no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ must be *unbounded* on ω, meaning that $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. Intuitively, for the path ω to be random, Sceptic must thus not be able to get arbitrarily rich by exploiting a pattern in ω by betting on its outcomes.

Definition 8.5. For any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, a path $\omega \in \Omega$ is R-*random* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ is unbounded on ω.

In the classical, precise-probabilistic randomness literature, these randomness notions are usually only defined for computable precise forecasting systems. In our imprecise-probabilistic setting, however, we choose to continue speaking of Martin-Löf, weak Martin-Löf and Schnorr randomness both when adopting computable and non-computable forecasting systems, where the non-computable forecasting systems aren't accessible by an oracle; the converse is typically assumed in the field of algorithmic randomness.[17] We do so for reasons of generality, because some results continue to hold for non-computable forecasting systems as well. This being said, it will turn out to be important whether a forecasting system is effectively implementable or not, as will become apparent in several propositions and theorems throughout this dissertation.

As we mentioned in the introduction to this chapter, the above martingale-theoretic randomness notions are all imprecise-probabilistic generalisations of precise-probabilistic martingale-theoretic randomness notions. Martin-Löf randomness was originally introduced by Per Martin-Löf in a precise-probabilistic measure-theoretic context [30], so our terminology in the present martingale-theoretic setting could come across as unjustified. However, as was proved (independently) by Peter Schnorr [1, 2] and Leonid Levin [3], the test-theoretic definition coincides with the martingale-theoretic one when restricting attention to non-degenerate (precise) computable forecasting systems; see the discussion in Section 14_{119} and Section 18_{168} further on for more details. Historically speaking, Martin-Löf randomness is considered to be the first satisfactory notion of randomness, and is arguably the most well-studied one. However, not everyone has been satisfied with this notion of randomness: Peter Schnorr formulated the following critique; in the quote

[17] The absence of this assumption is particularly useful in some of the results and discussions in Chapter ⚅$_{179}$. For most of the results in the other chapters, we could as well have considered non-computable forecasting systems that are accessible by an oracle; this would actually enable us to generalise some of our results from computable forecasting systems to arbitrary forecasting systems.

below, a (1)-test stands for a lower semicomputable martingale, and being weakly computable corresponds to being lower semicomputable.

> *"The algorithmic structure of a (1)-test F is not symmetrical. There's no reason why a martingale F should be weakly computable and $-F$ should not be so. Taking this into consideration we make the following definition." [1, p. 250]*

This led him to introduce two different precise-probabilistic martingale-theoretic notions of randomness, which are nowadays known as computable randomness and Schnorr randomness [1, 2]. Both notions take into account his critique by requiring that the tests should be both lower and upper semicomputable, which boils down to the tests being computable. Above, we introduced an imprecise-probabilistic generalisation of computable randomness, and we'll do the same below for Schnorr randomness. Recall that we've also already introduced yet another imprecise-probabilistic martingale-theoretic notion, which we called weak Martin-Löf randomness. Its precise-probabilistic counterpart is less well studied, and known under the name of *Hitchcock randomness* [32, 77].[18]

We continue by introducing an imprecise-probabilistic martingale-theoretic generalisation of Schnorr randomness, which differs from the previous randomness notions in the way Sceptic shouldn't be able to become arbitrarily rich. As mentioned above, the original precise-probabilistic notion was introduced by Peter Schnorr, and is based on the following consideration:

> *"Our considerations in Section 5 should have made clear that a reasonable concept of test function has to include the martingale property (2.2). Computability and the martingale property suffice to characterise effective tests. But which sequences are refused by an effective test? In analogy to (2.3) one would define that a sequence z does not withstand the test F if and only if $\limsup_n F(z(n)) = \infty$. However, if the sequence $F(z(n))$ increases so slowly that no one working with effective methods only would observe its growth, then the sequence z behaves as if it withstands the test F. The definition of $\mathcal{R}_F$ has to reflect this fact. That is, we have to make constructive the notion $\limsup_n F(z(n)) = \infty$." [1, p. 256]*

[18] When De Cooman & De Bock [36] started allowing for imprecise uncertainty models in the field of algorithmic randomness, they were unaware of the standard name given to this (non-standard) randomness notion. Their attention was however drawn to this randomness notion since it (also) allows for an interesting intersection property [36, Proposition 31]. In this dissertation, we choose to stick to their terminology, in order to remain consistent with their and our own work [36, 43, 44, 47, 48, 49].

So, for Schnorr randomness, to make constructive the notion of *unboundedness*, we require instead that Sceptic's running capital shouldn't be *computably unbounded* on ω for any implementable allowed betting strategy. More formally, we require that no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ should be *computably unbounded* on ω, where computably unbounded on ω means that $\limsup_{n\to\infty}[T(\omega_{1:n}) - \tau(n)] \geq 0$ for some real *growth function* $\tau\colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$, which is a non-negative real map that's

(i) computable;
(ii) non-decreasing, so $\tau(n+1) \geq \tau(n)$ for all $n \in \mathbb{N}_0$;
(iii) unbounded, so $\lim_{n\to\infty} \tau(n) = \infty$.[19]

Since any real *growth function* τ is unbounded, it expresses a computable rate at which T becomes unbounded on ω. Clearly, if $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ is computably unbounded on $\omega \in \Omega$, then it is also unbounded on ω.

Now, intuitively, and analogously to the definition for computable randomness, a path $\omega \in \Omega$ is considered to be S-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if Sceptic can adopt no computable (positive) betting strategy $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi) = \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ that allows her to get arbitrarily rich, but now at some computable rate.

Definition 8.6. A path $\omega \in \Omega$ is S-*random* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ is computably unbounded on ω.

Here too, in our imprecise-probabilistic setting, we continue to speak of Schnorr randomness, even when the forecasting systems are non-computable; we also continue to not assume that the non-computable forecasting systems are accessible by an oracle.

If the forecasting system φ is stationary in any of the above randomness notions $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, that is, if there's some credal set $C \in \mathscr{C}(\mathscr{X})$ such that $\varphi(s) = C$ for all $s \in \mathbb{S}$, then we'll often simply say that a path $\omega \in \Omega$ is R-random for the credal set C, instead of saying that it is R-random for the stationary forecasting system φ.

9 Basic properties of the martingale-theoretic randomness definitions

Let's have a first look at what properties are implied by our imprecise-probabilistic martingale-theoretic randomness notions. In a first part, we'll fix some forecasting system $\varphi \in \Phi(\mathscr{X})$, and describe the relations between the four aforementioned randomness notions, as well as examine how many paths are random for φ. In a second part, we keep some path $\omega \in \Omega$ fixed, and consider what forecasting systems make it R-random.

[19] Since the map τ is non-decreasing, its unboundedness is equivalent to $\lim_{n\to\infty} \tau(n) = \infty$.

So, we start by fixing some forecasting system $\varphi \in \Phi(\mathscr{X})$. It turns out there's an ordering on our four martingale-theoretic notions of randomness. To describe this ordering, we introduce, for every $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ and the forecasting system φ, the corresponding set of R-random paths $\Omega_{\mathrm{R}}(\varphi) := \{\omega \in \Omega : \omega \text{ is R-random for } \varphi\}$; they satisfy the following inclusions.

Proposition 9.1. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$. Then*

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{S}}(\varphi).$$ [20]

Proof. This is an immediate corollary of Definitions 8.552 and 8.6↶, Corollary 8.451 and the fact that computably unbounded implies unbounded. □

Thus, if a path $\omega \in \Omega$ is ML-random for a forecasting system φ, then it's also wML-, C- and S-random for φ. Consequently, for any given forecasting system φ, it's more difficult for a path $\omega \in \Omega$ to be ML-random than for it to be wML-, C- or S-random, and therefore there are at most as many paths that are ML-random as there are wML-, C- or S-random paths. This makes us say that ML-randomness is a *stronger* notion of randomness than wML-, C- and S-randomness. And similarly, *mutatis mutandis*, for the other randomness notions. Conversely, we say that S-randomness is a *weaker* notion of randomness than C-, wML- and ML-randomness, and similarly for the other randomness notions.

So, given that it's more difficult for a path $\omega \in \Omega$ to be ML-random than for it to be S-random, how many paths are then ML-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$? And, more generally, how many paths are R-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$? We answer this question by showing that any Forecaster who specifies a forecasting system $\varphi \in \Phi(\mathscr{X})$ is consistent in the sense that he believes himself to be well-calibrated: in the imprecise probability tree generated by his own forecasts, the set of R-random paths is almost sure, so he's almost sure that Sceptic won't be able to become arbitrarily rich by exploiting his—Forecaster's—forecasts.

Proposition 9.2. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ *and any forecasting system $\varphi \in \Phi(\mathscr{X})$. Then the set of* R*-random paths $\omega \in \Omega$ is almost sure for φ: $\underline{P}^{\varphi}(\Omega_{\mathrm{R}}(\varphi)) = 1$.*

Proof. By Proposition 9.1, it suffices to prove this property for R = ML, because then immediately

$$1 = \underline{P}^{\varphi}(\Omega_{\mathrm{ML}}(\varphi)) \overset{\mathrm{P3}_{35}}{\leq} \underline{P}^{\varphi}(\Omega_{\mathrm{R}}(\varphi)) \overset{\mathrm{P1}_{35}}{\leq} 1$$

[20] Examples can be found in the classical precise-probabilistic literature showing that the inclusions between these randomness notions are strict. In particular, the dissertation of Yongge Wang [78] contains an overview of old and novel results that show that (i) there's a path that's C-random for 1/2 but not ML-random for 1/2, and (ii) there's a path that's S-random for 1/2 but not CH-random for 1/2. It's only recently that an example has been given of a path that's wML-random for 1/2 but not ML-random for 1/2 [79]. For the attentive reader, I indeed don't know whether the inclusion between wML-randomness and C-randomness is strict.

for all $R \in \{ML, wML, C, S\}$, and hence, $\underline{P}^{\varphi}(\Omega_R(\varphi)) = 1$ for all $R \in \{ML, wML, C, S\}$.

By recalling that the set $\overline{\mathbb{T}}_{ML}(\varphi)$ is countable and by observing that $\Omega_{ML}(\varphi) = \{\omega \in \Omega\colon (\forall T \in \overline{\mathbb{T}}_{ML}(\varphi)) \limsup_{n\to\infty} T(\omega_{1:n}) < \infty\}$, the statement for $R = ML$ is immediate from Lemma 6.22$_{39}$. □

Consequently, for any $R \in \{ML, wML, C, S\}$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$, the R-random paths $\Omega_R(\varphi)$ are legion in a measure-theoretic sense. This also implies that Definitions 8.5$_{52}$ and 8.6$_{54}$ are meaningful, in the sense that every forecasting system $\varphi \in \Phi(\mathscr{X})$ has at least one R-random path.

Corollary 9.3. *Consider any* $R \in \{ML, wML, C, S\}$ *and any forecasting system* $\varphi \in \Phi(\mathscr{X})$. *Then there's at least one path* $\omega \in \Omega$ *that's* R*-random for* φ: $\Omega_R(\varphi) \neq \emptyset$.

Proof. This is an immediate corollary of Proposition 9.2$_{\curvearrowleft}$ and Lemma 6.19$_{38}$. □

Let's now fix some path $\omega \in \Omega$, and consider what forecasting systems make it R-random. As a converse to the previous corollary—which states that every forecasting system makes at least one path random—, there's for every path $\omega \in \Omega$ at least one forecasting system $\varphi \in \Phi(\mathscr{X})$ for which it's R-random, with $R \in \{ML, wML, C, S\}$. The reason is that all paths are R-random for the (maximally imprecise) *vacuous forecasting system* $\varphi_v \in \Phi(\mathscr{X})$, which is the stationary forecasting system defined by $\varphi_v(s) := C_v$ for all $s \in \mathbb{S}$. To understand why this perhaps surprising result holds, it suffices to realise that the test supermartingales $\overline{\mathbb{T}}(C_v)$ that correspond with C_v can never increase. These betting strategies therefore don't allow Sceptic to increase her capital, let alone become arbitrarily rich.

Proposition 9.4. *Consider any* $R \in \{ML, wML, C, S\}$. *All paths are* R*-random for the vacuous credal set* $C_v \in \mathscr{C}(\mathscr{X})$: $\Omega_R(C_v) = \Omega$.

Proof. It holds for every test supermartingale $T \in \overline{\mathbb{T}}_R(C_v)$ that $\Delta T(s)(x) \leq \max \Delta T(s) = \overline{E}_{C_v}(\Delta T(s)) \leq 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, and hence, T is bounded above by $T(\square) = 1$ on any path $\omega \in \Omega$. If we now invoke Definitions 8.5$_{52}$ and 8.6$_{54}$, we find that every path $\omega \in \Omega$ is R-random for C_v. □

Moreover, any path $\omega \in \Omega$ that's R-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, is not only also R-random for the vacuous forecasting system φ_v, but also R-random for any other forecasting system that's less informative—or more conservative—than φ. Consequently, the more precise a forecasting system is, the fewer R-random paths it has.

Proposition 9.5. *Consider any* $R \in \{ML, wML, C, S\}$ *and any forecasting systems* $\varphi, \varphi' \in \Phi(\mathscr{X})$. *If* $\varphi \subseteq \varphi'$, *then* $\Omega_R(\varphi) \subseteq \Omega_R(\varphi')$.

Proof. Since $\varphi \subseteq \varphi'$ by assumption, it follows from Proposition 6.4$_{29}$ that $\overline{\mathbb{T}}_R(\varphi') \subseteq \overline{\mathbb{T}}_R(\varphi)$, and hence, the result follows trivially from Definitions 8.5$_{52}$ and 8.6$_{54}$. □

When concentrating on stationary forecasting systems in particular, Propositions 9.4 and 9.5 tell us that every path $\omega \in \Omega$ is R-random for at least one credal set—the vacuous one $C_v = \mathcal{M}(\mathscr{X})$—and that if a path $\omega \in \Omega$ is R-random for a credal set $C \in \mathscr{C}(\mathscr{X})$, then it will also be R-random for any credal set $C' \in \mathscr{C}(\mathscr{X})$ for which $C \subseteq C'$. It's therefore natural to wonder whether every path $\omega \in \Omega$ has some *smallest* credal set C that makes it R-random, that is, such that $\omega \in \Omega_{\mathrm{R}}(C)$. This is the central topic of Ref. [47] where we only consider binary state spaces; we decided not to include these results in this dissertation both to limit the number of pages and to focus on results that hold for arbitrary finite state spaces.

We'll consider one more property of the randomness of a path $\omega \in \Omega$ in this section. To pave the way, we start by repeating that our four imprecise-probabilistic martingale-theoretic randomness notions define the randomness of a path $\omega \in \Omega$ with respect to a forecasting system $\varphi \in \Phi(\mathscr{X})$. Recall from Section 6.224 that, for every $s \in \mathbb{S}$, the forecasting system φ provides a description of a subject's uncertainty about the unknown outcome of $X_{|s|+1}$, given that he has observed the situation s. However, as the following two propositions show, not all forecasts that make up the forecasting system φ are important to the randomness of a path ω; the randomness of a path $\omega \in \Omega$ with respect to a forecasting system $\varphi \in \Phi(\mathscr{X})$ is preserved when changing φ on situations that aren't on ω, provided we restrict our attention to (non-degenerate) computable forecasting systems.

Proposition 9.6. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}\}$ *and any non-degenerate computable forecasting system* $\varphi \in \Phi(\mathscr{X})$. *If a path* $\omega \in \Omega$ *is* R*-random for* φ, *then it is* R*-random for any other computable forecasting system* $\varphi' \in \Phi(\mathscr{X})$ *for which* $\varphi'(\omega_{1:n}) = \varphi(\omega_{1:n})$ *for all* $n \in \mathbb{N}_0$.

Proof. Consider any computable forecasting system $\varphi' \in \Phi(\mathscr{X})$ for which $\varphi'(\omega_{1:n}) = \varphi(\omega_{1:n})$ for all $n \in \mathbb{N}_0$, and assume towards contradiction that there's some test supermartingale $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi')$ that's unbounded on ω. We can safely assume that $T'(s) \geq 2^{-2|s|^2}$ for all $s \in \mathbb{S}$: if $\mathrm{R} = \mathrm{ML}$, then simply consider the lower semicomputable test supermartingale $(T'+1)/2 \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$, and if $\mathrm{R} = \mathrm{wML}$, then this is immediate from Lemma 9.959 and Proposition 6.630. Since the forecasting systems $\varphi, \varphi' \in \Phi(\mathscr{X})$ are computable, there are two recursive rational maps $q, q' \colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s,n)\big)\big) \leq 2^{-n}$ and $d_{\mathrm{H}}\big(\varphi'(s), \mathrm{CH}\big(q'(s,n)\big)\big) \leq 2^{-n}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$. By Lemma 9.1160, there's a recursive rational map $q_{\mathrm{H}} \colon \mathbb{S} \times \mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ such that

$$\big|d_{\mathrm{H}}\big(\mathrm{CH}\big(q(s,n)\big), \mathrm{CH}\big(q'(s,n)\big)\big) - q_{\mathrm{H}}(s,n,N)\big| \leq 2^{-N} \text{ for all } s \in \mathbb{S} \text{ and } n, N \in \mathbb{N}. \quad (9.7)$$

Let the map $F' \colon \mathbb{S} \to \mathbb{Q}_{\geq 0}$ be defined, for all $s \in \mathbb{S}$, by

$$F'(s) := \begin{cases} \frac{1+2^{-|s|}}{2} & \text{if } q_{\mathrm{H}}(s_{1:k}, N_\varphi(s_{1:k})+3, N_\varphi(s_{1:k})+3) \leq 2^{-N_\varphi(s_{1:k})-1} \text{ for} \\ & \text{all } 0 \leq k \leq |s|-1 \\ 0 & \text{otherwise,} \end{cases} \quad (9.8)$$

with $N_\varphi \colon \mathbb{S} \to \mathbb{N}$ as in Lemma 9.12[60]. Since N_φ and q_{H} are recursive rational maps, the conditions in the above equation can be checked recursively, and hence, the rational map F' is recursive. Let $T \colon \mathbb{S} \to \mathbb{R} \colon s \mapsto T'(s)F'(s)$.

In a first step, we show that $T \in \overline{\mathbb{T}}(\varphi)$. T starts with unit capital since $T'(\square)F'(\square) = 1 \cdot {}^{(1+1)}/_{2} = 1$, and is non-negative since $T'(s) \geq 0$ and $F'(s) \geq 0$ for all $s \in \mathbb{S}$. It remains to prove the supermartingale property. Fix any $s \in \mathbb{S}$. We consider two cases. If $F'(s\,\cdot) = 0$, then $\overline{E}_\varphi(T(s\,\cdot)) = \overline{E}_\varphi(0) = 0 \leq T(s)$, where the second equality holds by C1[20]. Otherwise, that is, if $F'(s\,\cdot) = {}^{1+2^{-|s|-1}}/_{2}$, then the condition in Eq. (9.8)↶ is true for $0 \leq k \leq |s|$, and hence,

$$
\begin{aligned}
&d_{\mathrm{H}}\big(\varphi'(s_{1:k}), \varphi(s_{1:k})\big)\\
&\quad \leq d_{\mathrm{H}}\big(\varphi'(s_{1:k}), \mathrm{CH}\big(q'\big(s_{1:k}, N_\varphi(s_{1:k})+3\big)\big)\big)\\
&\qquad + d_{\mathrm{H}}\big(\mathrm{CH}\big(q'\big(s_{1:k}, N_\varphi(s_{1:k})+3\big)\big), \mathrm{CH}\big(q\big(s_{1:k}, N_\varphi(s_{1:k})+3\big)\big)\big)\\
&\qquad + d_{\mathrm{H}}\big(\mathrm{CH}\big(q\big(s_{1:k}, N_\varphi(s_{1:k})+3\big)\big), \varphi(s_{1:k})\big)\\
&\quad \overset{\text{Eq. (9.7)}}{\leq} 2^{-N_\varphi(s_{1:k})-3} + \Big(q_{\mathrm{H}}\big(s_{1:k}, N_\varphi(s_{1:k})+3, N_\varphi(s_{1:k})+3\big) + 2^{-N_\varphi(s_{1:k})-3}\Big)\\
&\qquad + 2^{-N_\varphi(s_{1:k})-3}\\
&\quad \leq 2^{-N_\varphi(s_{1:k})-3} + 2^{-N_\varphi(s_{1:k})-1} + 2^{-N_\varphi(s_{1:k})-3} + 2^{-N_\varphi(s_{1:k})-3}\\
&\quad \leq 2^{-N_\varphi(s_{1:k})} \text{ for all } 0 \leq k \leq |s|.
\end{aligned}
$$

By recalling that $T' \in \overline{\mathbb{T}}(\varphi')$ and that $T'(s) \geq 2^{-2|s|^2}$ for all $s \in \mathbb{S}$, it follows from Lemma 9.12(i)[60] that

$$
\overline{E}_{\varphi(s)}(T(s\,\cdot)) \overset{\text{C2}_{20}}{=} \frac{1+2^{-|s|-1}}{2}\overline{E}_{\varphi(s)}\big(T'(s\,\cdot)\big) \leq \frac{1+2^{-|s|}}{2}T'(s) = T(s) \text{ for all } s \in \mathbb{S},
$$

so we conclude that $T \in \overline{\mathbb{T}}(\varphi)$.

In a second step, we show that T is lower semicomputable if R = ML and that it's generated by a lower semicomputable multiplier process if R = wML. If R = ML, then T' is a lower semicomputable non-negative real process. Since F' is a recursive non-negative real process, which implies that it is lower semicomputable as well, it follows from Lemma 9.15[63] that T is lower semicomputable. Else, if R = wML, then T' is generated by a lower semicomputable multiplier process. By observing that F' is a recursive non-negative rational process such that $F'(\square) = 1$ and that, for any $s \in \mathbb{S}$, $F'(s) = 0$ if $F'(t) = 0$ for some $t \sqsubseteq s$, it follows from Lemma 9.16[63] that T is generated by a lower semicomputable multiplier process. We conclude that T is a lower semicomputable test supermartingale for φ if R = ML and that it's a test supermartingale for φ generated by a lower semicomputable multiplier process if R = wML.

In a third and last step, we show that T is unbounded on ω, then contradicting that ω is R-random for φ by Definition 8.5[52] and Lemma 9.17[64]. To do so, it suffices to show that $T(\omega_{1:n}) = T'(\omega_{1:n})^{(1+2^{-n})}/_{2}$ for all $n \in \mathbb{N}_0$, because then, since $\limsup_{n\to\infty} T'(\omega_{1:n}) = \infty$ by assumption, $\limsup_{n\to\infty} T(\omega_{1:n}) = \frac{1}{2}\limsup_{n\to\infty} T'(\omega_{1:n}) = \infty$. By recalling that $\varphi(\omega_{1:n}) = \varphi'(\omega_{1:n})$ for all $n \in \mathbb{N}_0$, we easily obtain for all $n \in \mathbb{N}_0$ that

$$
q_{\mathrm{H}}(\omega_{1:n}, N_\varphi(\omega_{1:n})+3, N_\varphi(\omega_{1:n})+3)
$$

$$
\begin{aligned}
&\overset{\text{Eq. (9.7)}_{57}}{\leq} d_{\mathrm{H}}\big(\mathrm{CH}\big(q\big(\omega_{1:n}, N_\varphi(\omega_{1:n})+3\big)\big), \mathrm{CH}\big(q'\big(\omega_{1:n}, N_\varphi(\omega_{1:n})+3\big)\big)\big) \\
&\qquad + 2^{-N_\varphi(\omega_{1:n})-3} \\
&\leq d_{\mathrm{H}}\big(\mathrm{CH}\big(q\big(\omega_{1:n}, N_\varphi(\omega_{1:n})+3\big)\big), \varphi(\omega_{1:n})\big) + d_{\mathrm{H}}\big(\varphi(\omega_{1:n}), \varphi'(\omega_{1:n})\big) \\
&\qquad + d_{\mathrm{H}}\big(\varphi'(\omega_{1:n}), \mathrm{CH}\big(q'(\omega_{1:n}, N(\omega_{1:n})+3)\big)\big) + 2^{-N_\varphi(\omega_{1:n})-3} \\
&\leq 2^{-N_\varphi(\omega_{1:n})-3} + 0 + 2^{-N_\varphi(\omega_{1:n})-3} + 2^{-N_\varphi(\omega_{1:n})-3} \\
&\leq 2^{-N_\varphi(\omega_{1:n})-1}.
\end{aligned}
$$

This implies that $F'(\omega_{1:n}) = (1+2^{-n})/2$ for all $n \in \mathbb{N}_0$ and therefore, indeed, that $T(\omega_{1:n}) = T'(\omega_{1:n})(1+2^{-n})/2$ for all $n \in \mathbb{N}_0$. □

Lemma 9.9. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$. For every lower semicomputable supermartingale multiplier D for φ, there's a lower semicomputable positive strict supermartingale multiplier D' for φ such that $D'^{\circledcirc}(s) \geq 2^{-2|s|^2}$ for all $s \in \mathbb{S}$ and such that, for any path $\omega \in \Omega$, $\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) = \infty$ if $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.*

Proof. Consider any lower semicomputable supermartingale multiplier D for φ. Let the gamble process $D' \colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ be defined by

$$D'(s) := \big(1 - 2^{-|s|-1}\big)D(s) + 2^{-|s|-2} \text{ for all } s \in \mathbb{S}.$$

Since $1 - 2^{-|s|-1} > 0$ and $2^{-|s|-2} > 0$ for all $s \in \mathbb{S}$, D' is lower semicomputable and positive because D is lower semicomputable and non-negative. Moreover, for any $s \in \mathbb{S}$,

$$\overline{E}_{\varphi(s)}(D'(s)) \overset{\text{C2}_{20},\text{C4}_{20}}{=} \big(1 - 2^{-|s|-1}\big)\overline{E}_{\varphi(s)}(D(s)) + 2^{-|s|-2} \leq \big(1 - 2^{-|s|-1}\big) + 2^{-|s|-2} < 1,$$

where the first inequality holds because D is a supermartingale multiplier for φ. We conclude that D' is a lower semicomputable positive strict supermartingale multiplier for φ.

Consider any $s \in \mathbb{S}$. Since $1 - 2^{-|t|-1} > 0$ and $D(t) \geq 0$ for all $t \in \mathbb{S}$, it follows that

$$
\begin{aligned}
D'^{\circledcirc}(s) &= \prod_{k=0}^{|s|-1} D'(s_{1:k})(s_{k+1}) \\
&\geq \prod_{k=0}^{|s|-1} 2^{-|s_{1:k}|-2} = 2^{\sum_{k=0}^{|s|-1}(-k-2)} \geq 2^{-|s|\cdot(|s|-1)-2|s|} = 2^{-|s|^2-|s|} \geq 2^{-2|s|^2}.
\end{aligned}
$$

Consider any path $\omega \in \Omega$ such that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. By Lemma 9.10↷ there's some $\epsilon > 0$ such that $\prod_{k=0}^{n-1}\big(1 - 2^{-k-1}\big) = \prod_{k=1}^{n}\big(1 - 2^{-k}\big) > \epsilon$ for all $n \in \mathbb{N}$, and therefore it holds for all $n \in \mathbb{N}$ that

$$
\begin{aligned}
\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) &= \limsup_{n\to\infty} \prod_{k=0}^{n-1} D'(\omega_{1:k})(\omega_{k+1}) \\
&\geq \limsup_{n\to\infty} \prod_{k=0}^{n-1} \big(1 - 2^{-k-1}\big) D(\omega_{1:k})(\omega_{k+1})
\end{aligned}
$$

$$\begin{aligned}&\geq \epsilon \limsup_{n\to\infty} \prod_{k=0}^{n-1} D(\omega_{1:k})(\omega_{k+1})\\&= \epsilon \limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty,\end{aligned}$$

completing the proof. □

Lemma 9.10. *Consider any $x \in [0,1)$. Then, $\prod_{n=1}^{k}(1-x^n) \geq \prod_{n=1}^{\infty}(1-x^n) > 0$ for all $k \in \mathbb{N}$.*

Proof. We start by observing that the sequence $\big(\prod_{n=1}^{m}(1-x^n)\big)_{m\in\mathbb{N}}$—which we'll also denote by $(c_m)_{m\in\mathbb{N}}$—is non-increasing and bounded below by zero, and hence, $\lim_{m\to\infty} c_m = \prod_{n=1}^{\infty}\big(1-x^n\big)$ is a well-defined non-negative real number and $c_k \geq \lim_{m\to\infty} c_m$ for all $k \in \mathbb{N}$.

We continue by showing that $\lim_{m\to\infty} c_m$ is a positive real number. Since $\ln(y) \geq 1-\frac{1}{y}$ for all $y \in \mathbb{R}_{>0}$, it holds for every $m \in \mathbb{N}$ that

$$\begin{aligned}\ln(c_m) = \sum_{n=1}^{m} \ln\big(1-x^n\big) &\geq \sum_{n=1}^{m}\left(1-\frac{1}{1-x^n}\right)\\&= \sum_{n=1}^{m} \frac{-x^n}{1-x^n} \geq \frac{-1}{1-x}\sum_{n=1}^{m} x^n = \frac{-x}{(1-x)^2} =: \alpha_x \in \mathbb{R}.\end{aligned}$$

Consequently, $c_m \geq \exp\alpha_x$ for all $m \in \mathbb{N}$, and hence, $\prod_{n=1}^{\infty}\big(1-x^n\big) = \lim_{m\to\infty} c_m \geq \exp\alpha_x > 0$. □

Lemma 9.11. *There's a single algorithm that, when provided with a code for two rational credal sets $C_{\mathrm{rat}}, C'_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, outputs a code for the computable real $d_H\big(C_{\mathrm{rat}}, C'_{\mathrm{rat}}\big)$.*

Proof. It's commonly known [63] that there's a single algorithm that upon the input of two finite sets of rational probability mass functions outputs—after a finite number of steps—the computable Hausdorff distance between the two convex polytopes generated by these sets, and hence, this result is immediate. □

Lemma 9.12. *Consider any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then there's a recursive natural map $N_\varphi \colon \mathbb{S} \to \mathbb{N}$ such that, for any forecasting system $\varphi' \in \Phi(\mathscr{X})$ and all $s \in \mathbb{S}$,*

(i) $\big(1+2^{-|s|-1}\big)\overline{E}_{\varphi(s)}\big(T'(s\,\cdot)\big) \leq \big(1+2^{-|s|}\big)T'(s)$ *if* $d_H\big(\varphi(t),\varphi'(t)\big) \leq 2^{-N_\varphi(t)}$ *for all $t \sqsubseteq s$ and $T' \in \overline{\mathbb{T}}(\varphi')$ such that $T'(s) \geq 2^{-2|s|^2}$;*

(ii) $\big(1+2^{-|s|-1}\big)\overline{E}_{\varphi'(s)}(T(s\,\cdot)) \leq \big(1+2^{-|s|}\big)T(s)$ *if* $d_H\big(\varphi(s),\varphi'(t)\big) \leq 2^{-N_\varphi(t)}$ *for all $t \sqsubseteq s$ and $T \in \overline{\mathbb{T}}(\varphi)$ such that $T(s) \geq 2^{-2|s|^2}$.*

Proof. Since φ is assumed to be non-degenerate and computable, consider the recursive natural maps $E_\varphi, C_\varphi \colon \mathbb{S} \to \mathbb{N}$ from Lemma 9.1362. Let $N \colon \mathbb{S} \to \mathbb{N}_0$ be defined as

$$N(s) := \min\left\{ n \in \mathbb{N}_0 \colon 2^{-n} \leq \frac{2^{-|s|-1-2|s|^2}}{\big(1+2^{-|s|-1}\big)\max C_\varphi(s\,\cdot)} \right\}$$

for all $s \in \mathbb{S}$. This map is recursive because C_φ is. Let $N_\varphi\colon \mathbb{S} \to \mathbb{N}\colon s \mapsto \max\{N(s), E_\varphi(s)\}$, which is clearly a recursive natural process. Now, fix any forecasting system $\varphi' \in \Phi(\mathscr{X})$.

To show (i), fix any $T' \in \overline{\mathbb{T}}(\varphi')$ and any $s \in \mathbb{S}$ such that $d_{\mathrm{H}}\big(\varphi(t), \varphi'(t)\big) \leq 2^{-N_\varphi(t)}$ for all $t \sqsubseteq s$ and such that $T'(s) \geq 2^{-2|s|^2}$. Since T' is positive, it holds that $\max T'(s\,\cdot) > 0$, and hence,

$$\begin{aligned}
&\big(1+2^{-|s|-1}\big)\overline{E}_{\varphi(s)}\big(T'(s\,\cdot)\big)\\
&\overset{\text{C2}_{20}}{=} \big(1+2^{-|s|-1}\big)\max T'(s\,\cdot)\overline{E}_{\varphi(s)}\bigg(\frac{T'(s\,\cdot)}{\max T'(s\,\cdot)}\bigg)\\
&\leq \big(1+2^{-|s|-1}\big)\max T'(s\,\cdot)\bigg[\overline{E}_{\varphi'(s)}\bigg(\frac{T'(s\,\cdot)}{\max T'(s\,\cdot)}\bigg)+2^{-N_\varphi(s)}\bigg]\\
&\overset{\text{C2}_{20}}{=} \big(1+2^{-|s|-1}\big)\overline{E}_{\varphi'(s)}(T'(s\,\cdot))+2^{-N_\varphi(s)}\big(1+2^{-|s|-1}\big)\max T'(s\,\cdot)\\
&\leq \big(1+2^{-|s|-1}\big)\overline{E}_{\varphi'(s)}(T'(s\,\cdot))+2^{-N(s)}\big(1+2^{-|s|-1}\big)\max T'(s\,\cdot)\\
&\leq \big(1+2^{-|s|-1}\big)T'(s)+\frac{2^{-|s|-1-2|s|^2}\big(1+2^{-|s|-1}\big)\max T'(s\,\cdot)}{\big(1+2^{-|s|-1}\big)\max C_\varphi(s\,\cdot)}\\
&\leq \big(1+2^{-|s|-1}\big)T'(s)+2^{-|s|-1-2|s|^2}\\
&\leq \big(1+2^{-|s|-1}\big)T'(s)+2^{-|s|-1}T'(s)\\
&= \big(1+2^{-|s|}\big)T'(s),
\end{aligned}$$

where the first inequality follows from Corollary 7.9[48], and the fourth inequality from Lemma 9.13: since $d_{\mathrm{H}}\big(\varphi(t), \varphi'(t)\big) \leq 2^{-N_\varphi(t)} \leq 2^{-E_\varphi(t)}$ for all $t \sqsubset sx$ with $x \in \mathscr{X}$, it holds by Lemma 9.13 that $T'(sx) \leq C_\varphi(sx)$ for all $x \in \mathscr{X}$, and hence, $\max T'(s\,\cdot) \leq \max C_\varphi(s\,\cdot)$.

To show (ii), fix any $T \in \overline{\mathbb{T}}(\varphi)$ and any $s \in \mathbb{S}$ such that $d_{\mathrm{H}}\big(\varphi(t), \varphi'(t)\big) \leq 2^{-N_\varphi(t)}$ for all $t \sqsubseteq s$ and such that $T(s) \geq 2^{-2|s|^2}$. Since T is positive, it holds that $\max T(s\,\cdot) > 0$, and hence,

$$\begin{aligned}
&\big(1+2^{-|s|-1}\big)\overline{E}_{\varphi'(s)}(T(s\,\cdot))\\
&\overset{\text{C2}_{20}}{=} \big(1+2^{-|s|-1}\big)\max T(s\,\cdot)\overline{E}_{\varphi'(s)}\bigg(\frac{T(s\,\cdot)}{\max T(s\,\cdot)}\bigg)\\
&\leq \big(1+2^{-|s|-1}\big)\max T(s\,\cdot)\bigg[\overline{E}_{\varphi(s)}\bigg(\frac{T(s\,\cdot)}{\max T(s\,\cdot)}\bigg)+2^{-N_\varphi(s)}\bigg]\\
&\overset{\text{C2}_{20}}{=} \big(1+2^{-|s|-1}\big)\overline{E}_{\varphi(s)}(T(s\,\cdot))+2^{-N_\varphi(s)}\big(1+2^{-|s|-1}\big)\max T(s\,\cdot)\\
&\leq \big(1+2^{-|s|-1}\big)\overline{E}_{\varphi(s)}(T(s\,\cdot))+2^{-N(s)}\big(1+2^{-|s|-1}\big)\max T(s\,\cdot)\\
&\leq \big(1+2^{-|s|-1}\big)T(s)+\frac{2^{-|s|-1-2|s|^2}\big(1+2^{-|s|-1}\big)\max T(s\,\cdot)}{\big(1+2^{-|s|-1}\big)\max C_\varphi(s\,\cdot)}\\
&\leq \big(1+2^{-|s|-1}\big)T(s)+2^{-|s|-1-2|s|^2}\\
&\leq \big(1+2^{-|s|-1}\big)T(s)+2^{-|s|-1}T(s)\\
&= \big(1+2^{-|s|}\big)T(s),
\end{aligned}$$

where the first inequality follows from Corollary 7.9[48], and the fourth inequality from Lemma 9.13: since $d_{\mathrm{H}}\big(\varphi(t),\varphi(t)\big)=0\leq 2^{-E_\varphi(t)}$ for all $t\sqsubset sx$ with $x\in\mathscr{X}$, it holds by Lemma 9.13 that $T(sx)\leq C_\varphi(sx)$ for all $x\in\mathscr{X}$, and hence, $\max T(s\cdot)\leq\max C_\varphi(s\cdot)$. □

Lemma 9.13. *For every non-degenerate computable forecasting system $\varphi\in\Phi(\mathscr{X})$ there are recursive natural maps $E_\varphi, C_\varphi\colon\mathbb{S}\to\mathbb{N}$, with $C_\varphi(\square)=1$, such that for every non-negative supermartingale $M\in\overline{\mathbb{M}}(\varphi')$, with $\varphi'\in\Phi(\mathscr{X})$, it holds for all $s\in\mathbb{S}$ that $M(s)\leq M(\square)C_\varphi(s)$ if $d_H\big(\varphi(t),\varphi'(t)\big)\leq 2^{-E_\varphi(t)}$ for all $t\sqsubset s$.*

Proof. Let $E_\varphi\colon\mathbb{S}\to\mathbb{N}$ be a recursive natural map as in Lemma 9.14, so $0<\overline{E}_{\varphi(s)}(\mathbb{I}_x)-2^{-E_\varphi(s)}$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$. Define the map $C'\colon\mathbb{S}\to\mathbb{R}$ by letting

$$C'(s)\coloneqq\prod_{k=0}^{|s|-1}\frac{1}{\overline{E}_{\varphi(s_{1:k})}(\mathbb{I}_{s_{k+1}})-2^{-E_\varphi(s_{1:k})}}\text{ for all }s\in\mathbb{S}.$$

This map is well-defined, real-valued and clearly positive. Clearly, $C'(\square)=1$. Since φ is computable and $\{\mathbb{I}_x\}_{x\in\mathscr{X}}$ is a finite set of rational gambles, it follows from Lemma 7.7[47] that the map $\mathbb{S}\times\mathscr{X}\to\mathbb{R}\colon(s,x)\mapsto\overline{E}_{\varphi(s)}(\mathbb{I}_x)$ is computable. Since subtracting $\mathbb{S}\to\mathbb{Q}\colon s\mapsto 2^{-E_\varphi(s)}$, taking the inverse and taking a finite product are computable operations, this implies that C' is computable as well. Let's now fix any non-negative supermartingale $M\in\overline{\mathbb{M}}(\varphi')$, with $\varphi'\in\Phi(\mathscr{X})$, and any $s\in\mathbb{S}$ such that $d_{\mathrm{H}}\big(\varphi(t),\varphi'(t)\big)\leq 2^{-E_\varphi(t)}$ for all $t\sqsubset s$, and prove that $M(s)\leq M(\square)C'(s)$. By invoking Lemma 7.8[48], we infer that

$$\overline{E}_{\varphi(t)}(\mathbb{I}_x)-\overline{E}_{\varphi'(t)}(\mathbb{I}_x)\leq\max_{f\in\mathscr{L}_1(\mathscr{X})}|\overline{E}_{\varphi(t)}(f)-\overline{E}_{\varphi'(t)}(f)|=d_{\mathrm{H}}\big(\varphi(t),\varphi'(t)\big)\leq 2^{-E_\varphi(t)}\quad\text{for all }t\sqsubset s\text{ and }x\in\mathscr{X},$$

which implies that $\overline{E}_{\varphi'(t)}(\mathbb{I}_x)\geq\overline{E}_{\varphi(t)}(\mathbb{I}_x)-2^{-E_\varphi(t)}>0$ for all $t\sqsubset s$ and $x\in\mathscr{X}$. Hence, for any $t\sqsubset s$ and $x\in\mathscr{X}$,

$$M(tx)\overset{\text{C2}_{20}}{=}\frac{\overline{E}_{\varphi'(t)}(M(tx)\mathbb{I}_x)}{\overline{E}_{\varphi'(t)}(\mathbb{I}_x)}\overset{\text{C5}_{20}}{\leq}\frac{\overline{E}_{\varphi'(t)}(M(t\cdot))}{\overline{E}_{\varphi'(t)}(\mathbb{I}_x)}\leq\frac{M(t)}{\overline{E}_{\varphi'(t)}(\mathbb{I}_x)}\leq\frac{M(t)}{\overline{E}_{\varphi(t)}(\mathbb{I}_x)-2^{-E_\varphi(t)}},$$

where the second inequality follows from the supermartingale property. A simple induction argument now shows that indeed $M(s)\leq M(\square)C'(s)$.

Since C' is a computable real map, there's a recursive rational map $q\colon\mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ such that $|C'(s)-q(s,n)|\leq 2^{-n}$ for all $s\in\mathbb{S}$ and $n\in\mathbb{N}$. Let $C_\varphi\colon\mathbb{S}\to\mathbb{N}$ be defined as $C_\varphi(\square)=1$ and $C_\varphi(s)\coloneqq\lceil q(s,1)+1/2\rceil$ for all $s\in\mathbb{S}\setminus\{\square\}$. By recalling that C' is positive and q is recursive, it's easy to see that C_φ is natural-valued, positive and recursive. Furthermore, we have that $M(\square)=M(\square)C_\varphi(\square)$ and that $M(s)\leq M(\square)C'(s)\leq M(\square)\big(q(s,1)+1/2\big)\leq M(\square)C_\varphi(s)$ for all $s\in\mathbb{S}\setminus\{\square\}$. □

Lemma 9.14. *For every non-degenerate computable forecasting system $\varphi\in\Phi(\mathscr{X})$ there's a recursive natural map $E_\varphi\colon\mathbb{S}\to\mathbb{N}$ such that $2^{-E_\varphi(s)}<\min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)$ for all $s\in\mathbb{S}$.*

Proof. Since φ is computable, $\{\mathbb{I}_x\}_{x\in\mathscr{X}}$ is a finite set of rational gambles and taking the minimum is a computable operation, it follows from Lemma 7.7[47] that the map $\mathbb{S}\to\mathbb{R}\colon s\mapsto \min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)$ is computable, and therefore lower semicomputable as well. Consequently, there's a recursive rational map $q\colon \mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ such that $q(s,\bullet)\nearrow \min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)$ for all $s\in\mathbb{S}$.

By the non-degeneracy of φ, $\overline{E}_{\varphi(s)}(\mathbb{I}_x)>0$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$, and hence, $\min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)>0$ for all $s\in\mathbb{S}$. This implies the existence of a recursive natural map $E\colon\mathbb{S}\to\mathbb{N}$ such that $2^{-E(s)}<\min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)$ for all $s\in\mathbb{S}$: for every $s\in\mathbb{S}$, find some $n\in\mathbb{N}$ for which $q(s,n)>0$ [you can do this in a finite number of steps since $q(s,\bullet)\nearrow\min_{x\in\mathscr{X}}\overline{E}_{\varphi(s)}(\mathbb{I}_x)>0$], and let $E(s)$ equal some natural number $N\in\mathbb{N}$ for which $2^{-N}<q(s,n)$. □

Lemma 9.15. *Consider any two lower semicomputable non-negative real processes $F_1,F_2\colon\mathbb{S}\to\mathbb{R}$. Then the non-negative real process $F\colon\mathbb{S}\to\mathbb{R}$ defined, for all $s\in\mathbb{S}$, by $F(s):=F_1(s)F_2(s)$ is lower semicomputable.*

Proof. Since F_1 and F_2 are lower semicomputable, there are two recursive rational maps $q,q'\colon\mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ such that $q(s,\bullet)\nearrow F_1(s)$ and $q'(s,\bullet)\nearrow F_2(s)$ for all $s\in\mathbb{S}$; by the non-negativity of both F_1 and F_2, we can safely assume that $q\geq 0$ and $q'\geq 0$ [otherwise, consider $\max\{q,0\}$ and $\max\{q',0\}$]. Consider the map $q^*\colon\mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ defined by

$$q^*(s,n)=q(s,n)q'(s,n)\text{ for all }s\in\mathbb{S}\text{ and }n\in\mathbb{N}.$$

Since q and q' are recursive non-negative non-decreasing rational maps, the map q^* is recursive, non-negative, non-decreasing and rational as well. Last, it's immediate that $\lim_{n\to\infty}q^*(s,n)=\lim_{n\to\infty}q(s,n)q'(s,n)=F_1(s)F_2(s)=F(s)$, and therefore we conclude that F is lower semicomputable. □

Lemma 9.16. *Consider any lower semicomputable multiplier process $D_1\colon\mathbb{S}\to\mathscr{L}(\mathscr{X})$ and any recursive non-negative rational process $F_2\colon\mathbb{S}\to\mathbb{R}$ such that $F_2(\square)=1$ and, for any $s\in\mathbb{S}$, $F_2(s)=0$ if $F_2(t)=0$ for some $t\sqsubseteq s$. Then there's a lower semicomputable multiplier process $D\colon\mathbb{S}\to\mathscr{L}(\mathscr{X})$ such that $D^{\circledcirc}(s):=F_2(s)D_1^{\circledcirc}(s)$ for all $s\in\mathbb{S}$.*

Proof. Since D_1 is lower semicomputable, there's some recursive rational map $q\colon\mathbb{S}\times\mathscr{X}\times\mathbb{N}\to\mathbb{Q}$ such that $q(s,x,\bullet)\nearrow D_1(s)(x)$ for all $s\in\mathbb{S}$ and $x\in\mathscr{X}$; by the non-negativity of D_1, we can safely assume that $q\geq 0$ [otherwise, consider $\max\{q,0\}$]. Since F_2 is recursive, non-negative and rational, it follows that the map $q'\colon\mathbb{S}\times\mathscr{X}\to\mathbb{R}$ defined by

$$q'(s,x):=\begin{cases}\frac{F_2(sx)}{F_2(s)} & \text{if }F_2(t)>0\text{ for all }t\sqsubseteq s\\ 0 & \text{otherwise}\end{cases}\quad\text{for all }s\in\mathbb{S}\text{ and }x\in\mathscr{X}$$

is well-defined, non-negative, rational and recursive.

Consider now the map $q^*\colon\mathbb{S}\times\mathscr{X}\times\mathbb{N}\to\mathbb{Q}$ defined by

$$q^*(s,x,n)=q(s,x,n)q'(s,x)\text{ for all }s\in\mathbb{S},\ x\in\mathscr{X}\text{ and }n\in\mathbb{N}.$$

Since q and q' are recursive, non-negative, rational maps and since q is non-decreasing in n, the map q^* is recursive, non-negative, non-decreasing in n and rational as well. For every $s \in \mathbb{S}$ and $x \in \mathscr{X}$, it furthermore holds that

$$\lim_{n\to\infty} q^*(s,x,n) = \lim_{n\to\infty} q(s,x,n)q'(s,x,) = \begin{cases} D_1(s)(x)\frac{F_2(sx)}{F_2(s)} & \text{if } F_2(t) > 0 \text{ for all } t \sqsubseteq s \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, the (non-negative) multiplier process $D\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ defined by

$$D(s)(x) := \begin{cases} D_1(s)(x)\frac{F_2(sx)}{F_2(s)} & \text{if } F_2(t) > 0 \text{ for all } t \sqsubseteq s \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}$$

is lower semicomputable. Finally, since $F_2(\square) = 1$, it holds for every $s \in \mathbb{S}$ that

$$\begin{aligned} F_2(s)D_1^{\circledcirc}(s) &= F_2(s)\prod_{k=0}^{|s|-1} D_1(s_{1:k})(s_{k+1}) \\ &= \begin{cases} \frac{F_2(s)}{F_2(\square)}\prod_{k=0}^{|s|-1} D_1(s_{1:k})(s_{k+1}) & \text{if } F_2(t) > 0 \text{ for all } t \sqsubseteq s \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \prod_{k=0}^{|s|-1} D_1(s_{1:k})(s_{k+1})\frac{F_2(s_{1:k+1})}{F_2(s_{1:k})} & \text{if } F_2(t) > 0 \text{ for all } t \sqsubseteq s \\ 0 & \text{otherwise} \end{cases} \\ &= \prod_{k=0}^{|s|-1} D(s_{1:k})(s_{k+1}) = D^{\circledcirc}(s). \end{aligned}$$

□

Lemma 9.17. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$. For every lower semicomputable multiplier process D that generates a test supermartingale $D^{\circledcirc}$ for φ, there's a lower semicomputable positive supermartingale multiplier D' for φ such that, for any path $\omega \in \Omega$, $\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) = \infty$ if $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.*

Proof. Consider any lower semicomputable multiplier process D that generates a test supermartingale $D^{\circledcirc}$ for φ. Since D is lower semicomputable, there's a recursive map $q\colon \mathbb{S} \times \mathscr{X} \times \mathbb{N} \to \mathbb{Q}$ such that $q(s,x,\bullet) \nearrow D(s)(x)$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$. Let the map $q'\colon \mathbb{S} \times \mathscr{X} \times \mathbb{N} \to \mathbb{Q}$ be defined by

$$q'(s,x,n) := \begin{cases} \max\{q(s,x,n),0\} & \text{if } q(t,y,n) > 0 \text{ for all } ty \sqsubseteq s \\ 0 & \text{otherwise} \end{cases}$$

$$\text{for all } s \in \mathbb{S},\ x \in \mathscr{X} \text{ and } n \in \mathbb{N}.$$

By construction, q' is non-negative. Moreover, also by construction, q' is non-decreasing in its third argument and recursive because q is non-decreasing in its third argument and recursive. Let the map $D'\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ be defined by $D'(s)(x) := \lim_{n\to\infty} q'(s,x,n)$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, which is clearly a lower semicomputable multiplier process. Then, by construction,

$$D'(s)(x) = \begin{cases} D(s)(x) & \text{if } D(t)(y) > 0 \text{ for all } ty \sqsubseteq s \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}.$$

In a next step, we'll show that D' is a supermartingale multiplier for φ. To this end, fix any $s \in \mathbb{S}$. If $D(t)(y) > 0$ for all $ty \sqsubseteq s$, then $D^{\circledcirc}(s) > 0$ and

$$D'(s) = D(s) = \frac{D^{\circledcirc}(s\,\cdot)}{D^{\circledcirc}(s)} = \frac{D^{\circledcirc}(s) + \Delta D^{\circledcirc}(s)}{D^{\circledcirc}(s)} = 1 + \frac{\Delta D^{\circledcirc}(s)}{D^{\circledcirc}(s)},$$

and hence,

$$\overline{E}_{\varphi(s)}(D'(s)) \overset{\text{C2}_{20},\text{C4}_{20}}{=} 1 + \frac{1}{D^{\circledcirc}(s)} \overline{E}_{\varphi(s)}(\Delta D^{\circledcirc}(s)) \leq 1,$$

where the inequality holds because $D^{\circledcirc}$ is a test supermartingale for φ. Otherwise, if $D(t)(y) = 0$ for some $ty \sqsubseteq s$, then $D'(s) = 0$, and hence, $\overline{E}_{\varphi(s)}(D'(s)) = 0$ due to C1$_{20}$. We conclude that D' is a lower semicomputable supermartingale multiplier for φ.

Consider any path $\omega \in \Omega$ such that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. Consequently, it holds that $D(\omega_{1:n})(\omega_{n+1}) > 0$ for all $n \in \mathbb{N}_0$, and therefore also $D'(\omega_{1:n})(\omega_{n+1}) = D(\omega_{1:n})(\omega_{n+1})$ for all $n \in \mathbb{N}_0$, which implies that $\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) = \infty$.

By invoking Lemma 9.9$_{59}$, we find that there's a lower semicomputable positive supermartingale multiplier D'' for φ such that $\limsup_{n\to\infty} D''^{\circledcirc}(\omega_{1:n}) = \infty$. □

For C- and S-randomness, we can drop the requirement of non-degeneracy: if a path $\omega \in \Omega$ is C-random, respectively S-random, for a computable forecasting system $\varphi \in \Phi(\mathscr{X})$, then it is C-random, respectively S-random, for any other computable forecasting system that specifies the same forecasts along ω.

Proposition 9.18. *Consider any* $\mathrm{R} \in \{\mathrm{C}, \mathrm{S}\}$ *and any computable forecasting system* $\varphi \in \Phi(\mathscr{X})$*. If a path* $\omega \in \Omega$ *is* R*-random for* φ*, then it is* R*-random for any other computable forecasting system* $\varphi' \in \Phi(\mathscr{X})$ *for which* $\varphi'(\omega_{1:n}) = \varphi(\omega_{1:n})$ *for all* $n \in \mathbb{N}_0$*.*

Proof. Assume towards contradiction that there's some computable test supermartingale $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi')$ that's (computably) unbounded on ω; we can safely assume that $T' > 0$ [otherwise, consider the computable test supermartingale $(T'+1)/2 \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi')$]. Let $T \colon \mathbb{S} \to \mathbb{R}$ be defined as

$$T(s) := T'(s) \prod_{k=0}^{|s|-1} \frac{\overline{E}_{\varphi'(s_{1:k})}(T'(s_{1:k}\,\cdot))}{\overline{E}_{\varphi(s_{1:k})}(T'(s_{1:k}\,\cdot))} \text{ for all } s \in \mathbb{S}.$$

Since $\varphi(\omega_{1:n}) = \varphi'(\omega_{1:n})$ for all $n \in \mathbb{N}_0$, it holds that $T(\omega_{1:n}) = T'(\omega_{1:n})$ for all $n \in \mathbb{N}_0$, and hence, T is (computably) unbounded on ω since T' is. So we're done if we can prove that $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$.

Since T' is computable and positive, and since the forecasting systems φ and φ' are computable, it follows from C1$_{20}$ and Lemma 7.7$_{47}$ that the real processes $\mathbb{S} \to \mathbb{R} \colon s \mapsto \overline{E}_{\varphi(s)}(T'(s\,\cdot))$ and $\mathbb{S} \to \mathbb{R} \colon s \mapsto \overline{E}_{\varphi'(s)}(T'(s\,\cdot))$ are positive and computable. This implies that the real process $\mathbb{S} \to \mathbb{R} \colon s \mapsto \prod_{k=0}^{|s|-1} \overline{E}_{\varphi'(s_{1:k})}(T'(s_{1:k}\,\cdot))/\overline{E}_{\varphi(s_{1:k})}(T'(s_{1:k}\,\cdot))$ is well-defined, positive and computable, and hence, T is positive and computable as the product of two computable positive real processes. Moreover, $T(\Box) = T'(\Box) = 1$.

So it only remains to check the supermartingale property to conclude that $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$. To this end, fix any $s \in \mathbb{S}$, and observe that

$$\begin{aligned}\overline{E}_{\varphi(s)}(T(s\,\cdot)) &\overset{\text{C2}_{20}}{=} \overline{E}_{\varphi(s)}(T'(s\,\cdot)) \prod_{k=0}^{|s|} \frac{\overline{E}_{\varphi'(s_{1:k})}(T'(s_{1:k}\,\cdot))}{\overline{E}_{\varphi(s_{1:k})}(T'(s_{1:k}\,\cdot))} \\ &= \overline{E}_{\varphi'(s)}(T'(s\,\cdot)) \prod_{k=0}^{|s|-1} \frac{\overline{E}_{\varphi'(s_{1:k})}(T'(s_{1:k}\,\cdot))}{\overline{E}_{\varphi(s_{1:k})}(T'(s_{1:k}\,\cdot))} \\ &\leq T'(s) \prod_{k=0}^{|s|-1} \frac{\overline{E}_{\varphi'(s_{1:k})}(T'(s_{1:k}\,\cdot))}{\overline{E}_{\varphi(s_{1:k})}(T'(s_{1:k}\,\cdot))} = T(s),\end{aligned}$$

where the inequality follows from the supermartingale property of T'. □

Propositions 9.6[57] and 9.18↶ above tell us that the randomness of a path $\omega \in \Omega$ with respect to a forecasting system $\varphi \in \Phi(\mathscr{X})$ only depends on the forecasts that the forecasting system φ specifies along the path, provided that we restrict our attention to (non-degenerate) computable forecasting systems. This is in line with Dawid's Weak Prequential Principle [8], which states that any criterion for assessing the 'agreement' between Forecaster's forecasts and Reality's outcomes should depend only on the actual observed sequences $(C_1, \ldots, C_n, \ldots) \in \mathscr{C}(\mathscr{X})^{\mathbb{N}}$ and $\omega = (x_1, \ldots, x_n, \ldots) \in \Omega$, and not on the strategies (if any) which might have produced these, such as a (non-degenerate) computable forecasting system $\varphi \in \Phi(\mathscr{X})$. As Vovk & Shen [9] show, it's possible to devise randomness notions for which this so-called 'Prequential Principle' is built-in; this will lead us to introduce several *prequential* imprecise-probabilistic randomness notions in Chapter ⚄[143] that, instead of defining the randomness of a path $\omega \in \Omega$ with respect to a forecasting system $\varphi \in \Phi(\mathscr{X})$, define the randomness of an infinite sequence $(C_1, x_1, \ldots, C_n, x_n, \ldots)$ of (rational) credal sets C_n and subsequent outcomes x_n.

10 Robustness of the martingale-theoretic randomness definitions ...

We continue by examining how robust our four imprecise-probabilistic martingale-theoretic notions are with respect to changes in Forecaster's forecasting system and Sceptic's allowed betting strategies: what changes can we make to the definitions without changing the set of R-random paths? More specifically, in Section 10.1↷, for any (non-degenerate computable) forecasting system $\varphi \in \Phi(\mathscr{X})$ and any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, we explore what other forecasting systems $\varphi' \in \Phi(\mathscr{X})$ have the same set of R-random paths, that is, what (other) forecasting systems $\varphi' \in \Phi(\mathscr{X})$ satisfy $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi')$. In Section 10.2[72], for any forecasting system $\varphi \in \Phi(\mathscr{X})$ and any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, we investigate to what extent we can change Sceptic's allowed betting strategies $\overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ and the way she shouldn't be able to get arbitrarily rich, without changing the notion of R-randomness.

10.1 ... in terms of changes to the forecasting systems

We'll start by restricting our attention to *non-degenerate computable* forecasting systems $\varphi \in \Phi(\mathscr{X})$. When considering the randomness of a path $\omega \in \Omega$ with respect to such a forecasting system φ, there are other ways we can change φ while preserving the randomness of ω, aside from changing φ on situations that aren't on ω [see Propositions 9.6$_{57}$ and 9.18$_{65}$]: we can replace φ by a (recursive) rational forecasting system. As the proposition below shows, there's for every non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$ some rational recursive forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that has the exact same set of random paths. So, in this sense, you could say that rational forecasting systems are enough to capture the essence of our martingale-theoretic randomness notions.

Proposition 10.1. *For every non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$ there's a recursive rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ such that, for any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, a path $\omega \in \Omega$ is R-random for φ if and only if it's R-random for φ_{rat}: $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi_{\mathrm{rat}})$.*

Proof. Since φ is computable, there's a recursive rational map $q\colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s,n)\big)\big) \leq 2^{-n}$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$. Since φ is non-degenerate as well, we can fix some recursive natural map $N_\varphi\colon \mathbb{S} \to \mathbb{N}$ with the properties as in Lemma 9.12$_{60}$. Let $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ be defined by $\varphi_{\mathrm{rat}}(s) \coloneqq \mathrm{CH}\big(q(s, N_\varphi(s))\big)$ for all $s \in \mathbb{S}$. This rational forecasting system is obviously recursive and $d_{\mathrm{H}}\big(\varphi(s), \varphi_{\mathrm{rat}}(s)\big) \leq 2^{-N_\varphi(s)}$ for all $s \in \mathbb{S}$.

To show that $\Omega_{\mathrm{R}}(\varphi) \supseteq \Omega_{\mathrm{R}}(\varphi_{\mathrm{rat}})$, consider any path $\omega \in \Omega$ that's R-random for φ_{rat} and assume towards contradiction that there's some test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ that's (computably) unbounded on ω. We can safely assume that $T(s) \geq 2^{-2|s|^2}$ for all $s \in \mathbb{S}$: if $\mathrm{R} \in \{\mathrm{ML}, \mathrm{C}, \mathrm{S}\}$, then simply consider the test supermartingale $(T+1)/2 \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$, and if $\mathrm{R} = \mathrm{wML}$, then this is immediate from Lemma 9.9$_{59}$ and Proposition 6.6$_{30}$. Define the map $T'\colon \mathbb{S} \to \mathbb{R}$ as

$$T'(s) \coloneqq T(s)\frac{1+2^{-|s|}}{2} \text{ for all } s \in \mathbb{S}. \tag{10.2}$$

In a first step, we prove that $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{\mathrm{rat}})$; in a second step, we prove that T' is (computably) unbounded on ω. T' starts with unit capital since $T(\Box)(1+1)/2 = 1$, and is positive since $T'(s) = T(s)(1+2^{-|s|})/2 \geq 2^{-2|s|^2}(1+2^{-|s|})/2 > 0$ for all $s \in \mathbb{S}$. To establish its supermartingale character, simply observe that by Lemma 9.12(ii)$_{60}$, since $d_{\mathrm{H}}\big(\varphi(s), \varphi_{\mathrm{rat}}(s)\big) \leq 2^{-N_\varphi(s)}$ and $T(s) \geq 2^{-2|s|^2}$ for all $s \in \mathbb{S}$ and since $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$, it holds that

$$\overline{E}_{\varphi_{\mathrm{rat}}(s)}(T'(s\,\cdot)) \overset{\mathrm{C2}_{20}}{=} \frac{1+2^{-|s|-1}}{2}\overline{E}_{\varphi_{\mathrm{rat}}(s)}(T(s\,\cdot)) \leq \frac{1+2^{-|s|}}{2}T(s) = T'(s) \text{ for all } s \in \mathbb{S},$$

so we conclude that $T' \in \overline{\mathbb{T}}(\varphi_{\mathrm{rat}})$. To also show that $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{\mathrm{rat}})$, in addition to T' being a test supermartingale for φ_{rat}, we also need to check its implementability. If $\mathrm{R} = \mathrm{ML}$, then T is a lower semicomputable real process. Since $F'\colon \mathbb{S} \to \mathbb{R}\colon s \mapsto (1+2^{-|s|})/2$ is a recursive non-negative real process, which implies that it is lower semicomputable as well, it follows from Eq. (10.2)↶ and Lemma 9.15₆₃ that T' is lower semicomputable. Else, if R = wML, then T is generated by a lower semicomputable positive multiplier process. By observing that F' is a recursive positive rational process such that $F'(\square) = 1$, it follows from Eq. (10.2)↶, Proposition 6.6₃₀ and Lemma 9.16₆₃ that the positive test supermartingale T' for φ_{rat} is generated by a lower semicomputable positive supermartingale multiplier for φ_{rat}. Otherwise, that is, if $\mathrm{R} \in \{\mathrm{C}, \mathrm{S}\}$, then T is a computable real process. By observing that F' is a computable real process, it follows from Eq. (10.2)↶ and the discussion in the first paragraph of Section 7.4₄₆ that T' is computable as the product of both processes. We conclude that $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{\mathrm{rat}})$.

In a second step, we consider two cases. If it holds that $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, then $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$ by assumption, which implies that $\limsup_{n\to\infty} T'(\omega_{1:n}) = \frac{1}{2}\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$, a contradiction. Otherwise, that is, if R = S, then by assumption there's some real growth function τ such that $\limsup_{n\to\infty}(T(\omega_{1:n}) - \tau(n)) \geq 0$. Consider the real growth function τ' defined by $\tau'(n) := \tau(n)/2$ for all $n \in \mathbb{N}_0$. It then holds that $\limsup_{n\to\infty}(T'(\omega_{1:n}) - \tau'(n)) = \frac{1}{2}\limsup_{n\to\infty}(T(\omega_{1:n}) - \tau(n)) \geq 0$, a contradiction.

To show that $\Omega_{\mathrm{R}}(\varphi) \subseteq \Omega_{\mathrm{R}}(\varphi_{\mathrm{rat}})$, simply reverse the roles of φ and φ_{rat} in the above line of reasoning [except for N_φ] and use Lemma 9.12(i)₆₀. □

We do away next with assuming the forecasting systems to be non-degenerate and computable. For this more (and most) general collection of forecasting systems, it turns out that the randomness of a path doesn't depend on a finite number of forecasts, that is, the randomness of a path $\omega \in \Omega$ for a forecasting system $\varphi \in \Phi(\mathscr{X})$ is preserved when changing the forecasts $\varphi(s) \in \mathscr{C}(\mathscr{X})$ in a finite number of situations $s \in \mathbb{S}$.

Proposition 10.3. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ *and any two forecasting system* $\varphi, \varphi' \in \Phi(\mathscr{X})$. *Then* $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi')$ *if* $\varphi(s) = \varphi'(s)$ *for all but finitely many* $s \in \mathbb{S}$.

Proof. We'll prove that $\Omega_{\mathrm{R}}(\varphi') \subseteq \Omega_{\mathrm{R}}(\varphi)$; in a similar way, it can be shown that $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi')$, leading us to conclude that $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi')$. To this end, consider any path $\omega \in \Omega_{\mathrm{R}}(\varphi')$ and any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$. By Definitions 8.5₅₂ and 8.6₅₄, we're done if we can prove that T remains (computably) bounded on ω, which is what we set out to do.

By assumption, there's only a finite number of situations $s \in \mathbb{S}$ for which $\varphi(s) \neq \varphi'(s)$, and therefore there are also only a finite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi'(s)}(\Delta T(s)) > 0$. The desired result is now immediate from Lemma 10.4. □

Lemma 10.4. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, *any two forecasting system* $\varphi, \varphi' \in \Phi(\mathscr{X})$, *any test supermartingale* $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$, *and any path* $\omega \in \Omega_{\mathrm{R}}(\varphi')$. *If* $\overline{E}_{\varphi'(s)}(T(s\,\cdot)) \leq T(s)$ *for all but finitely many situations* $s \in \mathbb{S}$, *then* T *remains bounded on* ω *if* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$ *and computably bounded on* ω *if* $\mathrm{R} = \mathrm{S}$.

Proof. By assumption, there's only a finite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi'(s)}(\Delta T(s)) > 0$. By Lemma 10.5, we can then fix natural numbers $N, K \in \mathbb{N}$ such that the test process $T' : \mathbb{S} \to \mathbb{R}$, defined by

$$T'(s) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K} T(s) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S},$$

is a test supermartingale for φ'. We continue by proving that $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi')$.

Observe that T' is positive if $\mathrm{R} \in \{\mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, because then $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi) \subseteq \mathscr{F}_{\mathrm{R}}$ is positive and because $K > 0$.

Let's now prove that T' is implementable in the same way as T is. If $\mathrm{R} = \mathrm{ML}$, then $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi) \subseteq \mathscr{F}_{\mathrm{ML}}$ is lower semicomputable, so it follows from Lemma 10.6 that T' is lower semicomputable as well. If $\mathrm{R} = \mathrm{wML}$, then $T \in \overline{\mathbb{T}}_{\mathrm{wML}}(\varphi) \subseteq \mathscr{F}_{\mathrm{wML}}$ is generated by a lower semicomputable multiplier process, so it follows from Lemma 10.771 that T' is generated by a lower semicomputable multiplier process as well, and hence, by Proposition 6.630, T' is generated by a lower semicomputable positive supermartingale multiplier for φ'. And finally, if $\mathrm{R} = \mathrm{C}$ or $\mathrm{R} = \mathrm{S}$, then $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi) = \overline{\mathbb{T}}_{\mathrm{S}}(\varphi) \subseteq \mathscr{F}_{\mathrm{C}} = \mathscr{F}_{\mathrm{S}}$ is a computable process, and it's therefore obvious that this is true for T' as well.

We conclude that, indeed, $T' \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi')$. We now consider two possibilities. If $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, then since ω is R-random for φ' by assumption, T' can't be unbounded on ω by Definition 8.552. Since also

$$\limsup_{n\to\infty} T'(\omega_{1:n}) < \infty \Rightarrow \limsup_{n\to\infty} \frac{T(\omega_{1:n})}{K} < \infty \overset{K>0}{\Rightarrow} \limsup_{n\to\infty} T(\omega_{1:n}) < \infty,$$

it then follows that T doesn't become unbounded on ω.

If $\mathrm{R} = \mathrm{S}$, then since ω is R-random for φ' by assumption, T' can't be computably unbounded on ω by Definition 8.654. Consider now any real growth function τ and an associated real growth function τ' defined by $\tau'(n) := \tau(n)/K$ for all $n \in \mathbb{N}_0$. It then holds that

$$\begin{aligned} \limsup_{n\to\infty} [T'(\omega_{1:n}) - \tau'(n)] < 0 &\Rightarrow \limsup_{n\to\infty} \Big[\frac{T(\omega_{1:n})}{K} - \frac{\tau(n)}{K}\Big] < 0 \\ &\overset{K>0}{\Rightarrow} \limsup_{n\to\infty} \Big[T(\omega_{1:n}) - \tau(n)\Big] < 0, \end{aligned}$$

and hence, since T' isn't computably unbounded on ω for the real growth function τ', T doesn't become computably unbounded on ω for τ. Since this holds for any real growth function τ, we conclude that T doesn't become computably unbounded on ω. □

Lemma 10.5. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$ and any non-negative test process F. If $\overline{E}_{\varphi(s)}(F(s\,\cdot)) \leq F(s)$ for all but finitely many situations $s \in \mathbb{S}$, then there are natural numbers $N, K \in \mathbb{N}$ such that the process $\tilde{F}$, defined by*

$$\tilde{F}(s) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K} F(s) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S},$$

is a test supermartingale for φ.

Proof. Assume that there's only a finite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi(s)}(F(s\cdot)) > F(s)$. Then we can fix some $N \in \mathbb{N}$ such that $\overline{E}_{\varphi(s)}(F(s\cdot)) \leq F(s)$ for all $s \in \mathbb{S}$ with $|s| > N$. Let $K \in \mathbb{N}$ be any positive natural number such that $K > F(s)$ for all $s \in \mathbb{S}$ with $|s| = N+1$, and consider the non-negative test process $\tilde{F}\colon \mathbb{S} \to \mathbb{R}$ defined by

$$\tilde{F}(s) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}F(s) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S}.$$

To prove that $\tilde{F}$ is a supermartingale for φ, we fix some $s \in \mathbb{S}$, and consider three mutually exclusive possibilities: $|s| < N$, $|s| = N$ and $|s| > N$. If $|s| < N$, then

$$\overline{E}_{\varphi(s)}\big(\Delta\tilde{F}(s)\big) = \overline{E}_{\varphi(s)}(0) \overset{\text{C1}_{20}}{=} 0.$$

If $|s| = N$, then

$$\overline{E}_{\varphi(s)}(\Delta\tilde{F}(s)) = \overline{E}_{\varphi(s)}\big(\tilde{F}(s\,\bullet) - \tilde{F}(s)\big) = \overline{E}_{\varphi(s)}\Big(\frac{1}{K}F(s\,\bullet) - 1\Big) \overset{\text{C5}_{20}}{\leq} \overline{E}_{\varphi(s)}(1-1) \overset{\text{C1}_{20}}{=} 0,$$

where the inequality holds because $K > F(t) \geq 0$ for all $t \in \mathbb{S}$ with $|t| = N+1$. Finally, if $|s| > N$, then

$$\overline{E}_{\varphi(s)}(\Delta\tilde{F}(s)) = \overline{E}_{\varphi(s)}\Big(\frac{1}{K}\Delta F(s)\Big) \overset{\text{C2}_{20}}{=} \frac{1}{K}\overline{E}_{\varphi(s)}(\Delta F(s)) \leq 0,$$

where in the second equality and final inequality, we also used the fact that $K > 0$. □

Lemma 10.6. *Consider any lower semicomputable real process F and any two natural numbers $N, K \in \mathbb{N}$. Then the real process $\tilde{F}$, defined by*

$$\tilde{F}(s) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}F(s) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S},$$

is lower semicomputable as well.

Proof. Since the real process F is lower semicomputable, there's a recursive rational map $q\colon \mathbb{S}\times\mathbb{N} \to \mathbb{Q}$ such that $q(s, n+1) \geq q(s, n)$ and $F(s) = \lim_{m\to\infty} q(s, m)$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$. Consider now the recursive rational map $\tilde{q}\colon \mathbb{S}\times\mathbb{N} \to \mathbb{Q}$ defined by

$$\tilde{q}(s, n) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}q(s, n) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S} \text{ and } n \in \mathbb{N}.$$

Then for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$,

$$\tilde{q}(s, n+1) = \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}q(s, n+1) & \text{if } |s| > N \end{cases} \geq \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}q(s, n) & \text{if } |s| > N \end{cases} = \tilde{q}(s, n)$$

and

$$\lim_{m\to\infty} \tilde{q}(s, m) = \begin{cases} 1 & \text{if } |s| \leq N \\ \lim_{m\to\infty} \frac{1}{K}q(s, m) & \text{if } |s| > N \end{cases} = \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K}F(s) & \text{if } |s| > N \end{cases} = \tilde{F}(s),$$

and therefore, $\tilde{F}$ is lower semicomputable as well. □

Lemma 10.7. *Consider any test process F that's generated by a lower semi-computable multiplier process, and any two natural numbers $N, K \in \mathbb{N}$. Then the test process $\tilde{F}$, defined by*

$$\tilde{F}(s) := \begin{cases} 1 & \text{if } |s| \leq N \\ \frac{1}{K} F(s) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S},$$

is generated by a lower semicomputable multiplier process as well.

Proof. Let D be the lower semicomputable multiplier process that generates F, meaning that $F = D^{\circledcirc}$.

Since D is lower semicomputable, there's a recursive rational map $q \colon \mathbb{S} \times \mathscr{X} \times \mathbb{N} \to \mathbb{Q}$ such that $q(s,x,n+1) \geq q(s,x,n)$ and $D(s)(x) = \lim_{m\to\infty} q(s,x,m)$ for all $s \in \mathbb{S}$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$. Since D is a multiplier process, it's non-negative, and hence, we can safely assume that the recursive rational map q is non-negative as well; otherwise, we just replace it by the recursive rational map $\max\{q, 0\}$. Moreover, since F is generated by the multiplier process D, it readily follows that $\tilde{F}$ is generated by the multiplier process $\tilde{D}$ defined by

$$\tilde{D}(s)(x) := \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K} F(sx) & \text{if } |s| = N \\ D(s)(x) & \text{if } |s| > N \end{cases} \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}.$$

So it suffices to prove that $\tilde{D}$ is lower semicomputable. To that end, consider the recursive rational map $\tilde{q} \colon \mathbb{S} \times \mathscr{X} \times \mathbb{N} \to \mathbb{Q}$ defined for all $s = x_{1:m} \in \mathbb{S}$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$ by

$$\tilde{q}(s,x,n) := \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K}\Big(\prod_{k=0}^{N-1} q(x_{1:k}, x_{k+1}, n)\Big) q(s,x,n) & \text{if } |s| = N \\ q(s,x,n) & \text{if } |s| > N. \end{cases}$$

Then for all $s = x_{1:m} \in \mathbb{S}$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$,

$$\begin{aligned} \tilde{q}(s,x,n+1) &= \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K}\Big(\prod_{k=0}^{N-1} q(x_{1:k}, x_{k+1}, n+1)\Big) q(s,x,n+1) & \text{if } |s| = N \\ q(s,x,n+1) & \text{if } |s| > N \end{cases} \\ &\geq \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K}\Big(\prod_{k=0}^{N-1} q(x_{1:k}, x_{k+1}, n)\Big) q(s,x,n) & \text{if } |s| = N \\ q(s,x,n) & \text{if } |s| > N \end{cases} \\ &= \tilde{q}(s,x,n), \end{aligned}$$

where the inequality holds because $K > 0$ and $q \geq 0$, and

$$\lim_{m\to\infty} \tilde{q}(s,x,m) = \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K} \lim_{m\to\infty}\Big(\prod_{k=0}^{N-1} q(x_{1:k}, x_{k+1}, m)\Big) q(s,x,m) & \text{if } |s| = N \\ \lim_{m\to\infty} q(s,x,m) & \text{if } |s| > N \end{cases}$$

$$= \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K}\Big(\prod_{k=0}^{N-1} D(x_{1:k})(x_{k+1})\Big) D(s)(x) & \text{if } |s| = N \\ D(s)(x) & \text{if } |s| > N \end{cases}$$

$$= \begin{cases} 1 & \text{if } |s| < N \\ \frac{1}{K} F(sx) & \text{if } |s| = N \\ D(s)(x) & \text{if } |s| > N \end{cases}$$

$$= \tilde{D}(s)(x),$$

so we see that $\tilde{D}$ is lower semicomputable, as needed. □

Another result that fits into this section, but which can be found in Section 20[192] instead, goes as follows: for any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$, there's a precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ such that $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(\varphi)$ [Theorem 20.1[193]]. Based on this result, you could jump to the conclusion that precise forecasting systems are enough to capture the essence of these randomness notions. As we'll explain in Chapter ⚅[179], more subtlety is required though when examining the necessity of allowing for imprecise forecasting systems in algorithmic randomness; we postpone the proof and discussion of this result to that chapter, where we believe it shows to better advantage.

10.2 ... in terms of changes to the betting strategies

Our imprecise-probabilistic martingale-theoretic randomness notions are not only to some extent robust with respect to changes in the forecasting system, as we explained in the previous section, but also reasonably robust with respect to (i) changing Sceptic's allowed betting strategies and (ii) the way she shouldn't be able to get arbitrarily rich, as we'll explain here. In what follows, for every $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, we typically start by showing how to *weaken* the conditions on (i) Sceptic's allowed betting strategies and (ii) the way she shouldn't be able to get arbitrarily rich, without changing the notion of R-randomness. We continue by proving such an equivalence result for a *stronger* set of conditions on (i) and (ii).

Martin-Löf randomness

For the notion of ML-randomness, we can extend Sceptic's set of betting strategies by not requiring initial unit capital, and by imposing boundedness below instead of non-negativity.

Proposition 10.8. *A path $\omega \in \Omega$ is* ML*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no lower semicomputable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ such that* $\limsup_{n\to\infty} M(\omega_{1:n}) = \infty$.

Proof. It clearly suffices to prove the 'only if'-part. To this end, assume the existence of a lower semicomputable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ such that $\limsup_{n\to\infty} M(\omega_{1:n}) = \infty$. Then we'll show that ω isn't ML-random for φ. Since M is bounded below, there's a natural $B \in \mathbb{N}$ such that $M + B > 0$. Let $A \in \mathbb{N}$ be such that $(M(\square)+B)/A \leq 1$, and let the real process $T\colon \mathbb{S} \to \mathbb{R}$ be defined by $T(\square) := 1$ and $T(s) := (M(s)+B)/A$ for all $s \in \mathbb{S} \setminus \{\square\}$. Clearly, $\limsup_{n\to\infty} T(\omega_{1:n}) = \frac{1}{A}\limsup_{n\to\infty} M(\omega_{1:n}) + \frac{B}{A} = \infty$, so we're done if we can show that $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$. Clearly, $T(\square) = 1$, $T \geq (M+B)/A > 0$ and T is lower semicomputable. Furthermore, for all $s \in \mathbb{S}$,

$$\overline{E}_{\varphi(s)}(T(s\,\cdot)) \overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{A}\overline{E}_{\varphi(s)}(M(s\,\cdot)) + \frac{B}{A} \leq \frac{M(s)+B}{A} \leq \begin{cases} 1 & \text{if } s = \square \\ T(s) & \text{otherwise} \end{cases} = T(s),$$

so we conclude that $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$. □

In the other direction, we can restrict Sceptic's allowed betting strategies by requiring positivity and satisfaction of the strict supermartingale property. Moreover, for a path $\omega \in \Omega$ to be ML-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, we can (merely) require that her running capital must never converge to infinity on ω for any implementable allowed betting strategy, in the sense that there's some allowed betting strategy $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$.

Proposition 10.9. *A path $\omega \in \Omega$ is* ML*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no lower semicomputable positive strict test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that* $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$.

Proof. By Definition 8.552, it clearly suffices to prove the 'if'-part. To this end, assume that ω isn't ML-random for φ, so assume the existence of a lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. Then we have to show that there is a lower semicomputable positive strict test supermartingale $T^* \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that $\lim_{n\to\infty} T^*(\omega_{1:n}) = \infty$.

In a first step, we consider the test supermartingale $T' \in \overline{\mathbb{T}}(\varphi)$ as defined in Lemma 10.10↷—for which $\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$—and prove that it is lower semicomputable. Since T is lower semicomputable, there's a recursive rational map $q\colon \mathbb{S}\times\mathbb{N} \to \mathbb{Q}$ such that $q(s,\bullet) \nearrow T(s)$ for all $s \in \mathbb{S}$. For any $k \in \mathbb{N}$, we consider the recursive rational map $q^{(k)}\colon \mathbb{S}\times\mathbb{N} \to \mathbb{Q}$, defined for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$ by

$$q^{(k)}(s,n) := \begin{cases} 2^k & \text{if } \max_{\ell\in\{0,\dots,|s|\}} q(s_{1:\ell},n) > 2^k \\ q(s,n) & \text{otherwise.} \end{cases}$$

By construction, $q^{(k)}$ is recursive uniformly in $k \in \mathbb{N}$ and non-decreasing in its second argument because q is recursive and non-decreasing in its second argument. For any $k \in \mathbb{N}$,

$$q^{(k)}(s,\bullet) \nearrow \begin{cases} 2^k & \text{if } \max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell}) > 2^k \\ T(s) & \text{otherwise,} \end{cases} = T^{(k)}(s),$$

with $T^{(k)}$ as defined in Lemma 10.10, and hence, $T^{(k)}$ is lower semicomputable uniformly in $k \in \mathbb{N}$. It is now immediate from Lemma 10.11 that T' is lower semicomputable as an infinite (weighted) sum of uniformly lower semicomputable non-negative maps $2^{-k}T^{(k)}$. We conclude that $T' \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$.

In a second step, we just observe that by Lemma 10.1276 there is a lower semicomputable positive strict test supermartingale T^* for φ such that $\lim_{n\to\infty} T^*(\omega_{1:n}) = \frac{1}{2}\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$. □

Lemma 10.10. *Consider any forecasting system φ and any (positive) test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$. Let $T^{(k)}\colon \mathbb{S} \to \mathbb{R}$ be defined as*

$$T^{(k)}(s) = \begin{cases} 2^k & \text{if } \max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell}) > 2^k \\ T(s) & \text{otherwise,} \end{cases} \quad \text{for all } k \in \mathbb{N} \text{ and } s \in \mathbb{S},$$

and let $T'\colon \mathbb{S} \to \mathbb{R}$ be defined as $T'(s) := \sum_{k=1}^{\infty} 2^{-k}T^{(k)}(s)$ for all $s \in \mathbb{S}$. Then T' is a (positive) test supermartingale for φ, and $\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$ for every path $\omega \in \Omega$ for which $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$.

Proof. We start by showing that $T^{(k)} \in \overline{\mathbb{T}}(\varphi)$ for every $k \in \mathbb{N}$. To this end, observe that $T^{(k)}$ is non-negative because T is non-negative, and $T^{(k)}(\square) = 1$ because $T(\square) = 1$; observe that $T^{(k)}$ is positive if T is positive. Furthermore, for every $s \in \mathbb{S}$, since $\overline{E}_{\varphi(s)}(\Delta T(s)) \leq 0$, it will follow that also $\overline{E}_{\varphi(s)}(\Delta T^{(k)}(s)) \leq 0$. To prove this, we consider two cases. The first case is that $T^{(k)}(s) = 2^k$. Since $T^{(k)}(sx) \leq 2^k$ for all $x \in \mathscr{X}$, it follows that $\Delta T^{(k)}(s) \leq 0$, and as a consequence, $\overline{E}_{\varphi(s)}(\Delta T^{(k)}(s)) \leq 0$ due to C120. The second case is that $T^{(k)}(s) = T(s) \neq 2^k$. This means that $\max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell}) \leq 2^k$. Then for all $x \in \mathscr{X}$, it follows that $T^{(k)}(sx) = T(sx)$ if $T(sx) \leq 2^k$ and $T^{(k)}(sx) = 2^k < T(sx)$ otherwise, and hence, in both cases: $T^{(k)}(sx) \leq T(sx)$. Since $T^{(k)}(s) = T(s)$, this implies that $\Delta T^{(k)}(s) \leq \Delta T(s)$, and therefore that $\overline{E}_{\varphi(s)}(\Delta T^{(k)}(s)) \leq \overline{E}_{\varphi(s)}(\Delta T(s)) \leq 0$ due to C520 and since T is a supermartingale for φ. In summary, $T^{(k)}$ is a (positive) test supermartingale for φ.

We continue by showing that $T' \in \overline{\mathbb{T}}(\varphi)$. Since $T^{(k)}(s)$ is non-negative and bounded above by $\max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell})$ for every $k \in \mathbb{N}$ and $s \in \mathbb{S}$, it follows that, for every $s \in \mathbb{S}$, $T'(s)$ is non-negative and bounded above by $\max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell})$, so $T'(s)$ is a well-defined and non-negative real number for every $s \in \mathbb{S}$; note that T' is positive if T is positive. Since $T^{(k)}(\square) = 1$ for all $k \in \mathbb{N}$, it follows that $T'(\square) = 1$. To prove the supermartingale property, we start by observing that, since $T'(s) = \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k}T^{(k)}(s)$ is well-defined and real for every $s \in \mathbb{S}$, it follows that for every $s \in \mathbb{S}$ and $x \in \mathscr{X}$:

$$\begin{aligned}\Delta T'(s)(x) = T'(sx) - T'(s) &= \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k}T^{(k)}(sx) - \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k}T^{(k)}(s) \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} 2^{-k}\Delta T^{(k)}(s)\end{aligned}$$

is well-defined and real. Since $\mathscr{X}$ is finite, this implies that $\{\sum_{k=1}^{n} 2^{-k}\Delta T^{(k)}(s)\}_{n\in\mathbb{N}_0}$ converges uniformly to $\Delta T'(s)$. Hence, for every $s \in \mathbb{S}$, since the upper expectation

$\overline{E}_{\varphi(s)}$ is continuous with respect to uniform convergence [C6[20]], it follows that

$$\begin{aligned}\overline{E}_{\varphi(s)}(\Delta T'(s)) &= \overline{E}_{\varphi(s)}\Big(\lim_{n\to\infty}\sum_{k=1}^{n} 2^{-k}\Delta T^{(k)}(s)\Big)\\ &\overset{\text{C6}_{20}}{=} \lim_{n\to\infty}\overline{E}_{\varphi(s)}\Big(\sum_{k=1}^{n} 2^{-k}\Delta T^{(k)}(s)\Big)\\ &\overset{\text{C2}_{20},\text{C3}_{20}}{\leq} \limsup_{n\to\infty}\sum_{k=1}^{n} 2^{-k}\overline{E}_{\varphi(s)}(\Delta T^{(k)}(s)) \leq 0,\end{aligned}$$

where the last inequality follows from the fact that $\overline{E}_{\varphi(s)}(\Delta T^{(k)}(s)) \leq 0$ for all $k \in \mathbb{N}$. We conclude that $T' \in \overline{\mathbb{T}}(\varphi)$.

Fix any $\omega \in \Omega$ for which $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. We finish this proof by showing that also $\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$. We'll do so by showing that for every $N \in \mathbb{N}_0$ there's a $K \in \mathbb{N}_0$ such that $T'(\omega_{1:n}) \geq N$ for all $n \geq K$. To this end, fix any $N \in \mathbb{N}_0$. Since $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$, there's a $K \in \mathbb{N}_0$ such that $T(\omega_{1:K}) > 2^N$, and hence, $T^{(k)}(\omega_{1:n}) = 2^k$ for all $k \in \{1,\dots,N\}$ and $n \geq K$. This implies that for all $n \geq K$, it holds that

$$T'(\omega_{1:n}) = \sum_{k\in\mathbb{N}} 2^{-k} T^{(k)}(\omega_{1:n}) \geq \sum_{k=1}^{N} 2^{-k} T^{(k)}(\omega_{1:n}) = N,$$

where the inequality holds because $T^{(k)} \geq 0$ for all $k \in \mathbb{N}$. □

Lemma 10.11. *Consider any non-negative lower semicomputable extended real map $r\colon \mathbb{N}\times\mathcal{D} \to [0,\infty]$. Then the map $r'\colon \mathcal{D} \to [0,\infty]\colon d \mapsto \sum_{k=1}^{\infty} r(k,d)$ is lower semicomputable.*

Proof. Start from a (recursive) code $q\colon \mathbb{N}\times\mathcal{D}\times\mathbb{N} \to \mathbb{Q}$ for the lower semicomputable map r, so $q(k,d,\bullet) \nearrow r(k,d)$ for all $(k,d) \in \mathbb{N}\times\mathcal{D}$; we can assume without loss of generality that q is non-negative everywhere [if it isn't, replace it by $\max\{q,0\}$]. Let the map $q'\colon \mathcal{D}\times\mathbb{N} \to \mathbb{Q}$ be defined as

$$q'(d,n) := \sum_{k=1}^{n} q(k,d,n) \text{ for all } d \in \mathcal{D} \text{ and } n \in \mathbb{N}.$$

This map is obviously rational and recursive because q is. It's non-decreasing in its second argument, because

$$q'(d,n+1) = \sum_{k=1}^{n+1} q(k,d,n+1) \geq \sum_{k=1}^{n+1} q(k,d,n) \geq \sum_{k=1}^{n} q(k,d,n) = q(d,n) \quad \text{for all } d \in \mathcal{D} \text{ and } n \in \mathbb{N}.$$

To conclude that the map r' is lower semicomputable, we'll show that $\lim_{n\to\infty} q'(d,n) = r'(d)$. To this end, fix any $d \in \mathcal{D}$. We start by observing that

$$\lim_{n\to\infty} q'(d,n) = \lim_{n\to\infty} \sum_{k=1}^{n} q(k,d,n) \leq \sum_{k=1}^{\infty} r(k,d) = r'(d).$$

There are now two distinct possibilities.

The first possibility is that $r'(d) = \infty$. Then, for every $N \in \mathbb{N}$ there's some $M \in \mathbb{N}$ such that $\sum_{k=1}^{M} r(k,d) \geq N+1$. Moreover, it follows from the assumptions that for every $k \in \{1,\dots,M\}$, there's some N_k such that $r(k,d) - \frac{1}{M} < q(k,d,N_k)$. If we now let $\overline{N} := \max\{M, N_1, \dots, N_M\}$, then

$$\begin{aligned}\lim_{n\to\infty} q'(d,n) \geq q'(d,\overline{N}) &= \sum_{k=1}^{\overline{N}} q(k,d,\overline{N}) \geq \sum_{k=1}^{M} q(k,d,\overline{N}) \geq \sum_{k=1}^{M} q(k,d,N_k) \\ &\geq \sum_{k=1}^{M} \left(r(k,d) - \frac{1}{M} \right) \geq N+1 - M\frac{1}{M} = N,\end{aligned}$$

and therefore $\lim_{n\to\infty} q'(d,n) = \infty = r'(d)$.

The second possibility is that $r'(d) \in \mathbb{R}$, which implies that $r(k,d) \in \mathbb{R}$ for all $k \in \mathbb{N}$. Then, for every $\epsilon > 0$ there's some $M \in \mathbb{N}$ such that $\sum_{k=1}^{M} r(k,d) \geq r'(d) - \frac{\epsilon}{2}$. Moreover, it follows from the assumptions that for every $k \in \{1,\dots,M\}$, there's some N_k such that $r(k,d) - \frac{\epsilon}{2}\frac{1}{M} < q(k,d,N_k)$. If we now let $\overline{N} := \max\{M, N_1, \dots, N_M\}$, then

$$\begin{aligned}\lim_{n\to\infty} q'(d,n) \geq q'(d,\overline{N}) &= \sum_{k=1}^{\overline{N}} q(k,d,\overline{N}) \geq \sum_{k=1}^{M} q(k,d,\overline{N}) \geq \sum_{k=1}^{M} q(k,d,N_k) \\ &\geq \sum_{k=1}^{M} \left(r(k,d) - \frac{\epsilon}{2}\frac{1}{M} \right) \geq r'(d) - \frac{\epsilon}{2} - M\frac{\epsilon}{2}\frac{1}{M} = r'(d) - \epsilon,\end{aligned}$$

and therefore $\lim_{n\to\infty} q'(d,n) = r'(d)$. □

Lemma 10.12. *Consider any forecasting system φ. For every lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ there is a lower semicomputable positive strict test supermartingale $T' \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that*

$$\liminf_{n\to\infty} T'(\omega_{1:n}) = \frac{1}{2} \liminf_{n\to\infty} T(\omega_{1:n})$$

and

$$\limsup_{n\to\infty} T'(\omega_{1:n}) = \frac{1}{2} \limsup_{n\to\infty} T(\omega_{1:n}).$$

Proof. Consider any lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ and let the real process $T' \colon \mathbb{S} \to \mathbb{R}$ be defined by $T'(s) := (T(s)+2^{-|s|})/2$ for all $s \in \mathbb{S}$. Clearly, $\liminf_{n\to\infty} T'(\omega_{1:n}) = \frac{1}{2}\liminf_{n\to\infty} T(\omega_{1:n})$ and $\limsup_{n\to\infty} T'(\omega_{1:n}) = \frac{1}{2}\limsup_{n\to\infty} T(\omega_{1:n})$, so we're done if we can show that T' is a lower semicomputable positive strict test supermartingale for φ. It's immediate that $T'(\square) = (1+1)/2 = 1$ and $T'(s) \geq 2^{-|s|}/2 > 0$ for all $s \in \mathbb{S}$, and T' is lower semicomputable because T is. Furthermore, for all $s \in \mathbb{S}$,

$$\begin{aligned}\overline{E}_{\varphi(s)}(T'(s\,\cdot)) &\overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{\overline{E}_{\varphi(s)}(T(s\,\cdot)) + 2^{-|s|-1}}{2} \\ &\leq \frac{T(s) + 2^{-|s|-1}}{2} < \frac{T(s) + 2^{-|s|}}{2} = T'(s),\end{aligned}$$

where the first inequality holds because $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$. □

As a consequence, whenever we restrict Sceptic's allowed betting strategies to a set that's smaller than the one in Proposition 10.8$_{72}$, but larger than the one in Proposition 10.9$_{73}$, and whenever we require the corresponding betting strategies not to converge to infinity instead of being bounded, we obtain a definition for ML-random sequences that is equivalent to Definition 8.5$_{52}$; similar remarks will hold for the other three randomness notions that we study below.

When restricting our attention to non-degenerate computable forecasting systems $\varphi \in \Phi(\mathscr{X})$, then ML-randomness allows for yet another interesting characterisation; as Corollary 14.24$_{129}$ shows,[21] and similarly to the precise-probabilistic version of ML-randomness [30, 32], there exists a so-called *universal* lower semicomputable test supermartingale $U \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ that conclusively tests the ML-randomness of any path $\omega \in \Omega$ for the forecasting system φ: ω is ML-random for φ if and only if $\lim_{n\to\infty} U(\omega_{1:n}) \neq \infty$. From Lemma 10.12$_{\curvearrowleft}$, it's furthermore clear that we can assume this universal lower semicomputable test supermartingale to be a positive strict test supermartingale, and hence, in this way, when restricting our attention to non-degenerate computable forecasting systems, we obtain an equivalent characterisation of ML-randomness that imposes even more restrictions on Sceptic's allowed betting strategies than in Proposition 10.9$_{73}$.

Weak Martin-Löf randomness

For the notion of wML-randomness, we can extend Sceptic's set of betting strategies by imposing non-negativity instead of positivity, and by not requiring satisfaction of the supermartingale multiplier property.

Proposition 10.13. *A path $\omega \in \Omega$ is* wML-*random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no lower semicomputable multiplier process D that generates a test supermartingale $D^{\circledcirc}$ for φ such that* $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.

Proof. It clearly suffices to prove the 'only if'-part. To this end, assume the existence of a lower semicomputable multiplier process D that generates a test supermartingale $D^{\circledcirc}$ for φ such that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. Then, by Definition 8.5$_{52}$ and Proposition 6.6$_{30}$, we have to prove that there's some lower semicomputable positive supermartingale multiplier D' for φ such that $\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) = \infty$, which is immediate from Lemma 9.17$_{64}$. □

We can also strengthen the conditions on the betting strategies Sceptic can choose from without changing the randomness notion. It turns out we can impose satisfaction of the strict supermartingale (multiplier) property.

[21] This forward reference could be perceived as awkward by some readers. However, all readers can rest assured that this paragraph only discusses this corollary, and isn't used further on in any proof.

Proposition 10.14. *A path $\omega \in \Omega$ is* wML*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no lower semicomputable positive strict supermartingale multiplier D for φ that generates a positive strict test supermartingale $D^{\circledcirc}$ for φ such that* $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.

Proof. It clearly suffices to prove the 'if'-part. To this end, assume that ω isn't wML-random for φ, so, by Definition 8.5₅₂ and Proposition 6.6₃₀, assume the existence of a lower semicomputable positive supermartingale multiplier D for φ such that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. Then, by Proposition 6.6₃₀, we have to prove that there's some lower semicomputable positive strict supermartingale multiplier D' for φ such that $\limsup_{n\to\infty} D'^{\circledcirc}(\omega_{1:n}) = \infty$, which is immediate from Lemma 9.9₅₉. □

Computable randomness

Analogously to ML-randomness, C-randomness is preserved when considering betting strategies that are bounded below, but that aren't necessarily non-negative, and neither necessarily start with unit capital.

Proposition 10.15. *A path $\omega \in \Omega$ is* C*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no computable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ such that* $\limsup_{n\to\infty} M(\omega_{1:n}) = \infty$.

Proof. It clearly suffices to prove the 'only if'-part. To this end, assume the existence of a computable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ such that $\limsup_{n\to\infty} M(\omega_{1:n}) = \infty$. Then we'll show that ω isn't C-random for φ. Since M is bounded below, there's a natural $B \in \mathbb{N}$ such that $M + B > 0$. Let $A \in \mathbb{N}$ be such that $(M(\square)+B)/A \leq 1$, and let the real process $T\colon \mathbb{S} \to \mathbb{R}$ be defined by $T(\square) := 1$ and $T(s) := (M(s)+B)/A$ for all $s \in \mathbb{S} \setminus \{\square\}$. Clearly, $\limsup_{n\to\infty} T(\omega_{1:n}) = \frac{1}{A}\limsup_{n\to\infty} M(\omega_{1:n}) + \frac{B}{A} = \infty$, so we're done if we can show that $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$. Clearly, $T(\square) = 1$, $T \geq (M+B)/A > 0$ and T is computable. Furthermore, for all $s \in \mathbb{S}$,

$$\overline{E}_{\varphi(s)}(T(s\cdot)) \overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{A}\overline{E}_{\varphi(s)}(M(s\cdot)) + \frac{B}{A} \leq \frac{M(s)+B}{A} \leq \begin{cases} 1 & \text{if } s = \square \\ T(s) & \text{otherwise} \end{cases} = T(s),$$

so we conclude that $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$. □

On the other hand, similarly to ML- and wML-randomness, we can require positivity and satisfaction of the strict supermartingale property; note that positivity is already part of our standard definition of C-randomness. Furthermore, in contrast with ML- and wML-randomness, we can impose recursiveness and rationality on the betting strategies without changing the notion of C-randomness. Moreover, as with ML-randomness, we can (also) require Sceptic's running capital never to converge to infinity, instead of never to be unbounded.

Proposition 10.16. *A path $\omega \in \Omega$ is* C*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's no recursive positive rational strict test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ such that* $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$.

Proof. By Definition 8.5₅₂, it clearly suffices to prove the 'if'-part. To this end, assume that ω isn't C-random for φ, so assume the existence of a computable positive test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ such that $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. Then we have to prove that there is a recursive positive rational strict test supermartingale $T^* \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ such that $\lim_{n\to\infty} T^*(\omega_{1:n}) = \infty$. By applying Lemma 10.17 to T, we know there's a recursive positive rational strict test supermartingale $T' \in \overline{\mathbb{T}}(\varphi)$ such that $|7T'(s) - T(s)| \leq 7 \cdot 2^{-|s|}$ for all $s \in \mathbb{S}$. Since $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$, it follows that $\limsup_{n\to\infty} T'(\omega_{1:n}) = \infty$. By applying Lemma 10.20↷ to T', it therefore follows that there's a computable positive test supermartingale $T'' \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ such that $\lim_{n\to\infty} T''(\omega_{1:n}) = \infty$. By again applying Lemma 10.17, this time to T'', we know there's a recursive positive rational strict test supermartingale $T^* \in \overline{\mathbb{T}}(\varphi)$ such that $|7T^*(s) - T''(s)| \leq 7$ for all $s \in \mathbb{S}$; since recursiveness implies computability, it's immediate that $T^* \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$. Since $\lim_{n\to\infty} T''(\omega_{1:n}) = \infty$, it follows that, indeed, $\lim_{n\to\infty} T^*(\omega_{1:n}) = \infty$. □

Lemma 10.17. *Fix any forecasting system $\varphi \in \Phi(\mathscr{X})$. For every computable test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$, there's a recursive positive rational strict test supermartingale $T' \in \overline{\mathbb{T}}(\varphi)$ such that $|7T'(s) - T(s)| \leq 7 \cdot 2^{-|s|}$ for all $s \in \mathbb{S}$.*

Proof. Consider any computable test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$. Since T is computable, there's some recursive rational map $q\colon \mathbb{S} \times \mathbb{N} \to \mathbb{Q}$ such that

$$|T(s) - q(s,N)| \leq 2^{-N} \quad \text{for all } s \in \mathbb{S} \text{ and } N \in \mathbb{N}. \tag{10.18}$$

Observe that, since $T(\square) = 1$, we can assume without loss of generality that $q(\square, 1) = 1$. Define the rational process $T'\colon \mathbb{S} \to \mathbb{Q}$ by letting

$$T'(s) = \frac{q(s,|s|+1) + 6 \cdot 2^{-|s|}}{7} \quad \text{for all } s \in \mathbb{S}.$$

Since the maps $|\bullet|$ and q are recursive, so is the rational process T'. Furthermore, it follows from Eq. (10.18) that

$$\left.\begin{aligned} q(sx,|sx|+1) &\leq T(sx) + \frac{1}{4} \cdot 2^{-|s|} \\ T(s) &\leq q(s,|s|+1) + \frac{1}{2} 2^{-|s|} \end{aligned}\right\} \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}. \tag{10.19}$$

Moreover, $T'(\square) = \frac{q(\square,1)+6}{7} = 1$, and the bottom inequality in Eq. (10.19) guarantees that T' is positive:

$$T'(s) = \frac{q(s,|s|+1) + 6 \cdot 2^{-|s|}}{7} \geq \frac{T(s) + 5 \cdot 2^{-|s|}}{7} \geq \frac{5 \cdot 2^{-|s|}}{7} > 0 \text{ for all } s \in \mathbb{S}.$$

Next, we show that T' is a strict supermartingale for φ. By combining the inequalities in Eq. (10.19), we find that for all $s \in \mathbb{S}$,

$$q(s\cdot,|s\cdot|+1) - q(s,|s|+1) \leq T(s\cdot) - T(s) + \frac{3}{4} \cdot 2^{-|s|},$$

and therefore also,

$$\Delta T'(s) = T'(s\cdot) - T'(s) = \frac{q(s\cdot,|s\cdot|+1) + 6 \cdot 2^{-|s\cdot|}}{7} - \frac{q(s,|s|+1) + 6 \cdot 2^{-|s|}}{7}$$

$$= \frac{q(s\cdot,|s\cdot|+1) - q(s,|s|+1) - 3\cdot 2^{-|s|}}{7}$$
$$\leq \frac{T(s\cdot) - T(s) + \frac{3}{4}\cdot 2^{-|s|} - 3\cdot 2^{-|s|}}{7} = \frac{\Delta T(s)}{7} - \frac{9\cdot 2^{-|s|}}{28}.$$

This implies that, indeed, for all $s \in \mathbb{S}$,

$$\overline{E}_{\varphi(s)}(\Delta T'(s)) \overset{\text{C5}_{20}}{\leq} \overline{E}_{\varphi(s)}\Big(\frac{\Delta T(s)}{7} - \frac{9\cdot 2^{-|s|}}{28}\Big) \overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{7}\overline{E}_{\varphi(s)}(\Delta T(s)) - \frac{9\cdot 2^{-|s|}}{28} < 0,$$

where the last strict inequality follows from the supermartingale inequality $\overline{E}_{\varphi(s)}(\Delta T(s)) \leq 0$.

This shows that T' is a recursive positive rational strict test supermartingale for φ. For the rest of the proof, consider that, by Eq. (10.18)↶, indeed

$$|7T'(s) - T(s)| = \big|q(s,|s|+1) + 6\cdot 2^{-|s|} - T(s)\big|$$
$$\leq 6\cdot 2^{-|s|} + \big|q(s,|s|+1) - T(s)\big|$$
$$\leq 6\cdot 2^{-|s|} + \frac{1}{2}2^{-|s|} \leq 7\cdot 2^{-|s|} \text{ for all } s \in \mathbb{S}. \qquad \square$$

Lemma 10.20. *Fix any forecasting system $\varphi \in \Phi(\mathscr{X})$. For every recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$ there's a computable positive test supermartingale $T' \in \overline{\mathbb{T}}(\varphi)$ such that $\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$ for every path $\omega \in \Omega$ for which $\limsup_{n\to\mathbb{N}_0} T(\omega_{1:n}) = \infty$.*

Proof. Consider any recursive positive rational test supermartingale $T \in \overline{\mathbb{T}}(\varphi)$. Let T' and $T^{(k)}$, with $k \in \mathbb{N}$, be defined as in Lemma 10.10₇₄. Then T' is a positive test supermartingale for φ, and $\lim_{n\to\infty} T'(\omega_{1:n}) = \infty$ for every path $\omega \in \Omega$ for which $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$.

We're done if we can show that T' is computable. To this end, observe that $T^{(k)}$ is rational and recursive uniformly in $k \in \mathbb{N}$ since T is rational and recursive. We now define the recursive map of rational numbers $q\colon \mathbb{S}\times\mathbb{N} \to \mathbb{Q}$ by

$$q(s,n) := \sum_{k=1}^{n} 2^{-k} T^{(k)}(s) \text{ for all } s \in \mathbb{S} \text{ and } n \in \mathbb{N}.$$

Next, we consider the recursive rational map $e\colon \mathbb{S}\times\mathbb{N} \to \mathbb{N}$ defined by

$$e(s,N) := N + \max_{\ell\in\{0,\dots,|s|\}} \lceil T(s_{1:\ell}) \rceil \text{ for all } N \in \mathbb{N} \text{ and } s \in \mathbb{S}.$$

Then for all $n, N \in \mathbb{N}$ and $s \in \mathbb{S}$, $n \geq e(s,N)$ implies that

$$\big|T'(s) - q(s,n)\big| = \Big|T'(s) - \sum_{k=1}^{n} 2^{-k} T^{(k)}(s)\Big|$$
$$= \sum_{k=n+1}^{\infty} 2^{-k} T^{(k)}(s)$$
$$\leq \sum_{k=n+1}^{\infty} 2^{-k} \max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell})$$

$$
\begin{aligned}
&= 2^{-n} \max_{\ell\in\{0,\dots,|s|\}} T(s_{1:\ell}) \\
&\leq 2^{-n} \max_{\ell\in\{0,\dots,|s|\}} \lceil T(s_{1:\ell}) \rceil \\
&\leq 2^{-e(s,N)} \max_{\ell\in\{0,\dots,|s|\}} \lceil T(s_{1:\ell}) \rceil \\
&= 2^{-N} \frac{\max_{\ell\in\{0,\dots,|s|\}} \lceil T(s_{1:\ell}) \rceil}{2^{\max_{\ell\in\{0,\dots,|s|\}} \lceil T(s_{1:\ell}) \rceil}} \leq 2^{-N},
\end{aligned}
$$

where the last inequality follows from the fact that $2^y \geq y$ if $y \geq 0$. Hence, T' is computable. □

Schnorr randomness

Similarly to C-randomness, we can replace the set of allowable betting strategies $\overline{\mathbb{T}}_S(\varphi)$ by the set of computable bounded below supermartingales for φ, without changing the set of S-random paths for φ. Moreover, for a path $\omega \in \Omega$ to be S-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, no such betting strategy M for φ should allow Sceptic to get arbitrarily rich at some computable rate—*up to some constant*—, as is specified below by writing that there should be no real growth function τ such that $\limsup_{n\to\infty}[M(\omega_{1:n}) - \tau(n)] > -\infty$.

Proposition 10.21. *A path $\omega \in \Omega$ is* S*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there are no computable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ and real growth function τ such that* $\limsup_{n\to\infty}[M(\omega_{1:n}) - \tau(n)] > -\infty$.

Proof. It clearly suffices to prove the 'only if'-part. To this end, assume the existence of a computable bounded below supermartingale $M \in \overline{\mathbb{M}}(\varphi)$ and a real growth function τ such that $\limsup_{n\to\infty}[M(\omega_{1:n}) - \tau(n)] > -\infty$. Then we have to show that ω isn't S-random for φ. Since M is bounded below, there's a natural $B \in \mathbb{N}$ such that $M + B > 0$ and $\limsup_{n\to\infty}[M(\omega_{1:n}) - \tau(n)] + B > 0$. Let $A \in \mathbb{N}$ be such that $(M(\square)+B)/A \leq 1$, let the real process $T\colon \mathbb{S} \to \mathbb{R}$ be defined by $T(\square) := 1$ and $T(s) := (M(s)+B)/A$ for all $s \in \mathbb{S} \setminus \{\square\}$, and let the real growth function τ' be defined by $\tau' = \tau/A$. Clearly,

$$
\limsup_{n\to\infty}\big[T(\omega_{1:n}) - \tau'(n)\big] = \frac{1}{A}\Big(\limsup_{n\to\infty}[M(\omega_{1:n}) - \tau(n)] + B\Big) > 0,
$$

so we're done if we can show that $T \in \overline{\mathbb{T}}_S(\varphi)$, because then there's a computable positive test supermartingale for φ that's computably unbounded on ω. Clearly, $T(\square) = 1$, $T \geq (M+B)/A > 0$ and T is computable. Furthermore, for all $s \in \mathbb{S}$,

$$
\overline{E}_{\varphi(s)}(T(s\cdot)) \overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{A}\overline{E}_{\varphi(s)}(M(s\cdot)) + \frac{B}{A} \leq \frac{M(s)+B}{A} \leq \begin{cases} 1 & \text{if } s = \square \\ T(s) & \text{otherwise} \end{cases} = T(s),
$$

so we conclude that $T \in \overline{\mathbb{T}}_S(\varphi)$. □

Again analogously to C-randomness, we can replace the set of betting strategies $\overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ for φ by recursive positive rational strict test supermartingales for φ. Moreover, we can require the real growth function to be natural-valued; a path $\omega \in \Omega$ is then S-random for φ if and only if there's no such test supermartingale T for φ that makes Sceptic infinitely rich with respect to some natural growth function, in the sense that there should be no natural growth function $\eta\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\limsup_{n\to\infty}\big[T(\omega_{1:n}) - \eta(n)\big] = \infty$.

Proposition 10.22. *A path $\omega \in \Omega$ is* S*-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there are no recursive positive rational strict test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ and* natural *growth function $\eta\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that* $\limsup_{n\to\infty}\big[T(\omega_{1:n}) - \eta(n)\big] = \infty$.

Proof. By Definition 8.5[52], it clearly suffices to prove the 'if'-part. To this end, assume that ω isn't S-random for φ, so assume the existence of a computable positive test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ and a real growth function τ such that $\limsup_{n\to\infty}[T(\omega_{1:n}) - \tau(n)] \geq 0$. Then we have to prove that there is a recursive positive rational strict test supermartingale $T' \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ and a natural growth function $\eta\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\limsup_{n\to\infty}\big[T'(\omega_{1:n}) - \eta(n)\big] = \infty$. By applying Lemma 10.17[79] to T, we know there's a recursive positive rational strict test supermartingale $T' \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ such that $|7T'(s) - T(s)| \leq 7\cdot 2^{-|s|}$ for all $s \in \mathbb{S}$. Since τ is computable, there's a recursive rational map $q\colon \mathbb{N}_0 \times \mathbb{N} \to \mathbb{Q}$ such that $|\tau(n) - q(n,N)| \leq 2^{-N}$ for all $N \in \mathbb{N}$. Let the natural map $\eta\colon \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\eta(0) := 0$ and

$$\eta(n+1) := \max\left\{\eta(n), \left\lfloor \frac{q(n+1,1)}{14} - 1 \right\rfloor\right\} \text{ for all } n \in \mathbb{N}_0.$$

This map is non-decreasing by definition, recursive because q is, and unbounded because

$$\lim_{n\to\infty} \eta(n) \geq \limsup_{n\to\infty} \frac{q(n,1)}{14} - 2 \geq \lim_{n\to\infty} \frac{\tau(n) - 1/2}{14} - 2 = \infty,$$

so we conclude that η is a natural growth function.

We're done now if we can show that $\limsup_{n\to\infty}\big[T'(\omega_{1:n}) - \eta(n)\big] = \infty$. To this end, observe that for all $n \in \mathbb{N}_0$:

$$\begin{aligned}
\eta(n+1) &= \max_{k\in\{1,\dots,n+1\}} \max\left\{0, \left\lfloor \frac{q(k,1)}{14} - 1 \right\rfloor\right\} \\
&\leq \max_{k\in\{1,\dots,n+1\}} \max\left\{0, \left\lfloor \frac{\tau(k) + 1/2}{14} - 1 \right\rfloor\right\} \\
&\leq \max_{k\in\{1,\dots,n+1\}} \max\left\{0, \left\lfloor \frac{\tau(k)}{14} \right\rfloor\right\} \\
&\leq \left\lfloor \frac{\tau(n+1)}{14} \right\rfloor \leq \frac{\tau(n+1)}{14},
\end{aligned} \tag{10.23}$$

where the third inequality holds by the non-decreasingness and non-negativity of τ. By recalling that $\limsup_{n\to\infty}[T(\omega_{1:n}) - \tau(n)] \geq 0$, we know there's a strictly increasing sequence of naturals $(n_i)_{i\in\mathbb{N}_0}$ such that $T(\omega_{1:n_i}) - \tau(n_i) \geq -1$ for all $i \in \mathbb{N}_0$.

Consequently,

$$\begin{aligned}
\limsup_{n\to\infty}\left[T'(\omega_{1:n})-\eta(n)\right] &\geq \limsup_{n\to\infty}\left[\frac{T(\omega_{1:n})}{7}-2^{-n}-\eta(n)\right]\\
&= \limsup_{n\to\infty}\left[\frac{T(\omega_{1:n})}{7}-\eta(n)\right]\\
&\overset{\text{Eq. (10.23)}}{\geq} \limsup_{n\to\infty}\left[\frac{T(\omega_{1:n})}{7}-\frac{\tau(n)}{14}\right]\\
&= \limsup_{n\to\infty}\left[\frac{T(\omega_{1:n})-\tau(n)}{7}+\frac{\tau(n)}{14}\right]\\
&\geq \limsup_{i\to\infty}\left[\frac{T(\omega_{1:n_i})-\tau(n_i)}{7}+\frac{\tau(n_i)}{14}\right]\\
&\geq -\frac{1}{7}+\limsup_{i\to\infty}\frac{\tau(n_i)}{14}=\infty.
\end{aligned}$$

□

Frequentist notions of randomness

What sequences do we consider to be random for a forecasting system $\varphi \in \Phi(\mathscr{X})$? In this chapter, we formally address and answer this question a second time by introducing a very general (imprecise-probabilistic) randomness notion that has a frequentist flavour. As a starting point for introducing this randomness notion, we'll consider von Mises' classical (precise-probabilistic) definition for randomness that appeared in Section 1_{2}: a path $\omega \in \Omega$ is *random* for a probability mass function $m \in \mathcal{M}(\mathscr{X})$ if

(i) the relative frequencies of every outcome $x \in \mathscr{X}$ along ω converge to the probability $m(x)$, that is, $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_x(\omega_k) = m(x)$ for all $x \in \mathscr{X}$;

(ii) every infinite subsequence from ω selected by an *admissible* selection process $S \in \mathcal{S}$ satisfies (i),

where it will be made precise further on in Section 11.1_{87} what it means for a selection process S to select an infinite subsequence from a path ω. In the above definition, we cannot allow every selection process $S \in \mathcal{S}$ to be admissible, because then there would for example be no binary path that's random for $1/2$ [25, Section 2]. Von Mises, however, left open what selection processes should be considered admissible.

As proven by Abraham Wald [23], random paths do exist when considering a *countable set of admissible selection processes* $\mathcal{S}^\infty \subseteq \mathcal{S}$; the corresponding precise-probabilistic randomness notion is called $\mathcal{S}^\infty$-randomness. In particular, if we let $\mathcal{S}^\infty$ be the countable set of computable or total computable selection processes—where the modifier 'total' will be explained further on in Section 11.1_{87}—, then we obtain the classical precise-probabilistic randomness notions known as *Church stochasticity* [27] and *weak Church* *stochasticity* [12], respectively.

To start lifting these randomness notions to an imprecise-probabilistic context, we'll make use of the following observation: condition (i) can be replaced by any of the two criteria below without changing the randomness notion.

(i') $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\omega_k) = E_m(f)$ for all $f \in \mathscr{L}(\mathscr{X})$

and

(i") $\lim_{n\to\infty} d\big(\frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\omega_k}, \{m\}\big) = 0$.

Proof. Fix any path $\omega \in \Omega$ and any probability mass function $m \in \mathcal{M}(\mathscr{X})$.

For (i)⇒(i'), observe that for any $f \in \mathscr{L}(\mathscr{X})$, since $f = \sum_{x\in\mathscr{X}} f(x)\mathbb{I}_x$:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(\omega_k) = \sum_{x\in\mathscr{X}} f(x) \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_x(\omega_k) = \sum_{x\in\mathscr{X}} f(x)m(x) = E_m(f).$$

For (i')⇒(i), simply let $f := \mathbb{I}_x$ for every $x \in \mathscr{X}$.

For (i)⇔(i"), note that $\mathbb{I}_x(y) = \mathbb{I}_y(x)$ for all $x, y \in \mathscr{X}$, and hence,

$$\begin{aligned}(\forall x \in \mathscr{X}) \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_x(\omega_k) = m(x) &\Leftrightarrow (\forall x \in \mathscr{X}) \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\omega_k}(x) = m(x)\\ &\Leftrightarrow \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\omega_k} = m\\ &\Leftrightarrow \lim_{n\to\infty} d\left(\frac{1}{n}\sum_{k=1}^{n} \mathbb{I}_{\omega_k}, \{m\}\right) = 0.\end{aligned}$$ □

Condition (i') requires that the running average of the outcomes along ω evaluated in a gamble $f \in \mathscr{L}(\mathscr{X})$ converges to the linear expectation $E_m(f)$ that's associated with m and f, whereas condition (i")—similarly to (i)—requires that the relative frequencies of the occurrence of outcomes along ω converge to the probability mass function m. It's these last two (equivalent) conditions that we'll examine and generalise in this chapter to obtain (equivalent) imprecise-probabilistic versions of $\mathcal{S}^\infty$-randomness, which we'll also call $\mathcal{S}^\infty$-randomness, and therefore of (weak) Church stochasticity as well.

This chapter is structured as follows. In Section 11↷, we formally introduce the imprecise-probabilistic notion of $\mathcal{S}^\infty$-randomness, and we do so by generalising the statements in (i') and (i"). We also discuss some of its properties, which are reminiscent of the results in Sections 9₅₄ and 10.1₆₇. Afterwards, in Section 12₁₀₂, we restrict our attention to Church and to weak Church stochasticity, and explain how these two frequentist randomness notions relate to the four aforementioned imprecise-probabilistic martingale-theoretic randomness notions. When restricting our attention to stationary forecasting systems, as we do in Section 12.3₁₀₆, we succeed in equipping (weak) Church stochasticity with an equivalent martingale-theoretic characterisation by considering a rather natural and simple class of implementable betting strategies.

11 $\mathcal{S}^\infty$-randomness

We start in Section 11.1 by generalising condition (i'), and thereby obtain the rather general imprecise-probabilistic frequentist randomness notion that we've been calling $\mathcal{S}^\infty$-randomness. In Section 11.292, we prove that this randomness notion has an equivalent characterisation in terms of a generalisation of condition (i"): a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if φ puts bounds on the relative frequencies of the outcomes along ω and some of its infinite subsequences. We end by discussing some properties of $\mathcal{S}^\infty$-randomness in Section 11.395.

11.1 (Equivalent) definition(s)

What does it mean for a selection process $S \in \mathcal{S}$ to select an (in)finite subsequence from a path $\omega \in \Omega$. For every $n \in \mathbb{N}_0$, the selection process S is said to *select* the entry ω_{n+1} along ω if $S(\omega_{1:n}) = 1$; when $S(\omega_{1:n}) = 0$, then S is said to merely scan or observe ω_{n+1}. We write $S(\omega)$ for the action of S along ω, that is, $S(\omega)$ is the sequence, finite or infinite, of all entries from ω selected by S in the order in which they appear along ω. In this way, S can be interpreted as a map $S\colon \Omega \to \Omega \cup \mathbb{S}$, which maps every path $\omega \in \Omega$ to a (possibly different) path $\omega' \in \Omega$ or a situation $s \in \mathbb{S}$. A selection process S is said to *accept* a path $\omega \in \Omega$ if $S(\omega) \in \Omega$, that is, if it *selects* an infinite subsequence of ω; an infinite subsequence ω' of ω is said to be *computably selectable* if there's some recursive selection process $S \in \mathcal{S}$ such that $S(\omega) = \omega'$. A selection process S is called *total* if it accepts all paths $\omega \in \Omega$, that is, if it maps every path $\omega \in \Omega$ to a possibly different path $\omega' \in \Omega$; it's called *partial* otherwise. An example of a total selection process is given by any temporal selection process $S \in \mathcal{S}$ for which $\sum_{k=0}^{\infty} S(k) = \infty$, with the identical selection process $S = 1$ of course being one of them. For any path $\omega \in \Omega$ and any countable set of selection processes $\mathcal{S}^\infty \subset \mathcal{S}$, we denote by $\mathcal{S}^\infty(\omega)$ the subset consisting of all selection processes $S \in \mathcal{S}^\infty$ that accept ω. In particular, for every path $\omega \in \Omega$, we collect the recursive selection processes that accept ω in the set $\mathcal{S}_{\mathrm{CH}}(\omega)$; the collection of all recursive selection processes is denoted by $\mathcal{S}_{\mathrm{CH}}$. Similarly, for every path $\omega \in \Omega$, we collect the recursive total selection processes that accept ω in the set $\mathcal{S}_{\mathrm{wCH}}(\omega)$, but we also just write $\mathcal{S}_{\mathrm{wCH}}$ since this set is path-independent.

To work towards an imprecise-probabilistic version of $\mathcal{S}^\infty$-randomness, observe that condition (ii), where (i') substitutes for (i), is equivalent to

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - E_m(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0 \text{ for all } f \in \mathscr{L}(\mathscr{X}) \text{ and } S \in \mathcal{S}^\infty(\omega). \tag{11.1}$$

Proof. It clearly suffices to prove the reverse implication. To this end, fix any path $\omega \in \Omega$, any probability mass function $m \in \mathcal{M}(\mathscr{X})$, any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω, and observe that

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - E_m(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\geq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - E_m(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&= -\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[-f(\omega_{k+1}) - E_m(-f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0,
\end{aligned}$$

where the first and last inequalities are immediate from Eq. (11.1)↶ by using the gambles f and $-f$, respectively. This, indeed, implies that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) f(\omega_{k+1})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} = E_m(f). \qquad \square$$

This criterium for $\mathcal{S}^\infty$-randomness in Eq. (11.1)↶ can be translated into a betting game which consists of a probability mass function $m \in \mathcal{M}(\mathscr{X})$, a single gamble $f \in \mathscr{L}(\mathscr{X})$ and a selection process $S \in \mathcal{S}^\infty$. Consider a sequential betting game as in Section 6.3[27]. There are again three players involved: Forecaster, Sceptic and Reality. Forecaster starts by specifying a probability mass function $m \in \mathcal{M}(\mathscr{X})$. Next, Sceptic chooses a gamble $f \in \mathscr{L}(\mathscr{X})$ and a selection process $S \in \mathcal{S}^\infty$, which determines for every situation $s \in \mathbb{S}$ whether she gambles $[S(s) = 1]$ or not $[S(s) = 0]$. By his specification of the probability mass function m, Forecaster is willing to sell the gamble f for his fair price $E_m(f)$, that is, he's willing to offer her the gamble $f - E_m(f)$. Afterwards, Reality reveals the successive outcomes $X_n = x_n$ at each successive *time instant* $n \in \mathbb{N}$, leading to the sequence $\omega = (x_1, \dots, x_n, \dots)$. At every time instant n, after Reality has revealed the outcome x_n, Sceptic plays the gamble $S(x_{1:n})[f(X_{n+1}) - E_m(f)]$, that is, she buys the gamble $f(X_{n+1})$ for Forecaster's fair price $E_m(f)$ if $S(x_{1:n}) = 1$, and refrains from betting otherwise $[S(x_{1:n}) = 0]$. Next, Reality reveals the subsequent outcome $X_{n+1} = x_{n+1} \in \mathscr{X}$ and the reward $S(x_{1:n})[f(x_{n+1}) - E_m(f)]$ goes to Sceptic. Now, Sceptic is said to have a winning strategy in this sequential betting game if she can come up with a gamble $f \in \mathscr{L}(\mathscr{X})$ and a selection process $S \in \mathcal{S}^\infty(\omega)$ such that she is guaranteed to have on average a (non-negligible) positive gain at arbitrarily large time instants by betting according to the above scheme, in the sense that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - E_m(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} > 0.$$

The path ω is then $\mathcal{S}^\infty$-random for φ if Sceptic has no such winning betting strategy, that is, if Eq. (11.1)↶ is satisfied.

We're now but one step away from an imprecise-probabilistic version of $\mathcal{S}^\infty$-randomness. Let's reconsider the above betting game. This time, Forecaster is allowed to start the betting game by specifying a forecasting system $\varphi \in \Phi(\mathscr{X})$ rather than a probability mass function $m \in \mathcal{M}(\mathscr{X})$. Next, Sceptic chooses a gamble $f \in \mathscr{L}(\mathscr{X})$ and a selection process $S \in \mathcal{S}^\infty$. In every situation $s \in \mathbb{S}$, by Forecaster's specification of the forecasting system φ, Sceptic is allowed to buy the gamble f for his minimum acceptable selling price $\overline{E}_{\varphi(s)}(f)$, so he's willing to offer her the gamble $f - \overline{E}_{\varphi(s)}(f)$ in s. Afterwards, Reality reveals the successive outcomes $X_n = x_n$ at each successive *time instant* $n \in \mathbb{N}$, leading to the sequence $\omega = (x_1, \dots, x_n, \dots)$. At every time instant n, after Reality has revealed the outcome x_n, Sceptic plays the gamble $S(x_{1:n})[f(X_{n+1}) - \overline{E}_{\varphi(x_{1:n})}(f)]$, that is, she buys the gamble $f(X_{n+1})$ for Forecaster's minimum acceptable selling price $\overline{E}_{\varphi(x_{1:n})}(f)$ in $x_{1:n}$ if $S(x_{1:n}) = 1$, and doesn't bet otherwise $[S(x_{1:n}) = 0]$. Next, Reality reveals the subsequent outcome $X_{n+1} = x_{n+1} \in \mathscr{X}$ and the reward $S(x_{1:n})[f(x_{n+1}) - \overline{E}_{\varphi(x_{1:n})}(f)]$ goes to Sceptic. The imprecise-probabilistic version of $\mathcal{S}^\infty$-randomness now builds upon the following idea: Sceptic is said to have a winning strategy in this betting game if she can come up with a gamble f and a selection process $S \in \mathcal{S}^\infty(\omega)$ such that she is guaranteed to have on average a (non-negligible) positive gain at arbitrarily large time instants by betting according to the above scheme, in the sense that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\left[f(\omega_{k+1}) - \overline{E}_{\varphi(\omega_{1:k})}(f)\right]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} > 0,$$

and the path ω is then considered $\mathcal{S}^\infty$-random for φ if Sceptic has no such winning betting strategy.

Definition 11.2. Consider any countable set of selection processes $\mathcal{S}^\infty$. Then a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-*random* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if for any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω:

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\left[f(\omega_{k+1}) - \overline{E}_{\varphi(\omega_{1:k})}(f)\right]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0.$$

As a continuation of our discussion in Section 10_{66}, we'll conclude this section by having a look at how robust the notion of $\mathcal{S}^\infty$-randomness is with respect to changes to the betting strategies and the forecasting system. We start by observing that $\mathcal{S}^\infty$-randomness has the below equivalent characterisation in terms of limit inferiors and conjugate lower expectations, which we'll use several times further on.

Proposition 11.3. *Consider any countable set of selection processes $\mathcal{S}^\infty$. Then a path $\omega \in \Omega$ is $\mathcal{S}^\infty$*-random *for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if for any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω:*

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\left[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\right]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0.$$

Proof. Fix any path $\omega \in \Omega$, any forecasting system $\varphi \in \Phi(\mathscr{X})$, any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω, and simply observe, using conjugacy, that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\Big[f(\omega_{k+1}) - \overline{E}_{\varphi(\omega_{1:k})}(f)\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0$$

$$\Leftrightarrow -\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\Big[f(\omega_{k+1}) - \overline{E}_{\varphi(\omega_{1:k})}(f)\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0$$

$$\Leftrightarrow \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[-f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(-f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0$$

Since f can be chosen arbitrarily, the result is immediate. □

It turns out that our notion of $\mathcal{S}^\infty$-randomness is also reasonably robust with respect to weakening the 'betting strategies': we can assume the gambles in Proposition 11.3 to be rational and to be part of $\mathscr{L}_1(\mathscr{X})$.

Proposition 11.4. *Consider any countable set of selection processes $\mathcal{S}^\infty$. Then a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-*random *for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if for any rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω:*

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0.$$

Proof. Due to Proposition 11.3, it clearly suffices to prove the 'if' part. We give a proof by contraposition. So assume that ω isn't $\mathcal{S}^\infty$-random, implying that there is some gamble $f \in \mathscr{L}(\mathscr{X})$ and real $\epsilon > 0$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -2\epsilon.$$

Let $O \in \mathbb{N}$ be any natural number such that $\max|f| + \epsilon \leq O$, and let $f' \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ be any rational gamble such that $f \leq f' \leq f + \epsilon$. Since clearly $(f'+O)/(2O) \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ and also $0 \leq (f+O)/(2O) \leq (f'+O)/(2O) \leq (f+\epsilon+O)/(2O) \leq 1$, it holds that $(f'+O)/(2O) \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$. Now simply observe that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\Big[\Big(\frac{f'(\omega_{k+1})+O}{2O}\Big) - \underline{E}_{\varphi(\omega_{1:k})}\Big(\frac{f'+O}{2O}\Big)\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})}$$

$$\overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{2O} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f'(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f')\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})}$$

$$\overset{\text{C5}_{20}}{\leq} \frac{1}{2O} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) + \epsilon - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})}$$

$$
\begin{aligned}
&= \frac{1}{2O} \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} + \frac{\epsilon}{2O} \\
&\leq -\frac{\epsilon}{2O} < 0,
\end{aligned}
$$

so we're done. □

Our notion of $\mathcal{S}^\infty$-randomness is not only to some extent robust with respect to changes in the 'betting strategies', as the above propositions show, but also with respect to changes in the forecasting system. It turns out that for every (computable) forecasting system $\varphi \in \Phi(\mathscr{X})$ there's a (recursive) rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that has the exact same set of $\mathcal{S}^\infty$-random paths. In this sense, you could say that rational forecasting systems suffice to capture the essence of this randomness notion.

Proposition 11.5. *Consider any countable set of selection processes $\mathcal{S}^\infty$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$. Then there's a rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ such that $\Omega_{\mathcal{S}^\infty}(\varphi) = \Omega_{\mathcal{S}^\infty}(\varphi_{\mathrm{rat}})$. Moreover, if φ is computable, then φ_{rat} can be assumed to be recursive.*

Proof. Let $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ be any rational forecasting system such that $d_{\mathrm{H}}\big(\varphi(s), \varphi_{\mathrm{rat}}(s)\big) \leq 2^{-|s|}$ for all $s \in \mathbb{S}$ [which is always possible by Lemma 5.6[18]]. Fix any path $\omega \in \Omega$, any rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$ and any selection process $\mathcal{S} \in \mathcal{S}^\infty(\omega)$ that accepts ω. Since $f \in \mathscr{L}_1(\mathscr{X})$, and therefore also $\max f - f \in \mathscr{L}_1(\mathscr{X})$, it follows from Corollary 7.9[48] and conjugacy that

$$
\begin{aligned}
0 &\leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \big|\underline{E}_{\varphi(\omega_{1:k})}(f) - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\big|}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \big|\underline{E}_{\varphi(\omega_{1:k})}(f) - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\big|}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&= \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \Big|\overline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(-f) - \overline{E}_{\varphi(\omega_{1:k})}(-f)\Big|}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\overset{\mathrm{C4}_{20}}{=} \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \Big|\overline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}\big(\max f - f\big) - \overline{E}_{\varphi(\omega_{1:k})}\big(\max f - f\big)\Big|}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) d_{\mathrm{H}}\big(\varphi_{\mathrm{rat}}(\omega_{1:k}), \varphi(\omega_{1:k})\big)}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) 2^{-k}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} 2^{-k}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\
&\leq 2 \limsup_{n\to\infty} \frac{1}{\sum_{k=0}^{n-1} S(\omega_{1:k})} = 0,
\end{aligned}
$$

where the last equality holds since S accepts ω. Consequently, there is for every $\epsilon > 0$ some $N \in \mathbb{N}$ such that

$$\left| \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \underline{E}_{\varphi(\omega_{1:k})}(f)}{\sum_{k=0}^{n-1} S(\omega_{1:k})} - \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \right| < \epsilon \text{ for all } n \geq N,$$

and hence, it follows from Proposition 11.4$_{90}$ that ω is $\mathcal{S}^\infty$-random for φ if and only if it's $\mathcal{S}^\infty$-random for φ_{rat}. We conclude that $\Omega_{\mathcal{S}^\infty}(\varphi) = \Omega_{\mathcal{S}^\infty}(\varphi_{\mathrm{rat}})$.

Moreover, if φ is computable, then there's a recursive map $q\colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathcal{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s,N)\big)\big) \leq 2^{-N}$ for all $s \in \mathbb{S}$ and $N \in \mathbb{N}$. So we can just let $\varphi_{\mathrm{rat}}(s) = \mathrm{CH}\big(q(s,|s|)\big)$ for all $s \in \mathbb{S}$. □

11.2 Alternative frequentist characterisation

In this section, we'll make the frequentist character of our imprecise-probabilistic version of $\mathcal{S}^\infty$-randomness perhaps (even) more explicit by proving an equivalent characterisation in terms of running frequencies based on a generalisation of condition (i"). The material in this section is based on a question by Alexander Shen, who asked for a characterisation in terms of running frequencies, which he deems more intuitive and natural. A path $\omega \in \Omega$ turns out to be $\mathcal{S}^\infty$-random for a forecasting system $\varphi \in \Phi(\mathcal{X})$ if and only if the frequencies of the outcomes $x \in \mathcal{X}$ along all infinite subsequences of ω selected by the selection processes in $\mathcal{S}^\infty(\omega)$ are bounded by φ.

To make this formal, we need some more notation and explanation. For every path $\omega \in \Omega$ and selection process $S \in \mathcal{S}^\infty(\omega)$ there is some smallest natural $N \in \mathbb{N}$ such that $\sum_{k=0}^{N-1} S(\omega_{1:k}) > 0$, which implies that $\sum_{k=0}^{n-1} S(\omega_{1:k}) > 0$ for all $n \geq N$. Consequently,

$$\left(\frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \mathbb{I}_{\omega_{k+1}}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \right)_{n \geq N}$$

is a well-defined sequence of gambles on $\mathcal{X}$, whose values for every $x \in \mathcal{X}$ are the corresponding relative numbers of occurrences along the infinite subsequence of ω selected by S; all these gambles are probability mass functions.

Given any collection of credal sets $(C_k)_{1 \leq k \leq n} \in \mathscr{C}(\mathcal{X})^n$, with $n \in \mathbb{N}$, and any collection of non-negative real weights $(\lambda_k)_{1 \leq k \leq n}$, their weighted *Minkowski sum* is given by $\sum_{k=1}^n \lambda_k C_k := \big\{\sum_{k=1}^n \lambda_k m_k \colon m_k \in C_k \text{ for all } 1 \leq k \leq n\big\}$; it's easy to verify that every such sum is still a credal set if $\sum_{k=1}^n \lambda_k = 1$ [80, Section 3.1]. In particular, for every $\omega \in \Omega$, $S \in \mathcal{S}^\infty(\omega)$, $\varphi \in \Phi(\mathcal{X})$ and the smallest natural $N \in \mathbb{N}$ for which $\sum_{k=0}^{N-1} S(\omega_{1:k}) > 0$, we'll consider the sequence of weighted Minkowski sums

$$\left(\frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \varphi(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \right)_{n \geq N},$$

which corresponds to the running averages of the credal sets that Forecasters' forecasting system specifies along the infinite subsequence of ω selected by S; every such Minkowski sum is a credal set.

With these objects at our disposal, we can more formally state our alternative frequentist characterisation for $\mathcal{S}^\infty$-randomness: a path $\omega \in \Omega$ turns out to be $\mathcal{S}^\infty$-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if the relative frequencies of occurrences—which are probability mass functions—converge to the running average of forecasts—which are credal sets—along all infinite subsequences of ω selected by the selection processes in $\mathcal{S}^\infty(\omega)$, and this, in the following sense.

Proposition 11.6. *Consider any countable set of selection processes $\mathcal{S}^\infty$. A path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if for all selection processes $S \in \mathcal{S}^\infty(\omega)$ that accept ω:*

$$\lim_{n\to\infty} d\left(\frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\mathbb{I}_{\omega_{k+1}}}{\sum_{k=0}^{n-1} S(\omega_{1:k})}, \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\varphi(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})}\right) = 0.$$

Proof. This is an immediate corollary of Proposition 11.389 and Proposition 11.7 below. □

This alternative characterisation is a rather straightforward imprecise-probabilistic generalisation of condition (i") combined with (ii).

This equivalent characterisation of $\mathcal{S}^\infty$-randomness in terms of frequencies is perhaps made (even) more intuitive when focusing on a stationary forecasting system C, because then it tells us that a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for the credal set C if and only if the relative frequencies of the outcomes along all infinite subsequences of ω selected by the selection processes in $\mathcal{S}^\infty(\omega)$ converge to C. An important point here is that convergence to C with respect to the distance d doesn't mean that these frequencies should necessarily converge to a probability mass function $m \in \mathcal{M}(\mathscr{X})$ in C. Loosely speaking, it means that these frequencies will eventually be contained in C but may continue to oscillate within C. When bringing conditions (i) and (ii) to mind, this also clearly shows that our notion of $\mathcal{S}^\infty$-randomness is indeed an imprecise-probabilistic generalisation of the classical precise-probabilistic one: a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for a probability mass function $m \in \mathcal{M}(\mathscr{X})$ if the relative frequencies of every outcome $x \in \mathscr{X}$ along all infinite subsequences of ω selected by the selection processes in $\mathcal{S}^\infty(\omega)$ converge to $m(x)$.

Our proof of the above equivalent alternative characterisation for $\mathcal{S}^\infty$-randomness relies on the proposition below, which doesn't require any implementability conditions for the selection processes.

Proposition 11.7. *Consider any path $\omega \in \Omega$, any forecasting system $\varphi \in \Phi(\mathscr{X})$, and any selection process $S \in \mathcal{S}$ that accepts ω. Then,*

$$\lim_{n\to\infty} d\left(\frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\mathbb{I}_{\omega_{k+1}}}{\sum_{k=0}^{n-1} S(\omega_{1:k})}, \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\varphi(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})}\right) = 0$$

if and only if

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0 \textit{ for all } f \in \mathscr{L}(\mathscr{X}).$$

Proof. For ease of notation, let

$$m_n := \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\mathbb{I}_{\omega_{k+1}}}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \text{ and } C_n := \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\varphi(\omega_{1:k})}{\sum_{k=0}^{n-1} S(\omega_{1:k})}$$

for all $n \in \mathbb{N}$ for which $\sum_{k=0}^{n-1} S(\omega_{1:k}) > 0$. By invoking Lemma 11.8 and conjugacy, we infer that, for all $f \in \mathscr{L}(\mathscr{X})$and $n \in \mathbb{N}$ for which $\sum_{k=0}^{n-1} S(\omega_{1:k}) > 0$,

$$E_{m_n}(f) = \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) f(\omega_{k+1})}{\sum_{k=0}^{n-1} S(\omega_{1:k})}$$

and

$$\underline{E}_{C_n}(f) = \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\underline{E}_{\varphi(\omega_{1:k})}(f)}{\sum_{k=0}^{n-1} S(\omega_{1:k})}.$$

For the 'if'-direction, assume *ex absurdo* the existence of some $\epsilon > 0$ such that $\limsup_{n\to\infty} d(m_n, C_n) > 2\epsilon$. This implies that there's some infinite subset of naturals $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that $d(m_{n_i}, C_{n_i}) > 2\epsilon$ for all $i \in \mathbb{N}$. By Lemma 11.9, there's for every $i \in \mathbb{N}$ some gamble $f_{n_i} \in \mathscr{L}_1(\mathscr{X})$ such that $\underline{E}_{C_{n_i}}(f_{n_i}) - E_{m_{n_i}}(f_{n_i}) > 2\epsilon$. Since the set $\mathscr{L}_1(\mathscr{X})$ is compact [81, Example 17.9(a)], there's some gamble $f \in \mathscr{L}_1(\mathscr{X})$ and some infinite set of naturals $\{n'_i\}_{i\in\mathbb{N}} \subseteq \{n_i\}_{i\in\mathbb{N}}$ such that $\max|f - f_{n'_i}| < \epsilon/2$ for all $i \in \mathbb{N}$ [81, Theorem 17.4]. Consequently, for all $i \in \mathbb{N}$,

$$\begin{aligned}\underline{E}_{C_{n'_i}}(f) - E_{m_{n'_i}}(f) &\overset{\text{C5}_{20}}{\geq} \underline{E}_{C_{n'_i}}(f_{n'_i} - \epsilon/2) - E_{m_{n'_i}}(f_{n'_i} + \epsilon/2)) \\ &\overset{\text{C4}_{20}}{=} \underline{E}_{C_{n'_i}}(f_{n'_i}) - E_{m_{n'_i}}(f_{n'_i}) - \epsilon \\ &> 2\epsilon - \epsilon = \epsilon,\end{aligned}$$

and hence, $\liminf_{n\to\infty}\big[E_{m_n}(f) - \underline{E}_{C_n}(f)\big] \leq -\epsilon < 0$.

For the 'only if'-direction, assume *ex absurdo* the existence of some $\epsilon > 0$ and $f \in \mathscr{L}(\mathscr{X})$ such that $\liminf_{n\to\infty}\big[E_{m_n}(f) - \underline{E}_{C_n}(f)\big] < -\epsilon$. This implies that there's some infinite subset of naturals $\{n_i\}_{i\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\underline{E}_{C_{n_i}}(f) - E_{m_{n_i}}(f) \geq \epsilon$ for all $i \in \mathbb{N}$. Let $N \in \mathbb{N}$ be such that $0 \leq (f+N)/2N \leq 1$. Then, for all $i \in \mathbb{N}$,

$$d(m_{n_i}, C_{n_i}) \geq \underline{E}_{C_{n_i}}\left(\frac{f+N}{2N}\right) - E_{m_{n_i}}\left(\frac{f+N}{2N}\right)$$

$$\overset{\text{C2}_{20},\text{C4}_{20}}{=} \frac{1}{2N}\Big[\underline{E}_{C_{n_i}}(f) - E_{m_{n_i}}(f)\Big] \geq \frac{\epsilon}{2N},$$

where the first inequality follows from Lemma 11.9. Hence, it holds that $\limsup_{n\to\infty} d(m_n, C_n) \geq \frac{\epsilon}{2N} > 0$. □

Lemma 11.8 ([80, Proposition 3])**.** *Consider any credal sets* $C, C' \in \mathscr{C}(\mathscr{X})$ *and any real* $0 \leq \lambda \leq 1$*. Then,* $\overline{E}_{\lambda C + (1-\lambda)C'}(f) = \lambda \overline{E}_C(f) + (1-\lambda)\overline{E}_{C'}(f)$ *for all* $f \in \mathscr{L}(\mathscr{X})$*.*

Lemma 11.9. *Consider any probability mass function* $m \in \mathcal{M}(\mathscr{X})$ *and any credal set* $C \in \mathscr{C}(\mathscr{X})$*. Then,* $d(m, C) = \max_{f\in\mathscr{L}_1(\mathscr{X})}\big(\underline{E}_C(f) - E_m(f)\big)$*.*

Proof. Consider any probability mass function $m' \in \mathcal{M}(\mathscr{X})$. Then it follows from Lemma 7.8[48] and the definition of the Hausdorff distance between credal sets that $\|m - m'\|_{\text{v}} = d_{\text{H}}\big(\{m\},\{m'\}\big) = \max_{f\in\mathscr{L}_1(\mathscr{X})}|\overline{E}_{\{m\}}(f) - \overline{E}_{\{m'\}}(f)| = \max_{f\in\mathscr{L}_1(\mathscr{X})}|E_m(f) - E_{m'}(f)|$. Hence,

$$\begin{aligned}
\|m - m'\|_{\text{v}} &= \max_{f\in\mathscr{L}_1(\mathscr{X})} \max\big\{E_{m'}(f) - E_m(f), E_m(f) - E_{m'}(f)\big\}\\
&= \max_{f\in\mathscr{L}_1(\mathscr{X})} \max\big\{E_{m'}(f) - E_m(f), E_{m'}(1-f) - E_m(1-f)\big\}\\
&= \max_{f\in\mathscr{L}_1(\mathscr{X})} \max_{g\in\{f,1-f\}} E_{m'}(g) - E_m(g)\\
&= \max_{f\in\mathscr{L}_1(\mathscr{X})} E_{m'}(f) - E_m(f),
\end{aligned}$$

where the last equality holds because, for all $f \in \mathscr{L}_1(\mathscr{X})$, also $1 - f \in \mathscr{L}_1(\mathscr{X})$. Consequently,

$$\begin{aligned}
d(m, C) &= \min_{m'\in C} \|m - m'\|_{\text{tv}}\\
&= \min_{m'\in C} \max_{f\in\mathscr{L}_1(\mathscr{X})} \big(E_{m'}(f) - E_m(f)\big)\\
&= \max_{f\in\mathscr{L}_1(\mathscr{X})} \min_{m'\in C} \big(E_{m'}(f) - E_m(f)\big)\\
&= \max_{f\in\mathscr{L}_1(\mathscr{X})} \big(\underline{E}_C(f) - E_m(f)\big),
\end{aligned}$$

where the third inequality holds by von Neumann's minimax theorem [82] because C and $\mathscr{L}_1(\mathscr{X})$ are compact convex sets, and because $E_{m'}(f) - E_m(f)$ is linear and therefore also continuous in $m' \in C$ and $f \in \mathscr{L}_1(\mathscr{X})$. □

11.3 Properties

It turns out that $\mathcal{S}^\infty$-randomness satisfies similar properties as our martingale-theoretic randomness notions in Section 9[54]; to state these properties, we'll denote the set of paths that are $\mathcal{S}^\infty$-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ by $\Omega_{\mathcal{S}^\infty}(\varphi) := \{\omega \in \Omega : \omega \text{ is } \mathcal{S}^\infty\text{-random for } \varphi\}$

Proposition 11.10. *Consider any countable set of selection processes* $\mathcal{S}^\infty$ *and any two forecasting systems* $\varphi, \varphi' \in \Phi(\mathscr{X})$*. Then,* *(i)* $\underline{P}^{\varphi}(\Omega_{\mathcal{S}^\infty}(\varphi)) = 1$; [almost all paths are random]

(ii) $\Omega_{\mathcal{S}^\infty}(\varphi) \neq \emptyset$; [non-emptiness]

(iii) $\Omega_{\mathcal{S}^\infty}(\varphi_{\mathrm{v}}) = \Omega$; [vacuity]

(iv) *if* $\varphi \subseteq \varphi'$, *then* $\Omega_{\mathcal{S}^\infty}(\varphi) \subseteq \Omega_{\mathcal{S}^\infty}(\varphi')$; [monotonicity]

Proof. Let's start by proving (i). By Proposition 11.4~90~, a path $\omega \in \Omega$ isn't $\mathcal{S}^\infty$-random for φ if there's a selection process $S \in \mathcal{S}^\infty$ that accepts ω, a rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$, and a rational $\epsilon \in (0,1)$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -\epsilon.$$

If this is the case, then Lemma 11.12~98~ and Proposition 6.6~30~ tell us that there's a positive test supermartingale $T_{S,\epsilon,f}$ such that $\limsup_{n\to\infty} T_{S,\epsilon,f}(\omega_{1:n}) = \infty$, with $S \in \mathcal{S}^\infty$, $\epsilon \in \mathbb{Q}_{>0} \cap (0,1)$ and $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$; to understand why we can assume without loss of generality that $T_{S,\epsilon,f}$ only depends on S, ϵ and f (and φ, which we consider fixed in the background), simply observe in Lemma 11.12~98~ that the choice of φ_{rat} only depends on ϵ and φ, and doesn't depend on ω, and that the choice of B only depends on ϵ and f, and doesn't depend on ω either. Since the sets $\mathcal{S}^\infty$, $\mathbb{Q}_{>0} \cap (0,1)$ and $\mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ are all countable, the collection $\mathbf{T}$ of all such test supermartingales for φ is countable as well. By observing that

$$A := \left\{\omega \in \Omega \colon (\forall T \in \mathbf{T}) \limsup_{n\to\infty} T(\omega_{1:n}) < \infty\right\} \subseteq \Omega_{\mathcal{S}^\infty}(\varphi),$$

it follows from Lemma 6.22~39~, P3~35~ and P1~35~ that

$$1 = \underline{P}^{\varphi}(A) \leq \underline{P}^{\varphi}\big(\Omega_{\mathcal{S}^\infty}(\varphi)\big) \leq 1.$$

(ii) is immediate from (i) and Lemma 6.19~38~.

To prove (iii), observe that for any path $\omega \in \Omega$, any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω:

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) f(\omega_{k+1})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq \min f = \underline{E}_{C_{\mathrm{v}}}(f).$$

Hence, it holds by Proposition 11.3~89~ that every path is $\mathcal{S}^\infty$-random for φ_{v}.

To prove (iv), consider any two forecasting systems $\varphi, \varphi' \in \Phi(\mathscr{X})$ such that $\varphi \subseteq \varphi'$ and any path $\omega \in \Omega_{\mathcal{S}^\infty}(\varphi)$. It follows from Eq. (5.8)~19~ that $\underline{E}_{\varphi(s)}(f) \geq \underline{E}_{\varphi'(s)}(f)$ for all $s \in \mathbb{S}$ and $f \in \mathscr{L}(\mathscr{X})$, and therefore, it holds for any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\Big[f(\omega_{k+1}) - \underline{E}_{\varphi'(\omega_{1:k})}(f)\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0,$$

where the last inequality holds because ω is assumed to be $\mathcal{S}^\infty$-random for φ [see Proposition 11.3~89~]. Hence, it holds by Proposition 11.3~89~ that ω is $\mathcal{S}^\infty$-random for φ'. □

In Lemma 11.12$_\curvearrowright$ below, which is used in the proof of the above proposition but is more generally applicable, and which is based on a result by De Cooman & De Bock [36, Lemma 22], we make use of so-called *almost computable* forecasting systems; this concept is especially useful in Propositions 12.2$_{103}$ and 12.3$_{104}$, and Corollary 12.4$_{105}$—which is in its turn used in Theorem 19.1$_{183}$—further on. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *almost computable* if for every $\epsilon > 0$ there's some recursive map $q\colon \mathbb{S} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s)\big)\big) \leq \epsilon$ for all $s \in \mathbb{S}$; we recall that a forecasting system $\varphi \in \Phi(\mathscr{X})$ is called computable if there's some recursive map $q\colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s,N)\big)\big) \leq 2^{-N}$ for all $s \in \mathbb{S}$ and $N \in \mathbb{N}$. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is called *almost computable for a selection process* $S \in \mathcal{S}$ if for every $\epsilon > 0$ there's some recursive map $q\colon \mathbb{S} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that, for all $s \in \mathbb{S}$, $d_{\mathrm{H}}\big(\varphi(s), \mathrm{CH}\big(q(s)\big)\big) \leq \epsilon$ if $S(s) = 1$; analogously, a rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ is called *recursive for a selection process* $S \in \mathcal{S}$ if there's some recursive map $q\colon \mathbb{S} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that, for all $s \in \mathbb{S}$, $\varphi_{\mathrm{rat}}(s) = \mathrm{CH}\big(q(s)\big)$ if $S(s) = 1$. Obviously, if a forecasting system $\varphi \in \Phi(\mathscr{X})$ is computable, then it is almost computable as well. The reverse direction doesn't hold, because—to give but one example—for every non-computable probability mass function $m \in \mathcal{M}(\mathscr{X})$ there's a sequence $q\colon \mathbb{N} \to \mathcal{M}_{\mathrm{rat}}(\mathscr{X})$ of rational probability mass functions such that $\lim_{n\to\infty} d_{\mathrm{H}}\big(m, q(n)\big) = 0$, and hence, the stationary forecasting system $\varphi_m \in \Phi(\mathscr{X})$ defined by $\varphi_m(s) := m$ for all $s \in \mathbb{S}$ is non-computable but almost computable. On the other hand, there are forecasting systems $\varphi \in \Phi(\mathscr{X})$ that are neither computable nor almost computable—as the following example shows—, and hence, we conclude that being almost computable is a strictly weaker condition than being computable and that being almost computable is not implied by being non-computable, making it a non-trivial implementability condition.

Example 11.11. Consider the binary state space $\mathscr{X} = \{0,1\}$, any non-recursive binary path $\omega \in \Omega$ and any two non-computable reals $p_1, p_2 \in [0,1]$ such that $p_1 < 1/4$ and $3/4 < p_2$. Let the temporal forecasting system $\varphi \in \Phi(\mathscr{X})$ be defined by

$$\varphi(n) := \begin{cases} p_1 & \text{if } \omega_{n+1} = 1 \\ p_2 & \text{if } \omega_{n+1} = 0 \end{cases} \text{ for all } n \in \mathbb{N}_0.$$

φ is obviously non-computable. It's also not almost computable. Indeed, assume towards contradiction that it is. This implies the existence of a recursive map $q\colon \mathbb{N}_0 \to \mathbb{Q}$ such that $|\varphi(n) - q(n)| < 1/4$ for all $n \in \mathbb{N}_0$. Consequently,

$$\omega_{n+1} = \begin{cases} 1 & \text{if } \varphi(n) = p_1 \\ 0 & \text{if } \varphi(n) = p_2 \end{cases} = \begin{cases} 1 & \text{if } q(n) < 1/2 \\ 0 & \text{if } q(n) > 1/2 \end{cases} \text{ for all } n \in \mathbb{N}_0,$$

so ω is recursive, which is the desired contradiction. ◇

If a forecasting system $\varphi \in \Phi(\mathscr{X})$ is almost computable (respectively almost recursive), then it's almost computable (respectively almost recursive) for any selection process $S \in \mathcal{S}$, and if a forecasting system $\varphi \in \Phi(\mathscr{X})$ is almost computable (respectively almost recursive) for a selection process $S \in \mathcal{S}$, then it's almost computable (respectively recursive) for any other selection process $S' \in \mathcal{S}$ that selects fewer situations, in the sense that $(S'(s) = 1 \Rightarrow S(s) = 1)$ for all $s \in \mathbb{S}$, which is equivalent to requiring that $S'(s) \leq S(s)$ for all $s \in \mathbb{S}$.

Lemma 11.12. *Consider any forecasting system $\varphi \in \Phi(\mathscr{X})$, any rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, any selection process $S \in \mathcal{S}$ and any rational $\epsilon \in (0,1)$. Let $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ be any rational forecasting system such that $d_H(\varphi(s), \varphi_{\mathrm{rat}}(s)) < \frac{\epsilon}{8}$ for all $s \in \mathbb{S}$ [which we know to exist by Lemma 5.6₁₈], and let $B \in \mathbb{N}$ be any natural number such that $\max_{x \in \mathscr{X}} |f(x)| + \frac{\epsilon}{8} \leq B$. Then the gamble process $D\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ defined by*

$$D(s)(x) \coloneqq 1 - \frac{\epsilon}{8B^2} S(s)\Big[f(x) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(s)}(f)\Big] \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}$$

is a positive supermartingale multiplier for φ such that, for every path $\omega \in \Omega$, if S accepts ω and

$$\liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -\epsilon, \tag{11.13}$$

then $\limsup_{n \to \infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. Moreover, if S is recursive and φ is almost computable for S, then—by the right choice of φ_{rat}—D can be assumed to be recursive, and thus $D^{\circledcirc} \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi) = \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$. If, in addition, S is total, then $D^{\circledcirc}$ can be assumed to be computably unbounded on ω instead of just unbounded on ω.

Proof. Let $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ be any rational forecasting system such that $d_{\mathrm{H}}(\varphi(s), \varphi_{\mathrm{rat}}(s)) < \frac{\epsilon}{8}$ for all $s \in \mathbb{S}$, which is always possible by Lemma 5.6₁₈. Since $f \in \mathscr{L}_1(\mathscr{X})$, and therefore also $\max f - f \in \mathscr{L}_1(\mathscr{X})$, it follows from Corollary 7.9₄₈ and conjugacy that

$$\begin{aligned}
\big|\underline{E}_{\varphi_{\mathrm{rat}}(s)}(f) - \underline{E}_{\varphi(s)}(f)\big| &= \Big|\overline{E}_{\varphi(s)}(-f) - \overline{E}_{\varphi_{\mathrm{rat}}(s)}(-f)\Big| \\
&\overset{\mathrm{C4}_{20}}{=} \Big|\overline{E}_{\varphi(s)}(\max f - f) - \overline{E}_{\varphi_{\mathrm{rat}}(s)}(\max f - f)\Big| \\
&< \frac{\epsilon}{8} \text{ for all } s \in \mathbb{S}.
\end{aligned} \tag{11.14}$$

If φ is assumed to be almost computable for S, then we can assume φ_{rat} to be recursive for S. Indeed, if φ is almost computable for S, then there's some recursive map $q\colon \mathbb{S} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that, for all $s \in \mathbb{S}$, $d_{\mathrm{H}}(\varphi(s), \mathrm{CH}(q(s))) < \frac{\epsilon}{8}$ if $S(s) = 1$, and hence, we just have to make $\varphi_{\mathrm{rat}}(s)$ equal $\mathrm{CH}(q(s))$ in all situations $s \in \mathbb{S}$ for which $S(s) = 1$.

Fix any $B \in \mathbb{N}$ such that $\max_{x\in\mathscr{X}}|f(x)| + \frac{\epsilon}{8} \le B$. Let the map $D\colon \mathbb{S} \to \mathscr{L}(\mathscr{X})$ be defined by

$$D(s)(x) := 1 - \frac{\epsilon}{8B^2} S(s)\Big[f(x) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(s)}(f)\Big] \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}.$$

We'll now show in a number of steps that D is a (recursive) positive supermartingale multiplier for φ for which $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.

To this end, we start by observing that D is a positive multiplier process. Indeed, since $\epsilon < 1$, $|S| \le 1$ and $|f(x) + \frac{\epsilon}{8} - \underline{E}_C(f)| \le |f(x)| + \frac{\epsilon}{8} + |\underline{E}_C(f)| \le |f(x)| + \frac{\epsilon}{8} + \max_{x\in\mathscr{X}}|f(x)| \le 2B \le 4B^2$ for all $x \in \mathscr{X}$ and $C \in \mathscr{C}(\mathscr{X})$, using C1$_{20}$ for the second inequality, it's immediate that, for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$,

$$D(s)(x) = 1 - \frac{\epsilon}{8B^2} S(s)\Big[f(x) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(s)}(f)\Big] \begin{cases} = 1 & \text{if } S(s) = 0 \\ > 1/2 & \text{if } S(s) = 1, \end{cases} \tag{11.15}$$

and hence, D is a positive multiplier process. We continue by showing that D is a supermartingale multiplier for φ. To this end, observe for any $s \in \mathbb{S}$ that

$$\begin{aligned}\overline{E}_{\varphi(s)}(D(s)) &= \overline{E}_{\varphi(s)}\bigg(1 - \frac{\epsilon}{8B^2} S(s)\Big[f + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(s)}(f)\Big]\bigg) \\ &\overset{\text{C2}_{20},\text{C4}_{20}}{=} \bigg(1 - \frac{\epsilon}{8B^2} S(s)\underbrace{\Big[\underline{E}_{\varphi(s)}(f) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(s)}(f)\Big]}_{>0 \text{ [Eq. (11.14)]}}\bigg) \begin{cases} = 1 & \text{if } S(s) = 0 \\ < 1 & \text{if } S(s) = 1, \end{cases}\end{aligned} \tag{11.16}$$

where the second equality also makes use of conjugacy, so we find that D is a positive supermartingale multiplier for φ. In the case that S is recursive and that φ is almost computable for S, we already know that we can assume that φ_{rat} is recursive for S, and hence, since ϵ is rational, B is natural and f is rational, it follows from Lemma 7.1$_{43}$ that D is recursive. Hence, $D^{\circledcirc}$ is recursive and therefore computable as well, which allows us to conclude from Proposition 6.6$_{30}$ that $D^{\circledcirc} \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi) = \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$.

It follows from Eq. (11.14) that, for any $\omega \in \Omega$ that is accepted by S and satisfies Eq. (11.13),

$$\begin{aligned}\liminf_{n\to\infty} &\frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \\ &\le \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} + \frac{\epsilon}{4} < -\frac{3\epsilon}{4}.\end{aligned} \tag{11.17}$$

Fix any ω that's accepted by S and that satisfies Eq. (11.13). Then, by Eq. (11.17), for any $m, M \in \mathbb{N}_0$, there's some $N > m$ such that $\sum_{k=0}^{N-1} S(\omega_{1:k}) \ge M$ and

$$\frac{\sum_{k=0}^{N-1} S(\omega_{1:k})\big[f(\omega_{k+1}) + \epsilon/8 - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{N-1} S(\omega_{1:k})} < -\frac{3\epsilon}{4}. \tag{11.18}$$

This will allow us to obtain a lower bound for $D^{\circledcirc}(\omega_{1:N})$. Since D is a positive multiplier process, it holds that $D^{\circledcirc}(\omega_{1:N}) = \exp(K)$, with

$$K := \sum_{k=0}^{N-1} \ln\Big(1 - \frac{\epsilon}{8B^2} S(\omega_{1:k})\Big[f(\omega_{k+1}) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\Big]\Big).$$

Since $\ln(1+x) \geq x - x^2$ for all $x > -1/2$, we infer from Eq. (11.15)↶ that

$$K \geq -\frac{\epsilon}{8B^2}\sum_{k=0}^{N-1} S(\omega_{1:k})\Big[f(\omega_{k+1}) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\Big] - \frac{\epsilon^2}{64B^4}\sum_{k=0}^{N-1} S(\omega_{1:k})^2[f(\omega_{k+1}) + \frac{\epsilon}{8} - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)]^2$$

and, also taking into account Eq. (11.18)↶, $S^2 = S$ and $\big[f(\omega_{k+1}) + \epsilon/8 - \underline{E}_{\varphi_{\mathrm{rat}}(\omega_{1:k})}(f)\big]^2 \leq 4B^2$,

$$\geq \frac{\epsilon}{8B^2}\frac{3\epsilon}{4}\sum_{k=0}^{N-1} S(\omega_{1:k}) - \frac{\epsilon^2}{16B^2}\sum_{k=0}^{N-1} S(\omega_{1:k}) = \frac{\epsilon^2}{32B^2}\sum_{k=0}^{N-1} S(\omega_{1:k}).$$

Hence,

$$D^{\circledcirc}(\omega_{1:N}) \geq \exp\Bigg(\frac{\epsilon^2}{32B^2}\sum_{k=0}^{N-1} S(\omega_{1:k})\Bigg) \geq \exp\Big(\frac{\epsilon^2}{32B^2}M\Big). \tag{11.19}$$

After recalling that the inequality above holds for any $M \in \mathbb{N}_0$ and for arbitrarily large well-chosen $N \in \mathbb{N}$, we conclude that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$.

If, in addition, we assume that S is total, then we know by Lemma 11.20 that there's some natural growth function $\tau_S \colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\sum_{k=0}^{n-1} S(\omega_{1:k}) \geq \tau_S(n)$ for all $n \in \mathbb{N}_0$. Hence, it follows from the discussion above, and Eq. (11.19) in particular, that for arbitrary large but well-chosen $N \in \mathbb{N}$:

$$D^{\circledcirc}(\omega_{1:N}) \geq \exp\Bigg(\frac{\epsilon^2}{32B^2}\tau_S(N)\Bigg).$$

Let the real growth function $\tau \colon \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ be defined as $\tau(n) := \exp\Big(\frac{\epsilon^2}{32B^2}\tau_S(n)\Big)$ for all $n \in \mathbb{N}_0$. By recalling that the inequality above holds for arbitrarily large well-chosen $N \in \mathbb{N}$, we conclude that $\limsup_{n\to\infty}\Big[D^{\circledcirc}(\omega_{1:n}) - \tau(n)\Big] \geq 0$. □

Lemma 11.20. *Consider any recursive total selection process $S \in \mathcal{S}_{\mathrm{wCH}}$. Then there's some natural growth function $\tau_S \colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\sum_{k=0}^{n-1} S(\omega_{1:k}) \geq \tau_S(n)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}_0$.*

Proof. Let $\tau_S \colon \mathbb{N}_0 \to \mathbb{N}_0$ be defined as $\tau_S(n) := \min_{s\in\mathscr{X}^n}\sum_{k=0}^{n-1} S(s_{1:k})$ for all $n \in \mathbb{N}_0$. τ_S is indeed non-negative and non-decreasing because $\tau_S(0) = 0$ and $\tau_S(n) \leq \min_{s\in\mathscr{X}^n}\sum_{k=0}^{n} S(s_{1:k}) = \min_{s\in\mathscr{X}^{n+1}}\sum_{k=0}^{n} S(s_{1:k}) = \tau_S(n+1)$ for all $n \in \mathbb{N}_0$. It's obviously also recursive since S is. So, in order to conclude that τ_S is a natural growth function, it only remains to prove that it's unbounded. To this end, assume towards contradiction the existence of some $N \in \mathbb{N}$ such that $\lim_{n\to\infty}\tau_S(n) \leq N$. For every $n \in \mathbb{N}_0$, let

$$A_n := \Bigg\{\omega \in \Omega \colon \sum_{k=0}^{n-1} S(\omega_{1:k}) \leq N\Bigg\}.$$

By assumption, these sets are non-empty. Moreover, observe that

$$A_n = \bigcup\Bigg\{[\![s]\!] \subseteq \Omega \colon s \in \mathscr{X}^n \text{ and } \sum_{k=0}^{n-1} S(s_{1:k}) \leq N\Bigg\} \text{ for all } n \in \mathbb{N}_0,$$

which implies that each A_n is clopen—and hence in particular closed—in the Cantor topology as a finite union of cylinder sets. These sets are also clearly nested, that is, $A_n \supseteq A_{n+1}$ for all $n \in \mathbb{N}_0$. Consequently, the family of non-empty closed sets $\{A_n\}_{n\in\mathbb{N}_0}$ has the finite intersection property, and hence, $\bigcap_{n\in\mathbb{N}_0} A_n \neq \emptyset$ [81, Theorem 17.4] by the compactness of Ω in the Cantor topology [81, Theorem 17.8]. So, there exists some path $\omega \in \Omega$ such that $\sum_{k=0}^{n-1} S(\omega_{1:k}) \leq N$ for all $n \in \mathbb{N}_0$, which contradicts that S is total.

By construction, it holds that $\sum_{k=0}^{n-1} S(s_{1:k}) \geq \tau_S(n)$ for all $s \in \mathscr{X}^n$ and $n \in \mathbb{N}_0$, and hence, $\sum_{k=0}^{n-1} S(\omega_{1:k}) \geq \tau_S(n)$ for all $\omega \in \Omega$ and $n \in \mathbb{N}_0$, which concludes this proof. □

Corollary 11.21. *Consider any path $\omega \in \Omega$, any rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, any recursive total selection process $S \in \mathcal{S}_{\mathrm{wCH}}(\omega)$ that accepts ω, any forecasting system $\varphi \in \Phi(\mathscr{X})$ that is almost computable for S, and any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$. *If $\omega \in \Omega_{\mathrm{R}}(\varphi)$, then*

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0. \tag{11.22}$$

Proof. This is immediate from Lemma 11.12$_{98}$, Definition 8.6$_{54}$ and Proposition 9.1$_{55}$. □

We pay special attention to Proposition 11.10(i)$_{96}$, which tells that for any forecasting system φ and any countable set of selection processes $\mathcal{S}^\infty$, the event that consist of all $\mathcal{S}^\infty$-random paths is almost sure for φ, and thus non-empty. Observe that this extends the work of Wald that we mentioned in Section 1$_2$. Moreover, the $\mathcal{S}^\infty$-random paths, which we now know to exist, will be used in the proof of Theorem 20.10$_{197}$.

We end this section by considering a particular property from Section 9$_{54}$ that we haven't (yet) translated to this frequentist setting, that is, that we didn't include in Proposition 11.10$_{95}$: if a path $\omega \in \Omega$ is R-random for a (non-degenerate) computable forecasting system $\varphi \in \Phi(\mathscr{X})$, with $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, then it's R-random as well for any other computable forecasting system $\varphi' \in \Phi(\mathscr{X})$ for which $\varphi'(\omega_{1:n}) = \varphi(\omega_{1:n})$ for all $n \in \mathbb{N}_0$—see Propositions 9.6$_{57}$ and 9.18$_{65}$. An analogous property trivially holds for $\mathcal{S}^\infty$-randomness, but now without any computability requirements on the forecasting system φ.

Proposition 11.23. *Consider any countable set of selection processes $\mathcal{S}^\infty$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$. If a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for φ, then it's $\mathcal{S}^\infty$-random for any other forecasting system $\varphi' \in \Phi(\mathscr{X})$ for which $\varphi'(\omega_{1:n}) = \varphi(\omega_{1:n})$ for all $n \in \mathbb{N}_0$.*

Proof. This is immediate from Definition 11.2$_{89}$. □

Hence, the notion of $\mathcal{S}^\infty$-randomness is also in line with Dawid's Weak Prequential Principle [8], which—as we recall from p. 66—requires that any criterion for assessing the 'agreement' between a forecasting system $\varphi \in \Phi(\mathscr{X})$ and a path $\omega \in \Omega$ should depend only on the forecasts that φ specifies along ω.

12 (Weak) Church randomness

If the countable set of (total) computable selection processes substitutes for $\mathcal{S}^\infty$, then we obtain an imprecise-probabilistic version of (weak) Church stochasticity, which we consider here. We provide and discuss their explicit definitions in Section 12.1, and examine in Section 12.2↷ how these two randomness notions relate to the four imprecise-probabilistic martingale-theoretic randomness notions from Section 8₅₀. When restricting our attention to stationary forecasting systems, in Section 12.3₁₀₆, we succeed in providing both randomness notions with a martingale-theoretic characterisation.

12.1 Definition

So, we'll pay special attention to the case where the countable set of selections processes coincides with the set $\mathcal{S}_{(\mathrm{w})\mathrm{CH}}$ consisting of all recursive (total) selection processes; recall from the introduction to Section 7₄₀ that these are sensible choices, because, as Alonzo Church argues, a path's randomness should be tested by 'effectively calculable' functions. The corresponding randomness notions will be called *Church* (CH) and *weak Church* (wCH) *randomness*.

Definition 12.1. A path $\omega \in \Omega$ is (w)CH-*random* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if for any gamble $f \in \mathscr{L}(\mathscr{X})$ and any recursive (total) selection process $S \in \mathcal{S}_{(\mathrm{w})\mathrm{CH}}(\omega)$ that accepts ω:

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\Big[f(\omega_{k+1}) - \overline{E}_{\varphi(\omega_{1:k})}(f)\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq 0.$$

Church and weak Church randomness are imprecise-probabilistic generalisations of classical randomness notions, which are in the classical precise-probabilistic literature better known under the name *Church stochasticity* [27] and *weak Church stochasticity* [12], respectively; this claim becomes (even) more explicit and intuitive when having a look at their alternative frequentist characterisation in Proposition 11.6₉₃. Historically, Church stochasticity is considered to be the earliest notion of randomness, which originally defined the randomness of an infinite binary sequence $\omega \in \{0,1\}^{\mathbb{N}}$ with respect to a precise probability model that assigns a fixed probability $p \in [0,1]$ to the outcome $X_n = 1$ for all $n \in \mathbb{N}$. According to this notion, an infinite binary sequence is Church random for p if the relative frequency of ones along every computably selectable infinite subsequence converges to p [23, 25]. This notion is typically called a stochasticity notion instead of a randomness notion because it's generally considered to be too weak to be called a randomness notion. This nowadays general belief has been best substantiated by Jean Ville [25, 28, 29], who pointed out the existence of a binary sequence that's

Church random for 1/2, and that satisfies the law of large numbers for that reason, but which fails to satisfy the law of the iterated logarithm, since the running frequency of ones along the sequence converges to 1/2 from below. For this reason, Jean Ville criticised Church's randomness definition, and argued that besides the law of large numbers, a random sequence also ought to satisfy other statistical laws [25, 28, 29]. Such discussions led to the development of many other notions of randomness, amongst which the ones introduced in Section 8_{50}.

Since the notion of weak Church randomness considers fewer selection processes than Church randomness does, it's clear that if a path $\omega \in \Omega$ is CH-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, then it's also wCH-random for φ, that is, $\Omega_{\mathrm{CH}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$ for all $\varphi \in \Phi(\mathscr{X})$, where $\Omega_{\mathrm{(w)CH}}(\varphi) := \{\omega \in \Omega \colon \omega \text{ is (w)CH-random for } \varphi\}$. This means that wCH-randomness is an even weaker notion of randomness than CH-randomness is, and hence, it's also considered too weak a randomness notion from a precise-probabilistic perspective.

That said, we choose to nevertheless speak of (weak) Church randomness instead of (weak) Church stochasticity for reasons of notational simplicity, as it allows us to say that a path $\omega \in \Omega$ is (w)CH-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, which is useful for stating properties that hold for all six randomness notions, as will for example be the case in Theorem 19.1_{183}. Moreover, here too, we choose to speak of (w)CH-randomness when adopting both computable and non-computable forecasting systems, where the non-computable forecasting systems aren't accessible by an oracle.

12.2 Relations

From the discussion after Definition $12.1_{\curvearrowleft}$, we know that $\Omega_{\mathrm{CH}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$, and that both Church and weak Church randomness are considered too weak a randomness notion from a precise-probabilistic perspective. How do they relate then to our four previously introduced imprecise-probabilistic martingale-theoretic randomness notions? If we restrict our attention to *almost computable* forecasting systems, then Church randomness turns out to be weaker than C-randomness, and therefore weaker than wML- and ML-randomness as well.

Proposition 12.2. *Consider any almost computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then*

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{CH}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi).$$

Proof. By Proposition 9.1_{55}, it suffices to prove that $\Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{CH}}(\varphi)$. To this end, consider any $\omega \in \Omega_{\mathrm{C}}(\varphi)$, and assume towards contradiction that $\omega \notin \Omega_{\mathrm{CH}}(\varphi)$. Definition $12.1_{\curvearrowleft}$ and Proposition 11.4_{90} then imply the existence of a recursive selection process $S \in \mathcal{S}_{\mathrm{CH}}(\omega)$ that accepts ω, a rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$ and a rational $\epsilon \in (0,1)$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -\epsilon.$$

From Lemma 11.12$_{98}$, since every almost computable forecasting system is almost computable for any (recursive) selection process $S \in \mathcal{S}$, we then infer the existence of a computable positive test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{C}}(\varphi)$ that's unbounded on ω, which is the desired contradiction. □

In particular, the above proposition holds for stationary forecasting systems C because every such stationary forecasting system is almost computable; this is immediate from the fact that the set of closed convex hulls of elements in $\mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ is dense in $\mathscr{C}(\mathscr{X})$ under the Hausdorff distance as is guaranteed by Lemma 5.6$_{18}$.

S-randomness doesn't fit in the relations of the above proposition; that is, S-randomness doesn't entail and is neither entailed by CH-randomness, even for almost computable forecasting systems. To show this, it suffices to prove that (i) there's an infinite binary sequence that's S-random for the ((almost) computable stationary) fair-coin forecasting system $\varphi_{1/2}$, but that isn't CH-random for $\varphi_{1/2}$, as has been done by Wang [78, Theorem 3.3.5(5)],[22] and to prove that (ii) there's an infinite binary sequence that's CH-random for the ((almost) computable stationary) fair-coin forecasting system $\varphi_{1/2}$, but that isn't S-random for $\varphi_{1/2}$, which follows immediately from results by Ville [28] and Schnorr [2] as has been explained by Wang [78, proof of Theorem 3.3.5(2)].

If, in addition, we restrict our attention to recursive *total* selection processes, that is, if we consider wCH-randomness, then S-randomness does turn out to entail wCH-randomness, and hence, all our four martingale-theoretic randomness notions entail wCH-randomness.

Proposition 12.3. *Consider any almost computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then*

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{S}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi).$$

Proof. By Proposition 9.1$_{55}$, it suffices to prove that $\Omega_{\mathrm{S}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$. To this end, consider any $\omega \in \Omega_{\mathrm{S}}(\varphi)$, and assume towards contradiction that $\omega \notin \Omega_{\mathrm{wCH}}(\varphi)$. Definition 12.1$_{102}$ and Proposition 11.4$_{90}$ then imply the existence of a recursive total selection process $S \in \mathcal{S}_{\mathrm{wCH}}(\omega)$ that accepts ω, a rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$ and a rational $\epsilon \in (0,1)$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\big[f(\omega_{k+1}) - \underline{E}_{\varphi(\omega_{1:k})}(f)\big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -\epsilon.$$

[22] Wang's definition of S-randomness uses martingales instead of supermartingales. It can however be easily proven that his definition coincides with ours for the fair-coin forecasting system.

From Lemma 11.12_{98}, since every almost computable forecasting system is almost computable for any (recursive total) selection process $S \in \mathcal{S}$, we then infer the existence of a computable positive test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{S}}(\varphi)$ that's computably unbounded on ω, which is the desired contradiction. □

Here too, the above proposition holds in particular for stationary forecasting systems.

We recall from Proposition 9.1_{55} and the discussion after Definition 12.1_{102} that $\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{S}}(\varphi)$ and $\Omega_{\mathrm{CH}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$ for all $\varphi \in \Phi(\mathscr{X})$. If we restrict our attention to almost computable forecasting systems, as has been explored in the two propositions above, then our two 'frequentist-flavoured' notions of randomness are furthermore related to our martingale-theoretic notions of randomness; we've summarised these results in the corollary below.

Corollary 12.4. *Consider any almost computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then*

$$\Omega_{\mathrm{ML}}(\varphi) \subseteq \Omega_{\mathrm{wML}}(\varphi) \subseteq \Omega_{\mathrm{C}}(\varphi) \begin{array}{c} \subsetneq \Omega_{\mathrm{CH}}(\varphi) \subsetneq \\ \subsetneq \Omega_{\mathrm{S}}(\varphi) \subsetneq \end{array} \Omega_{\mathrm{wCH}}(\varphi).$$ [23]

Proof. This is an immediate corollary of Propositions 9.1_{55}, 12.2_{103} and $12.3_{\frown}$. □

The relations $\Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{CH}}(\varphi)$ and $\Omega_{\mathrm{S}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$ in Corollary 12.4 do not hold for arbitrary forecasting systems $\varphi \in \Phi(\mathscr{X})$. Indeed, as our next example shows, there's a path $\omega \in \Omega$ and a precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$—that is not (almost) computable—such that ω is C-random for φ_{pr}—and therefore also S-random for φ_{pr}—, but that isn't wCH-random for φ_{pr}—and therefore also not CH-random for φ_{pr}. Hence, the relations $\Omega_{\mathrm{C}}(\varphi) \subseteq \Omega_{\mathrm{CH}}(\varphi)$ and $\Omega_{\mathrm{S}}(\varphi) \subseteq \Omega_{\mathrm{wCH}}(\varphi)$ do in general not hold for forecasting systems that aren't almost computable.

Example 12.5. Consider the binary state space $\mathscr{X} = \{0, 1\}$, and a path $\omega \in \mathscr{X}^{\mathbb{N}}$ that's C-random for $1/2$ [this is always possible by Corollary 9.3_{56}]. By Proposition 9.5_{56}, ω is also C-random for the interval forecast $[1/4, 3/4]$. We'll now come up with a precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$—that is not almost computable—for which ω is C- but not wCH-random. To this end, consider the countable set of selection processes $\mathcal{S}^{\infty} = \mathcal{S}^{1/2}_{\mathscr{F}_{\mathrm{C}}} \cup \{S = 1\}$ as defined in Section 20.1_{196} [see Eqs. $(20.7)_{196}$ and $(20.8)_{197}$], and consider any path $\varpi \in \Omega$ that's $\mathcal{S}^{\infty}$-random for $7/8$ [this is always possible by Proposition $11.10\mathrm{(ii)}_{96}$]; it

[23] Examples can be found in the classical precise-probabilistic literature showing that the inclusions between these randomness notions are strict; see also Footnote 20_{55}. In particular, the dissertation of Yongge Wang [78] contains an overview of old and novel results that show that (i) there's a path that's CH-random for 1/2 but not C-random for 1/2, (ii) there's a path that's S-random for 1/2 but not CH-random for 1/2, and (iii) there's a path that's CH-random for 1/2 but not S-random for 1/2. It follows from (ii) that there's a path that's wCH-random for 1/2 but not CH-random for 1/2, and it follows from (iii) that there's a path that's wCH-random for 1/2 but not S-random for 1/2.

follows from Definition 11.2 $_{89}$ that $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varpi_{k+1}={}^{7}\!/\!_{8}$ [let $S=1$ and f equal $\mathbb{I}_1$ and $-\mathbb{I}_1$]. Let $\varphi_{\mathrm{pr}}\in\Phi_{\mathrm{pr}}(\mathscr{X})$ be defined as

$$\varphi_{\mathrm{pr}}(s):=\begin{cases}3/4 & \text{if } \varpi_{|s|+1}=1\\ 1/4 & \text{if } \varpi_{|s|+1}=0\end{cases}\quad\text{for all } s\in\mathbb{S};\tag{12.6}$$

see also Eq. (20.6) $_{196}$. By Theorem 20.10 $_{197}$, $\Omega_{\mathrm{C}}([1/4,3/4])=\Omega_{\mathrm{C}}(\varphi_{\mathrm{pr}})$, and hence, ω is also C-random for the precise forecasting system φ_{pr}. So it only remains to show that ω is not wCH-random for φ_{pr}. Since ω is C-random for $1/2$, it follows from Corollary 12.4 that ω is also CH-random for $1/2$, and hence, $\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1}=1/2$ by Definition 12.1 $_{102}$ [let $S=1$ and f equal $\mathbb{I}_1$ and $-\mathbb{I}_1$]. Meanwhile,

$$\begin{aligned}
&\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\big(\omega_{k+1}-\varphi_{\mathrm{pr}}(\omega_{1:k})\big)\\
&\overset{\text{Eq. (20.6)}_{196}}{=}\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\omega_{k+1}-\left[\frac{1}{4}+\varpi_{k+1}\left(\frac{3}{4}-\frac{1}{4}\right)\right]\right)\\
&=-\frac{1}{4}+\liminf_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left(\omega_{k+1}-\frac{\varpi_{k+1}}{2}\right)\\
&=-\frac{1}{4}+\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1}-\frac{1}{2}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\varpi_{k+1}\\
&=-\frac{1}{4}+\frac{1}{2}-\frac{1}{2}\frac{7}{8}=-\frac{3}{16}<0,
\end{aligned}$$

so we conclude from Definition 12.1 $_{102}$ and Proposition 11.3 $_{89}$ [with $f=\mathbb{I}_1$ and $S=1$] that ω is not wCH-random for the precise forecasting system φ_{pr}. The precise forecasting system φ_{pr} is then necessarily not almost computable, because otherwise, since ω is C-random for φ_{pr}, it would follow from Corollary 12.4 that ω is wCH-random for φ_{pr} as well. ⋄

12.3 Alternative martingale-theoretic characterisation

When also restricting our attention to stationary forecasting systems, that is, to credal sets, then we can equip these two frequentist randomness notions with a characterisation in terms of a rather natural class of computable betting strategies that reminds of—and is similar to—our randomness definitions in Section 8 $_{50}$; the work in this section is based on analogous results that have been proved in a precise-probabilistic context [32, Section 7.4.3]. More specifically, for every credal set $C\in\mathscr{C}(\mathscr{X})$, we'll define a set of simple recursive supermartingales $\overline{\mathbb{T}}_{\mathrm{(w)CH}}(C)$, and prove that a path $\omega\in\Omega$ is (w)CH-random for φ if and only if no $T\in\overline{\mathbb{T}}_{\mathrm{(w)CH}}(C)$ is unbounded on ω.

To do so, we'll start by formally introducing these simple betting strategies. A supermartingale multiplier D for a credal set $C\in\mathscr{C}(\mathscr{X})$ is called *simple* if there's a rational positive gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, with $\overline{E}_C(f) < 1$, and a recursive selection process $S \in \mathcal{S}$ such that

$$D(s) = \begin{cases} 1 & \text{if } S(s) = 0 \\ f & \text{if } S(s) = 1 \end{cases} \text{ for all } s \in \mathbb{S};$$

it's additionally called *total* if S is total. Since the simple supermartingale multiplier D is clearly positive, it follows from Proposition 6.6$_{30}$ that the test supermartingale $D^{\circledcirc}$ generated by D is positive as well; since D is also clearly recursive, the test supermartingale $D^{\circledcirc}$ is recursive as well. We now use these (total) simple supermartingale multipliers to introduce the following two sets of betting strategies:

$\overline{\mathbb{T}}_{\mathrm{CH}}(C)$	all (recursive positive) test supermartingales for C generated by simple supermartingale multipliers;
$\overline{\mathbb{T}}_{\mathrm{wCH}}(C)$	all (recursive positive) test supermartingales for C generated by total simple supermartingale multipliers.

These sets of simple betting strategies now lead to the desired result.

Proposition 12.7. *A path $\omega \in \Omega$ is* (w)CH-*random for a credal set $C \in \mathscr{C}(\mathscr{X})$ if and only if no test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{(w)CH}}(C)$ is unbounded on ω.*

Proof. For the 'if'-direction, by Definition 12.1$_{102}$ and Proposition 11.4$_{90}$, assume towards contradiction the existence of some rational gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$, some (total) recursive selection process $S \in \mathcal{S}_{\mathrm{(w)CH}}(\omega)$ that accepts ω and some rational $\epsilon \in (0,1)$ such that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k})\left[f(\omega_{k+1}) - \underline{E}_C(f)\right]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} < -\epsilon.$$

Let C_{rat} be any rational credal set such that $d_{\mathrm{H}}(C, C_{\mathrm{rat}}) < \epsilon/8$ [which is always possible by Lemma 5.6$_{18}$], let $B \in \mathbb{N}$ be any natural number such that $\max_{x\in\mathscr{X}} |f(x)| + \frac{\epsilon}{8} \leq B$, and let the map $D\colon \mathbb{S} \to \mathbb{R}$ be defined by

$$D(s)(x) := 1 - \frac{\epsilon}{8B^2} S(s)\left[f(x) + \frac{\epsilon}{8} - \underline{E}_{C_{\mathrm{rat}}}(f)\right] \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathscr{X}.$$

Then it's immediate from Lemma 11.12$_{98}$ that $D^{\circledcirc}$ is a recursive positive test supermartingale for C generated by the positive supermartingale multiplier D and that $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \infty$. So we're done if we can show that $D^{\circledcirc} \in \overline{\mathbb{T}}_{\mathrm{(w)CH}}(C)$. Since D is a positive supermartingale multiplier for C, it only remains to show that it is simple (and total). To this end, observe for any $s \in \mathbb{S}$ that

$$D(s) = \begin{cases} 1 & \text{if } S(s) = 0 \\ 1 - \frac{\epsilon}{8B^2}\left(f + \frac{\epsilon}{8} - \underline{E}_{C_{\mathrm{rat}}}(f)\right) & \text{if } S(s) = 1 \end{cases} \text{ for all } s \in \mathbb{S}.$$

The gamble $f' \in \mathscr{L}(\mathscr{X})$ defined by $f' := 1 - \frac{\epsilon}{8B^2}\left(f + \frac{\epsilon}{8} - \underline{E}_{C_{\mathrm{rat}}}(f)\right)$ is positive by Eq. (11.15)$_{99}$ in the proof of Lemma 11.12$_{98}$ [with $\varphi_{\mathrm{rat}} \to C_{\mathrm{rat}}$], rational by

Lemma 7.1[43] and by the rationality of ϵ, B, f and C_{rat}, and satisfies $\overline{E}_C(f) < 1$ by Eq. (11.16)[99] in the proof of Lemma 11.12[98] [with $\varphi_{\mathrm{rat}} \to C_{\mathrm{rat}}$ and $\varphi \to C$] . By recalling that S is recursive, the above implies that D is a simple supermartingale multiplier for C, and that it's additionally total if S is total.

For the 'only if'-direction, assume towards contradiction the existence of a test supermartingale $T = D^{\circledcirc} \in \overline{\mathbb{T}}_{(\mathrm{w})\mathrm{CH}}(C)$ that's generated by a simple supermartingale multiplier D for C and that's unbounded on ω. Then there's a rational positive gamble $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, with $\overline{E}_C(f) < 1$, and a recursive (total) selection process $S \in \mathcal{S}$ such that $D(s) = 1 + S(s)(f-1)$ for all $s \in \mathbb{S}$. Since $\limsup_{n\to\infty} D^{\circledcirc}(\omega_{1:n}) = \limsup_{n\to\infty} T(\omega_{1:n}) = \infty$, we can assume that S accepts ω; otherwise there's some $M \in \mathbb{N}$ such that $D(\omega_{1:m}) = 1$ for all $m \geq M$, which prevents Sceptic from betting and thus from getting rich without bounds on ω. Since $D^{\circledcirc}$ is positive and unbounded on ω, we infer that

$$\begin{aligned}\limsup_{n\to\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) \ln(f(\omega_{k+1})) &= \limsup_{n\to\infty} \sum_{k=0}^{n-1} \ln\big(1 + S(\omega_{1:k})[f(\omega_{k+1}) - 1]\big) \\ &= \limsup_{n\to\infty} \ln\left(\prod_{k=0}^{n-1} \big(1 + S(\omega_{1:k})[f(\omega_{k+1}) - 1]\big)\right) \\ &= \limsup_{n\to\infty} \ln(D^{\circledcirc}(\omega_{1:n})) = \infty,\end{aligned}$$

and hence,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \ln(f(\omega_{k+1}))}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \geq 0, \tag{12.8}$$

because otherwise there's some $\epsilon > 0$ and $M \in \mathbb{N}$ such that

$$\begin{aligned}&\frac{\sum_{k=0}^{m-1} S(\omega_{1:k}) \ln(f(\omega_{k+1}))}{\sum_{k=0}^{m-1} S(\omega_{1:k})} < -\epsilon \text{ for all } m \geq M \\ &\quad\Rightarrow \sum_{k=0}^{m-1} S(\omega_{1:k}) \ln(f(\omega_{k+1})) < -\epsilon \sum_{k=0}^{m-1} S(\omega_{1:k}) \text{ for all } m \geq M \\ &\quad\Rightarrow \limsup_{n\to\infty} \sum_{k=0}^{n-1} S(\omega_{1:k}) \ln(f(\omega_{k+1})) = -\infty,\end{aligned}$$

a contradiction.

Since $\ln y \leq y - 1$ for all $y > 0$ and f is positive, we have in particular that $\ln f(x) \leq f(x) - 1$ for all $x \in \mathscr{X}$, which implies that $\overline{E}_C(\ln(f)) \leq \overline{E}_C(f-1) = \overline{E}_C(f) - 1 < 0$, using C5[20] for the first inequality and C4[20] for the equality. Consequently, it follows from Eq. (12.8) that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) \Big[\ln(f(\omega_{k+1})) - \overline{E}_C(\ln(f))\Big]}{\sum_{k=0}^{n-1} S(\omega_{1:k})} > 0.$$

By recalling that S is a recursive (total) selection process that accepts ω, we conclude that ω isn't (w)CH-random for C. □

The above result also confirms the relation between C- and CH-randomness in Proposition 12.2[103] for (almost computable) stationary forecasting systems, because obviously $\overline{\mathbb{T}}_{\mathrm{CH}}(C) \subseteq \overline{\mathbb{T}}_{\mathrm{C}}(C)$ for any $C \in \mathscr{C}(\mathscr{X})$, and hence, $\Omega_{\mathrm{C}}(C) \subseteq \Omega_{\mathrm{CH}}(C)$ for any $C \in \mathscr{C}(\mathscr{X})$.

⚃

Test-theoretic notions of randomness

Chapter ⚁$_{49}$ resolved around generalising the so-called *martingale-theoretic* approach to randomness by allowing for imprecise-probabilistic uncertainty models, where a path $\omega \in \Omega$ is considered to be martingale-theoretically random for a forecasting system if there's no specific type of supermartingale that becomes unbounded on ω in some specific way.

Of course, you can come up with many such martingale-theoretic randomness notions. What makes a randomness notion interesting then? When is its definition natural? For one thing, according to the classical precise-probabilistic literature, and as has also been discussed in Sections 8_{50} to 10_{66}, an interesting randomness notion should have an intuitive interpretation, it has to satisfy a number of interesting properties, and its definition should be reasonably robust with respect to changes (to the set of betting strategies). Another classical criterion is that there should be different ways to approach and define the algorithmic randomness notions,[24] besides the martingale-theoretic one [23, 31, 32]: via randomness tests [1, 30, 32], via Kolmogorov complexity [1, 30, 32, 83, 84], via order-preserving transformations of the event tree associated with a sequence of outcomes [1], via specific limit laws (such as Lévy's zero-one law) [85, 86], and so on.

In this chapter, we consider one of these alternative approaches, the randomness test approach. Intuitively speaking, in a classical precise-probabilistic setting, a path $\omega \in \Omega$ is regarded as test-theoretically random for a precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ if there's no computable way

[24] Although (w)CH-randomness isn't considered an 'interesting' randomness notion from a precise-probabilistic perspective, this of course reminds of our work in Sections 11_{87} and 12_{102} that equips (w)CH-randomness with several equivalent characterisations.

to specify a set of probability zero containing this path [9]. More precisely, the randomness of a path $\omega \in \Omega$ with respect to a precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ is tested as follows: a path ω is test-theoretically[25] random for φ_{pr} if it's impossible to specify—in some effectively implementable way—for every positive threshold $\delta > 0$ a set of paths that contains ω and that's small, in the sense that its probability is smaller than δ. In this definition, such a collection of sets is called a *null cover*—because the probability of its intersection is lower than every $\delta > 0$ and therefore zero—or a *test*, so randomness amounts to testing, for any such test, whether the path in question does *not* belong to its intersection.

In Section $13_{\curvearrowright}$, we start by showing how we can define specific tests—or null covers—involving credal sets that allow us to introduce two new flavours of so-called *test(-theoretic) randomness* for imprecise-probabilistic uncertainty models: one reminiscent of the original Martin-Löf approach, and another of the original Schnorr approach. We then proceed in Sections 14_{119} and 15_{136} to show that the test-theoretic notions of Martin-Löf and Schnorr randomness, respectively, are, under some computability and non-degeneracy conditions on the forecasting system, equivalent to the respective martingale-theoretic notions introduced in Chapter ⚁$_{49}$. We thus succeed in extending, to our more general imprecise probabilities context, earlier results by Schnorr [1] and Levin [4] showing that these test- and martingale-theoretic randomness notions are essentially equivalent for precise-probabilistic uncertainty models.[26] As a bonus, we use our argumentation in Sections 14.1_{120} and 14.2_{121} to prove in Section 14.3_{126} that there are so-called *universal* test supermartingales and *universal* randomness tests for our generalisations of Martin-Löf randomness.

We're actually not the first to allow for imprecise-probabilistic uncertainty models in test-theoretic approaches to algorithmic randomness. Another measure-theoretic notion of randomness that allows for imprecise-probabilistic (as well as non-computable) uncertainty models was put forward by Levin in 1973 and is nowadays known as *uniform randomness* [4, 5, 6]. This notion of uniform randomness allows for imprecision by considering so-called 'effectively compact classes of probability measures'. In Section 14.4_{130}, after proving the equivalence between our martingale- and test-theoretic versions of Martin-Löf randomness in Sections 14.1_{120} and 14.2_{121}, and after introducing the *universal* test in Section 14.3_{126}, we prove that our notion of

[25] We're well aware that the term 'test-theoretic' could be construed as somewhat misleading, because a martingale can also be considered as constituting a test. An alternative term for this approach that's sometimes used in the literature, is 'measure-theoretic', but that term isn't appropriate in the present imprecise probabilities context either, because as we'll see further on, the relevant objects involved are no longer probability measures but sublinear upper expectation functionals. We'll therefore stick to the term 'test-theoretic' when dealing with forecasting systems, for lack of a better alternative, and also use the term 'measure-theoretic' when dealing with measures.

[26] Schnorr in fact only proves this for the fair-coin forecasting system $\varphi_{1/2}$.

Martin-Löf test randomness for *computable* (imprecise-probabilistic) forecasting systems can be reinterpreted as a special case of *uniform randomness*. Together with the discussion in Sections 14.1$_{120}$ and 14.2$_{121}$, this then leads in effect to a previously non-existing martingale-theoretic account of uniform randomness, at least in the special case covered by our notion of Martin-Löf test randomness.

13 Test-theoretic randomness definitions

Let's turn to a 'test-theoretic', or *randomness test*, approach to defining Martin-Löf and Schnorr randomness for (imprecise-probabilistic) forecasting systems, which will be inspired by the existing corresponding notions for fair-coin, or more generally, computable precise forecasting systems [1, 3, 4, 30, 32]. In Sections 13.1 and 13.2$_{\curvearrowright}$, we'll respectively introduce the imprecise-probabilistic counterparts of Martin-Löf tests (ML-tests) and Schnorr tests (S-tests), which we'll then use in Section 13.3$_{119}$ to define two test-theoretic randomness notions: Martin-Löf test randomness (ML-test-randomness) and Schnorr test randomness (S-test-randomness).

13.1 Martin-Löf tests

Let's begin our discussion of Martin-Löf tests with a few notational conventions that will prove useful for the remainder of this chapter. With any subset A of $\mathbb{N}_0 \times \mathbb{S}$, we can associate a sequence A_n of subsets of $\mathbb{S}$, defined by

$$A_n := \{s \in \mathbb{S} \colon (n, s) \in A\} \text{ for all } n \in \mathbb{N}_0.$$

With each such A_n, we can associate the set of paths $[\![A_n]\!]$. If the set A is recursively enumerable, then we say that the $[\![A_n]\!]$ constitute a *computable sequence of effectively open sets*, as already introduced in Section 7.2$_{44}$.

Under the classical precise-probabilistic approach to algorithmic randomness, a sequence of global events $G_n \subseteq \Omega$ is now a Martin-Löf test [30] for a computable precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$—or the measure $\mu^{\varphi_{\mathrm{pr}}}$—if there's some recursively enumerable subset A of $\mathbb{N}_0 \times \mathbb{S}$ such that $G_n = [\![A_n]\!]$ and $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!]) = \mu^{\varphi_{\mathrm{pr}}}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$ [32, Definition 6.2.1]. The following definition trivially generalises this idea to our present—imprecise—context. It will lead in Section 13.3$_{119}$ further on to a suitable generalisation of Martin-Löf's randomness definition that allows for *imprecise-probabilistic* forecasting systems. Here too, we'll continue to speak of Martin-Löf tests also when φ is no longer precise, computable, or non-degenerate; in the classical precise-probabilistic literature, this approach where randomness tests—that are associated with a non-computable measure—have no access to the measure by an oracle is known as *Hippocratic* or *Blind randomness* [6, 87, 88].

Definition 13.1 (Martin-Löf test)**.** We call a sequence of global events $G_n \subseteq \Omega$ an ML-*test* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if there's some recursively enumerable subset A of $\mathbb{N}_0 \times \mathbb{S}$ such that for the associated computable sequence of effectively open sets $[\![A_n]\!]$, we have that $G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$.

We may always—and often will—assume without loss of generality that the subsets A_n of the event tree $\mathbb{S}$ that constitute the ML-test are *partial cuts*. Moreover, we can even assume the set A to be *recursive* rather than merely recursively enumerable, because there's actually a single algorithm that turns any recursively enumerable set $B \subseteq \mathbb{S}$ into a recursive partial cut $B' \subseteq \mathbb{S}$ such that $[\![B]\!] = [\![B']\!]$.[27] We refer to Ref. [32, Sec. 2.19] for discussion and proofs; see also the related discussions in Refs. [1, Korollar 4.10, p. 37] and [74, Lemma 2, Section 5.6].[28]

Corollary 13.2. *A sequence of global events G_n is an* ML*-test for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if there's some* recursive *subset A of $\mathbb{N}_0 \times \mathbb{S}$ such that A_n is a* partial cut, *$G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$.*

In what follows, we'll also use the term ML-*test* to refer to a subset A of $\mathbb{N}_0 \times \mathbb{S}$ that *represents* the ML-test G_n in the specific sense that $G_n = [\![A_n]\!]$ for all $n \in \mathbb{N}_0$. Due to Corollary 13.2, we can always assume such subsets A of $\mathbb{N}_0 \times \mathbb{S}$ to be recursive, and the corresponding A_n to be partial cuts. But we'll never assume that these simplifications are in place without explicitly saying so.

13.2 Schnorr tests

In order to propose a suitable generalisation of Schnorr's definition of a totally recursive sequential test [1, Def. (8.1), p. 63] for the (precise) fair-coin forecasting system $\varphi_{1/2}$ that associates a constant precise forecast $\varphi_{1/2}(s) := 1/2$ with each situation $s \in \mathbb{S}$, we need a few more notations. Starting from any subset A of $\mathbb{N}_0 \times \mathbb{S}$, we let

$$\left.\begin{aligned} A_n^{<\ell} &:= A_n \cap \{t \in \mathbb{S} \colon |t| < \ell\} \\ A_n^{\geq\ell} &:= A_n \cap \{t \in \mathbb{S} \colon |t| \geq \ell\} \end{aligned}\right\} \text{ for all } n, \ell \in \mathbb{N}_0. \tag{13.3}$$

In the important special case that A_n is a partial cut, the global event $[\![A_n]\!]$ is the disjoint union of the global events $[\![A_n^{<\ell}]\!]$ and $[\![A_n^{\geq\ell}]\!]$, implying that $\mathbb{I}_{[\![A_n]\!]} = \mathbb{I}_{[\![A_n^{<\ell}]\!]} + \mathbb{I}_{[\![A_n^{\geq\ell}]\!]}$.

Here as well, we'll continue to speak of S-tests also when φ is no longer the precise, computable and non-degenerate $\varphi_{1/2}$.

[27] When starting from a recursive subset $B \subseteq \mathbb{S}$, there's an easy way to obtain a recursive (sub)set $B' \subseteq \mathbb{S}$ such that $[\![B]\!] = [\![B']\!]$: simply consider the set $B' := \{s \in B \colon (\forall t \sqsubset s)\, t \notin B\} \subseteq B$.

[28] In truth, these references actually only consider binary state spaces. Nevertheless, we still chose to use them since the extension to arbitrary but finite state spaces is obvious and immediate.

Definition 13.4 (S-test)**.** We call a sequence of global events $G_n \subseteq \Omega$ an S-*test* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if there's some *recursive* subset A of $\mathbb{N}_0 \times \mathbb{S}$—called its *representation*—and some recursive map $e\colon \mathbb{N}_0{}^2 \to \mathbb{N}_0$—called its *tail bound*—such that $G_n = [\![A_n]\!]$ and $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and

$$\overline{P}^{\varphi}([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]) \leq 2^{-N} \text{ for all } (N,n) \in \mathbb{N}_0{}^2 \text{ and all } \ell \geq e(N,n). \tag{13.5}$$

As for the case of ML-tests, we can assume without loss of generality that the representation A is such that the A_n are partial cuts, at which point $[\![A_n]\!] \setminus [\![A_n^{<\ell}]\!] = [\![A_n^{\geq\ell}]\!]$ in Eq. (13.5). Moreover, we can assume without loss of generality that there's no dependence of the tail bound e on the index n of the $[\![A_n^{\geq\ell}]\!]$. The proposition below also shows that these simplifications can be implemented independently.

Proposition 13.6. *Consider any* S*-test* G_n *for a forecasting system* $\varphi \in \Phi(\mathscr{X})$ *with representation* $C \subseteq \mathbb{N}_0 \times \mathbb{S}$*. Then*

(i) *it also has a representation A such that $[\![A_n]\!] = [\![C_n]\!]$, $[\![A_n^{<l}]\!] = [\![C_n^{<l}]\!]$ and A_n is a partial cut for all $n, l \in \mathbb{N}_0$;*

(ii) *it has a tail bound e that doesn't depend on the index n of the $[\![C_n]\!] \setminus [\![C_n^{<\ell}]\!]$, meaning that $e(N,n) = e(N,n') =: e(N)$ for all $N, n, n' \in \mathbb{N}_0$, and that moreover is a growth function.*

Proof. By assumption, the representation C is a recursive subset of $\mathbb{N}_0 \times \mathbb{S}$ such that $G_n = [\![C_n]\!]$ and $\overline{P}^{\varphi}([\![C_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and such that there's some recursive map $e'\colon \mathbb{N}_0{}^2 \to \mathbb{N}_0$ such that $\overline{P}^{\varphi}([\![C_n]\!] \setminus [\![C_n^{<\ell}]\!]) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0{}^2$ and all $\ell \geq e'(N,n)$.

For the proof of the first statement, consider for any $n \in \mathbb{N}_0$, the set of situations

$$A_n := \{s \in C_n \colon (\forall t \sqsubset s)\, t \notin C_n\} \subseteq C_n,$$

which is clearly a partial cut and recursive uniformly in n. Of course, the corresponding $A := \{(n,s) \colon n \in \mathbb{N}_0 \text{ and } s \in A_n\} \subseteq C$ is then recursive. It follows readily from our construction that $[\![A_n]\!] = [\![C_n]\!]$ and $[\![A_n^{<\ell}]\!] = [\![C_n^{<\ell}]\!]$ for all $n, \ell \in \mathbb{N}_0$.

For the proof of the second statement, define $e\colon \mathbb{N}_0 \to \mathbb{N}_0$ by letting

$$e(N) := N + \max_{m=0}^{N} \max_{n=0}^{N} e'(m,n) \text{ for all } N \in \mathbb{N}_0.$$

Clearly, the map e is recursive because e' is. It's non-decreasing because

$$e(N+1) = N+1+\max_{m=0}^{N+1}\max_{n=0}^{N+1} e'(m,n) \geq N + \max_{m=0}^{N}\max_{n=0}^{N} e'(m,n) = e(N) \text{ for all } N \in \mathbb{N}_0,$$

and it's unbounded because $e(N) \geq N$ for all $N \in \mathbb{N}_0$. We conclude that e is a growth function. Now, fix any $N \in \mathbb{N}_0$ and $n \in \mathbb{N}_0$. Then there are two possibilities. The first is that $n \leq N$, and then for all $\ell \geq e(N)$ also $\ell \geq e'(N,n)$, and therefore, as we know from the beginning of this proof,

$$\overline{P}^{\varphi}([\![C_n]\!] \setminus [\![C_n^{<\ell}]\!]) \leq 2^{-N}.$$

The other possibility is that $n > N$, and then trivially for all $\ell \geq e(N)$

$$\overline{P}^{\varphi}\big(\llbracket C_n\rrbracket \setminus \llbracket C_n^{<\ell}\rrbracket\big) \leq \overline{P}^{\varphi}\big(\llbracket C_n\rrbracket\big) \leq 2^{-n} \leq 2^{-N},$$

where the first inequality follows from P3$_{35}$, and the penultimate one, as explained at the beginning of this proof, follows from the assumption. □

We'll also use the term S-*test* to refer to its representation A. So, an S-test is an ML-test with the additional property that it's always assumed to be recursive rather than merely recursively enumerable, and that the upper probabilities of its 'tail global events' converge to zero effectively. As indicated above, we can, and often will, assume that the sets A_n are partial cuts and that the tail bound is a univariate growth function. But we'll never assume that these simplifications are in place without explicitly saying so.

Let's now investigate our notion of an S-test in some more detail. First of all, we study how it relates to Schnorr's definition of a totally recursive sequential test [1, Def. (8.1), p. 63] for the (precise) fair-coin forecasting system $\varphi_{1/2}$.

Schnorr calls a recursive subset A of $\mathbb{N}_0 \times \mathbb{S}$ a *totally recursive sequential test* provided that $P^{\varphi_{1/2}}\big(\llbracket A_n\rrbracket\big) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$ and, *additionally*, the sequence of real numbers $P^{\varphi_{1/2}}\big(\llbracket A_n\rrbracket\big)$ is computable. Our additional condition (13.5)$_{\curvearrowleft}$ in Definition 13.4$_{\curvearrowleft}$ above therefore seems somewhat more involved than Schnorr's additional computability requirement for the sequence $P^{\varphi_{1/2}}(\llbracket A_n\rrbracket)$.

Let's now show, by means of Propositions 13.7 and 13.10$_{118}$ below, that that's only an illusion. Indeed, in Proposition 13.7 we show that our additional condition (13.5)$_{\curvearrowleft}$ implies the Schnorr-like additional computability requirement, even in the case of more general computable imprecise-probabilistic forecasting systems. And in Proposition 13.10$_{118}$, we prove that for general computable but *precise* forecasting systems the Schnorr-like additional requirement implies our additional effective convergence condition.

Proposition 13.7. *If $A \subseteq \mathbb{N}_0 \times \mathbb{S}$ is an* S*-test for a computable forecasting system $\varphi \in \Phi(\mathscr{X})$, then the $\overline{P}^{\varphi}\big(\llbracket A_n\rrbracket\big)$ constitute a computable sequence of real numbers.*

Proof of Proposition 13.7. Given the assumptions, an appropriate instantiation of our Workhorse Lemma 13.9$_{\curvearrowright}$ [with $\mathcal{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n,\ell,s) \in \mathbb{N}_0{}^2 \times \mathbb{S} \colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$] guarantees that the real map $(n,\ell) \mapsto \overline{P}^{\varphi}\big(\llbracket A_n^{<\ell}\rrbracket\big)$ is computable. Moreover, for all $n,\ell \in \mathbb{N}_0$,

$$\big|\overline{P}^{\varphi}\big(\llbracket A_n\rrbracket\big) - \overline{P}^{\varphi}\big(\llbracket A_n^{<\ell}\rrbracket\big)\big| = \overline{P}^{\varphi}\big(\llbracket A_n\rrbracket\big) - \overline{P}^{\varphi}\big(\llbracket A_n^{<\ell}\rrbracket\big) \leq \overline{P}^{\varphi}\big(\llbracket A_n\rrbracket \setminus \llbracket A_n^{<\ell}\rrbracket\big), \tag{13.8}$$

where the equality follows from P3$_{35}$, and the inequality follows from P2$_{35}$. Since A is an S-test, we know that it has a tail bound, so there's some recursive map $e \colon \mathbb{N}_0{}^2 \to \mathbb{N}_0$

such that $\overline{P}^{\varphi}(\llbracket A_n \rrbracket \setminus \llbracket A_n^{\geq \ell} \rrbracket) \leq 2^{-N}$ for all $(N,n) \in \mathbb{N}_0{}^2$ and all $\ell \geq e(N,n)$, and if we combine this with the inequality in Eq. (13.8), this leads to

$$\left|\overline{P}^{\varphi}(\llbracket A_n \rrbracket) - \overline{P}^{\varphi}(\llbracket A_n^{<\ell} \rrbracket)\right| \leq 2^{-N} \text{ for all } (N,n) \in \mathbb{N}_0{}^2 \text{ and all } \ell \geq e(N,n).$$

Since this tells us that the computable real map $(n,\ell) \mapsto \overline{P}^{\varphi}(\llbracket A_n^{<\ell} \rrbracket)$ converges effectively to the sequence of real numbers $\overline{P}^{\varphi}(\llbracket A_n \rrbracket)$, we conclude that $\overline{P}^{\varphi}(\llbracket A_n \rrbracket)$ is a computable sequence of real numbers. □

In the above proof, we made use of the following general and powerful lemma, various instantiations of which will help us through many a complicated argument further on.

Lemma 13.9 (Workhorse Lemma)**.** *Consider any computable forecasting system $\varphi \in \Phi(\mathscr{X})$, any countable set $\mathcal{D}$ whose elements can be encoded by the natural numbers, and any recursive set $C \subseteq \mathcal{D} \times \mathbb{N}_0 \times \mathbb{S}$ such that $|s| \leq p$ for all $(d,p,s) \in C$. Then $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | s)$ is a computable real uniformly in d, p and s, with $C_d^p := \{s \in \mathbb{S} \colon (d,p,s) \in C\}$ for all $p \in \mathbb{N}_0$ and $d \in \mathcal{D}$.*

Proof. We start by observing that C_d^p is a finite recursive set of situations, uniformly in d and p. Similarly,

$$C_d^{p\prime} := \{t \in \mathbb{S} \colon |t| = p \text{ and } C_d^p \sqsubseteq t\}$$

is clearly also a finite recursive set of situations, uniformly in d and p. Moreover, it's a partial cut.

Another important observation is that there are three mutually exclusive possibilities for any of the sets C_d^p and any $t \in \mathbb{S}$. The first possibility is that $C_d^p \sqsubseteq t$, which can be checked recursively, in the sense that there's a recursive map $q \colon \mathcal{D} \times \mathbb{N}_0 \times \mathbb{S} \to \{0,1\}$ such that, for all $(d,p,t) \in \mathcal{D} \times \mathbb{N}_0 \times \mathbb{S}$, $q(d,p,t) = 1$ if $C_d^p \sqsubseteq t$, and $q(d,p,t) = 0$ otherwise. In that case, we know from Corollary 6.15(i)[36] that $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | t) = 1$. The second possibility is that $t \parallel C_d^p$, which can be checked recursively as well. In that case, we know from Corollary 6.15(i)[36] that $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | t) = 0$. The third, final, and most involved possibility is that $t \sqsubset C_d^p$, which can also be checked recursively.

It's clear from this discussion that the computability of $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | s)$ is trivial when $s \parallel C_d^p$ or $C_d^p \sqsubseteq s$, so we'll from now on only pay attention to the case that $s \sqsubset C_d^p$. Since, obviously, $\llbracket C_d^{p\prime} \rrbracket = \llbracket C_d^p \rrbracket$ and in this case also $s \sqsubset C_d^{p\prime}$, we'll focus on the computability of $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | s)$.

For any $t \sqsupseteq s$ with $|t| = p$, we infer from the discussion above that $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) = 1$ if $t \in C_d^{p\prime}$ and $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) = 0$ otherwise. Clearly then, $\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t)$ is a computable real uniformly in d, p and t with $|t| = p$.

In a next step, we find by applying P5[36] that, for any $t \sqsupseteq s$ with $|t| = p - 1$,

$$\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t) = \overline{E}_{\varphi(t)}\Big(\overline{P}^{\varphi}(\llbracket C_d^{p\prime} \rrbracket | t\,\cdot)\Big),$$

which, by Lemma 7.7[47], is a computable real uniformly in d, p and t with $|t| = p - 1$, simply because φ is computable.

By applying P5[36] to situations $t \sqsupseteq s$ with successively smaller $|t|$, we eventually end up in the situation s after a finite number of steps, which implies that $\overline{P}^{\varphi}(\llbracket C_d^p \rrbracket | s)$ is a computable real, uniformly in d, p and s. □

The next proposition is concerned with the special case of precise forecasting systems $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$. We recall from Section 6.4₃₉ that the martingale-theoretic global upper and lower probabilities then coincide with the standard probability measure $\mu^{\varphi_{\mathrm{pr}}}$ associated with the local probability mass functions implicit in φ_{pr} on all (open) events $A \in \mathscr{B}(\Omega)$, and that for each partial cut K, the corresponding set of paths $[\![K]\!]$ is open in the Cantor topology, so $\underline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = \overline{P}^{\varphi_{\mathrm{pr}}}([\![K]\!]) = P^{\varphi_{\mathrm{pr}}}([\![K]\!])$. We'll use this fact implicitly and freely in the formulation and proof of the result below.

Proposition 13.10. *Consider an* ML-*test* G_n *for a computable* precise *forecasting system* $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$. *If the* $P^{\varphi_{\mathrm{pr}}}(G_n)$ *constitute a computable sequence of real numbers, then* G_n *is an* S-*test for* φ_{pr}.

Proof. By Corollary 13.2₁₁₄, we may assume without loss of generality that there's a recursive $A \subseteq \mathbb{N}_0 \times \mathbb{S}$ such that A_n is a partial cut, $G_n = [\![A_n]\!]$ and $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$. Assume that the $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$ constitute a computable sequence of real numbers. Then, by Definition 13.4₁₁₅, it suffices to prove that there's some recursive map $e \colon {\mathbb{N}_0}^2 \to \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \leq 2^{-N}$ for all $(N, n) \in {\mathbb{N}_0}^2$ and all $\ell \geq e(N, n)$.

To do so, we start by proving that the real map $(n, \ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is computable and that $\lim_{\ell \to \infty} P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) = 0$ for all $n \in \mathbb{N}_0$. First of all, observe that the computability of the forecasting system φ_{pr}, the recursive character of the finite partial cuts $A_n^{<\ell}$ and an appropriate instantiation of our Workhorse Lemma 13.9↶ [with $\mathcal{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n, \ell, s) \in {\mathbb{N}_0}^2 \times \mathbb{S} \colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$] allow us to infer that the real map $(n, \ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!])$ is computable. Since the forecasting system φ_{pr} is precise, and since $\mathbb{I}_{[\![A_n]\!]} = \mathbb{I}_{[\![A_n^{<\ell}]\!]} + \mathbb{I}_{[\![A_n^{\geq \ell}]\!]}$ for all $(n, \ell) \in {\mathbb{N}_0}^2$ due to A_n being a partial cut, we infer from Eq. (6.23)₄₀ that

$$P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) = P^{\varphi_{\mathrm{pr}}}([\![A_n]\!]) - P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!]). \tag{13.11}$$

Since $P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$ is a computable sequence of real numbers and $(n, \ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!])$ is a computable real map, it follows from Eq. (13.11) that $(n, \ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is a computable real map. Furthermore, since $\mathbb{I}_{[\![A_n^{<\ell}]\!]} \nearrow \mathbb{I}_{[\![A_n]\!]}$ point-wise as $\ell \to \infty$, it follows from P3₃₅ and P4₃₆ that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{<\ell}]\!]) \nearrow P^{\varphi_{\mathrm{pr}}}([\![A_n]\!])$ as $\ell \to \infty$, and therefore also that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \searrow 0$ as $\ell \to \infty$, for all $n \in \mathbb{N}_0$.

We're now ready to prove that there's some recursive map $e \colon {\mathbb{N}_0}^2 \to \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \leq 2^{-N}$ for all $(N, n) \in {\mathbb{N}_0}^2$ and all $\ell \geq e(N, n)$. Since $(n, \ell) \mapsto P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!])$ is a computable real map, there's some recursive rational map $q \colon {\mathbb{N}_0}^2 \times \mathbb{N} \to \mathbb{Q}$ such that

$$\left|P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) - q(n, \ell, N)\right| \leq 2^{-N} \text{ for all } (n, \ell, N) \in {\mathbb{N}_0}^2 \times \mathbb{N}. \tag{13.12}$$

Define the map $e \colon {\mathbb{N}_0}^2 \to \mathbb{N}_0$ by

$$e(N, n) := \min\left\{\ell \in \mathbb{N}_0 \colon q(n, \ell, N+2) < 2^{-(N+1)}\right\} \text{ for all } (N, n) \in {\mathbb{N}_0}^2. \tag{13.13}$$

Clearly, if we can prove that the set of natural numbers in the definition above is always non-empty, then the map e will be well-defined and recursive. To do so, fix any $(N, n) \in {\mathbb{N}_0}^2$, and observe that since $P^{\varphi_{\mathrm{pr}}}([\![A_n^{\geq \ell}]\!]) \searrow 0$ as $\ell \to \infty$, there always is some $\ell_o \in \mathbb{N}_0$ such that $P^{\varphi_{\mathrm{pr}}}(\llbracket A_n^{\geq \ell_o} \rrbracket) < 2^{-(N+2)}$. For this same ℓ_o, it then indeed follows from Eq. (13.12)↶ that

$$q(n, \ell_o, N+2) \leq P^{\varphi_{\mathrm{pr}}}(\llbracket A_n^{\geq \ell_o} \rrbracket) + 2^{-(N+2)} < 2^{-(N+2)} + 2^{-(N+2)} = 2^{-(N+1)}.$$

To complete the proof, consider any $n, N \in \mathbb{N}_0$ and any $\ell \geq e(N, n)$. Then, indeed,

$$\begin{aligned} P^{\varphi_{\mathrm{pr}}}(\llbracket A_n^{\geq \ell} \rrbracket) &\leq P^{\varphi_{\mathrm{pr}}}(\llbracket A_n^{\geq e(N,n)} \rrbracket) \leq q(n, e(N,n), N+2) + 2^{-(N+2)} \\ &< 2^{-(N+1)} + 2^{-(N+2)} < 2^{-N}, \end{aligned}$$

where the first inequality follows from $\ell \geq e(N, n)$ and P3$_{35}$, the second inequality follows from Eq. (13.12)↶, and the third inequality follows from Eq. (13.13)↶. □

13.3 Defining Martin-Löf and Schnorr test randomness

With the definitions of ML- and S-tests for a forecasting system at hand, we're now in a position to generalise both Martin-Löf's and Schnorr's definition for randomness using randomness tests, from fair-coin to imprecise-probabilistic forecasting systems. Intuitively, for both randomness notions, a path is test-theoretically random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if there's no computable way to specify a set of upper probability zero containing this path; after all, if a path is contained in a null set of paths that have some rare (but computably specifiable) property, then it indeed shouldn't be considered random. In the definition below, a computably specifiable set is taken to be the intersection of a collection of sets $\llbracket A_n \rrbracket$, with $n \in \mathbb{N}_0$, that constitute an ML-test or an S-test; the upper probability of this intersection is zero because it's bounded above by 2^{-n} for all $n \in \mathbb{N}_0$.

Definition 13.14 (Test randomness)**.** Consider a forecasting system $\varphi \in \Phi(\mathscr{X})$. Then we call a path $\omega \in \Omega$

(i) ML-*test-random* for φ if $\omega \notin \bigcap_{m \in \mathbb{N}_0} \llbracket A_m \rrbracket$, for all ML-tests A for φ;
(ii) S-*test-random* for φ if $\omega \notin \bigcap_{m \in \mathbb{N}_0} \llbracket A_m \rrbracket$, for all S-tests A for φ.

We want to show in the next two sections that for forecasting systems that are *computable* and satisfy a simple additional non-degeneracy condition, our 'test-theoretic' and 'martingale-theoretic' notions of both Martin-Löf and Schnorr randomness are equivalent.

14 Equivalence of Martin-Löf and Martin-Löf test randomness

Let's start by considering ML-randomness. The proof of our claim, in Theorem 14.1↷ below, that the 'test-theoretic' and 'martingale-theoretic' versions for this type of randomness are equivalent, follows the spirit of a reasonably similar proof in a paper on precise prequential Martin-Löf randomness by Vovk and Shen [9, Proof of Theorem 1]. It allows us to extend Schnorr's line of reasoning for this equivalence [1, Secs. 5–9] from fair-coin to non-degenerate computable imprecise-probabilistic forecasting systems.

Theorem 14.1. *Consider any path $\omega \in \Omega$ and any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then ω is* ML*-random for φ if and only if it's* ML*-test-random for φ.*

Proof. This is immediate from Propositions 14.2 and 14.4↷ below. □

14.1 Martin-Löf test randomness implies Martin-Löf randomness

We begin with the more easily proved side of the equivalence, which relies rather heavily on Ville's inequality.

Proposition 14.2. *Consider any path $\omega \in \Omega$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$. If ω is* ML*-test-random for φ then it's also* ML*-random for φ.*

Proof. We give a proof by contraposition. Assume that ω isn't ML-random for φ, which implies that there's some lower semicomputable test supermartingale T for φ that becomes unbounded on ω, so $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. Now, let's consider the set

$$A := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon T(s) > 2^n\} \subseteq \mathbb{N}_0 \times \mathbb{S}.$$

That T is a lower semicomputable test supermartingale implies, by Lemma 14.3(iii)&(i), that A is an ML-test for φ with

$$[\![A_m]\!] := \left\{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\right\} \text{ for all } m \in \mathbb{N}_0.$$

That $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$ then implies that $\omega \in [\![A_m]\!]$ for all $m \in \mathbb{N}_0$, so ω isn't ML-test-random for φ either. □

Lemma 14.3. *Consider any lower semicomputable test supermartingale T for φ, and let $A := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} \colon T(s) > 2^n\}$. Then*

(i) $[\![A_m]\!] = \{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\}$ *for all $m \in \mathbb{N}_0$;*
(ii) $\overline{P}^{\varphi}([\![A_m]\!]) \leq 2^{-m}$ *for all $m \in \mathbb{N}_0$;*
(iii) *A is an* ML*-test.*

Proof. We begin with the proof of (i). Since, by its definition, $[\![A_m]\!] = \bigcup\{[\![s]\!] \colon s \in A_m\}$, we have the following chain of equivalences for any $\varpi \in \Omega$:

$$\begin{aligned}\varpi \in [\![A_m]\!] &\Leftrightarrow (\exists s \in A_m)(\varpi \in [\![s]\!]) \Leftrightarrow (\exists s \in \mathbb{S})(\varpi \in [\![s]\!] \text{ and } (m,s) \in A)\\ &\Leftrightarrow (\exists s \in \mathbb{S})(\varpi \in [\![s]\!] \text{ and } T(s) > 2^m) \Leftrightarrow (\exists n \in \mathbb{N}_0) T(\varpi_{1:n}) > 2^m,\end{aligned}$$

proving (i).

Next, we turn to the proof of (ii). If we recall that T is a non-negative supermartingale for φ with $T(\square) = 1$ and let $C := 2^m > 0$ in Ville's inequality [Proposition 6.18$_{37}$], then we find, also taking into account (i) and P3$_{35}$, that indeed,

$$\overline{P}^{\varphi}([\![A_m]\!]) = \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) > 2^m\right\}\right)$$

$$\leq \overline{P}^{\varphi}\Big(\Big\{\varpi \in \Omega\colon \sup_{n\in\mathbb{N}_0} T(\varpi_{1:n}) \geq 2^m\Big\}\Big) \leq \frac{1}{2^m} T(\square) = 2^{-m}.$$

For (iii)↶, it now only remains to prove that the set $A = \{(n,s) \in \mathbb{N}_0 \times \mathbb{S}\colon T(s) > 2^n\}$ is recursively enumerable. The lower semicomputability of T implies that the set $\{(s,q) \in \mathbb{S} \times \mathbb{Q}\colon T(s) > q\}$ is recursively enumerable, and hence, the set A is recursively enumerable as well. □

14.2 Martin-Löf randomness implies Martin-Löf test randomness

We'll now tackle the converse result of Proposition 14.2↶, whose proof is definitely more involved.

Proposition 14.4. *Consider any path $\omega \in \Omega$ and any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. If ω is* ML*-random for φ then it's also* ML*-test-random for φ.*

Proof. Again, we give a proof by contraposition. Assume that ω isn't ML-test-random for φ. This implies that there's some ML-test A such that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. Then, by Lemma 14.5, there's a lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$. This tells us that, indeed, ω isn't ML-random for φ. □

Lemma 14.5. *Consider any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$ and any* ML*-test A for φ. Then there's a lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ such that, for any path $\omega \in \Omega$, $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$ if $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$.*

Proof. Consider any path $\omega \in \Omega$ and assume that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. The idea behind the proof is an altered, much simplified and stripped-down version of an argument borrowed in its essence from a different proof in a paper by Vovk and Shen about precise prequential Martin-Löf randomness [9, Proof of Theorem 1]. It's actually quite straightforward when we ignore its technical complexities: we'll use the ML-test A to construct a lower semicomputable test supermartingale W for φ that becomes unbounded on ω. Although it might not appear so at first sight from the way we go about it, this W is essentially obtained by summing the non-negative supermartingales $\overline{P}^{\varphi}([\![A_n]\!]|\bullet)$, each of which is 'fully turned on' as soon as the partial cut A_n is reached. The main technical difficulty will be to prove that this W is lower semicomputable, and we'll take care of this task in a roundabout way, in a number of lemmas [Lemmas 14.11₁₂₃–14.13₁₂₄ below].

Back to the proof now. Recall from Corollary 13.2₁₁₄ that we may assume without loss of generality that the set A is recursive and that the corresponding A_n are partial cuts. We also recall from Eq. (13.3)₁₁₄ the definition of the partial cuts $A_n^{<\ell} := \{s \in \mathbb{S}\colon (n,s) \in A \text{ and } |s| < \ell\} \subseteq A_n$, for all $n, \ell \in \mathbb{N}_0$, with $[\![A_n]\!] = \bigcup_{\ell\in\mathbb{N}_0} [\![A_n^{<\ell}]\!]$.

We begin by considering the real processes $W_n^\ell := \overline{P}^{\varphi}([\![A_n^{<\ell}]\!]|\bullet)$, where $n, \ell \in \mathbb{N}_0$. By Lemma 14.11₁₂₃, each W_n^ℓ is a non-negative computable supermartingale. We infer from P3₃₅ that $\overline{P}^{\varphi}([\![A_n]\!]) \geq \overline{P}^{\varphi}([\![A_n^{<\ell}]\!]) = W_n^\ell(\square)$, and therefore, also invoking Lemma 14.11(ii)₁₂₃ and the assumption that $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$, we get that

$$0 \leq W_n^\ell(\square) \leq 2^{-n}. \tag{14.6}$$

Next, fix any $s \in \mathbb{S}$ and any $\ell \in \mathbb{N}_0$, and let $W^\ell(s) := \frac{1}{2}\sum_{n=0}^\infty W_n^\ell(s)$. Observe that, since all the terms $W_n^\ell(s)$ are non-negative by Lemma 14.11(ii)↷, $W^\ell(s)$ is a non-negative extended real number. We first check that it is real-valued, as in principle, the defining series might converge to ∞. Combine Eq. (14.6)↶ and Lemma 9.13[62] [with $\varphi' \to \varphi$ and $C_\varphi \colon \mathbb{S} \to \mathbb{N}$ a recursive natural-valued process for which $C_\varphi(\square) = 1$] to find that:

$$0 \leq W_n^\ell(s) \leq W_n^\ell(\square) C_\varphi(s) \leq C_\varphi(s) 2^{-n} \text{ for all } n \in \mathbb{N}_0, \tag{14.7}$$

whence also

$$0 \leq W^\ell(s) = \frac{1}{2}\sum_{n=0}^\infty W_n^\ell(s) \leq C_\varphi(s), \tag{14.8}$$

which shows that $W^\ell(s)$ is bounded above, and therefore indeed real. Moreover, it follows from Lemma 14.11(ii)↷ that $W^\ell(s) \leq W^{\ell+1}(s)$ for all $\ell \in \mathbb{N}_0$, which guarantees that the limit $W(s) := \lim_{\ell\to\infty} W^\ell(s) = \sup_{\ell\in\mathbb{N}_0} W^\ell(s)$ exists as an extended real number. It's moreover real-valued, because we infer from taking the limit in Eq. (14.8) that also

$$0 \leq W(s) \leq C_\varphi(s). \tag{14.9}$$

We've thus defined a non-negative real process W, and we infer from Lemma 14.12↷ that W is a non-negative lower semicomputable supermartingale for φ. In addition, we infer from Eq. (14.9) that $0 \leq W(\square) \leq 1$.

Moreover, since $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$, we see that W is unbounded on ω. Indeed, consider any $n \in \mathbb{N}_0$, then since $\omega \in [\![A_n]\!]$, we can fix some $m_n \in \mathbb{N}_0$ such that $W_n^\ell(\omega_{1:m}) = 1$ for all $m, \ell \geq m_n$ [To see this, observe that $\omega \in [\![A_n]\!]$ first of all implies that there's some (unique) $O_n \in \mathbb{N}_0$ for which $\omega_{1:O_n} \in A_n$, and secondly that then $\omega_{1:O_n} \in A_n^{<\ell} \Leftrightarrow \ell > O_n$; so if $\ell \geq O_n + 1$ then $\omega_{1:m} \sqsupseteq A_n^{<\ell}$ for all $m \geq O_n$; now use Lemma 14.11(iii)↷ to find that then also $W_n^\ell(\omega_{1:m}) = 1$ for all $m \geq O_n$. Finally, let $m_n := O_n + 1$]. So, if we consider any $N \in \mathbb{N}_0$ and let $M_N := \max\{m_n \colon n \in \{0, 1, \dots, N\}\}$, then

$$W^\ell(\omega_{1:m}) \geq \frac{1}{2}\sum_{n=0}^N W_n^\ell(\omega_{1:m}) = \frac{1}{2}(N+1) \text{ for all } m, \ell \geq M_N,$$

and therefore also

$$W(\omega_{1:m}) \geq \frac{1}{2}(N+1) \text{ for all } m \geq M_N,$$

which shows that, in fact,

$$\lim_{m\to\infty} W(\omega_{1:m}) = \infty. \tag{14.10}$$

The relevant condition being $\overline{E}_{\varphi(\square)}(W(\square\,\cdot)) \leq W(\square)$, we see that replacing $W(\square) \leq 1$ by 1 doesn't change the supermartingale character of W, and doing so leads to a lower semicomputable test supermartingale for φ that's unbounded on ω. □

We want to draw attention to the fact that the test supermartingale W in Lemma 14.5↶ not only becomes unbounded but actually *converges to* ∞ on every path in the global event $\bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$ associated with the ML-test A. We'll come back to this in Section 14.3[126], when we show that ML-randomness for a non-degenerate computable forecasting system can be checked using a single (universal) lower semicomputable supermartingale; see in particular Corollary 14.24[129].

Lemma 14.11. *For any $n, \ell \in \mathbb{N}_0$, consider the real process W_n^ℓ, defined in the proof of Proposition 14.4*[121] *by $W_n^\ell := \overline{P}^\varphi(\llbracket A_n^{<\ell} \rrbracket | \bullet)$. Then the following statements hold:*

(i) $W_n^\ell(s) = \overline{E}_{\varphi(s)}(W_n^\ell(s\,\cdot))$ *for all* $s \in \mathbb{S}$;

(ii) $0 \leq W_n^\ell(s) \leq W_n^{\ell+1}(s) \leq 1$ *for all* $s \in \mathbb{S}$;

(iii) $W_n^\ell(s) = 1$ *for all* $s \sqsupseteq A_n^{<\ell}$;

(iv) *the real map* $(n, \ell, s) \mapsto W_n^\ell(s)$ *is computable.*

In particular, for all $n, \ell \in \mathbb{N}_0$, $W_n^\ell := \overline{P}^\varphi(\llbracket A_n^{<\ell} \rrbracket | \bullet)$ is a non-negative computable supermartingale for φ.

Proof. Statement (i) follows from P5[36].

The first and third inequalities in (ii) follow from P1[35]. The second inequality is a consequence of $A_n^{<\ell} \subseteq A_n^{<\ell+1}$ and the monotone character of the conditional lower expectation $\overline{P}^\varphi(\bullet|s)$ [use P3[35]].

Statement (iii) is an immediate consequence of Corollary 6.15(i)[36].

For the proof of (iv), consider that A is recursive and that the forecasting system φ is computable, and apply an appropriate instantiation of our Workhorse Lemma 13.9[117] [with $\mathcal{D} \to \mathbb{N}_0$, $d \to n$, $p \to \ell$ and $C \to \{(n, \ell, s) \in \mathbb{N}_0{}^2 \times \mathbb{S} \colon s \in A_n^{<\ell}\}$, and therefore $C_d^p \to A_n^{<\ell}$].

The rest of the proof is now immediate. □

Lemma 14.12. *The real process W, defined in the proof of Proposition 14.4*[121]*, is a non-negative lower semicomputable supermartingale for φ.*

Proof. First of all, recall from Eq. (14.9)↶ in the proof of Proposition 14.4[121] that W is indeed non-negative.

Next, define, for any $m, \ell \in \mathbb{N}_0$, the real process V_m^ℓ by letting $V_m^\ell(s) := \frac{1}{2}\sum_{n=0}^m W_n^\ell(s)$ for all $s \in \mathbb{S}$. It follows from Lemma 14.11(ii) that V_m^ℓ is non-negative. By Lemma 14.11(iv), the real map $(n, \ell, s) \mapsto W_n^\ell(s)$ is computable, so we see that so is the real map $(m, \ell, s) \mapsto V_m^\ell(s)$. Moreover, it's clear from the definition of the processes V_m^ℓ and W^ℓ that $V_m^\ell(s) \nearrow W^\ell(s)$ as $m \to \infty$, and that

$$\left|W^\ell(s) - V_m^\ell(s)\right| = \frac{1}{2}\sum_{n=m+1}^{\infty} W_n^\ell(s) \leq \frac{1}{2}C_\varphi(s)\sum_{n=m+1}^{\infty} 2^{-n} = \frac{1}{2}C_\varphi(s)2^{-m} \leq 2^{-m+L_{C_\varphi}(s)-1}$$

for all $\ell, m \in \mathbb{N}_0$ and all $s \in \mathbb{S}$,

where the first inequality follows from Eq. (14.7)↶ [with $C_\varphi \colon \mathbb{S} \to \mathbb{N}$ a recursive natural-valued process], and the second inequality is based on Lemma 14.13↷ and the notations introduced there [with L_{C_φ} recursive]. If we now consider the recursive map $e \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ defined by $e(N, s) := N + L_{C_\varphi}(s) - 1$, then we find that $|W^\ell(s) - V_m^\ell(s)| \leq 2^{-N}$ for all $(N, s) \in \mathbb{N}_0 \times \mathbb{S}$ and all $m \geq e(N, s)$, which guarantees that the real map $(\ell, s) \mapsto W^\ell(s)$ is computable.

Now, consider that for any $s \in \mathbb{S}$, $W^\ell(s) \nearrow W(s)$ as $\ell \to \infty$. Since we've just proved that $(\ell, s) \mapsto W^\ell(s)$ is a computable real map, we conclude that the process W is indeed lower semicomputable, as a point-wise limit of a non-decreasing computable sequence of computable real processes.

To complete the proof, we show that W is a supermartingale. It follows from C2

$$\overline{E}_{\varphi(s)}\big(\Delta V_m^\ell(s)\big)=\overline{E}_{\varphi(s)}\Big(\frac{1}{2}\sum_{n=0}^{m}\Delta W_n^\ell(s)\Big)\le\frac{1}{2}\sum_{n=0}^{m}\overline{E}_{\varphi(s)}\big(\Delta W_n^\ell(s)\big)\le 0\text{ for all }s\in\mathbb{S},$$

so V_m^ℓ is also a supermartingale. Since $V_m^\ell(s)\to W^\ell(s)$, we also find that $\Delta V_m^\ell(s)\to\Delta W^\ell(s)$ for all $s\in\mathbb{S}$. Since the gambles $\Delta V_m^\ell(s)$ are defined on the finite domain $\mathscr{X}$, this point-wise convergence also implies uniform convergence, so we can infer from C6$_{20}$ that

$$\overline{E}_{\varphi(s)}\big(\Delta W^\ell(s)\big)=\overline{E}_{\varphi(s)}\Big(\lim_{m\to\infty}\Delta V_m^\ell(s)\Big)=\lim_{m\to\infty}\overline{E}_{\varphi(s)}\big(\Delta V_m^\ell(s)\big)\le 0\text{ for all }s\in\mathbb{S}.$$

This shows that W^ℓ is also a supermartingale. And, since $W^\ell(s)\to W(s)$, we find that also $\Delta W^\ell(s)\to\Delta W(s)$ for all $s\in\mathbb{S}$. Since the gambles $\Delta W^\ell(s)$ are defined on the finite domain $\mathscr{X}$, this point-wise convergence also implies uniform convergence, so we can again infer from C6$_{20}$ that

$$\overline{E}_{\varphi(s)}(\Delta W(s))=\overline{E}_{\varphi(s)}\Big(\lim_{\ell\to\infty}\Delta W^\ell(s)\Big)=\lim_{\ell\to\infty}\overline{E}_{\varphi(s)}\big(\Delta W^\ell(s)\big)\le 0\text{ for all }s\in\mathbb{S}.$$

This shows that W is indeed a supermartingale. □

Lemma 14.13. *If the real process F is computable and $F\ge 1$, then there's a recursive map $L_F\colon\mathbb{S}\to\mathbb{N}$ such that $L_F\ge\log_2 F$, or equivalently, $F\le 2^{L_F}$.*

Proof. That F is computable implies that the non-negative process $\log_2 F$ is computable as well. That the non-negative real process $\log_2 F$ is computable means that there's some recursive map $q_F\colon\mathbb{S}\times\mathbb{N}\to\mathbb{Q}$ such that $|\log_2 F(s)-q_F(s,n)|\le 2^{-n}$ for all $(s,n)\in\mathbb{S}\times\mathbb{N}$, and therefore in particular that $|\log_2 F-q_F(\bullet,1)|\le 1$. Hence, $0\le\log_2 F\le 1+q_F(\bullet,1)\le 1+\lceil q_F(\bullet),1\rceil$ and $L_F:=1+\lceil q_F(\bullet,1)\rceil$ is a recursive and $\mathbb{N}$-valued process. □

Compared to the classical (precise) setting, *computability* alone isn't sufficient in the above proposition, as the following counterexample reveals. This is, essentially, a consequence of our preferring not to allow for extended real-valued test supermartingales; see also De Cooman and De Bock's discussion in Section 5.3 of Ref. [36].

Example 14.14. Consider the binary state space $\mathscr{X}=\{0,1\}$, any non-degenerate computable forecasting system $\varphi\in\Phi(\mathscr{X})$ and any path $\omega\in\mathscr{X}^{\mathbb{N}}$ that's ML-random for φ; that there always is such a path follows from Corollary 9.3$_{56}$. Let the degenerate forecasting system $\varphi_o\in\Phi(\mathscr{X})$ be defined by letting

$$\varphi_o(\square)(x):=\begin{cases}0&\text{if }x=\omega_1\\1&\text{if }x\neq\omega_1\end{cases}\text{ and }\varphi_o(s)(x):=\varphi(s)(x)$$

for all $s\in\mathbb{S}\setminus\{\square\}$ and $x\in\mathscr{X}$. We'll show that ω is ML-random but not ML-test-random for this modified forecasting system φ_o.

To show that ω isn't ML-test-random for φ_o, consider the recursive set $A := \bigcup_{n\in\mathbb{N}_0}\{(n,\omega_1)\} \subseteq \mathbb{N}_0 \times \mathbb{S}$, for which $A_n = \{\omega_1\}$ for all $n \in \mathbb{N}_0$, and therefore, obviously, $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. A is moreover an ML-test for φ_o, because, by E8$_{34}$, $\overline{P}^{\varphi_o}([\![A_n]\!]) = \overline{P}^{\varphi_o}([\![\omega_1]\!]) = \overline{E}^{\varphi_o}(\mathbb{I}_{[\![\omega_1]\!]}) = \overline{E}_{\varphi_o(\square)}(\mathbb{I}_{\omega_1}) = E_{\varphi_o(\square)}(\mathbb{I}_{\omega_1}) = \varphi_o(\square)(\omega_1) = 0$ for all $n \in \mathbb{N}_0$. Hence, ω can't be ML-test-random for φ_o.

To show that ω is ML-random for φ_o, assume towards contradiction that there's some lower semicomputable test supermartingale T_o for φ_o such that $\limsup_{n\to\infty} T_o(\omega_{1:n}) = \infty$. Fix any $M \in \mathbb{N}$ for which $\max\{T_o(1), T_o(0)\} < M$, and define the real process $T\colon \mathbb{S} \to \mathbb{R}$ by letting $T(\square) := 1$ and $T(s) := M^{-1}T_o(s)$ for all $s \in \mathbb{S} \setminus \{\square\}$; it's easy to check that T is a lower semicomputable test supermartingale for φ. Clearly, $\limsup_{n\to\infty} T(\omega_{1:n}) = \limsup_{n\to\infty} M^{-1}T_o(\omega_{1:n}) = \infty$, which contradicts the assumption that ω is ML-random for φ. ⋄

On the other hand, only imposing *non-degeneracy* isn't sufficient either in the above proposition: as the following counterexample shows, there's a positive—and therefore non-degenerate—non-computable precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ such that every path $\omega \in \Omega$ is ML-random for φ_{pr}, while no recursive path $\omega \in \Omega$ is ML-test-random for φ_{pr}; the construction below is based on techniques used in the proof of Theorem 20.1$_{193}$, which are in their turn based on notes by Alexander Shen.

Example 14.15. We'll construct such a positive non-computable precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$—for which every path is ML-random but for which no recursive path is ML-test-random—iteratively. Consider the binary state space $\mathscr{X} = \{0,1\}$, and let $(F_i)_{i\in\mathbb{N}}$ be an enumeration of all lower semicomputable non-negative processes; this is always possible by Lemma 7.6$_{46}$. We start by considering F_1. If $F_1(sx) > F_1(s)$ for an infinite number of $(s,x) \in \mathbb{S} \times \mathscr{X}$, then we fix some $s_1 \in \mathbb{S}$ such that $F_1(s_1x) > F_1(s_1)$ for some $x \in \mathscr{X}$. Let $\varphi_{\mathrm{pr}}(s_1)$ be equal to some positive probability mass function $m_1 \in \mathcal{M}(\mathscr{X})$ for which $E_{m_1}(F_1(s_1\,\cdot)) > F_1(s_1)$ [it's easy to infer from Eq. (5.2)$_{15}$ that this is always possible]. Otherwise, we let $s_1 := \square$. We continue by considering F_2. If $F_2(sx) > F_2(s)$ for an infinite number of $(s,x) \in \mathbb{S} \times \mathscr{X}$, then we (can) fix some $s_2 \in \mathbb{S}$ such that $|s_2| > |s_1| + 1$ and $F_2(s_2x) > F_2(s_2)$ for some $x \in \mathscr{X}$, and we let $\varphi_{\mathrm{pr}}(s_2)$ be equal to some positive probability mass function $m_2 \in \mathcal{M}(\mathscr{X})$ for which $E_{m_2}(F_2(s_2\,\cdot)) > F_2(s_2)$. Otherwise, we let $s_2 := s_1$. We continue by considering F_3. If $F_3(sx) > F_3(s)$ for an infinite number of $(s,x) \in \mathbb{S} \times \mathscr{X}$, then we (can) fix some $s_3 \in \mathbb{S}$ such that $|s_3| > |s_2| + 1$ and $F_3(s_3x) > F_3(s_3)$ for some $x \in \mathscr{X}$, and we let $\varphi_{\mathrm{pr}}(s_3)$ be equal to some positive probability mass function $m_3 \in \mathcal{M}(\mathscr{X})$ for which $E_{m_3}(F_3(s_3\,\cdot)) > F_3(s_3)$. Otherwise, we let $s_3 := s_2$. Repeat this procedure *ad infinitum* and let $\varphi_{\mathrm{pr}}(s)(x) := 1/2$ in all situations $s \in \mathbb{S}$ (and $x \in \mathscr{X}$) that haven't been assigned a probability mass function yet. Observe that, since either $s_{i+1} = s_i$ or $|s_{i+1}| > |s_i| + 1$ for all $i \in \mathbb{N}$, it holds for every $s \in \mathbb{S}$ that $\left|\{n \in \mathbb{N}_0 \colon 0 \leq n < |s| \text{ and } \varphi_{\mathrm{pr}}(s)(1) = 1/2\}\right| \geq \lfloor |s|/2 \rfloor$. In this way, we thus obtain a positive precise forecasting system φ_{pr} for which

$$(\exists^{\infty}(s,x)\in\mathbb{S}\times\mathscr{X})F_i(sx)>F_i(s)\\ \Rightarrow(\exists s\in\mathbb{S})E_{\varphi_{\mathrm{pr}}(s)}(F_i(s\,\cdot))>F_i(s)\text{ for all }i\in\mathbb{N}_0,\quad(14.16)$$

where $\exists^{\infty}$ expresses that there are infinitely many elements in the domain for which the property holds, and

$$P^{\varphi_{\mathrm{pr}}}(\llbracket s\rrbracket)\overset{\text{Eq. (6.23)}_{40}}{=}\prod_{k=0}^{|s|-1}\varphi_{\mathrm{pr}}(s_{1:k})(s_{k+1})\leq\left(\frac{1}{2}\right)^{\left\lfloor\frac{|s|}{2}\right\rfloor}\text{ for all }s\in\mathbb{S}.\qquad(14.17)$$

This ends the construction step.

Let's now first show that every path $\omega\in\mathscr{X}^{\mathbb{N}}$ is ML-random for φ_{pr}. Since every test supermartingale $T\in\overline{\mathbb{T}}_{\mathrm{ML}}(\varphi_{\mathrm{pr}})$ is a lower semicomputable non-negative process, it follows from the construction of φ_{pr} and Eq. (14.16) that there are only a finite number of $(s,x)\in\mathbb{S}\times\mathscr{X}$ such that $T(sx)>T(s)$, which implies that T is bounded above on every path $\omega\in\mathscr{X}^{\mathbb{N}}$. Hence, every path $\omega\in\mathscr{X}^{\mathbb{N}}$ is ML-random for φ_{pr}.

We continue by showing that no recursive path $\omega\in\mathscr{X}^{\mathbb{N}}$ can be ML-test-random for φ_{pr}. To this end, fix any recursive path $\omega\in\mathscr{X}^{\mathbb{N}}$, and let $A:=\bigcup_{n\in\mathbb{N}_0}\{(n,\omega_{1:2n})\}$. Clearly, A is a recursive subset of $\mathbb{N}_0\times\mathbb{S}$ and, by invoking Eq. (14.17), $P^{\varphi_{\mathrm{pr}}}(\llbracket A_n\rrbracket)=P^{\varphi_{\mathrm{pr}}}(\llbracket\omega_{1:2n}\rrbracket)\leq 2^{-n}$ for all $n\in\mathbb{N}_0$, which implies that A is an ML-test. Obviously, $\omega\in\bigcap_{m\in\mathbb{N}_0}\llbracket A_m\rrbracket$, so we indeed conclude that no recursive path $\omega\in\mathscr{X}^{\mathbb{N}}$ is ML-test-random for φ_{pr}.

We conclude the argument by showing that φ_{pr} is non-computable. To this end, assume towards contradiction that φ_{pr} is computable, and consider any recursive path $\omega\in\Omega$; ω is ML-random for φ_{pr} because all paths are. Since φ_{pr} is also positive, and hence, non-degenerate, it follows from Proposition 14.4₁₂₁ that ω is ML-test-random for φ_{pr}, a contradiction. ⋄

14.3 Universal Martin-Löf tests and universal lower semicomputable test supermartingales

In our definition of ML-randomness of a path $\omega\in\Omega$, *all* lower semicomputable test supermartingales $T\in\overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$ must remain bounded on ω. Similarly, for ω to be ML-test-random, we require that $\omega\notin\bigcap_{n\in\mathbb{N}_0}\llbracket A_n\rrbracket$ for *all* ML-tests A.

In his seminal paper [30], Martin-Löf proved that test randomness of a path can also be checked using a single, so-called *universal test*, which satisfies our conditions for an ML-test. A few years later, Schnorr proved in his doctoral thesis on algorithmic randomness for fair-coin forecasts that ML-randomness can also be checked using a single, so-called *universal*, lower semicomputable test supermartingale.

Let's now prove that something similar is still possible in our more general context. We begin by proving the existence of a universal ML-test.

Proposition 14.18. *Consider any computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then there's a so-called* universal ML-*test U for φ such that a path $\omega \in \Omega$ is* ML-*test-random for φ if and only if $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$.*

Proof of Proposition 14.18. By Corollary 7.3₄₄, we know there's a recursively enumerable set $A \subseteq \mathbb{N}_0{}^2 \times \mathbb{S}$ that contains all recursively enumerable sets $C \subseteq \mathbb{N}_0 \times \mathbb{S}$, in the sense that for every recursively enumerable set $C \subseteq \mathbb{N}_0 \times \mathbb{S}$ there's some $m_C \in \mathbb{N}_0$ such that $C = {}^{m_C}A$, with ${}^mA := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} : (m,n,s) \in A\}$ for all $m \in \mathbb{N}_0$. With every such ${}^mA$, we associate as usual the sets of situations ${}^mA_n$, defined for all $n \in \mathbb{N}_0$ by ${}^mA_n := \{s \in \mathbb{S} : (n,s) \in {}^mA\}$. For reasons explained after Definition 13.1₁₁₄, we can and will assume, without changing the map of global events $(m,n) \mapsto [\![{}^mA_n]\!]$, that all these sets ${}^mA_n$ are partial cuts and recursive uniformly in m and n; see also Corollary 13.2₁₁₄. For this A, we then have that for every recursively enumerable set $C \subseteq \mathbb{N}_0 \times \mathbb{S}$ there's some $m_C \in \mathbb{N}_0$ such that $[\![C_n]\!] = [\![{}^{m_C}A_n]\!]$ for all $n \in \mathbb{N}_0$.

As a first step in the proof, we show that there's a single finite algorithm for turning, for any given $m \in \mathbb{N}_0$, the corresponding recursive set ${}^mA$ into an ML-test ${}^mB$ for φ. Let ${}^mA_n^{<\ell} := \{s \in \mathbb{S} : (m,n,s) \in A, |s| < \ell\}$ for all $m,n,\ell \in \mathbb{N}_0$. It's clear from the construction that the finite sets ${}^mA_n^{<\ell}$ are recursive uniformly in m, n and ℓ. Observe that the computability of the forecasting system φ, the recursive character of the finite partial cuts ${}^mA_n^{<\ell}$ and an appropriate instantiation of our Workhorse Lemma 13.9₁₁₇ [with $\mathcal{D} \to \mathbb{N}_0{}^2$, $d \to (m,n)$, $p \to \ell$ and $C \to \{(m,n,\ell,s) \in \mathbb{N}_0{}^3 \times \mathbb{S} : s \in {}^mA_n^{<\ell}\}$, and therefore $C_d^p \to {}^mA_n^{<\ell}$] allow us to infer that the real map $(m,n,\ell) \mapsto \overline{P}^{\varphi}([\![{}^mA_n^{<\ell}]\!])$ is computable, meaning that there's some recursive rational map $q \colon \mathbb{N}_0{}^3 \times \mathbb{N} \to \mathbb{Q}$ such that

$$\left|\overline{P}^{\varphi}([\![{}^mA_n^{<\ell}]\!]) - q(m,n,\ell,N)\right| \leq 2^{-N} \text{ for all } m,n,\ell \in \mathbb{N}_0 \text{ and } N \in \mathbb{N}.$$

Observe that $q(m,n,\ell,n+2)$ is a rational approximation for $\overline{P}^{\varphi}([\![{}^mA_n^{<\ell}]\!])$ up to $2^{-(n+2)}$, since

$$\left|\overline{P}^{\varphi}([\![{}^mA_n^{<\ell}]\!]) - q(m,n,\ell,n+2)\right| \leq 2^{-(n+2)} \text{ for all } m,n,\ell \in \mathbb{N}_0. \tag{14.19}$$

Now consider the (obviously) recursive map $\lambda \colon \mathbb{N}_0{}^3 \to \mathbb{N}_0$, defined by

$$\lambda(m,n,\ell) := \max\Big\{p \in \{0,\dots,\ell\} : (\forall k \in \{0,\dots,p\})\, q(m,n,k,n+2) \leq 2^{-(n+1)} + 2^{-(n+2)}\Big\} \quad \text{for all } m,n,\ell \in \mathbb{N}_0. \tag{14.20}$$

Observe that $\lambda(m,n,0) = 0$, because

$$q(m,n,0,n+2) \leq \overline{P}^{\varphi}([\![{}^mA_n^{<0}]\!]) + 2^{-(n+2)} = \overline{P}^{\varphi}(\emptyset) + 2^{-(n+2)} = 2^{-(n+2)},$$

where the inequality follows from Eq. (14.19), and the last equality from P1₃₅; this argument also ensures that the map λ is indeed well-defined. Consequently, by construction,

$$q(m,n,\lambda(m,n,\ell),n+2) \leq 2^{-(n+1)} + 2^{-(n+2)} \text{ for all } m,n,\ell \in \mathbb{N}_0. \tag{14.21}$$

Also, the partial maps $\lambda(m,n,\bullet)$ are obviously non-decreasing.

Now let ${}^mB_n^{\ell} := {}^mA_n^{<\lambda(m,n,\ell)}$ for all $m,n,\ell \in \mathbb{N}_0$. It follows from Eqs. (14.19) and (14.21) that

$$\overline{P}^{\varphi}([\![{}^mB_n^{\ell}]\!]) = \overline{P}^{\varphi}([\![{}^mA_n^{<\lambda(m,n,\ell)}]\!]) \leq q(m,n,\lambda(m,n,\ell),n+2) + 2^{-(n+2)}$$

$$\leq \left(2^{-(n+1)} + 2^{-(n+2)}\right) + 2^{-(n+2)} = 2^{-n}.$$

We now use the sets ${}^mB_n^\ell$ in the obvious manner to define

$${}^mB_n \coloneqq \bigcup_{\ell\in\mathbb{N}_0} {}^mB_n^\ell \text{ and } {}^mB \coloneqq \bigcup_{n\in\mathbb{N}_0} \{n\} \times {}^mB_n, \text{ for all } m,n\in\mathbb{N}_0,$$

so the set ${}^mB \subseteq \mathbb{N}_0 \times \mathbb{S}$ is recursively enumerable as a countable union of finite sets $\{n\} \times {}^mB_n^\ell$ that are recursive uniformly in n and ℓ. Moreover, it follows from P3$_{35}$, P4$_{36}$ and the non-decreasing character of the partial map $\lambda(m,n,\bullet)$ that

$$\overline{P}^\varphi(\llbracket {}^mB_n \rrbracket) = \sup_{\ell\in\mathbb{N}_0} \overline{P}^\varphi(\llbracket {}^mB_n^\ell \rrbracket) \leq 2^{-n}, \text{ for all } m,n\in\mathbb{N}_0, \tag{14.22}$$

and therefore each ${}^mB$ is an ML-test for φ.

As a second step in the proof, we now show that any path $\omega\in\Omega$ is ML-test-random for φ if and only if $\omega \notin \bigcap_{n\in\mathbb{N}_0} \llbracket {}^mB_n \rrbracket$ for all $m\in\mathbb{N}_0$. Since each ${}^mB$ is an ML-test for φ, it suffices to show by Lemma 14.23↷ that for every recursively enumerable subset $C \subseteq \mathbb{N}_0 \times \mathbb{S}$ for which $\overline{P}^\varphi(\llbracket C_n \rrbracket) \leq 2^{-(n+1)}$ for all $n\in\mathbb{N}_0$, there's some $m_C\in\mathbb{N}_0$ such that $\llbracket C_n \rrbracket = \llbracket {}^{m_C}B_n \rrbracket$ for all $n\in\mathbb{N}_0$; this is what we now set out to do.

Since we assumed that C is recursively enumerable, we know that there's some $m_C \in \mathbb{N}_0$ such that $\llbracket C_n \rrbracket = \llbracket {}^{m_C}A_n \rrbracket$ for all $n \in \mathbb{N}_0$. This implies that $\overline{P}^\varphi(\llbracket {}^{m_C}A_n \rrbracket) = \overline{P}^\varphi(\llbracket C_n \rrbracket) \leq 2^{-(n+1)}$ for all $n\in\mathbb{N}_0$, so we see that for this m_C:

$$\begin{aligned} q(m_C,n,\ell,n+2) &\leq \overline{P}^\varphi(\llbracket {}^{m_C}A_n^{<\ell} \rrbracket) + 2^{-(n+2)} \\ &\leq \overline{P}^\varphi(\llbracket {}^{m_C}A_n \rrbracket) + 2^{-(n+2)} \\ &\leq 2^{-(n+1)} + 2^{-(n+2)} \text{ for all } n,\ell\in\mathbb{N}_0, \end{aligned}$$

where the first inequality follows from Eq. (14.19)↶, and the second inequality follows from P3$_{35}$. If we now look at the definition of the map λ in Eq. (14.20)↶, we see that $\lambda(m_C,n,\ell) = \ell$ for all $n,\ell\in\mathbb{N}_0$. Consequently,

$${}^{m_C}A_n = \bigcup_{\ell\in\mathbb{N}_0} {}^{m_C}A_n^{<\ell} = \bigcup_{\ell\in\mathbb{N}_0} {}^{m_C}A_n^{<\lambda(m_C,n,\ell)} = \bigcup_{\ell\in\mathbb{N}_0} {}^{m_C}B_n^\ell = {}^{m_C}B_n \text{ for all } n\in\mathbb{N}_0,$$

and therefore, indeed, $\llbracket C_n \rrbracket = \llbracket {}^{m_C}A_n \rrbracket = \llbracket {}^{m_C}B_n \rrbracket$ for all $n\in\mathbb{N}_0$.

As a third step in the proof, we show that we can combine the ML-tests ${}^mB$ for φ, with $m\in\mathbb{N}_0$, into a single ML-test U for φ. To this end, let $U_n \coloneqq \bigcup_{m\in\mathbb{N}_0} {}^mB_{n+m+1} = \bigcup_{m,l\in\mathbb{N}_0} {}^mB^l_{n+m+1}$ for all $n\in\mathbb{N}_0$. Then $U \coloneqq \bigcup_{n\in\mathbb{N}_0} \{n\} \times U_n$ is clearly recursively enumerable as a countably infinite union of finite sets $\{n\} \times {}^mB^l_{n+m+1}$ that are recursive uniformly in m, n and ℓ, given the construction in the first step of the proof. It's clear that

$$\begin{aligned} \overline{P}^\varphi(\llbracket U_n \rrbracket) &= \overline{P}^\varphi\Big(\bigcup_{m\in\mathbb{N}_0} \llbracket {}^mB_{n+m+1} \rrbracket\Big) = \sup_{k\in\mathbb{N}_0} \overline{P}^\varphi\Big(\bigcup_{m=0}^{k} \llbracket {}^mB_{n+m+1} \rrbracket\Big) \\ &\leq \sup_{k\in\mathbb{N}_0} \sum_{m=0}^{k} \overline{P}^\varphi(\llbracket {}^mB_{n+m+1} \rrbracket) \leq \sup_{k\in\mathbb{N}_0} \sum_{m=0}^{k} 2^{-(n+m+1)} = \sum_{m=0}^{\infty} 2^{-(n+m+1)} = 2^{-n}, \end{aligned}$$

where the second equality follows from P3$_{35}$ and P4$_{36}$, the first inequality follows from P2$_{35}$, and the second inequality holds by Eq. (14.22).We conclude that U is indeed an ML-test for φ.

We finish the argument, in a fourth and final step, by proving that any path $\omega \in \Omega$ is ML-test-random for φ if and only if $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$. To this end, consider any path $\omega \in \Omega$. For necessity, assume that ω is ML-test-random for φ. Then clearly also $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$ by Definition 13.14$_{119}$, since we've just proved that U is an ML-test for φ. For sufficiency, assume that $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$. To prove that ω is ML-test-random, we must prove, as argued above, that $\omega \notin \bigcap_{n\in\mathbb{N}_0} [\![{}^{m}B_n]\!]$ for all $m \in \mathbb{N}_0$. Assume towards contradiction that there's some $m_o \in \mathbb{N}_0$ such that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![{}^{m_o}B_n]\!]$. By construction, clearly, $[\![{}^{m_o}B_{n+m_o+1}]\!] \subseteq [\![U_n]\!]$ for all $n \in \mathbb{N}_0$. This implies that $\omega \in [\![U_n]\!]$ for all $n \in \mathbb{N}_0$, a contradiction. □

In the above proof, we made use of the following alternative characterisation of ML-test-randomness.

Lemma 14.23. *A path $\omega \in \Omega$ is* ML*-test-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if $\omega \notin \bigcap_{m\in\mathbb{N}_0} [\![C_m]\!]$ for all recursively enumerable subsets C of $\mathbb{N}_0 \times \mathbb{S}$ such that $\overline{P}^{\varphi}([\![C_n]\!]) \leq 2^{-(n+1)}$ for all $n \in \mathbb{N}_0$.*

Proof. It clearly suffices to prove the 'if' part. We give a proof by contraposition. So assume that ω isn't ML-test-random, meaning that there's some ML-test A for φ such that $\omega \in \bigcap_{m\in\mathbb{N}_0} [\![A_m]\!]$. Consider the recursively enumerable set $C \subseteq \mathbb{N}_0 \times \mathbb{S}$ defined by

$$C := \{(n,s) \in \mathbb{N}_0 \times \mathbb{S} : (n+1,s) \in A\},$$

then $C_n = A_{n+1}$, and therefore also $\overline{P}^{\varphi}([\![C_n]\!]) = \overline{P}^{\varphi}([\![A_{n+1}]\!]) \leq 2^{-(n+1)}$ for all $n \in \mathbb{N}_0$. Since $\bigcap_{n\in\mathbb{N}_0} [\![A_n]\!] \subseteq \bigcap_{n\in\mathbb{N}_0} [\![A_{n+1}]\!] = \bigcap_{n\in\mathbb{N}_0} [\![C_n]\!]$, we see that also $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![C_n]\!]$. □

We continue by proving the existence of a universal lower semicomputable supermartingale that, as mentioned in the discussion above Lemma 14.11$_{123}$ in Section 14$_{119}$, tends to infinity on every non-ML-random path $\omega \in \Omega$, instead of merely being unbounded there.

Corollary 14.24. *Consider any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then there's a so-called* universal *lower semicomputable test supermartingale T for φ such that any path $\omega \in \Omega$ isn't* ML*-(test-)random for φ if and only if $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$.*

Proof. Consider the universal ML-test U in Proposition 14.18$_{127}$. Lemma 14.5$_{121}$ then tells us there's a test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi)$—which we claim does the job—such that, for any path $\omega \in \Omega$, $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$ if $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$.

Indeed, consider any path $\omega \in \Omega$. Suppose that ω isn't ML-(test-)random for φ, then we know from (Theorem 14.1$_{120}$ and) Proposition 14.18$_{127}$ that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$, and therefore that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$. Conversely, suppose that $\lim_{n\to\infty} T(\omega_{1:n}) = \infty$. This tells us that ω isn't ML-random for φ (and therefore, by Proposition 14.2$_{120}$, not ML-test-random for φ either). □

14.4 The relation between uniform and Martin-Löf test randomness

Alexander Shen has recently pointed out to us that the idea of testing randomness for imprecise-probabilistic uncertainty models has been explored before. In 1973, Levin [4, 6] introduced a randomness test version of ML-randomness that allows for testing randomness for so-called *effectively compact classes of measures*, leading to a measure-theoretic randomness notion nowadays known as *uniform randomness*.

Below, we give a brief account of this notion of uniform randomness, and explain how our notion of ML-test-randomness, when restricted to *computable* forecasting systems, fits into that framework. To define uniform randomness, we first need to define a notion of effective compactness for sets of probability measures.

Effectively compact classes of probability measures

We denote by $\mathcal{M}(\Omega)$ the set of all probability measures over the Borel algebra $\mathcal{B}(\Omega)$, and recall from the discussion in Section 6.4₃₉ that every precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathcal{X})$ leads to a probability measure $\mu^{\varphi_{\mathrm{pr}}} \in \mathcal{M}(\Omega)$. Conversely, for any measure $\mu \in \mathcal{M}(\Omega)$, there's at least one precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathcal{X})$ such that $\mu = \mu^{\varphi_{\mathrm{pr}}}$, for instance—as is clear from our discussion on p. 26—the one defined by

$$\varphi_{\mathrm{pr}}(s)(x) := \begin{cases} \frac{\mu(\llbracket sx \rrbracket)}{\mu(\llbracket s \rrbracket)} & \text{if } \mu(\llbracket s \rrbracket) > 0 \\ 1/|\mathcal{X}| & \text{if } \mu(\llbracket s \rrbracket) = 0 \end{cases} \text{ for all } s \in \mathbb{S} \text{ and } x \in \mathcal{X}.$$

This tells us that we can for our present purposes identify probability measures and precise forecasting systems (although forecasting systems are more informative, as they provide 'full conditional information' [see also our discussion on p. 26]), in the sense that:

$$\mathcal{M}(\Omega) = \left\{\mu^{\varphi_{\mathrm{pr}}} : \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathcal{X})\right\}. \tag{14.25}$$

With any $b \Subset \mathbb{Q} \times (\mathcal{L}_{\mathrm{rat}}(\mathcal{X}) \cap \mathcal{L}_1(\mathcal{X})) \times \mathbb{S} \times \mathbb{Q}$, where '$\Subset$' is taken to mean 'is a finite subset of', we associate a so-called *basic open set* in the set of probability measures $\mathcal{M}(\Omega)$, denoted by $b(\Omega)$, and given by

$$b(\Omega) := \left\{\mu \in \mathcal{M}(\Omega) : u < \mu(f_s) < v \text{ for all } (u, f, s, v) \in b\right\},$$

where $\mu(f_s)$ denotes the integral $\int f_s(\omega)\,\mathrm{d}\mu(\omega)$, which is well-defined since f_s is Borel measurable; we recall that $f_s = \sum_{x \in \mathcal{X}} f(x) \mathbb{I}_{\llbracket sx \rrbracket} \in \mathcal{L}(\Omega)$, as has been defined on p. 33. So the basic open set $b(\Omega)$ consists of all probability measures that satisfy the finite collection of conditions characterised by b. We collect all generators b of basic open sets $b(\Omega)$ in the (countable) set $\mathcal{P}_{\mathrm{fin}}(\mathbb{Q} \times (\mathcal{L}_{\mathrm{rat}}(\mathcal{X}) \cap \mathcal{L}_1(\mathcal{X})) \times \mathbb{S} \times \mathbb{Q})$. In line with our discussion in Section 7.2₄₄ [with

$X \to \mathcal{M}(\Omega)$ and $\mathcal{D} \to \mathcal{P}_{\text{fin}}(\mathbb{Q} \times (\mathscr{L}_{\text{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})) \times \mathbb{S} \times \mathbb{Q})(\Omega)]$, a subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is then called *effectively open* if there's a recursively enumerable set $B \subseteq \mathcal{P}_{\text{fin}}(\mathbb{Q} \times (\mathscr{L}_{\text{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})) \times \mathbb{S} \times \mathbb{Q})$ such that $\bigcup_{b \in B} b(\Omega) = \mathcal{C}$. A subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is called *effectively closed* if $\mathcal{M}(\Omega) \setminus \mathcal{C}$ is effectively open. A subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is called *effectively compact* if it's compact—meaning that every open cover of $\mathcal{C}$ has a finite subcover—and if the set

$$\left\{B\colon B \Subset \mathcal{P}_{\text{fin}}(\mathbb{Q} \times (\mathscr{L}_{\text{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})) \times \mathbb{S} \times \mathbb{Q}) \text{ and } \bigcup_{b \in B} b(\Omega) \supseteq \mathcal{C}\right\}$$

of all finite open covers of $\mathcal{C}$ is recursively enumerable. By the argumentation around Proposition 5.5 in Ref. [6], it's immediate that a subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is effectively compact if and only if it's effectively closed.

How does the above exposition relate to Levin's notion of effective compactness for sets of probability measures, where he only considers the binary state space $\mathscr{X} = \{0, 1\}$? For our purposes, it suffices to notice that his basic open sets are a subset of ours: Levin considers finite sets of triples $b \Subset \mathbb{Q} \times \mathbb{S} \times \mathbb{Q}$ [6, Definition 5.3], with $b(\Omega) := \{\mu \in \mathcal{M}(\Omega)\colon u < \mu(s) < v \text{ for all } (u, s, v) \in b\}$, and every such triple $(u, s, v) \in \mathbb{Q} \times \mathbb{S} \times \mathbb{Q}$ obviously generates the same set of probability measures as the quadruple $(u, 1, s, v) \in \mathbb{Q} \times (\mathscr{L}_{\text{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})) \times \mathbb{S} \times \mathbb{Q}$. Consequently, if a subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is (effectively) compact in our topology, then it's also (effectively) compact in his topology, and if a subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is (effectively) closed in his topology, then it's also (effectively) closed in our topology. Vice versa, if a subset $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ is effectively compact in his topology, then it's also effectively closed in his topology by Proposition 5.5 in [6], which then implies that $\mathcal{C}$ is effectively closed in our topology, and thus effectively compact in our topology. We conclude that we consider the exact same collection of effectively compact sets of probability measures as Levin does.

With any forecasting system φ, as already mentioned on p. 26, we can associate a collection of *compatible* precise forecasting systems $\{\varphi_{\text{pr}}\colon \varphi_{\text{pr}} \in \Phi_{\text{pr}}(\mathscr{X}) \text{ and } \varphi_{\text{pr}} \in \varphi\}$, and therefore also, a collection of probability measures $\mathcal{C}[\varphi] := \{\mu^{\varphi_{\text{pr}}}\colon \varphi_{\text{pr}} \in \Phi_{\text{pr}}(\mathscr{X}) \text{ and } \varphi_{\text{pr}} \in \varphi\}$. We begin by uncovering a sufficient condition on φ for the corresponding collection of probability measures to be effectively compact.

Proposition 14.26. *Consider any computable forecasting system φ. Then the collection of probability measures $\mathcal{C}[\varphi] = \{\mu^{\varphi_{\text{pr}}}\colon \varphi_{\text{pr}} \in \Phi_{\text{pr}}(\mathscr{X}) \text{ and } \varphi_{\text{pr}} \in \varphi\}$ is effectively compact.*

Proof. It's equivalent to prove that $\{\mu^{\varphi_{\text{pr}}}\colon \varphi_{\text{pr}} \in \Phi_{\text{pr}}(\mathscr{X}) \text{ and } \varphi_{\text{pr}} \in \varphi\}$ is effectively closed, which we'll do by establishing the existence of a recursively enumerable set $B \subseteq \mathcal{P}_{\text{fin}}(\mathbb{Q} \times (\mathscr{L}_{\text{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})) \times \mathbb{S} \times \mathbb{Q})$ such that $\bigcup_{b \in B} b(\Omega) = \mathcal{M}(\Omega) \setminus \mathcal{C}[\varphi]$.

Since φ is computable, there's some recursive map $q\colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\text{fin}}(\mathcal{M}_{\text{rat}}(\mathscr{X}))$ such that $d_{\text{H}}(\varphi(s), \text{CH}(q(s, N))) \leq 2^{-N}$ for all $s \in \mathbb{S}$ and $N \in \mathbb{N}$. Let

$$B := \bigcup_{r\in\mathbb{Q}\cap(0,2), f\in\mathscr{L}_{\mathrm{rat}}(\mathscr{X})\cap\mathscr{L}_1(\mathscr{X}), s\in\mathbb{S}, n\in\mathbb{N}} \Big\{\Big\{(-1,1,s,r), \Big(r\big(\overline{E}_{\mathrm{CH}(q(s,n))}(f)+2^{-n}\big), f, s, 2\Big)\Big\}\Big\}.$$

This set is clearly recursively enumerable by Lemma 7.1[43].

To show that $\mathcal{M}(\Omega)\setminus\mathfrak{C}[\varphi]\subseteq\bigcup_{b\in B} b(\Omega)$, we start by proving that for any measure $\mu\in\mathcal{M}(\Omega)\setminus\mathfrak{C}[\varphi]$ there must be some $t\in\mathbb{S}$ such that $\mu([\![t]\!])>0$ and $\mu([\![t\cdot]\!])/\mu([\![t]\!])\notin\varphi(t)$. To this end, fix some compatible precise forecasting system $\varphi_{\mathrm{co}}\in\varphi$, and consider the precise (not necessarily computable) forecasting system φ'_{pr} defined by

$$\varphi'_{\mathrm{pr}}(s)(x) := \begin{cases} \frac{\mu([\![sx]\!])}{\mu([\![s]\!])} & \text{if } \mu([\![s]\!])>0 \\ \varphi_{\mathrm{co}}(s)(x) & \text{if } \mu([\![s]\!])=0 \end{cases} \text{ for all } s\in\mathbb{S} \text{ and } x\in\mathscr{X}.$$

By construction, as is clear from our discussion on p. 26, $\mu=\mu^{\varphi'_{\mathrm{pr}}}$. Since $\mu\in\mathcal{M}(\Omega)\setminus\mathfrak{C}[\varphi]$ by assumption, there's some $t\in\mathbb{S}$ such that $\varphi'_{\mathrm{pr}}(t)\notin\varphi(t)$. Since, for all $s\in\mathbb{S}$, $\varphi'_{\mathrm{pr}}(s)\in\varphi(s)$ if $\mu([\![s]\!])=0$, we infer that, indeed, $\mu([\![t]\!])>0$ and $\mu([\![t\cdot]\!])/\mu([\![t]\!])\notin\varphi(t)$.

By Lemma 14.31↷, since $\varphi'_{\mathrm{pr}}(t)\notin\varphi(t)$, and hence, $d\big(\varphi'_{\mathrm{pr}}(t),\varphi(t)\big)>0$, there's a gamble $f'\in\mathscr{L}_1(\mathscr{X})$ such that $\overline{E}_{\varphi(t)}(f')<E_{\varphi'_{\mathrm{pr}}(t)}(f')$. Let $\delta>0$ be any real such that $\overline{E}_{\varphi(t)}(f')+\delta<E_{\varphi'_{\mathrm{pr}}(t)}(f')$, and consider any rational gamble $f''\in\mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ such that $f'-\delta/2\leq f''\leq f'$. Let $f:=\max\{f'',0\}\in\mathscr{L}_{\mathrm{rat}}(\mathscr{X})$. Then, $0\leq f=\max\{f'',0\}\leq\max\{f',0\}\leq 1$, so we conclude that $f\in\mathscr{L}_{\mathrm{rat}}(\mathscr{X})\cap\mathscr{L}_1(\mathscr{X})$. Furthermore, $f'-\delta/2\leq f=\max\{f'',0\}\leq\max\{f',0\}=f'$, and hence,

$$\begin{aligned}\overline{E}_{\varphi(t)}(f) &\overset{\mathrm{C5}_{20}}{\leq} \overline{E}_{\varphi(t)}(f')<E_{\varphi'_{\mathrm{pr}}(t)}(f')-\delta \\ &\leq E_{\varphi'_{\mathrm{pr}}(t)}(f+\delta/2)-\delta \overset{\mathrm{C4}_{20}}{=} E_{\varphi'_{\mathrm{pr}}(t)}(f)-\delta/2<E_{\varphi'_{\mathrm{pr}}(t)}(f).\end{aligned} \tag{14.27}$$

As a result, since $\mu([\![t]\!])>0$,

$$\begin{aligned}\mu(f_t) &= \mu\Bigg(\sum_{x\in\mathscr{X}} f(x)\mathbb{I}_{[\![tx]\!]}\Bigg)=\sum_{x\in\mathscr{X}} f(x)\mu([\![tx]\!]) \\ &= \mu([\![t]\!])\sum_{x\in\mathscr{X}}\varphi'_{\mathrm{pr}}(t)(x)f(x)=\mu([\![t]\!])E_{\varphi'_{\mathrm{pr}}(t)}(f)\overset{\text{Eq. (14.27)}}{>}\mu([\![t]\!])\overline{E}_{\varphi(t)}(f),\end{aligned}$$

where the second equality follows from the properties of integrals. This implies that there's some real $\epsilon\in(0,1)$ such that $\mu(f_t)>\mu([\![t]\!])\overline{E}_{\varphi(t)}(f)+\epsilon$. Then, for this ϵ, there's a rational $r\in\mathbb{Q}\cap(0,2)$ such that

$$(-1<)0\leq\mu([\![t]\!])=\mu(1_t)<r<\mu([\![t]\!])+\epsilon/4<2 \tag{14.28}$$

and there's a natural $n\in\mathbb{N}$ such that $2^{-n}<\epsilon/8$, and for which then

$$0\overset{\mathrm{C1}_{20}}{\leq}\overline{E}_{\varphi(t)}(f)\leq\overline{E}_{\mathrm{CH}(q(t,n))}(f)+2^{-n}\leq\overline{E}_{\varphi(t)}(f)+2^{-n}+2^{-n}<\overline{E}_{\varphi(t)}(f)+\frac{\epsilon}{4},$$

where the second and third inequalities follow from Corollary 7.9[48] because $f\in\mathscr{L}_1(\mathscr{X})$. We then find that

$$0\leq r\Big(\overline{E}_{\mathrm{CH}(q(t,n))}(f)+2^{-n}\Big) \tag{14.29}$$

$$
\begin{aligned}
&< \Big(\mu(\llbracket t \rrbracket) + \frac{\epsilon}{4}\Big)\Big(\overline{E}_{\varphi(t)}(f) + \frac{\epsilon}{4}\Big) \\
&= \mu(\llbracket t \rrbracket)\overline{E}_{\varphi(t)}(f) + \mu(\llbracket t \rrbracket)\frac{\epsilon}{4} + \overline{E}_{\varphi(t)}(f)\frac{\epsilon}{4} + \frac{\epsilon^2}{16} \\
&\overset{\text{C1}_{20}}{\leq} \mu(\llbracket t \rrbracket)\overline{E}_{\varphi(t)}(f) + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon^2}{16} \\
&< \mu(\llbracket t \rrbracket)\overline{E}_{\varphi(t)}(f) + \epsilon \\
&< \mu(f_t) = \sum_{x \in \mathscr{X}} \mu(\llbracket tx \rrbracket) f(x) \leq \sum_{x \in \mathscr{X}} \mu(\llbracket tx \rrbracket) = \mu(\llbracket t \rrbracket) < 2.
\end{aligned} \tag{14.30}
$$

Due to Eqs. (14.28) and (14.30), we see that $\mu \in b_\mu$ for

$$
b_\mu = \Big\{(-1,1,t,r), \Big(r\big(\overline{E}_{\mathrm{CH}(q(t,n))}(f) + 2^{-n}\big), f, t, 2\Big)\Big\} \in B,
$$

and therefore that $\mu \in \bigcup_{b \in B} b(\Omega)$. So $\mathcal{M}(\Omega) \setminus \mathcal{C}[\varphi] \subseteq \bigcup_{b \in B} b(\Omega)$.

To also prove the converse inequality, namely that $\bigcup_{b \in B} b(\Omega) \subseteq \mathcal{M}(\Omega) \setminus \mathcal{C}[\varphi]$, consider any $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ such that $\varphi_{\mathrm{pr}} \in \varphi$. For any $f \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X}) \cap \mathscr{L}_1(\mathscr{X})$, $s \in \mathbb{S}$, $n \in \mathbb{N}$ and $r \in \mathbb{Q} \cap (0,2)$ such that $r > \mu^{\varphi_{\mathrm{pr}}}(\llbracket s \rrbracket) = \mu^{\varphi_{\mathrm{pr}}}(1_s)$, it follows that

$$
\begin{aligned}
\mu^{\varphi_{\mathrm{pr}}}(f_s) = \sum_{x \in \mathscr{X}} \mu^{\varphi_{\mathrm{pr}}}(\llbracket sx \rrbracket) f(x) &= \mu^{\varphi_{\mathrm{pr}}}(\llbracket s \rrbracket) \sum_{x \in \mathscr{X}} \varphi_{\mathrm{pr}}(s)(x) f(x) \\
&= \mu^{\varphi_{\mathrm{pr}}}(\llbracket s \rrbracket) E_{\varphi_{\mathrm{pr}}(s)}(f) \\
&\leq r\overline{E}_{\varphi(s)}(f) \leq r\Big(\overline{E}_{\mathrm{CH}(q(s,n))}(f) + 2^{-n}\Big),
\end{aligned}
$$

where the first inequality follows from C1[20] and Eq. (5.7)[19], and where the second inequality follows from Corollary 7.9[48]. This implies that, indeed, $\mu^{\varphi_{\mathrm{pr}}} \notin \bigcup_{b \in B} b(\Omega)$. □

Lemma 14.31. *Consider any probability mass function $m \in \mathcal{M}(\mathscr{X})$ and any credal set $C \in \mathscr{C}(\mathscr{X})$. Then, $d(m, C) \leq \max_{f \in \mathscr{L}_1(\mathscr{X})} E_m(f) - \overline{E}_C(f)$.*

Proof. It's immediate from Lemma 11.9[95] that

$$
\begin{aligned}
d(m,C) &= \max_{f \in \mathscr{L}_1(\mathscr{X})} \underline{E}_C(f) - E_m(f) = \max_{f \in \mathscr{L}_1(\mathscr{X})} E_m(-f) - \overline{E}_C(-f) \\
&\overset{\text{C4}_{20}}{=} \max_{f \in \mathscr{L}_1(\mathscr{X})} E_m(\max f - f) - \overline{E}_C(\max f - f) \leq \max_{f' \in \mathscr{L}_1(\mathscr{X})} E_m(f') - \overline{E}_C(f'),
\end{aligned}
$$

where the second equality follows from the conjugacy relationship, and where the inequality holds because $0 \leq \max f - f \leq 1$ for all $f \in \mathscr{L}_1(\mathscr{X})$. □

On the other hand, not every effectively compact set of probability measures is a collection that corresponds to a (computable) forecasting system. Consider, as a counterexample, the binary state space $\mathscr{X} = \{0,1\}$ and the set $Ber := \big\{\mu^{\varphi_{\mathrm{pr}}} : \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X}) \text{ and } (\exists p \in [0,1])(\forall s \in \mathbb{S})\varphi_{\mathrm{pr}}(s)(1) = p\big\}$ that consists of all Bernoulli (iid) probability measures. As is mentioned by Bienvenu et al. [6, Sec. 5.3], this set *Ber* is an example of an effectively compact set of measures.

But, there's no forecasting system φ for which it holds that $Ber = \mathcal{C}[\varphi]$. Indeed, consider any forecasting system φ for which $Ber \subseteq \mathcal{C}[\varphi]$. If we let $m_p \in \mathcal{M}(\mathscr{X})$ be defined by $(m_p(0), m_p(1)) = (1-p, p)$ for all $p \in (0,1)$, then necessarily $m_p \in \varphi(s)$ for all $p \in (0,1)$ and all $s \in \mathbb{S}$, which implies that $\varphi(s) = C_{\mathrm{v}}$ for all $s \in \mathbb{S}$. This means that φ can only be the vacuous forecasting system φ_{v}, for which, by Eq. $(14.25)_{130}$, $\mathcal{C}[\varphi_{\mathrm{v}}] = \mathcal{M}(\Omega) \neq Ber$.

We conclude in particular that *the collections of probability measures that correspond to computable forecasting systems constitute only a strict subset of the effectively compact sets of probability measures.*

Uniform randomness

Using this notion of effective compactness, we can now introduce uniform randomness, by associating tests with effectively compact classes of probability measures.

Definition 14.32 ([6, Def. 5.22])**.** We call a map $\psi\colon \Omega \to [0,+\infty]$ a $\mathcal{C}$-*test* for an effectively compact class of probability measures $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ if the set $\{\omega \in \Omega\colon \psi(\omega) > r\}$ is effectively open uniformly in $r \in \mathbb{Q}$ and if $\int \psi(\omega)\,\mathrm{d}\mu(\omega) \leq 1$ for all $\mu \in \mathcal{C}$.

A clarification is in order here. The conditions for a $\mathcal{C}$-test ψ require in particular that $\{\omega \in \Omega\colon \psi(\omega) > r\}$ should be open, and therefore belong to $\mathscr{B}(\Omega)$, for all rational r, implying that the map ψ is Borel measurable. This implies that the integral $\int \psi(\omega)\,\mathrm{d}\mu(\omega)$, which we'll also denote by $\mu(\psi)$, exists.

As explained in Section 2.2 of Ref. [6], the intuition behind $\mathcal{C}$-tests is as follows: when $\psi(\omega)$ is large, this means that the $\mathcal{C}$-test ψ finds a lot of 'regularities' in ω. When constructing a $\mathcal{C}$-test ψ, we are allowed to declare whatever we want as a 'regularity'. However, we shouldn't find too many 'regular' sequences on average: if we declare too many sequences to be 'regular', then the average $\int \psi(\omega)\,\mathrm{d}\mu(\omega)$ becomes too big for some measure $\mu \in \mathcal{C}$.

Going from tests to the corresponding randomness notion is now but a small step.

Definition 14.33 ([6, Defs. 5.2&5.22, Thm. 5.23])**.** Consider an effectively compact class of probability measures $\mathcal{C} \subseteq \mathcal{M}(\Omega)$. Then we call a path $\omega \in \Omega$ *uniformly random* for $\mathcal{C}$ if $\psi(\omega) < \infty$ for every $\mathcal{C}$-test ψ.

With the definition for uniform randomness now in place, we can show that our definition of ML-test-randomness for a computable forecasting system $\varphi \in \Phi(\mathscr{X})$ is a special case, where the effectively compact class $\mathcal{C}$ takes the specific form $\mathcal{C}[\varphi] = \{\mu^{\varphi_{\mathrm{pr}}}\colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X}) \text{ and } \varphi_{\mathrm{pr}} \in \varphi\}$.

Theorem 14.34. *Consider any computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then a path $\omega \in \Omega$ is* ML*-test-random for φ if and only if it's uniformly random for the effectively compact class of probability measures $\mathcal{C}[\varphi]$.*

Proof. For the 'only if'-direction, assume that there's some $\mathcal{C}[\varphi]$-test ψ such that $\psi(\omega)=\infty$. Then we must show that ω isn't ML-test-random for φ. First of all, that ψ is a $\mathcal{C}[\varphi]$-test implies in particular that $\{\varpi\in\Omega\colon\psi(\varpi)>r\}$ is effectively open uniformly in $r\in\mathbb{Q}$, meaning that there's some recursively enumerable subset $B\subseteq\mathbb{Q}\times\mathbb{S}$ such that, with $B_r:=\{s\in\mathbb{S}\colon(r,s)\in B\}$, $[\![B_r]\!]=\{\varpi\in\Omega\colon\psi(\varpi)>r\}$ for all $r\in\mathbb{Q}$. This in turn implies that $A:=\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon(2^n,s)\in B\}$ is a recursively enumerable subset of $\mathbb{N}_0\times\mathbb{S}$ such that, with $A_n:=\{s\in\mathbb{S}\colon(n,s)\in A\}$, $[\![A_n]\!]=\{\varpi\in\Omega\colon\psi(\varpi)>2^n\}$ for all $n\in\mathbb{N}_0$. By construction, $\psi>2^n\mathbb{I}_{[\![A_n]\!]}$ for all $n\in\mathbb{N}_0$. If we fix any $n\in\mathbb{N}_0$, then by assumption $\psi(\omega)=\infty>2^n$ and therefore $\omega\in[\![A_n]\!]$. Hence, $\omega\in\bigcap_{n\in\mathbb{N}_0}[\![A_n]\!]$, so we're done if we can prove that A is an ML-test for φ. We already know that A is recursively enumerable. Suppose towards contradiction that there's some $m\in\mathbb{N}_0$ such that $\overline{P}^{\varphi}([\![A_m]\!])>2^{-m}$ or, equivalently, $2^m\overline{P}^{\varphi}([\![A_m]\!])>1$. By Eq. (6.24)[40], since $[\![A_m]\!]\in\mathscr{B}(\Omega)$, it holds that $\overline{P}^{\varphi}([\![A_m]\!])=\sup_{\varphi_{\mathrm{pr}}\in\varphi}\mu^{\varphi_{\mathrm{pr}}}([\![A_m]\!])$, and therefore there's some precise $\varphi_{\mathrm{pr}}\in\varphi$ for which

$$1<2^m\mu^{\varphi_{\mathrm{pr}}}([\![A_m]\!])=\mu^{\varphi_{\mathrm{pr}}}(2^m\mathbb{I}_{[\![A_m]\!]})\leq\mu^{\varphi_{\mathrm{pr}}}(\psi),$$

where the last inequality follows from the properties of integrals because $\psi\geq2^m\mathbb{I}_{[\![A_m]\!]}$. This contradicts our assumption that ψ is a $\mathcal{C}[\varphi]$-test.

For the 'if'-direction, assume that $\omega\in\bigcap_{n\in\mathbb{N}_0}[\![A_n]\!]$ for some ML-test A for φ. If we let $C_n:=\bigcup_{m>n}A_m$ for all $n\in\mathbb{N}_0$, then clearly the set

$$\begin{aligned}C:=&\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon s\in C_n\}\\=&\{(n,s)\in\mathbb{N}_0\times\mathbb{S}\colon(\exists m>n)s\in A_m\}=\bigcup_{\substack{(m,n,s)\in\mathbb{N}_0{}^2\times\mathbb{S}:\\(m,s)\in A\text{ and }n<m}}\{(n,s)\}\end{aligned}$$

is recursively enumerable because A is, and the $[\![C_n]\!]$ therefore constitute a computable sequence of effectively open sets. Moreover, clearly $[\![C_0]\!]\supseteq[\![C_1]\!]\supseteq\dots$, and

$$\omega\in\bigcap_{n\in\mathbb{N}_0}[\![A_n]\!]\subseteq\bigcap_{n\in\mathbb{N}}[\![A_n]\!]\subseteq\bigcap_{n\in\mathbb{N}_0}\bigcup_{m>n}[\![A_m]\!]=\bigcap_{n\in\mathbb{N}_0}[\![C_n]\!],\tag{14.35}$$

where the second inclusion holds because $[\![A_{n+1}]\!]\subseteq\bigcup_{m>n}[\![A_m]\!]$ for all $n\in\mathbb{N}_0$. Now define the map $\psi\colon\Omega\to[0,+\infty]$ as $\psi(\omega):=\sum_{n\in\mathbb{N}}\mathbb{I}_{[\![C_n]\!]}(\omega)$ for all $\omega\in\Omega$. It follows from Eq. (14.35) that $\psi(\omega)=\infty$, so we're done if we can show that ψ is a $\mathcal{C}[\varphi]$-test.

It follows from the nestedness $[\![C_0]\!]\supseteq[\![C_1]\!]\supseteq\dots$ that $\{\omega\in\Omega\colon\psi(\omega)>n\}=[\![C_{n+1}]\!]$ for all $n\in\mathbb{N}_0$. Therefore, since the $[\![C_n]\!]$ constitute a computable sequence of effectively open sets, so do the $\{\omega\in\Omega\colon\psi(\omega)>n\}$. By observing that

$$\{\omega\in\Omega\colon\psi(\omega)>r\}=\begin{cases}\{\omega\in\Omega\colon\psi(\omega)>\lfloor r\rfloor\}=[\![C_{\lfloor r\rfloor+1}]\!]&\text{if }r\geq0\\\Omega=[\![\mathbb{S}]\!]&\text{if }r<0\end{cases}\text{ for all }r\in\mathbb{Q},$$

we infer that $\{\omega\in\Omega\colon\psi(\omega)>r\}$ is effectively open uniformly in $r\in\mathbb{Q}$. Furthermore, it holds for any $\varphi_{\mathrm{pr}}\in\varphi$ that

$$\begin{aligned}\mu^{\varphi_{\mathrm{pr}}}(\psi)&=\mu^{\varphi_{\mathrm{pr}}}\Big(\sum_{n\in\mathbb{N}}\mathbb{I}_{[\![C_n]\!]}\Big)\leq\mu^{\varphi_{\mathrm{pr}}}\Big(\sum_{n\in\mathbb{N}}\sum_{m>n}\mathbb{I}_{[\![A_m]\!]}\Big)\leq\sum_{n\in\mathbb{N}}\sum_{m>n}\mu^{\varphi_{\mathrm{pr}}}(\mathbb{I}_{[\![A_m]\!]})\\&=\sum_{n\in\mathbb{N}}\sum_{m>n}E^{\varphi_{\mathrm{pr}}}(\mathbb{I}_{[\![A_m]\!]})\leq\sum_{n\in\mathbb{N}}\sum_{m>n}\overline{E}^{\varphi}(\mathbb{I}_{[\![A_m]\!]})\leq\sum_{n\in\mathbb{N}}\sum_{m>n}2^{-m}=\sum_{n\in\mathbb{N}}2^{-n}=1,\end{aligned}$$

where the first two inequalities follow from the properties of integrals, the second equality follows from the discussion in Section 6.4[39], and the third inequality follows from Proposition 6.13[35]. □

15 Equivalence of Schnorr and Schnorr test randomness

Next, we turn to Schnorr randomness. Our argumentation that the 'test-theoretic' and 'martingale-theoretic' versions for this type of randomness are equivalent, in Theorem 15.1 below, adapts and simplifies a line of reasoning in Downey and Hirschfeldt's book [32, Thm. 7.1.7], with the aim of still making it work in our more general context. Here too, it allows us to extend Schnorr's argumentation [1, Secs. 5–9] for this equivalence from fair-coin to computable non-degenerate forecasting systems.

Theorem 15.1. *Consider any path $\omega \in \Omega$ and any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. Then ω is* S*-random for φ if and only if it's* S*-test-random for φ.*

Proof. This is immediate from Propositions 15.2 and 15.6↷ below. □

15.1 Schnorr test randomness implies Schnorr randomness

As was the case for ML-randomness, we begin with the implication that's easier to prove.

Proposition 15.2. *Consider any path $\omega \in \Omega$ and any forecasting system $\varphi \in \Phi(\mathscr{X})$. If ω is* S*-test-random for φ then it's* S*-random for φ.*

Proof. We give a proof by contraposition. Assume that ω isn't S-random for φ, then Proposition 10.22₈₂ implies that there's a recursive rational test supermartingale T and a natural growth function $\eta\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\limsup_{n\to\infty}[T(\omega_{1:n}) - \eta(n)] = \infty. \tag{15.3}$$

Drawing inspiration from Schnorr's proof [1, Satz (9.4), p. 73] and Downey and Hirschfeldt's simplified version of it [32, Thm. 7.1.7], we let

$$A := \left\{(n,t) \in \mathbb{N}_0 \times \mathbb{S}\colon T(t) \geq \eta(|t|) \geq 2^n\right\}. \tag{15.4}$$

Then A is a recursive subset of $\mathbb{N}_0 \times \mathbb{S}$ [because the inequalities in the expressions above are decidable, as all numbers involved are rational]. We also see that, for any $\varpi \in \Omega$,

$$\varpi \in [\![A_n]\!] \Leftrightarrow (\exists m \in \mathbb{N}_0)\varpi_{1:m} \in A_n \Leftrightarrow (\exists m \in \mathbb{N}_0)\big(T(\varpi_{1:m}) \geq \eta(m) \geq 2^n\big). \tag{15.5}$$

Hence, $[\![A_n]\!] \subseteq \{\varpi \in \Omega\colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^n\}$, so we infer from P3₃₅ and Ville's inequality [Proposition 6.18₃₇] that

$$\overline{P}^{\varphi}([\![A_n]\!]) \leq \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega\colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^n\right\}\right) \leq 2^{-n} \text{ for all } n \in \mathbb{N}_0.$$

This shows that A is an ML-test for φ.

Consider any $n \in \mathbb{N}_0$. Since η is non-decreasing, there's some $M \in \mathbb{N}_0$ such that $\eta(m) \geq 2^n$ for all $m \geq M$, and hence, it follows from Eq. (15.3) that there's some natural $M' > M$ such that $T(\omega_{1:M'}) \geq \eta(M') \geq 2^n$. Consequently, it's immediate from Eq. (15.5)↶ that $\omega \in \bigcap_{m\in\mathbb{N}_0} [\![A_m]\!]$. So we'll find that ω isn't S-test-random for φ, provided we can prove that A is an S-test.

To this end, we'll show that it has a tail bound. Define the map $e\colon \mathbb{N}_0{}^2 \to \mathbb{N}_0$ by letting $e(N,n) := \min\{k \in \mathbb{N}_0 : \eta(k) \geq 2^N\}$, for all $N, n \in \mathbb{N}_0$. Fix any $N, n \in \mathbb{N}_0$, then we infer from Eq. (15.4)↶ that

$$\varpi \in [\![A_n^{\geq \ell}]\!] \Leftrightarrow (\exists m \geq \ell)\, T(\varpi_{1:m}) \geq \eta(m) \geq 2^n), \text{ for all } \ell \in \mathbb{N}_0.$$

Hence, for all $\ell \geq e(N,n)$ and all $\varpi \in [\![A_n^{\geq \ell}]\!]$, there's some $m \geq \ell$ such that

$$T(\varpi_{1:m}) \geq \eta(m) \geq \eta(\ell) \geq \eta(e(N,n)) \geq 2^N,$$

which implies that $[\![A_n^{\geq \ell}]\!] \subseteq \{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^N\}$. P3$_{35}$ and Ville's inequality [Proposition 6.18$_{37}$] then guarantee that, for all $\ell \geq e(N,n)$, since $[\![A_n]\!] \setminus [\![A_n^{<\ell}]\!] \subseteq [\![A_n^{\geq \ell}]\!]$,

$$\overline{P}^{\varphi}([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]) \leq \overline{P}^{\varphi}([\![A_n^{\geq \ell}]\!]) \leq \overline{P}^{\varphi}\left(\left\{\varpi \in \Omega \colon \sup_{m\in\mathbb{N}_0} T(\varpi_{1:m}) \geq 2^N\right\}\right) \leq 2^{-N}. \quad \square$$

15.2 Schnorr randomness implies Schnorr test randomness

Non-degeneracy and computability of the forecasting system are enough to guarantee that the converse implication also holds. That neither non-degeneracy nor computability is a sufficient condition can be shown by essentially the same counter-examples as in the case of ML-randomness; we refrain from spelling them out explicitly, because that would essentially imply giving the exact same counter-examples.

Proposition 15.6. *Consider any path $\omega \in \Omega$ and any non-degenerate computable forecasting system $\varphi \in \Phi(\mathscr{X})$. If ω is* S*-random for φ then it's* S*-test-random for φ.*

Proof. For this converse result too, we give a proof by contraposition. Assume that ω isn't S-test-random for φ, which implies that there's some S-test A for φ such that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. It follows from Proposition 13.6$_{115}$ that we may assume without loss of generality that the sets of situations A_n are partial cuts for all $n \in \mathbb{N}_0$. We'll now use this A to construct a computable test supermartingale that's computably unbounded on ω.

We infer from Lemma 15.12$_{139}$ that there's some growth function ς such that

$$\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}([\![A_n^{\geq \varsigma(k)}]\!]) \leq 2^{-k} \text{ for all } k \in \mathbb{N}_0. \tag{15.7}$$

We use this growth function ς to define the following maps, all of which are non-negative supermartingales for φ, by P1$_{35}$, P5$_{36}$ and C2$_{20}$:

$$Z_{n,k}\colon \mathbb{S} \to \mathbb{R}\colon s \mapsto 2^k \overline{P}^{\varphi}([\![A_n^{\geq \varsigma(k)}]\!] | s), \text{ for all } n, k \in \mathbb{N}_0.$$

Since the computable forecasting system φ was assumed to be non-degenerate, Lemma 9.13$_{62}$ [with $\varphi' \to \varphi$ and $C_\varphi\colon \mathbb{S} \to \mathbb{N}$ a recursive natural-valued process for which $C_\varphi(\square) = 1$] now implies that

$$0 \leq Z_{n,k}(s) \leq Z_{n,k}(\square)C_\varphi(s) = 2^k\overline{P}^\varphi\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket\big)C_\varphi(s) \text{ for all } s \in \mathbb{S}. \tag{15.8}$$

If we also define the (possibly extended) real process $Z \coloneqq \frac{1}{2}\sum_{n,k\in\mathbb{N}_0} Z_{n,k}$, then we infer from Equations (15.7)↶ and (15.8) that

$$0 \leq Z(s) = \frac{1}{2}\sum_{n,k\in\mathbb{N}_0} Z_{n,k}(s) \leq \frac{1}{2}C_\varphi(s)\sum_{n,k\in\mathbb{N}_0} 2^k\overline{P}^\varphi\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket\big) \leq \frac{1}{2}C_\varphi(s)\sum_{k\in\mathbb{N}_0} 2^{-k} = C_\varphi(s) \text{ for all } s \in \mathbb{S}. \tag{15.9}$$

This guarantees that Z is real-valued, and that, moreover, $Z(\square) \leq 1$.

Now, fix any $s \in \mathbb{S}$. Then we readily see that $\frac{1}{2}\sum_{n=0}^N\sum_{\ell=0}^L Z_{n,\ell}(s) \nearrow Z(s)$ and therefore also $\frac{1}{2}\sum_{n=0}^N\sum_{\ell=0}^L \Delta Z_{n,\ell}(s) \to \Delta Z(s)$ as $N, L \to \infty$. Since the gambles $\Delta Z_{n,\ell}(s)$ and $\Delta Z(s)$ are defined on the finite domain $\mathscr{X}$, this point-wise convergence also implies uniform convergence, so we can infer from C6$_{20}$ and

$$\overline{E}_{\varphi(s)}\Big(\frac{1}{2}\sum_{n=0}^N\sum_{\ell=0}^L \Delta Z_{n,\ell}(s)\Big) \leq \frac{1}{2}\sum_{n=0}^N\sum_{\ell=0}^L \overline{E}_{\varphi(s)}(\Delta Z_{n,\ell}(s)) \leq 0,$$

which is implied by C2$_{20}$, C3$_{20}$ and the supermartingale character of the $Z_{n,\ell}$, that also

$$\overline{E}_{\varphi(s)}(\Delta Z(s)) = \lim_{N,L\to\infty} \overline{E}_{\varphi(s)}\Big(\frac{1}{2}\sum_{n=0}^N\sum_{\ell=0}^L \Delta Z_{n,\ell}(s)\Big) \leq 0. \tag{15.10}$$

This tells us that Z is a non-negative supermartingale for φ. It follows from Lemma 15.13$_{140}$ that Z is also computable.

The relevant condition being $\overline{E}_{\varphi(\square)}(Z(\square\,\cdot)) \leq Z(\square)$, we see that replacing $Z(\square) \leq 1$ by 1 doesn't change the supermartingale character of Z, and doing so leads to a computable test supermartingale Z' for φ.

To show that this Z' is computably unbounded on ω, we take two steps.

In a first step, we fix any $n \in \mathbb{N}_0$. Since $\omega \in \bigcap_{m\in\mathbb{N}_0}\llbracket A_m\rrbracket$, and since the A_m were assumed to be partial cuts, there's some (unique) $\ell_n \in \mathbb{N}_0$ such that $\omega_{1:\ell_n} \in A_n$. This tells us that if $\ell \leq \ell_n$, then also $\omega_{1:\ell_n} \in A_n^{\geq\ell}$, and therefore, by Corollary 6.15(i)$_{36}$, that $\overline{P}^\varphi\big(\llbracket A_n^{\geq\ell}\rrbracket|\omega_{1:\ell_n}\big) = 1$ for all $\ell \leq \ell_n$. Hence,

$$\overline{P}^\varphi\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket|\omega_{1:\ell_n}\big) = 1 \text{ for all } k \in \mathbb{N}_0 \text{ such that } \varsigma(k) \leq \ell_n.$$

Let's now define the map $\varsigma^\sharp\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that $\varsigma^\sharp(\ell) \coloneqq \sup\{k \in \mathbb{N}_0\colon \varsigma(k) \leq \ell\}$ for all $\ell \in \mathbb{N}_0$, where we use the convention that $\sup\emptyset = 0$. It's clear that $\varsigma^\sharp$ is a growth function. Moreover, as soon as $\ell_n \geq \varsigma(0)$, we find that, in particular, $\varsigma(k) \leq \ell_n$ for $k = \varsigma^\sharp(\ell_n)$. Hence,

$$\overline{P}^\varphi\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket|\omega_{1:\ell_n}\big) = 1 \text{ for } k = \varsigma^\sharp(\ell_n), \text{ if } \ell_n \geq \varsigma(0).$$

This leads us to the conclusion that for all $n \in \mathbb{N}_0$, there's some $\ell_n \in \mathbb{N}_0$ such that

$$Z'(\omega_{1:\ell_n}) \geq Z(\omega_{1:\ell_n}) \geq \frac{1}{2}Z_{n,\varsigma^\sharp(\ell_n)}(\omega_{1:\ell_n}) = 2^{\varsigma^\sharp(\ell_n)-1} \text{ if } \ell_n \geq \varsigma(0). \tag{15.11}$$

Since $\varsigma^\sharp$ is a growth function, so is the map $\rho\colon \mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$\rho(m) := 2^{\varsigma^\sharp(m)-1} \text{ for all } m \in \mathbb{N}_0.$$

We'll therefore be done if we can now show that the sequence ℓ_n is unbounded as $n \to \infty$, because the inequality in Eq. (15.11) will then guarantee that

$$\limsup_{m\to\infty}[Z'(\omega_{1:m}) - \rho(m)] \geq 0,$$

so the computable test supermartingale Z' is computably unbounded on ω.

Proving that ℓ_n is unbounded as $n \to \infty$ is therefore our second step. To accomplish this, we use the assumption that φ is non-degenerate. Assume, towards contradiction, that there's some natural number B such that $\ell_n \leq B$ for all $n \in \mathbb{N}_0$. The non-degenerate character of φ implies that $\overline{E}_{\varphi(s)}(\mathbb{I}_x) > 0$ for all $s \in \mathbb{S}$ and $x \in \mathscr{X}$, which implies in particular that there's some real $\delta \in (0,1)$ such that $\overline{E}_{\varphi(\omega_{1:k})}(\mathbb{I}_{\omega_{k+1}}) \geq \delta$ for all non-negative integers $k \leq B$, as they are finite in number. But this implies that

$$2^{-n} \geq \overline{P}^\varphi([\![A_n]\!]) \geq \overline{P}^\varphi([\![\omega_{1:\ell_n}]\!]) = \prod_{k=0}^{\ell_n-1} \overline{E}_{\varphi(\omega_{1:k})}(\mathbb{I}_{\omega_{k+1}}) \geq \delta^{\ell_n} \geq \delta^B \text{ for all } n \in \mathbb{N}_0,$$

where the first inequality follows from the properties of an S-test, the second inequality from $[\![\omega_{1:\ell_n}]\!] \subseteq [\![A_n]\!]$ and P3[35], the equality from Proposition 6.16[36], and the fourth inequality from $1 > \delta > 0$ and $\ell_n \leq B$. However, since $1 > \delta > 0$ and $B \in \mathbb{N}$, there's always some $n \in \mathbb{N}_0$ such that $2^{-n} < \delta^B$, which is the desired contradiction. □

Lemma 15.12. *Consider any* S*-test A for a computable forecasting system $\varphi \in \Phi(\mathscr{X})$, such that the corresponding A_n are partial cuts for all $n \in \mathbb{N}_0$. Then there's some growth function $\varsigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that*

$$\sum_{n=0}^{\infty} 2^k \overline{P}^\varphi\big([\![A_n^{\geq\varsigma(k)}]\!]\big) \leq 2^{-k} \text{ for all } k \in \mathbb{N}_0.$$

Proof. Proposition 13.6(ii)[115] guarantees that there's a growth function $e\colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\overline{P}^\varphi\big([\![A_n^{\geq\ell}]\!]\big) = \overline{P}^\varphi\big([\![A_n]\!] \setminus [\![A_n^{<\ell}]\!]\big) \leq 2^{-N} \text{ for all } N, n \in \mathbb{N}_0 \text{ and all } \ell \geq e(N),$$

where the equality holds because the A_n are assumed to be partial cuts. Let $\varsigma\colon \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varsigma(k) := \max_{n=0}^{2k+1} e(2k+2+n)$ for all $k \in \mathbb{N}_0$. Clearly, ς is recursive because e is. It follows from the non-decreasingness and unboundedness of e that ς is non-decreasing, since

$$\varsigma(k+1) = \max_{n=0}^{2k+3} e(2k+4+n) \geq \max_{n=0}^{2k+1} e(2k+2+n) = \varsigma(k) \text{ for all } k \in \mathbb{N}_0,$$

and that ς is unbounded, since $\varsigma(k) \geq e(2k+2)$ for all $k \in \mathbb{N}_0$. So we conclude that ς is a growth function.

Now, for any $k \in \mathbb{N}_0$, we find that, indeed,

$$\sum_{n=0}^{\infty} 2^k \overline{P}^\varphi\big([\![A_n^{\geq\varsigma(k)}]\!]\big) = 2^k \sum_{n=0}^{2k+1} \overline{P}^\varphi\big([\![A_n^{\geq\varsigma(k)}]\!]\big) + 2^k \sum_{n=2k+2}^{\infty} \overline{P}^\varphi\big([\![A_n^{\geq\varsigma(k)}]\!]\big)$$

$$\overset{\text{P3}_{35}}{\leq} 2^k \sum_{n=0}^{2k+1} \overline{P}^{\varphi}(\llbracket A_n^{\geq e(2k+2+n)} \rrbracket) + 2^k \sum_{n=2k+2}^{\infty} \overline{P}^{\varphi}(\llbracket A_n \rrbracket)$$
$$\leq 2^k \sum_{n=0}^{2k+1} 2^{-(2k+2+n)} + 2^k \sum_{n=2k+2}^{\infty} 2^{-n}$$
$$= 2^{-(k+1)} \sum_{n=0}^{2k+1} 2^{-(n+1)} + 2^{-(k+1)}$$
$$\leq 2^{-(k+1)} + 2^{-(k+1)} = 2^{-k}. \qquad \square$$

Lemma 15.13. *The non-negative supermartingale Z in the proof of Proposition 15.6$_{137}$ is computable.*

Proof. We use the notations in the proof of Proposition 15.6$_{137}$. We aim at obtaining a computable real map that converges effectively to Z. First of all, for any $p \in \mathbb{N}_0$,

$$Z(s) = \frac{1}{2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} Z_{n,k}(s) = \frac{1}{2} \sum_{k=0}^{p} \sum_{n=0}^{\infty} Z_{n,k}(s) + \underbrace{\frac{1}{2} \sum_{k=p+1}^{\infty} \sum_{n=0}^{\infty} Z_{n,k}(s)}_{R_1(p,s)},$$

where

$$|R_1(p,s)| = R_1(p,s) = \frac{1}{2} \sum_{k=p+1}^{\infty} \sum_{n=0}^{\infty} Z_{n,k}(s)$$
$$\leq \frac{1}{2} \sum_{k=p+1}^{\infty} \sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}(\llbracket A_n^{\geq \varsigma(k)} \rrbracket) C_{\varphi}(s) = \frac{1}{2} C_{\varphi}(s) \sum_{k=p+1}^{\infty} \Big(\sum_{n=0}^{\infty} 2^k \overline{P}^{\varphi}(\llbracket A_n^{\geq \varsigma(k)} \rrbracket) \Big)$$
$$\leq C_{\varphi}(s) \frac{1}{2} \sum_{k=p+1}^{\infty} 2^{-k} = C_{\varphi}(s) 2^{-(p+1)} \leq 2^{-(p+1-L_{C_{\varphi}}(s))}.$$

In this chain of (in)equalities, the first inequality follows from Eq. (15.8)$_{138}$ and the non-degeneracy of φ, and the second inequality follows from Eq. (15.7)$_{137}$. The last inequality is based on Lemma 14.13$_{124}$ [with $C_{\varphi} \colon \mathbb{S} \to \mathbb{N}$ a recursive natural-valued process] and the notations introduced there. If we therefore define the recursive map $e_1 \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ by $e_1(N,s) := N + L_{C_{\varphi}}(s)$ for all $(N,s) \in \mathbb{N}_0 \times \mathbb{S}$ [with $L_{C_{\varphi}}$ recursive by Lemma 14.13$_{124}$], then we find that

$$|R_1(p,s)| \leq 2^{-(N+1)} \text{ for all } (N,s) \in \mathbb{N}_0 \times \mathbb{S} \text{ and all } p \geq e_1(N,s).$$

Next, we consider any $p, q \in \mathbb{N}_0$ and look at

$$\frac{1}{2} \sum_{k=0}^{p} \sum_{n=0}^{\infty} Z_{n,k}(s) = \frac{1}{2} \sum_{k=0}^{p} \sum_{n=0}^{q} Z_{n,k}(s) + \underbrace{\frac{1}{2} \sum_{k=0}^{p} \sum_{n=q+1}^{\infty} Z_{n,k}(s)}_{R_2(p,q,s)},$$

where

$$|R_2(p,q,s)| = R_2(p,q,s) = \frac{1}{2} \sum_{k=0}^{p} \sum_{n=q+1}^{\infty} Z_{n,k}(s)$$

$$\leq \frac{1}{2}\sum_{k=0}^{p}\sum_{n=q+1}^{\infty}2^k\overline{P}^{\varphi}\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket\big)C_{\varphi}(s) \leq \frac{1}{2}C_{\varphi}(s)2^p\sum_{k=0}^{p}\Big(\sum_{n=q+1}^{\infty}\overline{P}^{\varphi}\big(\llbracket A_n\rrbracket\big)\Big)$$
$$\leq C_{\varphi}(s)2^{p-1}(p+1)\sum_{n=q+1}^{\infty}2^{-n} = C_{\varphi}(s)2^{p-q-1}(p+1) \leq 2^{2p-q-1+L_{C_{\varphi}}(s)}.$$

In this chain of (in)equalities, the first inequality follows from Eq. (15.8)$_{138}$ and the non-degeneracy of φ, the second inequality follows from P3$_{35}$ since $\llbracket A_n^{\geq\varsigma(k)}\rrbracket \subseteq \llbracket A_n\rrbracket$ for all $k,n\in\mathbb{N}_0$, and the third inequality follows from the assumption that A is an S-test. The fourth inequality is based on Lemma 14.13$_{124}$ and the notations introduced there, and the fact that $p+1\leq 2^p$ for all $p\in\mathbb{N}_0$. If we therefore define the recursive map $e_3\colon {\mathbb{N}_0}^2\times\mathbb{S}\to\mathbb{N}_0$ by $e_3(p,N,s) := N+2p+L_{C_{\varphi}}(s)$ for all $(p,N,s)\in{\mathbb{N}_0}^2\times\mathbb{S}$ [recall that $L_{C_{\varphi}}$ is recursive by Lemma 14.13$_{124}$], then we find that

$$|R_2(p,q,s)| \leq 2^{-(N+1)} \text{ for all } (p,N,s)\in{\mathbb{N}_0}^2\times\mathbb{S} \text{ and } q\geq e_3(p,N,s).$$

Now, consider the recursive map $e_2\colon \mathbb{N}_0\times\mathbb{S}\to\mathbb{N}_0$ defined by $e_2(N,s) := e_3(e_1(N,s),N,s)$ for all $(N,s)\in\mathbb{N}_0\times\mathbb{S}$, and let

$$V_N(s) := \frac{1}{2}\sum_{k=0}^{e_1(N,s)}\sum_{n=0}^{e_2(N,s)}Z_{n,k}(s) \text{ for all } N\in\mathbb{N}_0 \text{ and } s\in\mathbb{S}.$$

Since the real map $(n,k,s)\mapsto Z_{n,k}(s)$ is computable by Lemma 15.14, it follows that the real map $(N,s)\mapsto V_N(s)$ is computable as well, since by definition each $V_N(s)$ is a finite sum of real numbers that are computable uniformly in n, k and s, and the finite number of terms to be included are fully determined by the recursive maps e_1 and e_2 as a function of N and s. From the argumentation above, we infer that

$$\begin{aligned}|Z(s)-V_N(s)| &= |R_1(e_1(N,s),s)+R_2(e_1(N,s),e_2(N,s),s)|\\ &\leq |R_1(e_1(N,s),s)|+|R_2(e_1(N,s),e_2(N,s),s)|\\ &\leq 2^{-(N+1)}+2^{-(N+1)} = 2^{-N} \text{ for all } s\in\mathbb{S} \text{ and } N\in\mathbb{N}_0,\end{aligned}$$

proving that Z is indeed computable. □

Lemma 15.14. *For the non-negative supermartingales $Z_{n,k}$ defined in the proof of Proposition 15.6$_{137}$, the real map $(n,k,s)\mapsto Z_{n,k}(s)$ is computable.*

Proof. We use the notations and assumptions in the proof of Proposition 15.6$_{137}$. Clearly, it suffices to prove that the real map $(n,k,s)\mapsto Z_{n,k}(s)2^{-k} = \overline{P}^{\varphi}\big(\llbracket A_n^{\geq\varsigma(k)}\rrbracket|s\big)$ is computable. If we let

$$A_n^{k,\ell} := A_n^{<\ell}\cap A_n^{\geq\varsigma(k)} = \{s\in A_n\colon \varsigma(k)\leq|s|<\ell\},$$

then $\llbracket A_n^{k,\ell}\rrbracket \subseteq \llbracket A_n^{\geq\varsigma(k)}\rrbracket$ and the global events $\llbracket A_n^{k,\ell}\rrbracket$ and $\llbracket A_n^{\geq\ell}\rrbracket$ are disjoint for all $\ell,n,k\in\mathbb{N}_0$, because the A_n have been assumed to be partial cuts. Moreover,

$$\llbracket A_n^{k,\ell}\rrbracket \subseteq \llbracket A_n^{\geq\varsigma(k)}\rrbracket \left\{\begin{array}{ll} = \llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket & \text{if } \ell>\varsigma(k)\\ \subseteq \llbracket A_n^{\geq\ell}\rrbracket = \llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket & \text{if } \ell\leq\varsigma(k)\end{array}\right\} \subseteq \llbracket A_n^{k,\ell}\rrbracket\cup\llbracket A_n^{\geq\ell}\rrbracket, \tag{15.15}$$

where the last equality holds because then $\llbracket A_n^{k,\ell} \rrbracket = \emptyset$. By Lemma 15.17, there's some recursive map $\tilde{e} \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ such that $\overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | s) \leq 2^{-N}$ for all $(N, n, s) \in \mathbb{N}_0{}^2 \times \mathbb{S}$ and all $\ell \geq \tilde{e}(N, s)$. This allows us to infer that

$$\begin{aligned}\overline{P}^{\varphi}(\llbracket A_n^{k,\ell} \rrbracket | s) \leq \overline{P}^{\varphi}(\llbracket A_n^{\geq \varsigma(k)} \rrbracket | s) &\leq \overline{P}^{\varphi}(\llbracket A_n^{k,\ell} \rrbracket \cup \llbracket A_n^{\geq \ell} \rrbracket | s) \\ &\leq \overline{P}^{\varphi}(\llbracket A_n^{k,\ell} \rrbracket | s) + \overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | s) \\ &\leq \overline{P}^{\varphi}(\llbracket A_n^{k,\ell} \rrbracket | s) + 2^{-N} \text{ for all } N, k, n \in \mathbb{N}_0 \text{ and } s \in \mathbb{S} \text{ and } \ell \geq \tilde{e}(N, s),\end{aligned} \tag{15.16}$$

where the first two inequalities follow from Eq. (15.15)↶ and P3$_{35}$, and the third inequality follows from P2$_{35}$, because $\llbracket A_n^{k,\ell} \rrbracket$ and $\llbracket A_n^{\geq \ell} \rrbracket$ are disjoint. Now, the sets $A_n^{k,\ell}$ are recursive uniformly in n, k and ℓ, and it also holds that $|s| < \ell$ for all $s \in A_n^{k,\ell}$ and $n, k, \ell \in \mathbb{N}_0$. Hence, the real map $(n, k, \ell, s) \mapsto \overline{P}^{\varphi}(\llbracket A_n^{k,\ell} \rrbracket | s)$ is computable by an appropriate instantiation of our Workhorse Lemma 13.9$_{117}$ [with $\mathcal{D} \to \mathbb{N}_0{}^2$, $d \to (n, k)$, $p \to \ell$ and $C \mapsto \{(n, k, \ell, s) \in \mathbb{N}_0{}^3 \times \mathbb{S} \colon s \in A_n^{k,\ell}\}$, and therefore $C_d^p \to A_n^{k,\ell}$], because the forecasting system φ is computable as well. The inequalities in Eq. (15.16) tell us that this computable real map converges effectively to the real map $(n, k, s) \mapsto \overline{P}^{\varphi}(\llbracket A_n^{\geq \varsigma(k)} \rrbracket | s)$, which is therefore computable as well. □

Lemma 15.17. *Consider any* $\mathbb{S}$*-test* A *for a non-degenerate computable forecasting system* $\varphi \in \Phi(\mathscr{X})$*, such that the corresponding* A_n *are partial cuts for all* $n \in \mathbb{N}_0$*. Then there's some recursive map* $\tilde{e} \colon \mathbb{N}_0 \times \mathbb{S} \to \mathbb{N}_0$ *such that its partial maps* $\tilde{e}(\bullet, s)$ *are growth functions for all* $s \in \mathbb{S}$*, and such that*

$$\overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | s) \leq 2^{-N} \textit{ for all } (N, n, s) \in \mathbb{N}_0{}^2 \times \mathbb{S} \textit{ and all } \ell \geq \tilde{e}(N, s).$$

Proof. Proposition 13.6(ii)$_{115}$ guarantees that there's a growth function $e \colon \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket) = \overline{P}^{\varphi}(\llbracket A_n \rrbracket \setminus \llbracket A_n^{< \ell} \rrbracket) \leq 2^{-M} \text{ for all } M, n \in \mathbb{N}_0 \text{ and all } \ell \geq e(M),$$

where the equality holds because the A_n are assumed to be partial cuts. Since the real process $\overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | \bullet)$ is a non-negative supermartingale by Corollary 6.15$_{36}$, we infer from the non-degeneracy of φ, Lemma 9.13$_{62}$ [with $\varphi' \to \varphi$ and $C_\varphi \geq 1$ computable] and Lemma 14.13$_{124}$ [with L_{C_φ} recursive] that

$$0 \leq \overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | s) \leq \overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket) C_\varphi(s) \leq 2^{-M} C_\varphi(s) \leq 2^{-M + L_{C_\varphi}(s)} \quad \text{for all } (M, n) \in \mathbb{N}_0{}^2 \text{ and all } \ell \geq e(M).$$

It's therefore clear that if we let

$$\tilde{e}(N, s) \coloneqq e(N + L_{C_\varphi}(s)) \text{ for all } (N, s) \in \mathbb{N}_0 \times \mathbb{S},$$

then

$$\overline{P}^{\varphi}(\llbracket A_n^{\geq \ell} \rrbracket | s) \leq 2^{-N} \text{ for all } (N, n, s) \in \mathbb{N}_0{}^2 \times \mathbb{S} \text{ and all } \ell \geq \tilde{e}(N, s).$$

This map $\tilde{e}$ is recursive because the maps e and L_{C_φ} are. Furthermore, for any fixed s in $\mathbb{S}$, $\tilde{e}(\bullet, s)$ is clearly non-decreasing and unbounded, because e is. □

A prequential approach to martingale- and test-theoretic randomness

Let's quickly recapitulate what we did in the previous three chapters. We considered an infinite sequence $X_1, X_2, X_3, \dots$ of variables $X_k \in \mathscr{X}$, let a subject put forward a so-called *forecasting system* $\varphi \in \Phi(\mathscr{X})$ that describes his (conditional) uncertainty about the unknown outcome of the variable X_{n+1} given the observation $(X_1, \dots, X_n) = s \in \mathbb{S}$, and examined when an infinite outcome sequence $(x_1, x_2, x_3, \dots)$ 'agrees with' a subject's specification of a forecasting system. In other words, the central question throughout this dissertation has been: 'what sequences do we consider to be random for a forecasting system?'. We've answered this question by adopting a *martingale-theoretic, frequentist* and *test-theoretic* approach to randomness, and in doing so, we let go of the classical assumption that a subject is always able to associate (conditional) precise probabilities m_s with every $s \in \mathbb{S}$. In our framework, a subject is also allowed to forecast sets of probabilities $C \in \mathscr{C}(\mathscr{X})$, which we've been calling credal sets; we'll call this approach, which involves specifying a forecast $C_s \in \mathscr{C}(\mathscr{X})$ for all possible $s \in \mathbb{S}$, the *standard* approach to forecasting.

In a number of papers [7, 8], Philip Dawid and Volodya Vovk have questioned whether it's always natural or even possible for a subject to specify such a (precise) forecasting system. Consider for example a weather forecaster who provides a daily probability for rain in the next 24 hours. His forecasts are based on the rain history he has actually observed (as well as other information), and he isn't required to provide forecasts for all rain histories that might have been or might be. Dawid and Vovk provided a practical way to solve this problem by putting forward their so-called *prequential forecasting* framework.

In a number of groundbreaking papers [9, 89], this prequential forecasting idea was applied to questions related to algorithmic randomness. Instead of defining what it means for an infinite binary sequence of outcomes, such as $(0,1,1,0,0,1,0,\dots)$, to be random for a forecasting system, their authors came up with randomness notions that consider the randomness of an infinite binary sequence of outcomes only with respect to the probability mass functions that are actually forecast along the sequence, that is, they defined what it means for an infinite sequence $(m_1, x_1, m_2, x_2, \dots, m_k, x_k, \dots)$ of precise forecasts $m_k \in \mathcal{M}(\mathscr{X})$ and subsequent outcomes $x_k \in \{0,1\}$ to be random. They did this using a *martingale*- as well as a *test-theoretic* approach.

In this chapter, not only do we question whether it's always natural or even possible for a subject to specify a precise forecasting system—as we have been doing in the previous chapters—, we also call into doubt whether a subject is always able to put forward precise forecasts m_k in a prequential context. Consider for example the weather forecasting set-up introduced before, and let's assume that it's the weather forecaster's task to predict rainfall in some remote mountain valley. Since the weather is known to be very unreliable in this kind of setting, the weather forecaster had better be very careful when making such predictions, especially in the beginning of his endeavour, when only limited data might be at hand. In such a scenario, it might be better for him to resort to a less committal description of his rainfall-related uncertainty that better reflects his limited knowledge, which for instance he could do by specifying only bounds on probabilities—a credal set—instead of precise probabilities. To deal with this concern, we intend to build upon the previously introduced prequential approach to randomness, and extend the precise forecasts $m_k \in \mathcal{M}(\mathscr{X})$ to so-called imprecise forecasts $C_k \in \mathscr{C}(\mathscr{X})$. In doing so, we extend the range of applicability of both Dawid and Vovk's and our own work [see Chapters ⚁$_{49}$ and ⚃$_{111}$]: we'll define what it means for an infinite sequence $(C_1, x_1, C_2, x_2, \dots)$ of forecasts C_k and subsequent outcomes x_k to be random—both in a martingale- and a test-theoretic way—, prove that these two randomness notions coincide, and compare the resulting prequential *imprecise-probabilistic* randomness definition(s) to our previously introduced *standard* imprecise-probabilistic generalisation of Martin-Löf randomness [see Definition 8.5$_{52}$].[29]

Let's recapitulate, as well as clarify, the terminology we'll adopt further on, which is based on three distinctions that can be made in approaching the study of randomness. The first possible distinction is between *standard* and *prequential* approaches: do we consider the randomness of an infinite

[29] In our prequential imprecise-probabilistic approach to randomness, we thus focus on modifications of our notion of ML-(test-)randomness. It's however rather straightforward to apply (many of) the lines of reasoning in this chapter to the other martingale- and test-theoretic randomness notions we've introduced in Sections 8$_{50}$ and 13$_{113}$.

sequence of outcomes with respect to a given forecasting system, or do we consider the randomness of an infinite sequence of forecasts followed by outcomes? The second distinction is between *(precise-)probabilistic* and *imprecise-probabilistic* approaches: do we consider point forecasts or their extension to more general imprecise forecasts? And the third possible distinction is between *martingale-theoretic* and *test-theoretic* approaches: do we approach the randomness of a sequence of outcomes by using the impossibility of a gambling system that allows one to become unboundedly rich by betting on them, or rather by using a collection of so-called randomness tests that the sequence must pass in other to qualify for being called random?

So far, prequential randomness has been studied in a precise-probabilistic setting [9, 89]. Here, we propose to extend the precise-probabilistic discussion of prequential randomness to an imprecise-probabilistic one, and to do this for both the martingale-theoretic and the test-theoretic approaches.

We have structured this chapter as follows. In Section 16, we reconsider the infinite betting game as introduced in Section 6.3$_{27}$, and explain how to move from the standard approach to a prequential one. This allows us in Section 17$_{150}$ to formally introduce our *prequential* imprecise-probabilistic martingale-theoretic notion of randomness—which we'll call *game-randomness*; our terminology follows Dawid & Vovk [8] and Vovk & Shen [9]. We compare the definitions and properties of our standard and prequential martingale-theoretic randomness notions in Section 17.3$_{153}$. In particular, we show that both notions coincide when imposing some mild (computability) conditions on the uncertainty models involved; see also Figure 15.1$_{\curvearrowleft}$. In Section 18$_{168}$, we introduce a prequential imprecise-probabilistic test-theoretic notion of randomness, which we'll call *prequential test-randomness*. We prove that it coincides with game-randomness, and thereby extend earlier results on the equivalence of martingale-theoretic and test-theoretic approaches in prequential precise-probabilistic randomness [9, Corollary 1]; again, see also Figure 15.1$_{\curvearrowleft}$. This will obviously also allow us to carry over to prequential test-randomness all the properties derived for game-randomness in Section 17.4$_{161}$.

16 Sequential and prequential games

Consider Frank Deboosere—a famous Belgian weather forecaster—whose daily job consists in making good forecasts about whether the sun will or won't shine on the next day. This corresponds to a binary option space: $\mathscr{X} = \{☼, ☹\}$. We formalise his forecasting task in the following forecasting protocol:

FOR $n = 1, 2, 3, \dots$:
  Forecaster Frank announces $C_n \in \mathscr{C}(\mathscr{X})$.
  Reality announces $x_n \in \mathscr{X}$.

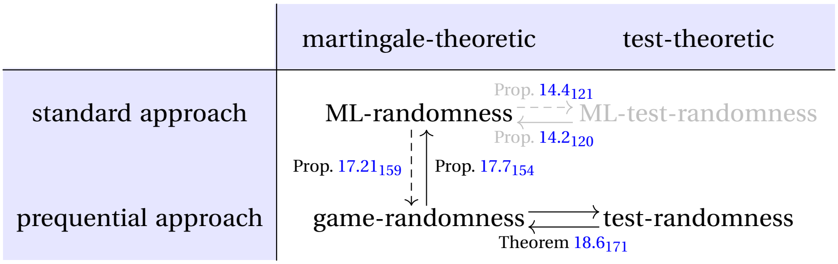


Figure 15.1. We consider three randomness notions: Martin-Löf randomness, game-randomness and test-randomness. In Section 17.3153, we show that game-randomness implies Martin-Löf randomness, and that these notions coincide under some mild computability conditions on the uncertainty models. In Section 18.2170, we show that game-randomness and test-randomness coincide, without extra conditions on the uncertainty models.

Here, Reality is imagined to be another player, who gets to determine whether or not the sun shines on day n. At each step $n \in \mathbb{N}$ in the protocol, C_n expresses Frank's beliefs about X_n after observing the outcomes $(x_1, \dots, x_{n-1})$.

Clearly, Frank can do a good or a bad forecasting job. For example, if he forecasts $\mathbb{I}_{☼} \in \mathcal{M}(\mathscr{X})$ at every time step, but it rains every day, then we might be inclined to say he's doing a bad job. But if he forecasts $1/2$ at every time step and it rains half of the time, then we might (perhaps) agree he's doing a good job. This brings us again to the central question of this chapter: when do we say that Frank makes good predictions, or more specifically, that his forecasts $(C_1, \dots, C_n, \dots)$ 'agree with' the outcomes $(x_1, \dots, x_n, \dots)$? The field of (prequential) algorithmic randomness tries to answer this question by defining what it means for an infinite sequence $(C_1, x_1, \dots, C_n, x_n, \dots)$ of forecasts C_n and subsequent outcomes x_n to be 'random'.

16.1 The standard approach

At the start of Chapter 49, the above question took the following different shape: 'What sequences do we consider to be random for a forecasting system $\varphi \in \Phi(\mathscr{X})$?'. On the standard approach, it's thus assumed that Forecaster Frank's forecasts in the protocol mentioned above can be derived from a so-called *forecasting system*: Frank not only has to specify forecasts $C_n := C_{(x_1,\dots,x_{n-1})}$ to express his beliefs about X_n *after* observing the actual outcomes $(x_1, \dots, x_{n-1})$, but he also has to specify forecasts $C_s \in \mathscr{C}(\mathscr{X})$ for all possible situations $s \in \mathbb{S}$ that could in principle occur or have occurred: he has to specify what we've called a forecasting system.

We see that, in this context, it's more natural to talk about the randomness of a path $\omega \in \Omega$ for a forecasting system φ, rather than for a sequence of forecasts $(C_1, \dots, C_n, \dots)$.

In Chapter ⚁$_{49}$, to answer this randomness question, a third player called Sceptic tested the correspondence between Forecaster Frank's forecasting system φ and Reality's outcomes; in this chapter, we'll identify Sceptic with Frank's colleague Sabine Hagedoren—who is another famous Belgian weather forecaster. In the context of Chapter ⚁$_{49}$, her job consisted in engaging in a betting game: if Frank does a good forecasting job, Sabine shouldn't be able to tremendously increase her capital in the long run, by gambling on the next outcome using bets that Frank makes available in each successive situation s by his specification of the forecast $\varphi(s)$. We then called a path $\omega \in \Omega$ *random* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if Sabine can't come up with an implementable betting strategy $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi)$ that makes her rich without bounds along ω [see Definitions 8.5$_{52}$ and 8.6$_{54}$]. In this chapter, we'll focus on lower semicomputable betting strategies, and hence, in the standard setting, we'll restrict our attention to ML-randomness.

But for now, let's (temporarily) forget about forecasting systems again, and have a first look at how to devise a notion of randomness for an infinite sequence $(C_1, x_1, \dots, C_n, x_n, \dots)$ of forecasts C_n and subsequent outcomes x_n when adopting a *prequential* perspective.

16.2 The prequential approach

We start by introducing a bit of notation. An infinite sequence $(C_1, x_1, \dots, C_n, x_n, \dots) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets C_n and subsequent outcomes x_n is called a *prequential path* and generically denoted by υ.[30] Such a prequential path is a possible instantiation of an infinite sequence of moves in the above-mentioned forecasting protocol, where at each step n, Forecaster Frank emits a credal set C_n for the next outcome variable $X_n \in \mathscr{X}$, which Reality subsequently determines to be x_n.

An infinite sequence of rational credal sets $(C_1, \dots, C_n, \dots) \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^{\mathbb{N}}$ is generically denoted by ζ. A finite sequence of credal sets and outcomes $(C_1, x_1, \dots, C_n, x_n) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, where we let $(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* := \bigcup_{k \in \mathbb{N}_0} (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^k$, is called a *prequential situation*, is generically denoted by v and has length $|\mathrm{v}| = n$. A finite sequence of rational credal sets $(C_1, \dots, C_n) \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^*$, with $\mathscr{C}_{\mathrm{rat}}(\mathscr{X})^* := \bigcup_{k \in \mathbb{N}_0} (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}))^k$, is generically denoted by c and has length $|c| = n$. For any $k \in \mathbb{N}_0$, $\upsilon_{1:k} = (C_1, x_1, \dots, C_k, x_k)$, and similarly for infinite sequences of rational credal sets $\zeta \in \mathscr{C}(\mathscr{X})^{\mathbb{N}}$ and for prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ with $k \leq |\mathrm{v}|$. Furthermore, for any $k \in \mathbb{N}_0$, $\upsilon_k = (C_k, x_k)$, and similarly for infinite sequences of rational credal sets $\zeta \in \mathscr{C}(\mathscr{X})^{\mathbb{N}}$, for finite sequences of rational credal sets $c \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^*$

[30] We'll limit ourselves to rational forecasts in this prequential setting and draw attention to this restriction by using a subscript 'rat'; a rational forecasting system is for example denoted by φ_{rat}, and—as already announced—the set of all rational forecasts by $\mathscr{C}_{\mathrm{rat}}(\mathscr{X})$. In Section 17$_{150}$, we'll provide some explanation and motivation for this restriction.

with $k \leq |c|$, and for prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ with $k \leq |\mathrm{v}|$. The empty prequential situation $\mathrm{v}_{1:0} = ()$ is denoted also by $\square$.

For ease of notation, we won't differentiate between $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and $(\zeta, \omega) \in \mathscr{C}(\mathscr{X})^{\mathbb{N}} \times \Omega$. In the same spirit, we won't differentiate between $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $(c, s) \in \bigcup_{n \in \mathbb{N}_0} (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}))^n \times \mathscr{X}^n$. If a prequential path $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ doesn't allow zero probability jumps, i.e., if $\overline{E}_{\zeta_n}(\mathbb{I}_{\omega_n}) > 0$ for all $n \in \mathbb{N}$, meaning that there is no $n \in \mathbb{N}$ such that Reality chose the outcome ω_n that Forecaster Frank has given a precise probability zero for $[\overline{E}_{\zeta_n}(\mathbb{I}_{\omega_n}) = 0]$, then we call v a *non-degenerate* prequential path; we call it a *degenerate* prequential path otherwise, and collect all degenerate prequential paths in the set $(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}_{\mathrm{deg}}$. Analogously, we call a prequential situation $\mathrm{v} = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ *non-degenerate* if $\overline{E}_{c_m}(\mathbb{I}_{s_m}) > 0$ for all $1 \leq m \leq |s|$, and *degenerate* otherwise; we collect all degenerate prequential situations in the set $(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}$.

The concatenation of a finite sequence of rational credal sets $c \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^*$ and a rational credal set $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ is denoted by cC_{rat}, and the concatenation of a prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, a rational credal set $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and an outcome $x \in \mathscr{X}$ by $\mathrm{v}C_{\mathrm{rat}}x$. In this way, for any $\mathrm{v} = (c, s) = (C_1, x_1, \ldots, C_n, x_n) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and $x \in \mathscr{X}$, we have that $\mathrm{v}C_{\mathrm{rat}}x = (C_1, x_1, \ldots, C_n, x_n, C_{\mathrm{rat}}, x) = (cC_{\mathrm{rat}}, sx) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$.

Consider any two prequential situations $\mathrm{v}, \mathrm{v}' \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. We say that v *precedes* v', and write $\mathrm{v} \sqsubseteq \mathrm{v}'$, if v is a precursor of v'; we may then also write $\mathrm{v}' \sqsupseteq \mathrm{v}$ and say that v' *follows* v. We say that v *strictly precedes* v', and write $\mathrm{v} \sqsubset \mathrm{v}'$, if $\mathrm{v} \sqsubseteq \mathrm{v}'$ and $\mathrm{v} \neq \mathrm{v}'$. If both $\mathrm{v} \not\sqsubseteq \mathrm{v}'$ and $\mathrm{v}' \not\sqsubseteq \mathrm{v}$, then we say that v and v' are *incomparable*, and we write $\mathrm{v} \parallel \mathrm{v}'$. For any set of prequential situations $V \subseteq (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, we write $\mathrm{v} \sqsubseteq V$ if v precedes some member of V, we write $V \sqsubseteq \mathrm{v}$ if v has some precursor in V, we write $\mathrm{v} \sqsubset V$ if v is the precursor of some member of V but doesn't follow any member of V, and we write $\mathrm{v} \parallel V$ if v is incomparable with all members of V. For any prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$, we say that v *goes through* v, and write $\mathrm{v} \sqsubseteq v$, if there's some $n \in \mathbb{N}_0$ such that $\mathrm{v} = v_{1:n}$; if we let $[\![\mathrm{v}]\!] := \{v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}} \colon \mathrm{v} \sqsubseteq v\}$, then v goes through v if and only if $v \in [\![\mathrm{v}]\!]$. Similarly, for any $\zeta \in \mathscr{C}(\mathscr{X})^{\mathbb{N}}$ and $c \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^*$, we say that ζ *goes through* c, and write $c \sqsubseteq \zeta$, if there's some $n \in \mathbb{N}_0$ such that $c = \zeta_{1:n}$. For any set of prequential situations $V \subseteq (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, we let $[\![V]\!] := \bigcup_{\mathrm{v} \in V} [\![\mathrm{v}]\!]$; we remark that $[\![(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}]\!] = (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}_{\mathrm{deg}}$.

In the prequential setting, it's not assumed that Forecaster Frank's forecasts are produced by some underlying forecasting system. Instead, as is (re)presented in the protocol on p. 145, he's allowed to produce forecasts on the fly, so there's no need for Frank to provide forecasts in all situations that could occur. To test whether Frank is doing a good job, Sceptic Sabine here too engages in a betting game, only now she has to be able to specify an allowed change in capital for all possible successions of rational credal sets (that could have been chosen by Frank) and outcomes (that could have been revealed by Reality), that is, she has to specify a possible change in capital for all prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and all possibly following rational credal sets $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$.

We'll assume that Sceptic Sabine starts with unit capital. In order to specify a strategy, for any prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and any rational credal set $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, she needs to select a gamble $\sigma(\mathrm{v}, C_{\mathrm{rat}}) \in \mathscr{L}(\mathscr{X})$ that's made available to her by Forecaster Frank's specification of the credal set C_{rat}, that is, to select an uncertain change of capital $\sigma(\mathrm{v}, C_{\mathrm{rat}}) \in \mathscr{L}(\mathscr{X})$ for which $\overline{E}_{C_{\mathrm{rat}}}(\sigma(\mathrm{v}, C_{\mathrm{rat}})) \leq 0$, meaning that he expects her to have non-positive gain. After n successive steps, where Forecaster has put forward the successive rational credal sets $c = (C_1, \ldots, C_n)$ and where Reality has produced the successive outcomes $s = (x_1, \ldots, x_n)$, which results into the prequential situation $\mathrm{v} = (c, s)$, Sabine's *accumulated capital* $F(\mathrm{v})$ according to such a strategy is

$$F(\mathrm{v}) = 1 + \sum_{k=0}^{n-1} \sigma(\mathrm{v}_{1:k}, c_{k+1})(s_{k+1}),$$

with

$$\sigma(\mathrm{v}, C_{\mathrm{rat}}) = F(\mathrm{v}C_{\mathrm{rat}}\,\cdot) - F(\mathrm{v}) =: \Delta F(\mathrm{v}C_{\mathrm{rat}}\,\cdot).$$

Furthermore, we'll again prohibit Sabine from borrowing money, meaning that $F \geq 0$. In summary then, her prequential betting strategies, which we'll refer to as test superfarthingales, are formalised as follows; as announced in the introduction to this chapter, we borrow the underlying idea as well as the terminology from Dawid & Vovk [8] and Vovk & Shen [9].

Definition 16.1. We call *superfarthingale* any real-valued map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ such that $\overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) \leq F(\mathrm{v})$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, and we collect all superfarthingales in the set $\overline{\mathbb{F}}$. We call a non-negative superfarthingale $F \geq 0$ such that $F(\square) = 1$ a *test superfarthingale*.

There's a specific feature of superfarthingales that we'll use a couple of times: for any superfarthingale $F \in \overline{\mathbb{F}}$ and any non-degenerate prequential situation $\mathrm{v} = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, the capital $F(\mathrm{v})$ is bounded above by the initial capital $F(\square)$ and the rational forecasts c that are specified along v, and this in the following sense.

Lemma 16.2. *For any non-degenerate prequential situation $v = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and any non-negative superfarthingale $F \in \overline{\mathbb{F}}$:*

$$F(v) \leq \prod_{k=1}^{|s|} \frac{1}{\overline{\overline{E}}_{c_k}(\mathbb{I}_{s_k})} F(\square).$$

Proof. The proof is similar to that of Lemma 9.13[62]. Consider any non-degenerate prequential situation $\mathrm{v}C_{\mathrm{rat}}x \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. Then $\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x) > 0$ and

$$F(\mathrm{v}C_{\mathrm{rat}}x) = \frac{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x)}{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x)} F(\mathrm{v}C_{\mathrm{rat}}x) \overset{\mathrm{C2}_{20}}{=} \frac{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x F(\mathrm{v}C_{\mathrm{rat}}x))}{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x)}$$

$$\overset{\text{C5}_{20}}{\leq} \frac{1}{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x)} \overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) \leq \frac{1}{\overline{E}_{C_{\mathrm{rat}}}(\mathbb{I}_x)} F(\mathrm{v}),$$

where the first inequality also makes use of the non-negativity of F, and where the last inequality follows from the superfarthingale property. A simple induction argument now leads to the desired result. □

Here too, for (interesting) random prequential paths to exist, we'll restrict Sabine's betting strategies to a countable set. To obtain a sensible prequential notion of randomness, we'll do so by imposing lower semicomputability.

17 A prequential martingale-theoretic approach

In this section, we'll introduce and discuss two prequential notions of randomness. In Section 17.1, we use our notion of ML-randomness [see Definition 8.5_{52}] to define what it means for a prequential path to be *(prequentially)* ML-*random*; forecasting systems are still in the picture and take centre stage here. In Section 17.2_{153}, we do away with forecasting systems and test the randomness of a prequential path via lower semicomputable test superfarthingales; the corresponding genuinely prequential randomness notion will be called *game-randomness*. We compare both notions in Section 17.3_{153}—where we show that they coincide when restricting our attention to (non-degenerate) recursive rational forecasting systems—, and derive a number of properties of game-randomness in Section 17.4_{161}.

Before introducing these prequential imprecise-probabilistic and martingale-theoretic notions of randomness, which are inspired by Vovk and Shen's precise-probabilistic work [9], let's now first argue why we'll restrict our attention to *rational* forecasts in this prequential context, as we already mentioned in Footnote 30_{147}. First of all, compared to their approach in Ref. [9], it allows us to employ a technically less involved version of implementability that results in simpler proofs. Secondly, and perhaps more importantly, we intend to compare our standard and prequential notions of randomness—culminating in an equivalence result [see Theorem 17.24_{161}] that holds under the restriction of *recursive* (and hence rational) forecasting systems—, and, as has been shown in Proposition 10.1_{67}, rational credal sets are enough to capture the essence of ML-randomness in the standard setting when restricting our attention to non-degenerate *computable* forecasting systems.

17.1 A standard prequential martingale-theoretic approach: prequential Martin-Löf randomness

We can give a more prequential flavour to the previously introduced notion of ML-randomness [see Definition 8.5_{52}], but before doing so, we want and have to introduce some more notation and terminology. With any infinite sequence of outcomes $\omega \in \Omega$ and with any forecasting system $\varphi \in \Phi(\mathscr{X})$, we associate the infinite sequence of forecasts $\varphi[\omega] := (\varphi(\omega_{1:0}), \varphi(\omega_{1:1}), \varphi(\omega_{1:2}), \dots)$. Similarly, we associate with any finite sequence of outcomes $s \in \mathbb{S}$ and with any forecasting system $\varphi \in \Phi(\mathscr{X})$ the finite sequence of forecasts $\varphi[s] := (\varphi(s_{1:0}), \varphi(s_{1:1}), \dots, \varphi(s_{1:|s|-1}))$. This allows us to check the *compatibility* of a forecasting system $\varphi \in \Phi(\mathscr{X})$ with a given infinite sequence $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of forecasts and outcomes, in the sense that φ should emit the same forecasts based on the observed outcomes ω in v as the forecasts ζ that are present in v: we say that φ is *compatible* with v if $\varphi[\omega] = \zeta$, that is, if $\varphi(\omega_{1:n}) = \zeta_{n+1}$ for all $n \in \mathbb{N}_0$; an equivalent condition for compatibility of φ with $v = (\zeta, \omega)$ is clearly given by

$$(\forall s \in \mathbb{S})(s \sqsubseteq \omega \Rightarrow \varphi[s] \sqsubseteq \zeta). \tag{17.1}$$

The compatibility between a forecasting system $\varphi \in \Phi(\mathscr{X})$ and a prequential path $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is illustrated in Figure 17.2$_{\curvearrowright}$.

If the forecasting system φ produces more conservative forecasts along ω compared to ζ, that is, if $\zeta_{n+1} \subseteq \varphi(\omega_{1:n})$ for all $n \in \mathbb{N}_0$, then we say that φ is more *conservative* (or less *informative*) on $v = (\zeta, \omega)$. Similarly, we say that a forecasting system φ is *compatible* with a prequential situation $\mathrm{v} = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{*}$ if $\varphi(s_{1:n}) = c_{n+1}$ for all $0 \leq n \leq |\mathrm{v}| - 1$.
It will be useful in what follows to also observe that

$$(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{*} = \bigcup_{\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})} \big\{(\varphi_{\mathrm{rat}}[s], s) \colon s \in \mathbb{S}\big\} \tag{17.3}$$

and

$$(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}} = \bigcup_{\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})} \big\{(\varphi_{\mathrm{rat}}[\omega], \omega) \colon \omega \in \Omega\big\}; \tag{17.4}$$

this tells us that we can use the rational forecasting systems to 'cover' the set of all prequential situations, as well as the set of all prequential paths: every prequential path has at least one compatible rational forecasting system.

The notion of ML-randomness [see Definition 8.5$_{52}$] can now be adapted to this new prequential context as follows. It leads to a martingale-theoretic definition of prequential randomness that still involves the use of forecasting systems.

Definition 17.5. We'll call a sequence $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes *(prequentially)* ML-*random* if ω is ML-random for all rational forecasting systems $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that are compatible with v.

To provide this (prequential) randomness notion with an interpretation, consider again the prequential forecasting protocol introduced in Section 16$_{145}$; recall that it's Sceptic Sabine's job to test whether Forecaster Frank's (rational) forecasts $\zeta = (C_1, \dots, C_n, \dots)$ 'agree with' Reality's

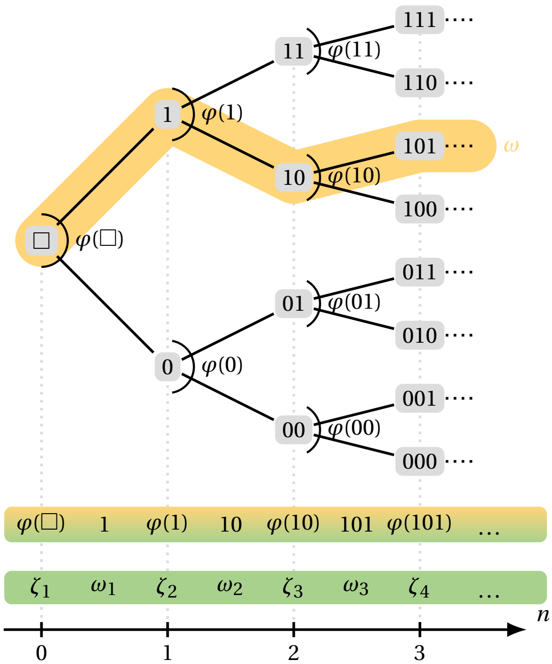


Figure 17.2. The forecasting system φ is represented in the imprecise probability tree, the path $\omega \in \Omega$ that constitutes the prequential path $\upsilon = (\zeta, \omega)$ is depicted in yellow, the prequential path $\upsilon = (\zeta, \omega)$ is depicted in green, and the prequential path $(\varphi[\omega]_1, \omega_1, \varphi[\omega]_2, \omega_2, \varphi[\omega]_3, \omega_3, \ldots)$ that originates from the specification of the forecasting system φ and the path ω is depicted by a merging of yellow and green. The forecasting system φ is then compatible with the prequential path $\upsilon = (\zeta, \omega)$ if and only if $\varphi(\square) = \zeta_1$, $\varphi(1) = \zeta_2$, $\varphi(10) = \zeta_3$, etc.

outcomes $\omega = (\omega_1, \ldots, \omega_n, \ldots)$. In line with the standard approach to algorithmic randomness, we'll assume that Frank has a (rational) forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ fixed in the background, which then determines the (rational) forecasts $C_n \in \mathscr{C}_{\text{rat}}(\mathscr{X})$, with $n \in \mathbb{N}$, he puts forward. In contradistinction with the standard approach, we won't assume that Sabine has any knowledge about Frank's forecasting system φ_{rat}—even if the forecasting system is computable. To make up for her lack of knowledge, she's allowed to test the possible randomness of a prequential path $\upsilon = (\zeta, \omega) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ by putting forward any (implementable) betting strategy $T \in \overline{\mathbb{T}}_{\text{ML}}(\varphi_{\text{rat}})$ that is allowed by a (rational) forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ that's compatible with υ; the prequential path $\upsilon = (\zeta, \omega)$ is then considered random if every such betting strategy remains bounded on ω.

17.2 A fully prequential approach: game-randomness

To obtain a truly prequential imprecise-probabilistic martingale-theoretic notion of randomness, in the sense that it doesn't involve the intervention of forecasting systems, we mimic Vovk and Shen's approach [9], and proceed by imposing lower semicomputability on Sabine's prequential betting strategies—which we called test superfarthingales. Contrary to their approach, we won't allow the test superfarthingales to be infinite-valued as a way to express that degenerate prequential paths—which allow zero probability jumps—shouldn't be random; instead, we explicitly require that random prequential paths should be non-degenerate.

Definition 17.6. We call a sequence $\upsilon = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes *game-random* if it's non-degenerate and if all lower semicomputable test superfarthingales $F \in \overline{\mathbb{F}}$ satisfy $\limsup_{n \to \infty} F(\upsilon_{1:n}) < \infty$.

In the following section, we intend to explore how the two new prequential randomness notions compare: how does (prequential) Martin-Löf randomness compare to game-randomness? In particular, we'll be able to show that both definitions result in (almost-)equivalent randomness notions when we restrict our attention to non-degenerate *recursive* rational forecasting systems on the standard approach. This endeavour can be seen as a continuation (and generalisation) of the discussion in Section 4 of Ref. [9], where Vovk and Shen prove that precise-probabilistic versions of these definitions coincide for non-degenerate computable forecasting systems.[31]

Afterwards, we'll compare a few basic properties for both imprecise-probabilistic randomness notions, where we'll be especially concerned with whether (and which) computability restrictions (on sequences of rational forecasts) are necessary for these properties to hold.

17.3 Equivalence of (prequential) Martin-Löf and game-randomness

So, let's start comparing these randomness notions: in a first subsection, we'll show that game-randomness implies (prequential) Martin-Löf randomness—without any implementability requirements on the rational forecasts—, and in a second subsection, we'll show that (prequential) Martin-Löf randomness implies game-randomness when imposing some arguably mild implementability conditions on the rational forecasts at hand.

Game-randomness implies (prequential) Martin-Löf randomness

As the following result shows, any prequential path that's game-random is also prequentially Martin-Löf random. Game-randomness is therefore at

[31] Vovk and Shen allow for real-valued precise forecasts, rather than our rational-valued interval forecasts.

least as *strong* a randomness notion as prequential Martin-Löf randomness; note that no recursiveness is imposed on the rational forecasting systems.

Proposition 17.7. *Consider any infinite sequence of rational forecasts and outcomes $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ that's game-random. Then the infinite sequence of outcomes ω is* ML*-random for any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that's compatible with v, meaning that v is prequentially* ML*-random.*

Proof. Consider any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that's compatible with the game-random prequential path $v = (\zeta, \omega)$ (which is non-degenerate by assumption), meaning that $v = (\varphi_{\mathrm{rat}}[\omega], \omega)$, and assume towards contradiction that there's some lower semicomputable test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{ML}}(\varphi_{\mathrm{rat}})$ such that $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$; note that by Definition 8.5$_{52}$ and Proposition 10.9$_{73}$ we may assume without loss of generality that T is positive everywhere. We'll now construct a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ in such a way that $F'(\varphi_{\mathrm{rat}}[s], s) = T(s)$ for all situations $s \in \mathbb{S}$ for which the corresponding prequential situation $(\varphi_{\mathrm{rat}}[s], s)$ is non-degenerate, and for which then of course $\limsup_{n\to\infty} F'(v_{1:n}) = \limsup_{n\to\infty} F'(\varphi_{\mathrm{rat}}[\omega_{1:n}], \omega_{1:n}) = \limsup_{n\to\infty} T(\omega_{1:n}) = \infty$.

Define the map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ by letting $F(c,s) := T(s)$ for all $(c,s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, which is clearly lower semicomputable because T is. By construction, $F(\varphi_{\mathrm{rat}}[\bullet], \bullet)\colon \mathbb{S} \to \mathbb{R}$ is a positive test supermartingale for φ_{rat} because T is. Invoking Lemma 17.8, we then indeed find that there's some lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that $F'(\varphi_{\mathrm{rat}}[s], s) = T(s)$ for all $s \in \mathbb{S}$ for which $(\varphi_{\mathrm{rat}}[s], s)$ is non-degenerate. □

In the above proof, we use the following lemma, which is slightly more general then what we need here, but which entails a corollary that will help us prove in Section 17.4$_{161}$ the existence of a so-called *universal* test superfarthingale.

Lemma 17.8. *There's a single algorithm that, upon the input of a code for a lower semicomputable map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$, outputs a code for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that*

(i) *$F'(v) = 0$ for all degenerate prequential situations $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$;*

(ii) *for any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ such that the map $F(\varphi_{\mathrm{rat}}[\bullet], \bullet)\colon \mathbb{S} \to \mathbb{R}$ is a positive test supermartingale for φ_{rat}, it holds that $F'(\varphi_{\mathrm{rat}}[s], s) = F(\varphi_{\mathrm{rat}}[s], s)$ for all situations $s \in \mathbb{S}$ for which the corresponding prequential situation $(\varphi_{\mathrm{rat}}[s], s)$ is non-degenerate.*

Proof. Start from a code for the lower semicomputable map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$. By Lemma 7.5$_{45}$, we can invoke a single algorithm that outputs a code $q\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathbb{N} \to \mathbb{Q}$ for F such that $q(\mathrm{v}, \bullet) \nearrow F(\mathrm{v})$ and $q(\mathrm{v}, n) < q(\mathrm{v}, n+1)$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $n \in \mathbb{N}$. We'll now use the code q to construct a code q' for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ that satisfies the requirements of the lemma.

Let $q' \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathbb{N} \to \mathbb{Q}$ be defined by $q'(\square, n) \coloneqq 1$ and

$$q'(\mathrm{v}C_{\mathrm{rat}}x, n) \coloneqq \begin{cases} \max\big(A(\mathrm{v}, C_{\mathrm{rat}}, x, n) \cup \{0\}\big) & \text{if } \mathrm{v}C_{\mathrm{rat}}x \text{ is non-degenerate} \\ 0 & \text{otherwise} \end{cases}$$
$$\text{for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*,\ C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}),\ x \in \mathscr{X} \text{ and } n \in \mathbb{N}, \quad (17.9)$$

where the map $A \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X} \times \mathbb{N} \to \big\{Q \subseteq \mathbb{Q} \colon |Q| < \infty\big\}$ is defined for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$ by

$$A(\mathrm{v}, C_{\mathrm{rat}}, x, n) \coloneqq \Big\{q(\mathrm{v}C_{\mathrm{rat}}x, m) \colon 0 \leq m \leq n,\ 0 \leq q(\mathrm{v}C_{\mathrm{rat}}\,\cdot, m) \text{ and}$$
$$\overline{E}_{C_{\mathrm{rat}}}(q(\mathrm{v}C_{\mathrm{rat}}\,\cdot, m)) \leq q'(\mathrm{v}, n)\Big\}. \quad (17.10)$$

First of all, let's have a look at why both maps are well-defined; we'll do so via forward propagation. Start by observing that $q'(\square, n)$ is well-defined for all $n \in \mathbb{N}$, and fix any $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and $x \in \mathscr{X}$. For every $n \in \mathbb{N}$, the definition of $A(\square, C_{\mathrm{rat}}, x, n)$ only makes use of q [which is given] and $q'(\square, n)$ [which is already defined]. Hence, $A(\square, C_{\mathrm{rat}}, x, n)$ and $q'(C_{\mathrm{rat}}x, n)$ are well-defined for all $n \in \mathbb{N}$. Via continued forward propagation, it's then immediate that $A(\mathrm{v}, C_{\mathrm{rat}}, x, n)$ and $q'(\mathrm{v}C_{\mathrm{rat}}x, n)$ are well-defined for any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$. By construction, since the map A outputs finite sets of rationals, the map q' is non-negative and rational. Since q is a recursive map, and since the inequalities in Eq. (17.10) are decidable [use Lemma 7.1[43]], it isn't too difficult to see that the map A and the map q' are recursive. Moreover, since q is in particular non-decreasing in its second argument, it follows readily that

$$q'(\mathrm{v}C_{\mathrm{rat}}x, n) \leq \max\{q(\mathrm{v}C_{\mathrm{rat}}x, n), 0\}$$
$$\text{for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*,\ C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}),\ x \in \mathscr{X} \text{ and } n \in \mathbb{N}. \quad (17.11)$$

Secondly, the map q' is non-decreasing in its second argument, as we now show by induction on its first argument. We start by observing that, trivially, $q'(\square, n) \leq q'(\square, n+1)$ for all $n \in \mathbb{N}$. For the induction step, fix any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, $x \in \mathscr{X}$ and $n \in \mathbb{N}$, and assume that—this is the induction hypothesis—$q'(\mathrm{v}, n) \leq q'(\mathrm{v}, n+1)$. We then have to show that also $q'(\mathrm{v}C_{\mathrm{rat}}x, n) \leq q'(\mathrm{v}C_{\mathrm{rat}}x, n+1)$. This is trivial when $\mathrm{v}C_{\mathrm{rat}}x$ is degenerate; when $\mathrm{v}C_{\mathrm{rat}}x$ is non-degenerate, it follows readily from the fact that $A(\mathrm{v}, C_{\mathrm{rat}}, x, n) \subseteq A(\mathrm{v}, C_{\mathrm{rat}}, x, n+1)$, which is itself immediately verified from Eq. (17.10).

Thirdly, for any $n \in \mathbb{N}$, the map $q'(\bullet, n) \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{Q}$ is a test superfarthingale. To prove this, we may clearly concentrate on the superfarthingale condition. Fix any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and $n \in \mathbb{N}$, and infer from Eq. (17.10) that

$$(\exists x \in \mathscr{X})A(\mathrm{v}, C_{\mathrm{rat}}, x, n) = \emptyset \Leftrightarrow (\forall x \in \mathscr{X})A(\mathrm{v}, C_{\mathrm{rat}}, x, n) = \emptyset,$$

so we only need to consider two mutually exclusive possibilities. The first possibility is that $(\forall x \in \mathscr{X})A(\mathrm{v}, C_{\mathrm{rat}}, x, n) = \emptyset$. Then $q'(\mathrm{v}C_{\mathrm{rat}}\,\cdot, n) = 0$, so $\overline{E}_{C_{\mathrm{rat}}}(q'(\mathrm{v}C_{\mathrm{rat}}\,\cdot, n)) = \overline{E}_{C_{\mathrm{rat}}}(0) = 0 \leq q'(\mathrm{v}, n)$, where the second equality follows from C1[20]. The second possibility is that $A(\mathrm{v}, C_{\mathrm{rat}}, x, n)$ is non-empty for every $x \in \mathscr{X}$, so there's some $m \in \{1, \dots, n\}$ for which $\overline{E}_{C_{\mathrm{rat}}}(q(\mathrm{v}C_{\mathrm{rat}}\,\cdot, m)) \leq q'(\mathrm{v}, n)$ and $q(\mathrm{v}C_{\mathrm{rat}}\,\cdot, m) = \max(A(\mathrm{v}, C_{\mathrm{rat}}, \cdot, n) \cup \{0\})$, where the equality also follows from the already established fact that the map q is increasing in its second argument. Since it follows from Eq. (17.9)↶ that $\max(A(\mathrm{v}, C_{\mathrm{rat}}, \cdot, n) \cup \{0\}) \geq q'(\mathrm{v}C_{\mathrm{rat}}\cdot, n)$, where the inequality takes into account that there may be some $x \in \mathscr{X}$ such that $\mathrm{v}C_{\mathrm{rat}}x$ is degenerate, we find that, in this second case also,

$$\overline{E}_{C_{\mathrm{rat}}}(q'(\mathrm{v}C_{\mathrm{rat}}\cdot, n)) \leq \overline{E}_{C_{\mathrm{rat}}}(\max(A(\mathrm{v}, C_{\mathrm{rat}}, \cdot, n) \cup \{0\})) = \overline{E}_{C_{\mathrm{rat}}}(q(\mathrm{v}C_{\mathrm{rat}}\cdot, m)) \leq q'(\mathrm{v}, n),$$

where the first inequality follows from C5$_{20}$. So, indeed, $q'(\bullet, n)$ is a test superfarthingale.

As a fourth and final preliminary step, we can now infer from Lemma 16.2$_{149}$ that for every non-degenerate prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ there's some $B_{\mathrm{v}} \in \mathbb{R}$ such that $q'(\mathrm{v}, n) \leq B_{\mathrm{v}}$ for all $n \in \mathbb{N}$. Consequently, $q'(\mathrm{v}, \bullet)$ is bounded above for every $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, because it also holds that $q'(\mathrm{v}, n) = 0$ for all degenerate prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}$ and $n \in \mathbb{N}$.

With this set-up phase completed, let the map $F' \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$ be defined through $q'(\mathrm{v}, \bullet) \nearrow F'(\mathrm{v})$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. Since $F \geq 0$, we infer from Eq. (17.11)↶ that then

$$F'(\mathrm{v}) \leq F(\mathrm{v}) \text{ for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \setminus \{\Box\}. \tag{17.12}$$

In our preliminary set-up, we have already established that there's an algorithm that, upon input of the code q for the lower semicomputable map F, outputs a code q' for the map F'. This map F' is well-defined, real-valued, non-negative and lower semicomputable due to the non-decreasingness, boundedness, non-negativity and recursiveness of q' respectively. Moreover, $F'(\Box) = 1$. It results that we only need to check the superfarthingale property explicitly in order to conclude that F' is a lower semicomputable test superfarthingale. Fix, to this end, any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$. If we recall that the map $q'(\bullet, n) \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ is a test superfarthingale for every $n \in \mathbb{N}$, we immediately infer from C6$_{20}$ and the real-valuedness of F' that, indeed, $\overline{E}_{C_{\mathrm{rat}}}(F'(\mathrm{v}C_{\mathrm{rat}}\cdot)) = \lim_{n\to\infty} \overline{E}_{C_{\mathrm{rat}}}(q'(\mathrm{v}C_{\mathrm{rat}}\cdot, n)) \leq \lim_{n\to\infty} q'(\mathrm{v}, n) = F'(\mathrm{v})$.

To complete the proof, we show that F' satisfies the conditions (i)$_{154}$ and (ii)$_{154}$. For (i)$_{154}$, fix any degenerate prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and note that then, by construction, $q'(\mathrm{v}, n) = 0$ for all $n \in \mathbb{N}$. Hence, indeed, $F'(\mathrm{v}) = 0$.

For (ii)$_{154}$, fix any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$, consider the map $T \colon \mathbb{S} \to \mathbb{R}$ defined by $T(s) := F(\varphi_{\mathrm{rat}}[s], s)$ for all $s \in \mathbb{S}$, and assume that T is a positive test supermartingale for φ_{rat}. Then we must show that $F'(\varphi_{\mathrm{rat}}[s], s) = T(s)$ for all $s \in \mathbb{S}$ for which the prequential situation $(\varphi_{\mathrm{rat}}[s], s)$ is non-degenerate. We know that $F'(\varphi_{\mathrm{rat}}[\Box], \Box) = F'(\Box) = 1 = T(\Box)$, and it already follows from Eq. (17.12) that $F'(\varphi_{\mathrm{rat}}[s], s) \leq F(\varphi_{\mathrm{rat}}[s], s) = T(s)$ for all $s \in \mathbb{S} \setminus \{\Box\}$, so we'll concentrate on the converse inequality. Assume towards contradiction that there's some $t \in \mathbb{S} \setminus \{\Box\}$ for which $(\varphi_{\mathrm{rat}}[t], t)$ is non-degenerate and $F'(\varphi_{\mathrm{rat}}[t], t) < T(t)$, implying that there's some $\epsilon > 0$ such that

$$q'((\varphi_{\mathrm{rat}}[t], t), n) + \epsilon < T(t) \text{ for all } n \in \mathbb{N}. \tag{17.13}$$

We'll use an induction argument to show that this is impossible.

Intuitively, the induction argument works as follows. We begin by observing that $q'(\Box, n) = 1 = T(\Box)$ for all $n \in \mathbb{N}$. As induction step, we assume that

$T(t_{1:k}) - \epsilon_k < q'((\varphi_{\mathrm{rat}}[t_{1:k}], t_{1:k}), N)$ for some $N \in \mathbb{N}$, $0 \leq k < |t|$ and $0 < \epsilon_k < \epsilon$, and prove that there's then also some $N' \in \mathbb{N}$ and $\epsilon_k < \epsilon_{k+1} < \epsilon$ for which $T(t_{1:k+1}) - \epsilon_{k+1} < q'((\varphi_{\mathrm{rat}}[t_{1:k+1}], t_{1:k+1}), N')$. Via forward propagation, this will allow us to conclude that $T(t) - \epsilon < T(t) - \epsilon_{|t|} < q'((\varphi_{\mathrm{rat}}[t], t), N'')$ for some $N'' \in \mathbb{N}$. But, to pull this argument off, we first need to do some preparatory work.

Taking into account that it follows from the assumptions that, for all $n \in \mathbb{N}$,

$$q((\varphi_{\mathrm{rat}}[t], t), \bullet) \nearrow T(t) > 0 \text{ and } q((\varphi_{\mathrm{rat}}[t], t), n) < q((\varphi_{\mathrm{rat}}[t], t), n+1), \tag{17.14}$$

we now claim that there are $\epsilon_0, \epsilon_1, \dots, \epsilon_{|t|} \in \mathbb{R}$ and $n_0, n_1, \dots, n_{|t|} \in \mathbb{N}$ such that

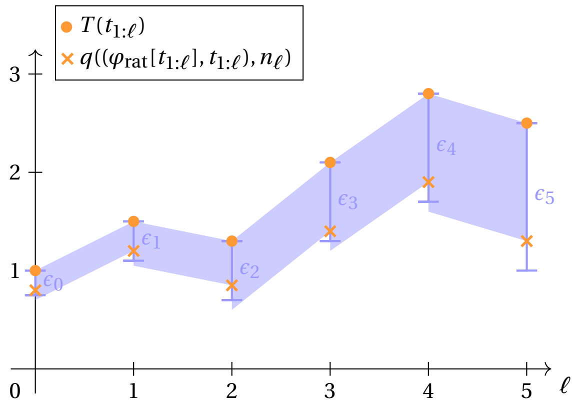
$$0 < \epsilon_0 < \epsilon_1 < \cdots < \epsilon_{|t|} < \epsilon \tag{17.15}$$

$$T(t_{1:k}) < q((\varphi_{\mathrm{rat}}[t_{1:k}], t_{1:k}), n_k) + \epsilon_k \qquad \text{for all } k \in \{0, 1, \dots, |t|\} \tag{17.16}$$

$$0 \leq q((\varphi_{\mathrm{rat}}[t_{1:k}]\varphi_{\mathrm{rat}}(t_{1:k}), t_{1:k}\cdot), n_{k+1}) \qquad \text{for all } k \in \{0, 1, \dots, |t|-1\} \tag{17.17}$$

$$q((\varphi_{\mathrm{rat}}[t_{1:k}]\varphi_{\mathrm{rat}}(t_{1:k}), t_{1:k}\cdot), n_{k+1}) + \epsilon_k < T(t_{1:k}\cdot) \qquad \text{for all } k \in \{0, 1, \dots, |t|-1\}. \tag{17.18}$$

The argument establishing the claim starts with $k := |t|$, finding ϵ_k such that (17.15) is satisfied [which is trivially possible], and then finding n_k such that (17.16) and (17.17) are satisfied [which is possible given the assumptions (17.14)]. We then move to $k := |t| - 1$, finding an ϵ_k such that (17.15) [again trivially possible] and (17.18) [made possible by the assumptions (17.14)] are satisfied, and then finding an n_k such that (17.16) and (17.17) are satisfied [which is again possible given the assumptions (17.14)]. We continue this procedure until ϵ_1 and n_1 have been defined, and finish by finding ϵ_0 such that (17.15) [again trivially possible] and (17.18) [made possible by the assumptions (17.14)] are satisfied, and then finding an n_0 such that (17.16) is satisfied [which is again possible given the assumptions (17.14)]; these conditions are depicted below for some situation $t \in \mathbb{S}$ for which $|t| = 5$.



Now, let $N := \max\{n_0, n_1, \dots, n_{|t|}\}$. To start the induction argument, observe that, trivially, $q'(\Box, N) = 1 > T(\Box) - \epsilon_0$. For the induction step, we fix any $k \in \{0, 1, \dots, |t|-1\}$

and assume that $q'((\varphi_{\text{rat}}[t_{1:k}], t_{1:k}), N) > T(t_{1:k}) - \epsilon_k$ [this is the induction hypothesis]. It then follows that

$$\begin{aligned}
&\overline{E}_{\varphi_{\text{rat}}(t_{1:k})}\big(q((\varphi_{\text{rat}}[t_{1:k}]\varphi_{\text{rat}}(t_{1:k}), t_{1:k}\cdot), n_{k+1})\big) \\
&\qquad \le \overline{E}_{\varphi_{\text{rat}}(t_{1:k})}(T(t_{1:k}\cdot) - \epsilon_k) && \text{[use Eq. (17.18) and C5}_{20}\text{]} \\
&\qquad = \overline{E}_{\varphi_{\text{rat}}(t_{1:k})}(T(t_{1:k}\cdot)) - \epsilon_k && \text{[use C4}_{20}\text{]} \\
&\qquad \le T(t_{1:k}) - \epsilon_k && [T \text{ supermartingale}] \\
&\qquad \le q'((\varphi_{\text{rat}}[t_{1:k}], t_{1:k}), N),
\end{aligned}$$

where the last inequality follows from the induction hypothesis. Hence, by Eqs. (17.10)$_{155}$ and (17.17),

$$q((\varphi_{\text{rat}}[t_{1:k+1}], t_{1:k+1}), n_{k+1}) \in A((\varphi_{\text{rat}}[t_{1:k}], t_{1:k}), \varphi_{\text{rat}}(t_{1:k}), t_{k+1}, N),$$

which implies that

$$\begin{aligned}
q'((\varphi_{\text{rat}}[t_{1:k+1}], t_{1:k+1}), N) &\ge \max A((\varphi_{\text{rat}}[t_{1:k}], t_{1:k}), \varphi_{\text{rat}}(t_{1:k}), t_{k+1}, N) \\
&\ge q((\varphi_{\text{rat}}[t_{1:k+1}], t_{1:k+1}), n_{k+1}) \\
&> T(t_{1:k+1}) - \epsilon_{k+1}, && \text{[use Eq. (17.16)]}
\end{aligned}$$

where the first inequality follows from (17.10)$_{155}$ since $(\varphi_{\text{rat}}[t], t)$ is non-degenerate. Repeating this argument until we reach $k = |t| - 1$, we eventually find that $q'((\varphi_{\text{rat}}[t], t), N) > T(t) - \epsilon_{|t|} > T(t) - \epsilon$, which is the desired contradiction with Eq. (17.13)$_{156}$. □

The following result is now immediate if we also take into account Eq. (17.3)$_{151}$, which guarantees that for any prequential situation $v = (c, s) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$, there's some rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ such that $c = \varphi_{\text{rat}}[s]$.

Corollary 17.19. *There's a single algorithm that, upon the input of a code for a lower semicomputable map $F\colon (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$, outputs a code for a lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that, for all prequential situations $v \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$,*

(i) *$F'(v) = 0$ if v is degenerate;*

(ii) *$F'(v) = F(v)$ if v is non-degenerate and F is a positive test superfarthingale.*

Proof. Consider any lower semicomputable map $F\colon (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$. We claim that the lower semicomputable test superfarthingale F' from Lemma 17.8$_{154}$ does the job. To show so, it clearly suffices to focus on (ii). To this end, let's assume that F is a positive test superfarthingale and fix any non-degenerate prequential situation $v = (c, s) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$, and any rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ such that $c = \varphi_{\text{rat}}[s]$ [which we know to exist]. Then it follows from Lemma 17.20 that the map $F(\varphi_{\text{rat}}[\bullet], \bullet)\colon \mathbb{S} \to \mathbb{R}$ is a positive test supermartingale, and hence, by Lemma 17.8(ii)$_{154}$, $F'(v) = F'(\varphi_{\text{rat}}[s], s) = F(\varphi_{\text{rat}}[s], s) = F(v)$. □

Lemma 17.20. *Consider any test superfarthingale $F \in \overline{\mathbb{F}}$ and any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$. Then the map $T\colon \mathbb{S} \to \mathbb{R}$ defined by $T(s) := F(\varphi_{\mathrm{rat}}[s], s)$ for all $s \in \mathbb{S}$ is a test supermartingale for φ_{rat}.*

Proof. Obviously, it holds that $T(\Box) = 1$ and $T \geq 0$ because $F(\Box) = 1$ and $F \geq 0$. Furthermore, for any $s \in \mathbb{S}$, it follows from the superfarthingale condition that $\overline{E}_{\varphi_{\mathrm{rat}}(s)}(T(s\,\cdot)) = \overline{E}_{\varphi_{\mathrm{rat}}(s)}(F(\varphi_{\mathrm{rat}}[s\,\cdot], s\,\cdot)) = \overline{E}_{\varphi_{\mathrm{rat}}(s)}(F(\varphi_{\mathrm{rat}}[s]\varphi_{\mathrm{rat}}(s), s\,\cdot) \leq F(\varphi_{\mathrm{rat}}[s], s) = T(s)$, so we conclude that $T \in \overline{\mathbb{T}}(\varphi_{\mathrm{rat}})$. □

(Prequential) Martin-Löf randomness implies game-randomness

Conversely, as our next result shows, any path $\omega \in \Omega$ that is ML-random with respect to some rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ also leads to a game-random prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$, provided we impose recursiveness on the forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and non-degeneracy on the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$.

Proposition 17.21. *Consider any recursive rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and any path $\omega \in \Omega$. If ω is* ML*-random for φ_{rat} and $(\varphi_{\mathrm{rat}}[\omega], \omega)$ is non-degenerate, then the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$ is game-random.*

Proof. Since we assumed the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$ to be non-degenerate, assume towards contradiction that there's some lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ such that $\limsup_{n\to\infty} F(\varphi_{\mathrm{rat}}[\omega_{1:n}], \omega_{1:n}) = \infty$. Let $T\colon \mathbb{S} \to \mathbb{R}$ be defined by $T(s) := F(\varphi_{\mathrm{rat}}[s], s)$ for all $s \in \mathbb{S}$, then $\limsup_{n\to\infty} T(\omega_{1:n}) = \infty$. So we're done if we can show that $T \in \overline{\mathbb{T}}(\varphi_{\mathrm{rat}})$ and that T is lower semicomputable. It's immediate from Lemma 17.20 that $T \in \overline{\mathbb{T}}(\varphi_{\mathrm{rat}})$. Since F is assumed to be lower semicomputable, there's some recursive rational map $q\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathbb{N} \to \mathbb{Q}$ such that $q(\mathrm{v}, \bullet) \nearrow F(\mathrm{v})$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. Let the rational map $q'\colon \mathbb{S} \times \mathbb{N} \to \mathbb{Q}$ be defined as $q'(s, n) := q((\varphi_{\mathrm{rat}}[s], s), n)$ for all $s \in \mathbb{S}$ and $n \in \mathbb{N}$. This is a recursive map since φ_{rat} is assumed to be a recursive rational forecasting system. By construction, $q'(s, \bullet) = q((\varphi_{\mathrm{rat}}[s], s), \bullet) \nearrow F(\varphi_{\mathrm{rat}}[s], s) = T(s)$ for all $s \in \mathbb{S}$, and therefore T is lower semicomputable. □

As an immediate corollary, we then also have that prequential ML-randomness implies game-randomness, provided we impose similar restrictions.

Corollary 17.22. *Consider any recursive rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and any path $\omega \in \Omega$. If $(\varphi_{\mathrm{rat}}[\omega], \omega)$ is prequentially* ML*-random and non-degenerate, then it's game-random as well.*

Proof. If the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$ is prequentially ML-random, then ω is ML-random for the recursive rational forecasting system φ_{rat} by definition, and hence, this result is immediate from Proposition 17.21. □

The following example shows that the recursiveness requirement for the rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ in the previous corollary can't be dropped, so game-randomness is a *strictly stronger* randomness notion than prequential ML-randomness, since there's at least one prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ that's prequentially ML-random but not game-random.

Example 17.23. Consider the binary state space $\mathscr{X} = \{0,1\}$ and any path $\omega \in \Omega$ such that $(1/2, \omega)$ is game-random [this is always possible by Proposition 17.28_{163}]. By Proposition 17.7_{154}, $(1/2, \omega)$ is then also prequentially ML-random, and hence, ω is ML-random for the (stationary) fair-coin forecasting system $\varphi_{1/2}$. The path ω contains an infinite number of zeros and an infinite number of ones, because otherwise $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1} = 1$ or $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1}\omega_{k+1} = 0$, which implies that ω isn't wCH-random for $\varphi_{1/2}$, and therefore also not ML-random for $\varphi_{1/2}$ by Proposition 12.2_{103}. From Proposition 34 in [36] (which tells us that any recursive path that has infinitely many zeroes and infinitely many ones is only ML-random for the vacuous interval forecast $[0,1]$, and not for any other interval forecast), it follows that the path ω is then necessarily non-recursive. Consider the temporal rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ defined by

$$\varphi_{\mathrm{rat}}(n) := \begin{cases} [0, 1/2] & \text{if } \omega_{n+1} = 1 \\ [1/2, 1] & \text{if } \omega_{n+1} = 0 \end{cases} \text{ for all } n \in \mathbb{N}_0,$$

which is non-recursive since ω is.

We claim that the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$ is prequentially ML-random. To see this, assume towards contradiction that there's some rational forecasting system $\varphi'_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that is compatible with $(\varphi_{\mathrm{rat}}[\omega], \omega)$—implying that $1/2 \in \varphi'_{\mathrm{rat}}(\omega_{1:n})$ for all $n \in \mathbb{N}_0$—such that ω isn't ML-random for φ'_{rat}. Let the rational forecasting system $\varphi''_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ be defined by

$$\varphi''_{\mathrm{rat}}(s) := \begin{cases} 1/2 & \text{if } s \sqsubseteq \omega \\ \varphi'_{\mathrm{rat}}(s) & \text{otherwise.} \end{cases} \text{ for all } s \in \mathbb{S}.$$

By construction, $\varphi''_{\mathrm{rat}}[\omega] = 1/2$ and $\varphi''_{\mathrm{rat}} \subseteq \varphi'_{\mathrm{rat}}$ [because $1/2 \in \varphi'_{\mathrm{rat}}(\omega_{1:n})$ for all $n \in \mathbb{N}_0$]. Then, by Proposition 9.5_{56}, ω isn't ML-random for φ''_{rat} either, and hence, $(\varphi''_{\mathrm{rat}}[\omega], \omega) = (1/2, \omega)$ isn't prequentially ML-random.

Nevertheless, the prequential path $(\varphi_{\mathrm{rat}}[\omega], \omega)$ isn't game-random. To see this, define the test superfarthingale $F \in \overline{\mathbb{F}}$ inductively by letting $F(\square) := 1$ and, for all $\mathrm{v} \in (\mathscr{I}_{\mathrm{rat}} \times \mathscr{X})^*$, $I_{\mathrm{rat}} \in \mathscr{I}_{\mathrm{rat}}$ and $x \in \mathscr{X}$,

$$F(\mathrm{v} I_{\mathrm{rat}} x) := \begin{cases} 2F(\mathrm{v}) & \text{if } I_{\mathrm{rat}} = [0, 1/2] \text{ and } x = 1 \\ 2F(\mathrm{v}) & \text{if } I_{\mathrm{rat}} = [1/2, 1] \text{ and } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

To show that F is indeed a test superfarthingale, it clearly suffices to check the superfarthingale property. To this end, fix any $\mathrm{v} \in (\mathscr{I}_{\mathrm{rat}} \times \mathscr{X})^*$ and $I_{\mathrm{rat}} \in$

$\mathscr{I}_{\text{rat}}$. If $I_{\text{rat}} = [0, 1/2]$, then $F(\text{v}I_{\text{rat}}\,\cdot) = 2F(\text{v})\mathbb{I}_1$, and hence, $\overline{E}_{I_{\text{rat}}}(F(\text{v}I_{\text{rat}}\,\cdot)) = \max_{p\in[0,1/2]} 2F(\text{v})\,p = F(\text{v})$. If $I_{\text{rat}} = [1/2, 1]$, then $F(\text{v}I_{\text{rat}}\,\cdot) = 2F(\text{v})\mathbb{I}_0$, and hence, $\overline{E}_{I_{\text{rat}}}(F(\text{v}I_{\text{rat}}\,\cdot)) = \max_{p\in[1/2,1]} 2F(\text{v})(1-p) = F(\text{v})$. Otherwise, $F(\text{v}I_{\text{rat}}\,\cdot) = 0$, and hence, by C1$_{20}$, $\overline{E}_{I_{\text{rat}}}(F(\text{v}I_{\text{rat}}\,\cdot)) = 0 \leq F(\text{v})$.

Moreover, F is clearly recursive and therefore lower semicomputable, and $F(\varphi_{\text{rat}}[\omega_{1:n}], \omega_{1:n}) = 2^n$ for all $n \in \mathbb{N}_0$. As a result, we find that $\limsup_{n\to\infty} F(\varphi_{\text{rat}}[\omega_{1:n}], \omega_{1:n}) = \infty$, so $(\varphi_{\text{rat}}[\omega], \omega)$ can't be game-random. ⋄

By combining the last two propositions, we obtain conditions under which ML- and game-randomness coincide; they mimic the ones in Corollary 1 of Ref. [9], which are required to obtain a similar equivalence in Vovk and Shen's precise-probabilistic setting. Under such conditions, but now also in our more general imprecise-probabilistic setting, the standard and prequential approaches to algorithmic randomness again turn out to be not that different.

Theorem 17.24. *Consider any non-degenerate recursive rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$. Then any path $\omega \in \Omega$ is* ML*-random for φ_{rat} if and only if the prequential path $(\varphi_{\text{rat}}[\omega], \omega) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random.*

This also shows that if a path $\omega \in \Omega$ is ML-random for a non-degenerate recursive rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$, then only the forecasts $\varphi_{\text{rat}}[\omega]$ that are produced along ω matter, since the path ω is also ML-random for any other non-degenerate recursive rational forecasting system $\varphi'_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ such that $\varphi'_{\text{rat}}[\omega] = \varphi_{\text{rat}}[\omega]$; this corollary complements Proposition 9.6$_{57}$ and the discussion below Proposition 9.18$_{65}$. Hence, this result is again in line with Dawid's Weak Prequential Principle [8], which states that any criterion for assessing the 'agreement' between Forecaster Frank and Reality should depend only on the actual observed sequences $\zeta = (C_1, \ldots, C_n, \ldots) \in \mathscr{C}(\mathscr{X})^{\mathbb{N}}$ and $\omega = (x_1, \ldots, x_n, \ldots) \in \Omega$, and not on the strategies (if any) which might have produced these, such as a non-degenerate recursive rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$ for which $\varphi_{\text{rat}}[\omega] = \zeta$.

By recalling our work in Section 14$_{119}$, and more specifically the conditions under which ML-randomness and ML-test-randomness coincide [see Theorem 14.1$_{120}$,], it's immediate from the above theorem that we can provide game-randomness with a test-theoretic characterisation under the standard approach to randomness.

Corollary 17.25. *Consider any non-degenerate recursive rational forecasting system $\varphi_{\text{rat}} \in \Phi_{\text{rat}}(\mathscr{X})$. Then any path $\omega \in \Omega$ is* ML*-test-random for φ_{rat} if and only if the prequential path $(\varphi_{\text{rat}}[\omega], \omega) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random.*

17.4 Properties

Let's now prove a number of interesting properties of game-randomness. As a first property, similarly as for (non-prequential, or in other words, standard) precise-probabilistic ML-randomness [30, 32], we mention (and prove) the existence of a so-called *optimal* lower semicomputable test superfarthingale $O \in \overline{\mathbb{F}}$ that conclusively tests the game-randomness of any non-degenerate prequential path $\upsilon \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$; a lower semicomputable test superfarthingale O is called *optimal* if for every lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ there's some real number $c > 0$ such that $cO(\mathrm{v}) \geq F(\mathrm{v})$ for all non-degenerate prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$.

Theorem 17.26. *There's an optimal lower semicomputable test superfarthingale $O \in \overline{\mathbb{F}}$.*

Proof. By Lemma 7.6[46], there's some uniformly lower semicomputable sequence of maps $f_n \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$ that contains every lower semicomputable map $f \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, +\infty]$. The sequence $(f_n)_{n \in \mathbb{N}}$ contains all lower semicomputable positive test superfarthingales $F \in \overline{\mathbb{F}}$, so it follows from Corollary 17.19[158] that there's some uniformly lower semicomputable sequence of test superfarthingales $F_n \in \overline{\mathbb{F}}$ such that for every positive test superfarthingale $F' \in \overline{\mathbb{F}}$ there's some $N \in \mathbb{N}$ such that

$$F_N(\mathrm{v}) = \begin{cases} F'(\mathrm{v}) & \text{if v is non-degenerate} \\ 0 & \text{if v is degenerate} \end{cases} \quad \text{for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*.$$

Let $O \colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ be defined by $O(\mathrm{v}) := \sum_{n=1}^{\infty} 2^{-n} F_n(\mathrm{v})$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. Since $F_n \geq 0$ and $F_n(\square) = 1$ for all $n \in \mathbb{N}$, it follows that O is well-defined (although possibly infinite), $O \geq 0$ and $O(\square) = 1$. To check that O is real-valued, fix any prequential situation $\mathrm{v} = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. If v is degenerate, then $O(\mathrm{v}) = 0$ because $F_n(\mathrm{v}) = 0$ for all $n \in \mathbb{N}$ by Corollary 17.19[158]. If v is non-degenerate, then we infer from Lemma 16.2[149] that there's some real number $B_{\mathrm{v}} \in \mathbb{R}$ such that $F_n(\mathrm{v}) \leq B_{\mathrm{v}}$ for all $n \in \mathbb{N}$, and therefore $O(\mathrm{v}) \leq \sum_{n=1}^{\infty} 2^{-n} B_{\mathrm{v}} = B_{\mathrm{v}}$. Since O equals an infinite sum of uniformly lower semicomputable non-negative maps $2^{-n} F_n$, it follows from Lemma 10.11[75] that O is lower semicomputable. To show that O is a superfarthingale, fix any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and any $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, and observe that

$$\begin{aligned}
\overline{E}_{C_{\mathrm{rat}}}(O(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) &= \lim_{k \to \infty} \overline{E}_{C_{\mathrm{rat}}}\Big(\sum_{n=1}^{k} 2^{-n} F_n(\mathrm{v}C_{\mathrm{rat}}\,\cdot)\Big) && \text{[use C6[20]]} \\
&\leq \lim_{k \to \infty} \sum_{n=1}^{k} 2^{-n} \overline{E}_{C_{\mathrm{rat}}}(F_n(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) && \text{[use C2[20] and C3[20]]} \\
&\leq \sum_{n=1}^{\infty} 2^{-n} F_n(\mathrm{v}) \\
&= O(\mathrm{v}),
\end{aligned}$$

where the first equality also uses the real-valuedness of O and the non-negativity of the F_n, and where the second inequality uses the superfarthingale character of F_n. We conclude that O is a lower semicomputable test superfarthingale.

We claim that O is an *optimal* lower semicomputable test superfarthingale. Consider any lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$, then $\frac{F+1}{2}$ is clearly a lower semicomputable positive test superfarthingale. We then know, with the notations explained in the beginning of the proof, that there's some $N \in \mathbb{N}$ such that

$$F_N(\mathrm{v}) = \begin{cases} \frac{F(\mathrm{v})+1}{2} & \text{if v is non-degenerate} \\ 0 & \text{if v is degenerate} \end{cases} \quad \text{for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*.$$

Let $c := 2^{N+1}$. Then, for every non-degenerate prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$,

$$cO(\mathrm{v}) = 2^{N+1} \sum_{n=1}^{\infty} 2^{-n} F_n(\mathrm{v}) \geq 2^{N+1} 2^{-N} F_N(\mathrm{v}) = 2\frac{F(\mathrm{v})+1}{2} \geq F(\mathrm{v}),$$

where the first inequality holds by the non-negativity of the test superfarthingales $F_n \in \overline{\mathbb{F}}$. □

Proposition 17.27. *Consider any optimal lower semicomputable test superfarthingale $O \in \overline{\mathbb{F}}$ and any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$. Then v is game-random if and only if* $\limsup_{n\to\infty} O(v_{1:n}) < \infty$.

Proof. The 'only if'-part is obvious: if v is game-random, then $\limsup_{n\to\infty} F(v_{1:n}) < \infty$ for all lower semicomputable test superfarthingales $F \in \overline{\mathbb{F}}$, and therefore also for O. For the 'if'-part, assume towards contradiction that there's some lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ such that $\limsup_{n\to\infty} F(v_{1:n}) = \infty$. We then know that there's a constant $c > 0$ such that $cO(\mathrm{v}) \geq F(\mathrm{v})$ for all non-degenerate prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, and hence, since v is non-degenerate by assumption, $\limsup_{n\to\infty} O(v_{1:n}) \geq \frac{1}{c} \limsup_{n\to\infty} F(v_{1:n}) = \infty$. □

For standard ML-randomness, where the emphasis lies on the compatibility between a path and a forecasting system, we have that, for every forecasting system $\varphi \in \Phi(\mathscr{X})$, there's at least one path $\omega \in \Omega$ that's ML-random for φ [see Corollary 9.3$_{56}$]. In the prequential setting, we have an analogous result for sequences of rational forecasts $\zeta \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^{\mathbb{N}}$ and sequences of outcomes $\omega \in \Omega$.

Proposition 17.28. *For every infinite sequence of rational credal sets $\zeta \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^{\mathbb{N}}$ there's at least one path $\omega \in \Omega$ such that $(\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random.*

Proof. Consider the universal superfarthingale O from Theorem 17.26↶. Assume that the path ω has been defined up to $n \geq 0$ entries such that $1 = O(\square) \geq O(\zeta_{1:1}, \omega_{1:1}) \geq \cdots \geq O(\zeta_{1:n}, \omega_{1:n})$ and $\overline{E}_{\zeta_m}(\mathbb{I}_{\omega_m}) > 0$ for all $1 \leq m \leq n$; so the prequential situation $(\zeta_{1:n}, \omega_{1:n})$ is non-degenerate. Let $\mathscr{X}_{\zeta_{n+1}} \subseteq \mathscr{X}$ be the set containing all outcomes $x \in \mathscr{X}$ for which $\max_{m \in \zeta_{n+1}} m(x) = \overline{E}_{\zeta_{n+1}}(\mathbb{I}_x) > 0$. The set $\mathscr{X}_{\zeta_{n+1}}$ is non-empty, because otherwise

$$1 \overset{\mathrm{C1}_{20}}{=} \overline{E}_{\zeta_{n+1}}(1) = \overline{E}_{\zeta_{n+1}}\Big(\sum_{x \in \mathscr{X}} \mathbb{I}_x\Big) \overset{\mathrm{C3}_{20}}{\leq} \sum_{x \in \mathscr{X}} \overline{E}_{\zeta_{n+1}}(\mathbb{I}_x) = 0.$$

By the superfarthingale property, we know that there's always some $y \in \mathscr{X}_{\zeta_{n+1}}$ such that

$$\begin{aligned}O(\zeta_{1:n},\omega_{1:n}) &\geq \overline{E}_{\zeta_{n+1}}(O(\zeta_{1:n+1},\omega_{1:n}\cdot))\\&= \max_{m\in\zeta_{n+1}}\sum_{x\in\mathscr{X}} m(x)O(\zeta_{1:n+1},\omega_{1:n}x)\\&= \max_{m\in\zeta_{n+1}}\sum_{x\in\mathscr{X}_{\zeta_{n+1}}} m(x)O(\zeta_{1:n+1},\omega_{1:n}x)\\&\geq \min_{x\in\mathscr{X}_{\zeta_{n+1}}} O(\zeta_{1:n+1},\omega_{1:n}x) = O(\zeta_{1:n+1},\omega_{1:n}y),\end{aligned}$$

where the second inequality makes use of the non-negativity of probability mass functions, and then let $\omega_{n+1} := y$. Observe that the prequential situation $(\zeta_{1:n+1},\omega_{1:n+1})$ is non-degenerate. Invoking the Axiom of dependent choice, we obtain a non-degenerate prequential path $v = (\zeta,\omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ such that $\limsup_{n\to\infty} O(v_{1:n}) \leq 1$. □

In the next proposition and theorems, the required computability conditions on sequences of rational forecasts (in this prequential setting) differ from the ones on forecasting systems that are needed to obtain similar results in the standard setting [see Chapter ⚀49] and in the standard precise-probabilistic setting [32, 74]. Recall for example that any path $\omega \in \Omega$ that's ML-random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ is also ML-random for any other more conservative forecasting system [see Proposition 9.5 56]. Meanwhile, for a similar result to hold in the prequential setting, we need to restrict our attention to sequences of rational forecasts that are not only more conservative, but that also have a compatible recursive rational forecasting system.

Proposition 17.29. *Consider any recursive rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and any game-random prequential path $v = (\zeta,\omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$. If φ_{rat} is more conservative on v, then $(\varphi_{\mathrm{rat}}[\omega],\omega)$ is game-random as well.*

Proof. Consider any recursive rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that's more conservative on v. We can always consider a rational forecasting system $\varphi'_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ that's compatible with v such that $\varphi'_{\mathrm{rat}} \subseteq \varphi_{\mathrm{rat}}$; note that we don't require computability/recursiveness here and that $v = (\varphi'_{\mathrm{rat}}[\omega],\omega)$ is non-degenerate since v is game-random. By Proposition 17.7 154, we then know that ω is ML-random for φ'_{rat}. Consequently, by Proposition 9.5 56, since $\varphi'_{\mathrm{rat}} \subseteq \varphi_{\mathrm{rat}}$, ω is also ML-random for φ_{rat}. Since $v = (\varphi'_{\mathrm{rat}}[\omega],\omega)$ is non-degenerate and $\varphi'_{\mathrm{rat}} \subseteq \varphi_{\mathrm{rat}}$, the prequential path $(\varphi_{\mathrm{rat}}[\omega],\omega)$ is non-degenerate, and hence, by Proposition 17.21 159, $(\varphi_{\mathrm{rat}}[\omega],\omega)$ is game-random too. □

The computability/recursiveness requirement on the rational forecasting system φ_{rat} in Proposition 17.29 is not only sufficient, but also (rather) necessary. This follows immediately from Example 17.23 160: the prequential path $(1/2,\omega)$ is game-random, while for the more conservative but non-recursive forecasting system φ_{rat}, the prequential path $(\varphi_{\mathrm{rat}}[\omega],\omega)$ isn't.

There are also prequential properties for which the required computability/recursiveness conditions on the forecasts are less, rather than more, stringent; the remainder of the results in this section will all deal with such conditions. If we restrict our attention for example to the standard precise-probabilistic setting, then the ML-randomness of a path $\omega \in \Omega$ with respect to a computable measure is preserved under so-called almost-everywhere computable, measure-preserving maps; this property is known as *randomness conservation*, see for example [74, Theorem 123] and [90, Theorem 1]. In our prequential setting, we have a result that's similar in spirit, but for a different kind of maps and without any computability requirement on the infinite sequence of rational credal sets: if a prequential path v is game-random, then every so-called computably selected infinite subsequence is game-random as well.

To formalise what we mean by 'computably selected', we introduce the notion of a *selection function* $S\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \to \{0,1\}$, which for any prequential path $v = (\zeta,\omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and any $n \in \mathbb{N}$ selects $v_n = (\zeta_n,\omega_n) \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X}$ if and only if $S(v_{1:n-1},\zeta_n) = 1$. Under this interpretation, it's natural to associate with every selection function S the map $S^{\subseteq}\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ from prequential situations to prequential situations that upon input of a prequential situation $\mathrm{v} = (c,s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ outputs a subsequence $\mathrm{v}' = \mathrm{v}_{n_1}\mathrm{v}_{n_2}\dots\mathrm{v}_{n_\ell}$ of length $\ell = \sum_{k=0}^{|\mathrm{v}|-1} S(\mathrm{v}_{1:k},c_{k+1})$ that's defined by $n_j \coloneqq \min\{n \in \mathbb{N}\colon \sum_{k=0}^{n-1} S(\mathrm{v}_{1:k},c_{k+1}) = j\}$ for all $1 \leq j \leq \ell$. Observe that $S^{\subseteq}(\Box) = \Box$ by construction. Clearly, if S is recursive, then so is $S^{\subseteq}$.

For any prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and any selection function S, if $\sum_{k=0}^{\infty} S(v_{1:k},\zeta_{k+1}) = \infty$, then S can be seen as mapping v to an infinite subsequence of v, which we'll denote by $S(v)$, and whose prequential situations are described by the $S^{\subseteq}(v_{1:n})$ for $n \in \mathbb{N}_0$; if S is recursive, then we say that it *computably selects* the infinite subsequence $S(v)$ from v. It turns out that game-randomness is preserved under recursive selection functions.

Theorem 17.30. *Consider any infinite sequence of rational credal sets and outcomes $v = (\zeta,\omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and any recursive selection function $S\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \to \{0,1\}$ such that $\sum_{k=0}^{\infty} S(v_{1:k},\zeta_{k+1}) = \infty$. If v is game-random, then so is $S(v)$.*

Proof. Assume towards contradiction that $S(v)$ isn't game-random. Since v is assumed to be game-random, it's non-degenerate, and therefore so is $S(v)$, by virtue of it being an infinite subsequence of v. Consequently, there's some lower semicomputable test superfarthingale $F' \in \overline{\mathbb{F}}$ such that $\limsup_{n\to\infty} F'(S(v)_{1:n}) = \infty$. Define the map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ by letting

$$F(\mathrm{v}) \coloneqq F'(S^{\subseteq}(\mathrm{v})) \text{ for all } \mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*.$$

Clearly, $\limsup_{n\to\infty} F(v_{1:n}) = \limsup_{n\to\infty} F'(S^{\subseteq}(v_{1:n})) = \limsup_{m\to\infty} F'(S(v)_{1:m}) = \infty$, so we're done if we can prove that F is a lower semicomputable test superfarthingale. We start by observing that $F(\square) = F'(S^{\subseteq}(\square)) = F'(\square) = 1$ and that $F(\mathrm{v}) = F'(S^{\subseteq}(\mathrm{v})) \geq 0$ for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$. Since S is recursive and since F' is lower semicomputable, it's immediate that F is lower semicomputable as well. It therefore only remains to prove the superfarthingale property. To this end, fix any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$. If $S(\mathrm{v}, C_{\mathrm{rat}}) = 0$, then $S^{\subseteq}(\mathrm{v}C_{\mathrm{rat}}x) = S^{\subseteq}(\mathrm{v})$ for all $x \in \mathscr{X}$, and hence,

$$\overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) = \overline{E}_{C_{\mathrm{rat}}}(F'(S^{\subseteq}(\mathrm{v}C_{\mathrm{rat}}\,\cdot))) = \overline{E}_{C_{\mathrm{rat}}}(F'(S^{\subseteq}(\mathrm{v}))) = \overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v})) = F(\mathrm{v}),$$

where the last equality follows from C1[20]. Otherwise, that is, if $S(\mathrm{v}, C_{\mathrm{rat}}) = 1$, then $S^{\subseteq}(\mathrm{v}C_{\mathrm{rat}}x) = S^{\subseteq}(\mathrm{v})C_{\mathrm{rat}}x$ for all $x \in \mathscr{X}$, and therefore

$$\overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) = \overline{E}_{C_{\mathrm{rat}}}(F'(S^{\subseteq}(\mathrm{v}C_{\mathrm{rat}}\,\cdot))) = \overline{E}_{C_{\mathrm{rat}}}(F'(S^{\subseteq}(\mathrm{v})C_{\mathrm{rat}}\,\cdot)) \leq F'(S^{\subseteq}(\mathrm{v})) = F(\mathrm{v}),$$

where the inequality follows from the superfarthingale character of F'. □

In the standard setting, when restricting our attention to almost computable forecasting systems $\varphi \in \Phi(\mathscr{X})$, the frequency of the outcomes along a ML-random path is bounded by the forecasting system [see Definition 12.1[102] and Propositions 11.3[89] and 12.2[103]]. In the prequential setting, we have a similar result, but (again) without any computability requirement on the infinite sequence of rational forecasts. Observe that, when dealing with precise rational probability mass functions, the statement below results in and simplifies to the perhaps more familiar expression $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big(f(\omega_k) - E_{m_k}(f)\big) = 0$, with $f \in \mathscr{L}(\mathscr{X})$.

Theorem 17.31. *Consider any infinite sequence of rational credal sets and outcomes $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$. If v is game-random, then*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \big(f(\omega_k) - \underline{E}_{\zeta_k}(f)\big) \geq 0 \textit{ for any } f \in \mathscr{L}(\mathscr{X}).$$

The proof below uses a similar line of reasoning as in the proof of Lemma 11.12[98], which is in its turn based on a result by De Cooman & De Bock [36, Lemma 22].

Proof. Assume towards contradiction that there's some gamble $f \in \mathscr{L}(\mathscr{X})$ and rational number $\epsilon \in \mathbb{Q}$, with $0 < \epsilon < 1$, such that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big(f(\omega_k) - \underline{E}_{\zeta_k}(f)\Big) < -2\epsilon.$$

Let $f' \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ be any rational gamble such that $f \leq f' \leq f + \epsilon$. It then follows that

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big(f'(\omega_k) - \underline{E}_{\zeta_k}(f')\Big) \overset{\text{C5}_{20}}{\leq} \liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big(f(\omega_k) + \epsilon - \underline{E}_{\zeta_k}(f)\Big) < -\epsilon. \quad (17.32)$$

Fix any $B \in \mathbb{N}$ such that $2\max_{x\in\mathscr{X}}|f(x)| < B$. Define the map $F\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X})\times\mathscr{X})^* \to \mathbb{R}$ by letting

$$F(\mathrm{v}) := \prod_{k=1}^{|\mathrm{v}|}\Big(1-\frac{\epsilon}{2B^2}\big[f'(s_k)-\underline{E}_{c_k}(f')\big]\Big) \text{ for all } \mathrm{v}=(c,s)\in(\mathscr{C}_{\mathrm{rat}}(\mathscr{X})\times\mathscr{X})^*.$$

We'll now show in a number of steps that F is a lower semicomputable test superfarthingale for which $\limsup_{n\to\infty} F(\upsilon_{1:n}) = \infty$, implying that υ can't be game-random.

Trivially, $F(\square) = 1$. Since $\epsilon < 1$ and $|f'(x) - \underline{E}_{C_{\mathrm{rat}}}(f')| < B \leq B^2$ for all $x \in \mathscr{X}$ and $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, using C1$_{20}$ for the first inequality, it's immediate that $1 - \frac{\epsilon}{2B^2}\big[f'(x) - \underline{E}_{C_{\mathrm{rat}}}(f')\big] > 1/2$ for all $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and $x \in \mathscr{X}$, and hence, F is also positive. Moreover, for any $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X})\times\mathscr{X})^*$ and $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, we have that

$$\begin{aligned}
\overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) &= F(\mathrm{v})\overline{E}_{C_{\mathrm{rat}}}\Big(1+\frac{\epsilon}{2B^2}\big[\underline{E}_{C_{\mathrm{rat}}}(f')-f'\big]\Big) && [\text{use C2}_{20}]\\
&= F(\mathrm{v})\Big[1+\frac{\epsilon}{2B^2}\overline{E}_{C_{\mathrm{rat}}}(\underline{E}_{C_{\mathrm{rat}}}(f')-f')\Big] && [\text{use C2}_{20}\text{ and C4}_{20}]\\
&= F(\mathrm{v})\Big[1+\frac{\epsilon}{2B^2}\Big(\underline{E}_{C_{\mathrm{rat}}}(f')+\overline{E}_{C_{\mathrm{rat}}}(-f')\Big)\Big] && [\text{use C4}_{20}]\\
&= F(\mathrm{v}),
\end{aligned}$$

where the last equality follows from the conjugacy relationship, so we find that F is a test superfarthingale. From the rational-valuedness of the forecasts $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, the rational-valuedness of the gamble $f' \in \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$, Lemma 7.1$_{43}$ and the conjugacy relationship it follows that F is recursive, and therefore lower semicomputable as well. We conclude that F is a lower semicomputable test superfarthingale.

By the assumptions, amongst which Eq. (17.32)$_\curvearrowleft$, there's for any $n \in \mathbb{N}_0$ some $N > n$ such that

$$\frac{1}{N}\sum_{k=1}^{N}\Big(f'(\omega_k)-\underline{E}_{\zeta_k}(f')\Big) < -\epsilon. \tag{17.33}$$

This will allow us to obtain a lower bound for $F(\upsilon_{1:N})$. By recalling that $1 - \frac{\epsilon}{2B^2}\big[f'(x) - \underline{E}_{C_{\mathrm{rat}}}(f')\big] > 1/2$ for all $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and $x \in \mathscr{X}$, it holds that $F(\upsilon_{1:N}) = \exp(K)$, with

$$K := \sum_{k=1}^{N}\ln\Big(1-\frac{\epsilon}{2B^2}\big[f'(\omega_k)-\underline{E}_{\zeta_k}(f')\big]\Big).$$

Since $\ln(1+x) \geq x - x^2$ for all $x > -1/2$, we infer that

$$K \geq -\frac{\epsilon}{2B^2}\sum_{k=1}^{N}\Big(f'(\omega_k)-\underline{E}_{\zeta_k}(f')\Big) - \frac{\epsilon^2}{4B^4}\sum_{k=1}^{N}\Big(f'(\omega_k)-\underline{E}_{\zeta_k}(f')\Big)^2$$

and, also taking into account Eq. (17.33) and $\Big(f'(\omega_k)-\underline{E}_{\zeta_k}(f')\Big)^2 \leq B^2$,

$$\geq \frac{\epsilon^2}{2B^2}N - \frac{\epsilon^2}{4B^2}N = \frac{\epsilon^2}{4B^2}N.$$

Hence,

$$F(\upsilon_{1:N}) \geq \exp\Big(\frac{\epsilon^2}{4B^2}N\Big).$$

After recalling that the inequality above holds for arbitrarily large well-chosen $N \in \mathbb{N}$, we conclude that $\limsup_{n\to\infty} F(v_{1:n}) = \infty$, contradicting the assumed game-randomness of v. □

When combining the two theorems above, it's immediately clear that if a prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random, then all its computably selectable infinite subsequences—including the prequential path itself—satisfy the above frequentist conditions; in spirit, this result also generalises Dawid's ideas on calibration in Ref. [91].

Corollary 17.34. *Consider any infinite sequence of rational credal sets and outcomes $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and any recursive selection function $S\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \to \{0,1\}$ such that $\sum_{k=0}^{\infty} S(v_{1:k}, \zeta_{k+1}) = \infty$. If v is game-random, then $S(v) = (\zeta', \omega') \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random as well, and therefore*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \Big(f(\omega'_k) - \underline{E}_{\zeta'_k}(f) \Big) \geq 0 \textit{ for any } f \in \mathscr{L}(\mathscr{X}).$$

18 A prequential test-theoretic approach

As mentioned in the introduction of this chapter, we also want to equip our prequential martingale-theoretic randomness notion with a test-theoretic characterisation. Recall from Chapter ⚃$_{111}$ that classically, in a standard precise-probabilistic test-theoretic setting, the randomness of a path $\omega \in \Omega$ with respect to a *precise* forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ is tested by constructing so-called null covers: a path ω is ML-random for φ_{pr} if it's impossible to specify—in some effectively implementable way—for every positive threshold $\delta > 0$ a set of paths that contains ω and that's small, in the sense that its probability is smaller than δ.

In Chapter ⚃$_{111}$, we've used global (conditional) upper probabilities to define a generalised test-theoretic notion of (standard and imprecise-probabilistic) ML-randomness that allows for testing a path's randomness for forecasting systems instead of merely for measures; recall that a ML-*test* for a forecasting system $\varphi \in \Phi(\mathscr{X})$ is a recursively enumerable subset $A \subseteq \mathbb{N}_0 \times \mathbb{S}$ such that $\overline{P}^{\varphi}([\![A_n]\!]) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, and a path $\omega \in \Omega$ is then considered to be ML-*test-random* for φ if there's no ML-test A such that $\omega \in \bigcap_{n\in\mathbb{N}_0} [\![A_n]\!]$. Theorem 14.1$_{120}$ shows that ML-randomness and ML-test-randomness lead to equivalent notions of randomness when restricting our attention to non-degenerate computable forecasting systems, in the sense that both definitions then have the same set of random paths—which extends the classical equivalence result for standard precise-probabilistic ML-randomness as proved independently by Schnorr and Levin [1, 2, 4].

To import these ideas into our imprecise-probabilistic and prequential setting, we need some way to express what it means for a set of *prequential* paths to be small.

18.1 Definition: test-randomness

In the present section, where we adopt a *prequential* test-theoretic setting, we'll test the randomness of an infinite sequence of rational credal sets and outcomes $\upsilon \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ via adopting some kind of *prequential* null covers: a prequential path $\upsilon \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ will be called *(prequentially) test-random* if it's impossible to specify—in some effectively implementable way—for every positive threshold $\delta > 0$ a set of prequential paths that contains υ and whose 'upper probability' according to every rational forecasting system is smaller than δ in the following sense.

Definition 18.1. We call *prequential test* any sequence of sets of prequential situations $V_n \subseteq (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$, with $n \in \mathbb{N}_0$, such that V_n is recursively enumerable uniformly in $n \in \mathbb{N}_0$ and such that for all $n \in \mathbb{N}_0$ and all $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$:

$$\overline{P}^{\varphi_{\mathrm{rat}}}\Big(\big[\!\big[\{s \in \mathbb{S}\colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}\big]\!\big]\Big) \leq 2^{-n}. \tag{18.2}$$

This definition is again inspired by Vovk and Shen's approach [9, Definition 2].[32]

A path $\upsilon \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ will then be considered random if it passes all prequential tests in the following sense.

Definition 18.3. We call a sequence $\upsilon \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes *(prequentially) test-random* if $\upsilon \notin \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$ for all prequential tests $(V_n)_{n\in\mathbb{N}_0}$.

Let's show how such prequential tests can be interpreted. Consider again the prequential forecasting protocol introduced in Section 16_{145}, and recall that Sceptic Sabine wants to test whether Forecaster Frank is doing a good forecasting job, that is, whether his forecasts $(C_1, \dots, C_n, \dots)$ 'agree with' Reality's outcomes $(\omega_1, \dots, \omega_n, \dots)$. As is typically done in the standard setting, we'll assume that Frank draws (rational) forecasts $C_n \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$, with $n \in \mathbb{N}$, from some underlying forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$, but we'll do away with the assumption that Sabine is aware of, or has access to, Frank's forecasting system φ_{rat}—even if the forecasting system is computable. To test

[32] At first sight, our present notion of a prequential test could seem to differ (greatly) from the one put forward in Ref. [9]; it's, however, completely similar in spirit. It merely appears to be different because we adopt Martin-Löf-style tests instead of integral-style tests [92, Sections 4.5.6 and 4.5.7] and a less complicated notion of effective implementability. The similarity between our and their test-theoretic tests becomes (more) apparent when taking a look at the equivalent so-called *set representation* of their tests, which can be found in the proof of Lemma 3 in Ref. [9].

the possible randomness of a prequential path $(C_1, \omega_1, \dots, C_n, \omega_n, \dots)$, we'll require her to specify a test strategy beforehand, but thus without knowing what interval forecasts $(C_1, \dots, C_n, \dots)$ will be output by Frank, nor what forecasting system φ_{rat} they stem from. To deal with her lack of knowledge, as Eq. (18.2)$_{\curvearrowleft}$ suggests, she outputs a set of prequential paths that for every fixed forecasting system corresponds with a ML-test—where the forecasting system is 'accessible' in the way of Eq. (18.2)$_{\curvearrowleft}$.

In this way, a prequential test can thus be seen as a collection of ML-tests, one for each rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$. And, as the following proposition shows, a prequential path $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is then prequentially test-random if and only if the path ω passes all such (standard and imprecise-probabilistic) ML-tests that are associated with prequential tests and with rational forecasting systems that are compatible with v, and which are exactly the forecasting systems Frank could have drawn from.

Proposition 18.4. *A sequence $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes is* (prequentially) test-random *if and only if*

$$\omega \notin \bigcap_{n \in \mathbb{N}_0} \left[\!\left[\left\{ s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n \right\} \right]\!\right]$$

for all forecasting systems φ_{rat} that are compatible with v and all prequential tests $(V_n)_{n \in \mathbb{N}_0}$.

Proof. Consider any prequential path $v = (\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$. Since we've already established in the context of Eqs. (17.1)$_{151}$, (17.3)$_{151}$ and (17.4)$_{151}$ that any such prequential path has compatible forecasting systems, and since $\varphi_{\mathrm{rat}}[\omega] = \zeta$ for any such compatible forecasting system φ_{rat}, it's obvious that v is (prequentially) test-random if and only if

$$v = (\varphi_{\mathrm{rat}}[\omega], \omega) \notin \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!] \tag{18.5}$$

for all prequential tests $(V_n)_{n \in \mathbb{N}_0}$ and all rational forecasting systems φ_{rat} that are compatible with $v = (\zeta, \omega)$, in the sense that $\varphi_{\mathrm{rat}}[\omega] = \zeta$. Now simply observe that

$$\begin{aligned}
(\varphi_{\mathrm{rat}}[\omega], \omega) \in [\![V_n]\!] &\Leftrightarrow (\exists (c,s) \in V_n)\big((c,s) \sqsubseteq (\varphi_{\mathrm{rat}}[\omega], \omega)\big) \\
&\Leftrightarrow (\exists (c,s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*)\big((c,s) \sqsubseteq (\varphi_{\mathrm{rat}}[\omega], \omega) \text{ and } (c,s) \in V_n\big) \\
&\Leftrightarrow (\exists s \in \mathbb{S})\big((\varphi_{\mathrm{rat}}[s], s) \sqsubseteq (\varphi_{\mathrm{rat}}[\omega], \omega) \text{ and } (\varphi_{\mathrm{rat}}[s], s) \in V_n\big) \\
&\Leftrightarrow (\exists s \in \mathbb{S})\big(s \sqsubseteq \omega \text{ and } (\varphi_{\mathrm{rat}}[s], s) \in V_n\big) \\
&\Leftrightarrow \omega \in \left[\!\left[\left\{ s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n \right\} \right]\!\right].
\end{aligned}$$

□

18.2 Equivalence of game- and test-randomness

Besides the arguably natural interpretation we've given to it, this prequential and test-theoretic randomness notion also has a number of interesting properties. Indeed, as the following theorem shows, game-randomness and prequential test-randomness turn out to be equivalent, so prequential test-randomness has all the properties that game-randomness does. We stress that this equivalence holds without any computability requirements on the forecasts; this is different from the analogous equivalence result for standard imprecise-probabilistic Martin-Löf martingale- and test-theoretic randomness we've established in Theorem 14.1120, which requires non-degenerate computable forecasting systems.

Theorem 18.6. *A sequence $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes is game-random if and only if it's (prequentially) test-random.*

Proof. For sufficiency, assume that v isn't game-random. Then there are two possibilities.

The first possibility is that v is degenerate, so we can fix some $N \in \mathbb{N}_0$ such that $v_{1:N} = (c, s) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ is degenerate. Now let $V_n := \{v_{1:N}\}$ for all $n \in \mathbb{N}_0$, which clearly defines a sequence of sets of prequential situations that is recursively enumerable uniformly in $n \in \mathbb{N}_0$. Furthermore, for any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$, we find that

$$\begin{aligned}\overline{P}^{\varphi_{\mathrm{rat}}}\Big(\big[\!\big[\{s \in \mathbb{S}\colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}\big]\!\big]\Big) &= \begin{cases}\overline{P}^{\varphi_{\mathrm{rat}}}([\![s]\!]) & \text{if } \varphi_{\mathrm{rat}}[s] = c\\ \overline{P}^{\varphi_{\mathrm{rat}}}(\emptyset) & \text{otherwise}\end{cases}\\ &= \begin{cases}\prod_{k=1}^{N} \overline{E}_{c_k}(\mathbb{I}_{s_k}) & \text{if } \varphi_{\mathrm{rat}}[s] = c\\ 0 & \text{otherwise}\end{cases}\\ &= 0 \leq 2^{-n},\end{aligned}$$

where the second equality follows from Proposition 6.1636 and P135, and where the last equality holds since, by the degeneracy of $v_{1:N} = (c, s)$, there's some $1 \leq m \leq N$ such that $\overline{E}_{c_m}(\mathbb{I}_{s_m}) = 0$. So, we conclude that the sequence V_n is a prequential test. Moreover, by construction, $v \in [\![v_{1:N}]\!] = \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$, so we can conclude that v isn't test-random.

The second possibility is that there's some lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ such that $\limsup_{n \to \infty} F(v_{1:n}) = \infty$. Then, by Lemma 18.7, there's a prequential test $(V_n)_{n \in \mathbb{N}_0}$ such that $v \in \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$, so we can again conclude that v isn't test-random.

For necessity, assume that v isn't test-random. This means that there's a prequential test $(V_n)_{n \in \mathbb{N}_0}$ such that $v \in \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$. Then, by Lemma 18.8↷, there's a lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ such that $\limsup_{n \to \infty} F(v_{1:n}) = \infty$ if v is non-degenerate, so we can conclude that v isn't game-random. □

Lemma 18.7. *Consider any lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$. Then there's a prequential test $(V_n)_{n \in \mathbb{N}_0}$ such that, for any sequence $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ of rational credal sets and outcomes, $v \in \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$ if $\limsup_{n \to \infty} F(v_{1:n}) = \infty$.*

Proof. Let $V_n := \{\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*\colon F(\mathrm{v}) > 2^n\}$ for all $n \in \mathbb{N}_0$. The lower semicomputability of F implies that the set $\{(\mathrm{v}, q) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \times \mathbb{Q}\colon F(\mathrm{v}) > q\}$ is recursively enumerable, and hence, V_n is recursively enumerable uniformly in

$n \in \mathbb{N}_0$. Moreover, for any rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$, the map $F(\varphi_{\mathrm{rat}}[\bullet], \bullet)\colon \mathbb{S} \to \mathbb{R}$ is a (non-negative) test supermartingale for φ_{rat} by Lemma 17.20$_{159}$. Consequently, by Ville's inequality [Proposition 6.18$_{37}$], it holds for any $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and $n \in \mathbb{N}_0$ that

$$\begin{aligned}\overline{P}^{\varphi_{\mathrm{rat}}}\Big(\big[\!\big[\{s \in \mathbb{S}\colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}\big]\!\big]\Big) &= \overline{P}^{\varphi_{\mathrm{rat}}}\Big(\big[\!\big[\{s \in \mathbb{S}\colon F(\varphi_{\mathrm{rat}}[s], s) > 2^n\}\big]\!\big]\Big) \\ &= \overline{P}^{\varphi_{\mathrm{rat}}}\Big(\Big\{\omega \in \Omega\colon \sup_{m \in \mathbb{N}_0} F(\varphi_{\mathrm{rat}}[\omega_{1:m}], \omega_{1:m}) > 2^n\Big\}\Big) \\ &\overset{\mathrm{P3}_{35}}{\leq} \overline{P}^{\varphi_{\mathrm{rat}}}\Big(\Big\{\omega \in \Omega\colon \sup_{m \in \mathbb{N}_0} F(\varphi_{\mathrm{rat}}[\omega_{1:m}], \omega_{1:m}) \geq 2^n\Big\}\Big) \\ &\leq 2^{-n} F(\varphi_{\mathrm{rat}}[\square], \square) = 2^{-n} F(\square) = 2^{-n},\end{aligned}$$

so we can conclude that the sequence V_n is a prequential test. Now, fix any prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ and assume that $\limsup_{n\to\infty} F(v_{1:n}) = \infty$. Then we see that $v \in [\![V_n]\!]$ for all $n \in \mathbb{N}_0$, so $v \in \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$. □

Lemma 18.8. *Consider any prequential test* $(V_n)_{n \in \mathbb{N}_0}$. *Then there's a lower semicomputable test superfarthingale* $F \in \overline{\mathbb{F}}$ *such that, for any non-degenerate sequence* $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ *of rational credal sets and outcomes,* $\lim_{n\to\infty} F(v_{1:n}) = \infty$ *if* $v \in \bigcap_{n \in \mathbb{N}_0} [\![V_n]\!]$.

Proof. Let's assume that $\bigcap_{n \in \mathbb{N}_0} [\![V_n]\!] \neq \emptyset$; this lemma is trivially true otherwise. This implies that $V_n \neq \emptyset$ for all $n \in \mathbb{N}_0$. Consequently, since a prequential test is recursively enumerable uniformly in $n \in \mathbb{N}_0$, there's some recursive map $q\colon \mathbb{N}_0 \times \mathbb{N}_0 \to (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ such that $V_n = \{q(n,k)\colon k \in \mathbb{N}_0\}$ for all $n \in \mathbb{N}_0$. Define the finite sets $V_n^\ell := \{q(n,k)\colon 0 \leq k \leq \ell\}$ for all $n, \ell \in \mathbb{N}_0$, so $V_n = \lim_{\ell\to\infty} V_n^\ell$ for all $n \in \mathbb{N}_0$. They are increasing(ly nested) in ℓ, and they are recursive uniformly in n and ℓ. In the remainder of this proof, we'll use these sets to construct an appropriate lower semicomputable test superfarthingale F. This will take several steps, so bear with us.

In a first step, we fix any $n, \ell \in \mathbb{N}_0$, and define $W_n^\ell\colon (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{Q}$ as follows. Let $W_n^\ell(\mathrm{v}) := 1$ in all prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ for which $V_n^\ell \sqsubseteq \mathrm{v}$, and let $W_n^\ell(\mathrm{v}) := 0$ for all prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ for which $\mathrm{v} \parallel V_n^\ell$. By the finiteness of the set of prequential situations V_n^ℓ, it then only remains to define the values of $W_n^\ell(\mathrm{v})$ in the finite number of prequential situations $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ for which $\mathrm{v} \sqsubset V_n^\ell$. To guarantee that W_n^ℓ satisfies the superfarthingale property, we'll complete the construction via backward propagation: for every $\mathrm{v} \sqsubset V_n^\ell$, if W_n^ℓ has already been defined for all prequential situations v' for which $\mathrm{v} \sqsubset \mathrm{v}' \sqsubseteq V_n^\ell$—which are finite in number as well—, we let

$$W_n^\ell(\mathrm{v}) := \max_{C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}), x \in \mathscr{X}\colon \mathrm{v}C_{\mathrm{rat}}x \sqsubseteq V_n^\ell} \overline{E}_{C_{\mathrm{rat}}}\big(W_n^\ell(\mathrm{v}C_{\mathrm{rat}}\,\cdot)\big). \tag{18.9}$$

The map W_n^ℓ satisfies a number of properties. By construction, W_n^ℓ satisfies the superfarthingale property and is therefore a superfarthingale. Furthermore, it's easy to verify that W_n^ℓ is non-decreasing in ℓ [because the V_n^ℓ are] and rational-valued [use Eq. (5.7)$_{19}$], and that $0 \leq W_n^\ell \leq 1$ [use C1$_{20}$]. By the uniform recursiveness and finiteness of the V_n^ℓ and the rationality of the credal sets, it follows from Lemma 7.1$_{43}$ that W_n^ℓ is recursive uniformly in n and ℓ. By construction, we also have that $W_n^\ell(\mathrm{v}) =$

1 for all $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ for which $V_n^\ell \sqsubseteq \mathrm{v}$. A less obvious final property is that $W_n^\ell(\square) \leq 2^{-n}$, and we'll spend some time proving this, before proceeding to the second step.

We start by observing that for every prequential situation $\mathrm{v} \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*$ there's some rational credal set $C_{\mathrm{v}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ such that $W_n^\ell(\mathrm{v}) = \overline{E}_{C_{\mathrm{v}}}(W_n^\ell(\mathrm{v}C_{\mathrm{v}}\,\cdot))$. Indeed, if $V_n^\ell \sqsubseteq \mathrm{v}$, then also $V_n^\ell \sqsubseteq \mathrm{v}C_{\mathrm{rat}}x$ for all $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and all $x \in \mathscr{X}$, so

$$W_n^\ell(\mathrm{v}) = 1 \overset{\mathrm{C1}_{20}}{=} \overline{E}_{C_{\mathrm{rat}}}(1) = \overline{E}_{C_{\mathrm{rat}}}(W_n^\ell(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) \text{ for all } C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}).$$

If $\mathrm{v} \parallel V_n^\ell$, then also $\mathrm{v}C_{\mathrm{rat}}x \parallel V_n^\ell$ for all $C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ and all $x \in \mathscr{X}$, so

$$W_n^\ell(\mathrm{v}) = 0 \overset{\mathrm{C1}_{20}}{=} \overline{E}_{C_{\mathrm{rat}}}(0) = \overline{E}_{C_{\mathrm{rat}}}(W_n^\ell(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) \text{ for all } C_{\mathrm{rat}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X}).$$

Finally, if $\mathrm{v} \sqsubset V_n^\ell$, then by Eq. (18.9)↶ there's some $C_{\mathrm{v}} \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ such that $W_n^\ell(\mathrm{v}) = \overline{E}_{C_{\mathrm{v}}}(W_n^\ell(\mathrm{v}C_{\mathrm{v}}\,\cdot))$.

Based on this observation, we now construct a special rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ by forward propagation. For a start, we know that there's some $C_\square \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ such that $W_n^\ell(\square) = \overline{E}_{C_\square}(W_n^\ell(C_\square,\cdot))$; so fix any and let $\varphi_{\mathrm{rat}}(\square) := C_\square$. For every $x \in \mathscr{X}$, we know in a next step that there's some $C_x \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ such that $W_n^\ell(C_\square, x) = \overline{E}_{C_x}\big(W_n^\ell(C_\square C_x, x\,\cdot)\big)$; so fix any and let $\varphi_{\mathrm{rat}}(x) := C_x$ and therefore also $\varphi_{\mathrm{rat}}[x] = C_\square$, which guarantees that $W_n^\ell(\varphi_{\mathrm{rat}}[x], x) = \overline{E}_{\varphi_{\mathrm{rat}}(x)}\big(W_n^\ell(\varphi_{\mathrm{rat}}[x\,\cdot], x\,\cdot)\big)$. Next, for every $y \in \mathscr{X}$, we know in a next step that there's some $C_y \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})$ such that $W_n^\ell(C_\square C_x, xy) = \overline{E}_{C_y}\big(W_n^\ell(C_\square C_x C_y, xy\,\cdot)\big)$; so fix any and let $\varphi_{\mathrm{rat}}(xy) := C_y$ and therefore also $\varphi_{\mathrm{rat}}[xy] = C_\square C_x$, which guarantees that $W_n^\ell(\varphi_{\mathrm{rat}}[xy], xy) = \overline{E}_{\varphi_{\mathrm{rat}}(xy)}\big(W_n^\ell(\varphi_{\mathrm{rat}}[xy\,\cdot], xy\,\cdot)\big)$. And so on. In this way, by forward propagation, we can construct a rational forecasting system $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ such that

$$W_n^\ell(\varphi_{\mathrm{rat}}[s], s) = \overline{E}_{\varphi_{\mathrm{rat}}(s)}\big(W_n^\ell(\varphi_{\mathrm{rat}}[s\,\cdot], s\,\cdot)\big) \text{ for all } s \in \mathbb{S}. \tag{18.10}$$

We now let $U_n^\ell := \Big\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n^\ell\Big\}$ and $N_n^\ell := \max\Big\{|s| \colon s \in U_n^\ell\Big\}$. By construction, it holds for all $s \in \mathscr{X}^{N_n^\ell}$ that

$$\begin{aligned} W_n^\ell(\varphi_{\mathrm{rat}}[s], s) &= \begin{cases} 1 & \text{if } V_n^\ell \sqsubseteq (\varphi_{\mathrm{rat}}[s], s) \\ 0 & \text{otherwise, so if } (\varphi_{\mathrm{rat}}[s], s) \parallel V_n^\ell \end{cases} \\ &= \begin{cases} 1 & \text{if } U_n^\ell \sqsubseteq s \\ 0 & \text{otherwise, so if } s \parallel U_n^\ell \end{cases} \\ &= \overline{P}^{\varphi_{\mathrm{rat}}}\Big(\llbracket U_n^\ell \rrbracket | s\Big), \end{aligned} \tag{18.11}$$

where the last equality holds by Corollary 6.15(i)36. For any situation $s \in \mathscr{X}^{N_n^\ell - 1}$, it now follows that

$$\begin{aligned} W_n^\ell(\varphi_{\mathrm{rat}}[s], s) &= \overline{E}_{\varphi_{\mathrm{rat}}(s)}\big(W_n^\ell(\varphi_{\mathrm{rat}}[s\,\cdot], s\,\cdot)\big) && \text{[use Eq. (18.10)]} \\ &= \overline{E}_{\varphi_{\mathrm{rat}}(s)}\Big(\overline{P}^{\varphi_{\mathrm{rat}}}\Big(\llbracket U_n^\ell \rrbracket | s\,\cdot\Big)\Big) && \text{[use Eq. (18.11)]} \\ &= \overline{P}^{\varphi_{\mathrm{rat}}}\Big(\llbracket U_n^\ell \rrbracket | s\Big). && \text{[use P5}_{36}\text{]} \end{aligned}$$

Via continued backward propagation, we then find that, indeed,

$$W_n^\ell(\square) = \overline{P}^{\varphi_{\text{rat}}}(\llbracket U_n^\ell \rrbracket) = \overline{P}^{\varphi_{\text{rat}}}\Big(\llbracket \big\{s \in \mathbb{S} \colon (\varphi_{\text{rat}}[s], s) \in V_n^\ell\big\} \rrbracket\Big) \\ \leq \overline{P}^{\varphi_{\text{rat}}}\Big(\llbracket \big\{s \in \mathbb{S} \colon (\varphi_{\text{rat}}[s], s) \in V_n\big\} \rrbracket\Big) \leq 2^{-n},$$

where the first inequality follows from P3$_{35}$ and the last inequality holds because $(V_n)_{n \in \mathbb{N}_0}$ is a prequential test.

In a second step, let $W_n \colon (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ be defined as $W_n(\text{v}) := \lim_{\ell \to \infty} W_n^\ell(\text{v})$ for all $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $n \in \mathbb{N}_0$. Since W_n^ℓ is non-negative, rational-valued, non-decreasing in ℓ and bounded above by 1, we can conclude that W_n is non-negative, well-defined, real-valued, and bounded above by 1. Since W_n^ℓ is rational-valued, recursive uniformly in n and ℓ, $W_n^\ell(\text{v}) \leq W_n^{\ell+1}(\text{v})$ for all $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $n, \ell \in \mathbb{N}_0$, and $W_n(\text{v}) = \lim_{\ell \to \infty} W_n^\ell(\text{v})$ for all $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $n \in \mathbb{N}_0$, it holds by definition that W_n is lower semicomputable uniformly in $n \in \mathbb{N}_0$. By recalling that $W_n^\ell(\text{v}) = 1$ for all $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ for which $V_n^\ell \sqsubseteq \text{v}$ and that $V_n = \lim_{\ell \to \infty} V_n^\ell$, it also follows from the non-decreasingness of W_n^ℓ in ℓ that $W_n(\text{v}) = 1$ for all $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ such that $V_n \sqsubseteq \text{v}$. Furthermore, for any $n \in \mathbb{N}_0$, $W_n(\square) \leq 2^{-n}$ because $W_n^\ell(\square) \leq 2^{-n}$ for all $\ell \in \mathbb{N}_0$. For any $n \in \mathbb{N}_0$, since $\overline{E}_{C_{\text{rat}}}(W_n^\ell(\text{v}C_{\text{rat}}\,\cdot)) \leq W_n^\ell(\text{v})$ [since W_n^ℓ is a superfarthingale] for all $\ell \in \mathbb{N}_0$, $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $C_{\text{rat}} \in \mathscr{C}_{\text{rat}}(\mathscr{X})$, it follows from the boundedness of W_n^ℓ and the continuity of the upper expectation operator $\overline{E}_C$ with respect to pointwise (and therefore uniform) convergence [use C6$_{20}$] that

$$\overline{E}_{C_{\text{rat}}}(W_n(\text{v}C_{\text{rat}}\,\cdot)) \leq W_n(\text{v}) \text{ for all } \text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \text{ and } C_{\text{rat}} \in \mathscr{C}_{\text{rat}}(\mathscr{X}), \tag{18.12}$$

and so we can conclude that W_n is a non-negative superfarthingale.

In a third step, we let $F \colon (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \to \mathbb{R}$ be defined as

$$F(\text{v}) := \begin{cases} \frac{1}{2}\sum_{n=0}^\infty W_n(\text{v}) & \text{if v is non-degenerate,} \\ 0 & \text{if v is degenerate,} \end{cases} \quad \text{for all } \text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*.$$

It follows from the non-negativity of W_n that F is well-defined (although possibly infinite) and non-negative. Since $W_n(\square) \leq 2^{-n}$ for all $n \in \mathbb{N}_0$, we have that $F(\square) \leq \frac{1}{2}\sum_{n=0}^\infty 2^{-n} \leq 1$. To check that F is real-valued as claimed, fix any situation $\text{v} = (c, s) \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$. If v is degenerate, then $F(\text{v}) = 0$. Otherwise, that is, if v is non-degenerate, then we infer from Lemma 16.2$_{149}$ that there's some real number $B_{\text{v}} \in \mathbb{R}$ such that $W_n(\text{v}) \leq B_{\text{v}} W_n(\square)$ for all $n \in \mathbb{N}_0$, and therefore $F(\text{v}) \leq \frac{B_{\text{v}}}{2}\sum_{n=0}^\infty W_n(\square) \leq \frac{B_{\text{v}}}{2}\sum_{n=0}^\infty 2^{-n} = B_{\text{v}}$, so here too, $F(\text{v}) \in \mathbb{R}$. By Lemma 10.11$_{75}$, the map $(\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^* \to [0, \infty] \colon \text{v} \mapsto \frac{1}{2}\sum_{n=0}^\infty W_n(\text{v})$ is lower semicomputable as it equals an infinite sum of non-negative maps $W_n/2$ that are lower semicomputable uniformly in $n \in \mathbb{N}_0$, and hence, since it's decidable whether a prequential situation $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ is non-degenerate or not [use Lemma 7.1$_{43}$], F is lower semicomputable as well. Furthermore, F is a superfarthingale. To show this, we fix any $\text{v} \in (\mathscr{C}_{\text{rat}}(\mathscr{X}) \times \mathscr{X})^*$ and $C_{\text{rat}} \in \mathscr{C}_{\text{rat}}(\mathscr{X})$. There are two distinct possibilities. If v is degenerate, then $\text{v}C_{\text{rat}}x$ is degenerate as well for every $x \in \mathscr{X}$, so $\overline{E}_{C_{\text{rat}}}(F(\text{v}C_{\text{rat}}\,\cdot)) = \overline{E}_{C_{\text{rat}}}(0) \overset{\text{C1}_{20}}{=} 0 = F(\text{v})$. If v is non-degenerate, let $\mathscr{X}_{C_{\text{rat}}} \subseteq \mathscr{X}$ be the set containing all outcomes $x \in \mathscr{X}$ for which $\max_{m \in C_{\text{rat}}} m(x) = \overline{E}_{C_{\text{rat}}}(\mathbb{I}_x) > 0$. Then $\text{v}C_{\text{rat}}x$ is non-degenerate if $x \in \mathscr{X}_{C_{\text{rat}}}$, and degenerate otherwise. Hence, it follows from Eq. (18.12)↶ that

$$
\begin{aligned}
\overline{E}_{C_{\mathrm{rat}}}(F(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) &= \max_{m\in C_{\mathrm{rat}}} \sum_{x\in\mathscr{X}} m(x)F(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \max_{m\in C_{\mathrm{rat}}} \sum_{x\in\mathscr{X}_{C_{\mathrm{rat}}}} m(x)F(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \max_{m\in C_{\mathrm{rat}}} \sum_{x\in\mathscr{X}_{C_{\mathrm{rat}}}} m(x)\lim_{k\to\infty}\frac{1}{2}\sum_{n=0}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \max_{m\in C_{\mathrm{rat}}} \lim_{k\to\infty} \sum_{x\in\mathscr{X}_{C_{\mathrm{rat}}}} m(x)\frac{1}{2}\sum_{n=0}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}x)\\
&\leq \max_{m\in C_{\mathrm{rat}}} \sup_{k\in\mathbb{N}_0} \sum_{x\in\mathscr{X}_{C_{\mathrm{rat}}}} m(x)\frac{1}{2}\sum_{n=0}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \sup_{k\in\mathbb{N}_0} \max_{m\in C_{\mathrm{rat}}} \sum_{x\in\mathscr{X}_{C_{\mathrm{rat}}}} m(x)\frac{1}{2}\sum_{n=0}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \sup_{k\in\mathbb{N}_0} \max_{m\in C_{\mathrm{rat}}} \sum_{x\in\mathscr{X}} m(x)\frac{1}{2}\sum_{n=0}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}x)\\
&= \sup_{k\in\mathbb{N}_0} \overline{E}_{C_{\mathrm{rat}}}\Big(\frac{1}{2}\sum_{n=1}^{k} W_n(\mathrm{v}C_{\mathrm{rat}}\,\cdot)\Big)\\
&\leq \sup_{k\in\mathbb{N}_0} \frac{1}{2}\sum_{n=0}^{k} \overline{E}_{C_{\mathrm{rat}}}(W_n(\mathrm{v}C_{\mathrm{rat}}\,\cdot)) && [\text{use C2}_{20}\text{ and C3}_{20}]\\
&\leq \sup_{k\in\mathbb{N}_0} \frac{1}{2}\sum_{n=1}^{k} W_n(\mathrm{v}) && [\text{use Eq. (18.12)↶}]\\
&= \frac{1}{2}\sum_{n=0}^{\infty} W_n(\mathrm{v}) = F(\mathrm{v}). && [W_n \geq 0]
\end{aligned}
$$

Finally, replace $F(\square) \leq 1$ by 1; this doesn't affect the superfarthingale character of F, nor its lower semicomputability or non-negativity. We conclude that the map F thus constructed is a lower semicomputable test superfarthingale.

In a fourth and final step, we complete the proof by fixing any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ such that $v \in \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$ and by showing that F is unbounded on v. Since $v \in \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$, there's for every $n \in \mathbb{N}_0$ some $m_n \in \mathbb{N}_0$ such that $V_n \sqsubseteq v_{1:m}$ and therefore $W_n(v_{1:m}) = 1$ for all $m \geq m_n$. Fix any $N \in \mathbb{N}_0$ and let $M_N \coloneqq \max\{m_0, \dots, m_N\} \in \mathbb{N}_0$. Then it follows that $W_n(v_{1:m}) = 1$ for all $m \geq M_N$ and all $0 \leq n \leq N$. Hence, since v is non-degenerate,

$$
F(v_{1:m}) = \frac{1}{2}\sum_{n=0}^{\infty} W_n(v_{1:m}) \geq \frac{1}{2}\sum_{n=0}^{N} W_n(v_{1:m}) = \frac{1}{2}\sum_{n=0}^{N} 1 = \frac{N+1}{2} \text{ for all } m \geq M_N,
$$

where the first equality holds by the non-degeneracy of v, and where the first inequality follows from the non-negativity of the W_n. Hence, $\lim_{n\to\infty} F(v_{1:n}) = \infty$. □

18.3 Properties

The equivalence between game- and (prequential) test-randomness allows to easily carry over a number of properties derived in Section 17[150]. We especially mention the existence of a so-called *universal* prequential test that conclusively tests the test-randomness of any prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$.

Corollary 18.13. *There's a* universal *prequential test* $(U_n)_{n\in\mathbb{N}_0}$ *with the property that any prequential path* $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ *is (prequentially) test-random if and only if* $v \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!]$.

Proof. By Theorem 17.26[162] and Proposition 17.27[163], there's a lower semicomputable test superfarthingale $F \in \overline{\mathbb{F}}$ such that any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random if and only if $\limsup_{n\to\infty} F(v_{1:n}) < \infty$. Consequently, by Lemma 18.7[171], there's a prequential test $(V_n)_{n\in\mathbb{N}_0}$ such that, for any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$, $v \in \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$ if v isn't game-random. On the other hand, for any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$, it holds by Theorem 18.6[171] that $v \notin \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$ if v is game-random, and hence, any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random if and only if $v \notin \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$. Remember that $(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}$ denotes the set of all degenerate prequential situations, and let $U_n \coloneqq V_n \cup (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}$ for all $n \in \mathbb{N}_0$. Note that $(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^*_{\mathrm{deg}}$ is a recursive set, and therefore the sequence U_n is recursively enumerable uniformly in $n \in \mathbb{N}_0$, because the sequence V_n is. Furthermore, for any $\varphi_{\mathrm{rat}} \in \Phi_{\mathrm{rat}}(\mathscr{X})$ and $n \in \mathbb{N}_0$, it holds that

$$
\begin{aligned}
&\overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in U_n\}]\!]\Big) \\
&\quad= \overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}]\!] \cup [\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate}\}]\!]\Big) \\
&\quad\overset{\text{P2}_{35}}{\leq} \overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}]\!]\Big) \\
&\qquad\qquad + \overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate}\}]\!]\Big) \\
&\quad\overset{\text{P4}_{36}}{=} \overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \in V_n\}]\!]\Big) \\
&\qquad\qquad + \lim_{n\to\infty} \overline{P}^{\varphi_{\mathrm{rat}}}\Big([\![\{s \in \mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate and } |s| \leq n\}]\!]\Big) \\
&\quad\overset{\text{P2}_{35}}{\leq} 2^{-n} + \lim_{n\to\infty} \sum_{s\in\mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate and } |s|\leq n} \overline{P}^{\varphi_{\mathrm{rat}}}([\![s]\!]) \\
&\quad= 2^{-n} + \sum_{s\in\mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate}} \overline{P}^{\varphi_{\mathrm{rat}}}([\![s]\!]) \overset{\text{Prop. 6.16}_{36}}{=} 2^{-n} + 0 = 2^{-n},
\end{aligned}
$$

where the second equality makes use of the fact that the sequence of (global) gambles

$$\Big(\mathbb{I}_{[\![\{s\in\mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate and } |s|\leq n\}]\!]}\Big)_{n\in\mathbb{N}_0}$$

is non-decreasing and converges pointwise to the (global) gamble

$$\mathbb{I}_{[\![\{s\in\mathbb{S} \colon (\varphi_{\mathrm{rat}}[s], s) \text{ is degenerate }\}]\!]} \in \mathscr{L}(\Omega).$$

We conclude that the sequence U_n constitutes a prequential test.

Moreover, for any prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$, we have that

$$
\begin{aligned}
v \notin \bigcap_{n\in\mathbb{N}_0} [\![U_n]\!] &\Leftrightarrow v \notin \Big(\bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]\Big) \cup \Big[\!\!\Big[(\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{*}_{\mathrm{deg}}\Big]\!\!\Big] \\
&\Leftrightarrow v \notin \Big(\bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]\Big) \cup (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}_{\mathrm{deg}} \\
&\Leftrightarrow \Big(v \notin \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!] \text{ and } v \notin (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}_{\mathrm{deg}}\Big) \\
&\Leftrightarrow v \text{ is game-random} \\
&\Leftrightarrow v \text{ is test-random}, && \text{[use Theorem } 18.6_{171}\text{]}
\end{aligned}
$$

where the penultimate equivalence makes use of Definition 17.6_{153} and the fact that any non-degenerate prequential path $v \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random if and only if $v \notin \bigcap_{n\in\mathbb{N}_0} [\![V_n]\!]$. □

How (im)precise are randomness notions?

In the previous chapters, we generalised several martingale-theoretic, frequentist and test-theoretic randomness notions by allowing for imprecise forecasting systems. In this imprecise-probabilistic approach to randomness, many properties that were initially proven in a precise-probabilistic setting continued to hold for (*non-*)*computable* imprecise forecasting systems: every forecasting system makes at least one path random [Propositions 9.3$_{56}$ and 11.10(ii)$_{96}$], the randomness of a path with respect to a (computable) (non-degenerate) forecasting system only depends on the forecasts that are specified along the path [Propositions 9.6$_{57}$, 9.18$_{65}$ and 11.23$_{101}$], the betting strategies used to define computable randomness can be assumed to be rational-valued and recursive [Proposition 10.16$_{78}$], computable randomness entails Church randomness for computable forecasting systems [Proposition 12.2$_{103}$], and the notions of Martin-Löf and Martin-Löf test randomness coincide for non-degenerate computable forecasting systems, to give only a few examples. Interestingly, some other properties don't have a precise-probabilistic counterpart, and seem to stem from the use of imprecise-probabilistic uncertainty models: all paths are random for the vacuous credal set [Propositions 9.4$_{56}$ and 11.10(iii)$_{96}$], and if a path is random for a forecasting system, then it's always random for any other forecasting system that's less informative [Propositions 9.5$_{56}$ and 11.10(iv)$_{96}$]. These inherently imprecise-probabilistic properties that we've seen so far seem to be only few in number though. In this chapter, we'll explore and show what other such properties hold for our randomness notions, and we'll study whether allowing for imprecise uncertainty models (and letting go of computable uncertainty models) changes our view on and understanding of random sequences.

To do so, it's instructive to draw inspiration from the *measure-theoretic* notion of *uniform randomness* in which—as we mentioned in Chapter ⚃$_{111}$—imprecise-probabilistic (as well as non-computable) uncertainty models have long been adopted by Levin in 1973 [4, 5, 6]; this notion of randomness has been well-studied, and has properties that provide an answer to the above question (in a test-theoretic setting). Indeed, we recall that the notion of uniform randomness allows for imprecision by considering so-called 'effectively compact classes of probability measures'; moreover, for every such effectively compact class of measures, a path is uniformly random with respect to the class if and only if it's uniformly random with respect to some (not necessarily computable nor fixed) probability measure in the considered class [6, Definition 5.9, Theorem 5.23 and Remark 5.24],[33] which is obviously an inherently imprecise-probabilistic property that has no (interesting) precise-probabilistic counterpart.

Basically, in what follows, we'll investigate whether our imprecise-probabilistic notions of randomness satisfy a similar property: is a path $\omega \in \Omega$ random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if it is random for some (not necessarily fixed) compatible precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$? In Section 19$_{\curvearrowright}$, where we restrict our attention to *computable* precise forecasting systems $\varphi_{\mathrm{pr}} \in \Phi(\mathscr{X})$, we give a negative answer to this question by showing that for every stationary forecasting system $\varphi \in \mathscr{C}(\mathscr{X})$ with 'non-zero diameter' there is a path $\omega \in \Omega$ that's random for φ, but that isn't random for any computable forecasting system whose 'highest imprecision is smaller than' that of φ, which includes all *computable* precise forecasting systems $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$. This leads us to say that randomness is *inherently imprecise*, because there are paths that are random for an imprecise forecasting system, but not for any computable precise forecasting system.

In Section 20$_{192}$, where we restrict our attention to martingale-theoretic randomness notions, we reveal that the computability assumption on $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ is crucial for the previous claim, because we obtain a positive answer when letting go of computability. In fact, we actually succeed in proving a stronger property: we show that for every forecasting system $\varphi \in \Phi(\mathscr{X})$ there's some fixed compatible (not necessarily computable) precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ such that a path $\omega \in \Omega$ is random for φ if and only if it's random for φ_{pr}; when the forecasting system φ is stationary and has 'non-zero diameter', then this precise forecasting system φ_{pr} is necessarily non-stationary and non-computable, due to the result in the previous paragraph. This result also indicates the value of allowing for non-computable forecasting systems in our martingale-theoretic notions of randomness—where these uncertainty

[33] For the sake of clarity, it remains to note that the singleton consisting of a measure is a compact class of measures, and that the (not necessarily computable) measure in the above statement is accessible by an oracle in the sense of Definition 5.9 in Ref. [6].

models aren't accessible by an oracle—, since doing so allows us to prove this natural property. This property furthermore readily implies the weaker property that we mentioned before: a path $\omega \in \Omega$ is random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if it's random for at least one compatible (not necessarily computable) precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$. Hence, in addition to the fact that our martingale-theoretic and Levin's measure-theoretic notion of randomness coincide for computable non-degenerate forecasting systems [Theorems 14.1_{120} and 14.34_{134}], they also carry similar properties. We also complement this result with a reflection on what it tells us about allowing for imprecision and non-computability in a martingale-theoretic approach to algorithmic randomness.

So, for the results in this chapter, (not) allowing for non-computable uncertainty models is of crucial importance. If we do, we seem to be able to replace imprecise uncertainty models by precise ones; if we don't, then randomness is inherently imprecise. We leave aside whether using non-computable uncertainty models—that aren't accessible by an oracle—in algorithmic randomness is a defensible choice in general; we merely let go of the (classical) computability restriction on uncertainty models, and investigate what happens if we do so. Nevertheless, in Section 21_{199}, we do argue why computable uncertainty models are to be favoured from a practical point of view, thus leading to an argument in favour of allowing for imprecision. We do so by complementing the main results in Sections 19 and 20_{192} with a discussion of their possible implications for a prospective statistics based on imprecise probabilities.

19 Randomness is inherently imprecise

Let's start examining how allowing for imprecision changes our understanding of random sequences. We kick off this endeavour by questioning whether imprecise-probabilistic uncertainty models are really needed to capture a path's randomness. Is it for example 'easier' to capture the randomness of some paths by imprecise forecasting systems? Are there paths whose randomness can only be described by imprecise uncertainty models?

To start the discussion, we consider the binary state space $\mathscr{X} = \{0,1\}$ and fix any $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$. Recall from Propositions 9.5_{56} and $11.10(\mathrm{iv})_{96}$ that if a path $\omega \in \Omega$ is R-random for some precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$, then it's also R-random for any other forecasting system that's less informative. Hence, in particular, for any real numbers $p, q \in [0,1]$ such that $p < q$, if ω is R-random for the temporal precise forecasting system $\varphi_{p,q}$, defined by

$$\varphi_{p,q}(s) := \begin{cases} p & \text{if } |s| \text{ is odd} \\ q & \text{if } |s| \text{ is even} \end{cases} \quad \text{for all } s \in \mathbb{S},$$

then it's also R-random for the interval forecast $[p, q]$. So we see that its least conservative stationary outer approximation $[p, q]$ can be used as a simpler—because stationary—yet imprecise alternative for $\varphi_{p,q}$. In many cases, this procedure of replacing a precise non-stationary forecasting system by a simpler stationary imprecise one will result in a larger set of R-random paths, and therefore leads to a less informative description of our uncertainty about ω. As argued by De Cooman and De Bock in Ref. [36, Section 10]: this observation might lead to the suspicion that all instances of randomness with respect to a stationary imprecise uncertainty model can be 'explained away' as a mere consequence of randomness with respect to non-stationary but precise uncertainty models. This would imply that the imprecision in a forecasting system isn't essential, and can always be dismissed as a mere artefact, a simple effect of using a stationary representation that isn't powerful enough to allow for the ideal representation, which must be, one would suspect, always precise but non-stationary.

However, as also argued by De Cooman and De Bock in Ref. [36, Section 10], this suspicion is misguided when focusing on computable forecasting systems: interval forecasts don't merely serve as an alternative for non-stationary *computable* precise forecasting systems. Indeed, their Theorem 37 in Ref. [36] shows that, for binary state spaces, there's at least one path $\omega \in \Omega$ that's R-random for $[p, q]$, with R $\in$ {ML, wML, C, S}, but not R-random for any (more) precise (possibly non-stationary) computable forecasting system $\varphi \in \Phi(\mathscr{X})$. This result led De Cooman and De Bock [36, Section 10] to claim that R-*randomness is inherently imprecise*, with R $\in$ {ML, wML, C, S}, because the randomness of the paths ω in their Theorem 37 can only be captured by an imprecise forecasting system, and can't be explained away as an effect of oversimplification. Moreover, as they explain as well, the imprecision involved is non-negligible, and can be made arbitrarily large, because besides excluding the possibility of randomness of such paths for precise computable forecasting systems, they also can't be random for any computable forecasting system whose highest imprecision is smaller than that of the original, stationary one.

In this dissertation, we strengthen their result by proving that it continues to hold for arbitrary finite state spaces as well as for frequentist and test-theoretic notions of randomness: for every R $\in$ {ML, wML, C, S, CH, wCH} and every credal set $C \in \mathscr{C}(\mathscr{X})$ with non-zero diameter there's at least one path $\omega \in \Omega$ that's R-random for C but not for any forecasting system $\varphi \in \Phi(\mathscr{X})$ whose highest imprecision is smaller than that of C. It remains to formalise what we mean by 'non-zero diameter' and 'highest imprecision smaller than'.

The *diameter* dia: $\mathscr{C}(\mathscr{X}) \to \mathbb{R}_{\geq 0}$ of a credal set $C \in \mathscr{C}(\mathscr{X})$ is defined as $\mathrm{dia}(C) := \max_{m,m' \in C} \|m - m'\|_{\mathrm{tv}}$. So, $\mathrm{dia}(C) > 0$ if and only if C consists of more than one probability mass function. A forecasting system $\varphi \in \Phi(\mathscr{X})$ is said to have *highest imprecision smaller than* some credal set $C \in \mathscr{C}(\mathscr{X})$ if $\sup_{s \in \mathbb{S}} \mathrm{dia}\big(\varphi(s)\big) < \mathrm{dia}(C)$.

Theorem 19.1. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}, \mathrm{CH}, \mathrm{wCH}\}$ *and any credal set* $C \in \mathscr{C}(\mathscr{X})$*. Then there's a path* $\omega \in \Omega$ *that's* R*-random for the credal set* C*, but that's never* R*-random for any computable forecasting system* $\varphi \in \Phi(\mathscr{X})$ *whose highest imprecision is smaller than that of* C*, in the specific sense that* $\sup_{s \in \mathbb{S}} \mathit{dia}\big(\varphi(s)\big) < \mathit{dia}(C)$.

Proof. This statement is trivially true when $\mathrm{dia}(C) = 0$, because there are no forecasting systems $\varphi \in \Phi(\mathscr{X})$ for which $\sup_{s \in \mathbb{S}} \mathrm{dia}\big(\varphi(s)\big) < 0$ and because we know from Corollary 9.3[56] and 11.10(ii)[96] that there's at least one path that is R-random for C. So, let's assume that $\mathrm{dia}(C) > 0$; this implies that $|\mathscr{X}| > 1$, because $\mathrm{dia}(C) = \mathrm{dia}(\{1\}) = 0$ for all $C \in \mathscr{C}(\mathscr{X})$ if $|\mathscr{X}| = 1$.

To start the argument, fix any recursive map $\lambda \colon \mathbb{N} \to \mathbb{N}$ such that for each $\ell \in \mathbb{N}$ there are infinitely many $n \in \mathbb{N}$ that are mapped to ℓ, meaning that $\lambda(n) = \ell$; consider for example the recursive map that corresponds to the sequence $(1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \dots)$.

We also let $\varphi_1, \varphi_2, \dots, \varphi_\ell, \dots$ be any enumeration of the (countably many) computable forecasting systems whose highest imprecision is smaller than that of the credal set C, in the specific sense that $\sup_{s \in \mathbb{S}} \mathrm{dia}\big(\varphi_\ell(s)\big) < \mathrm{dia}(C)$ for all $\ell \in \mathbb{N}$. To understand why there are indeed countable many such forecasting systems, we note that by Lemma 7.2[44] there are only countably many computable forecasting systems since there are only countably many (partial) recursive maps $q \colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ [this is the required upper bound] and observe that every rational probability mass function $m_{\mathrm{rat}} \in \mathcal{M}_{\mathrm{rat}}(\mathscr{X})$—of which there are countably many because $|\mathscr{X}| > 1$ by assumption—has diameter zero and corresponds to a unique stationary recursive (and therefore computable) forecasting system [this is the required lower bound].

Consider now any $\ell \in \mathbb{N}$, or in other words, any such computable forecasting system φ_ℓ. Let $0 < \epsilon_\ell < 1$ be any rational number [which there always is] such that

$$\sup_{s \in \mathbb{S}} \mathrm{dia}\big(\varphi_\ell(s)\big) + 20\epsilon_\ell < \mathrm{dia}(C). \tag{19.2}$$

For any given $\ell \in \mathbb{N}$, we now fix some finite number of rational probability mass functions $\{{}^\ell m_1, \dots, {}^\ell m_{i_\ell}\} \subseteq \mathcal{M}_{\mathrm{rat}}(\mathscr{X})$ such that $d_{\mathrm{H}}\big(C, \mathrm{CH}\big(\{{}^\ell m_1, \dots, {}^\ell m_{i_\ell}\}\big)\big) < \epsilon_\ell$ [which is always possible by Lemma 5.6[18]]. Let $\{{}^\ell m'_1, \dots, {}^\ell m'_{i_\ell}\} \subseteq \mathcal{M}(\mathscr{X})$ be any finite set of probability mass functions such that ${}^\ell m'_i \in C$ and $d\big({}^\ell m_i, {}^\ell m'_i\big) < \epsilon_\ell$ for all $1 \le i \le i_\ell$ [it's immediate from the definition of the Hausdorff distance that this is possible]. Consider any $N_\ell \in \mathbb{N}$ such that $2^{-N_\ell} < \epsilon_\ell$. Since the forecasting system φ_ℓ is computable, we know that there's some recursive map $q \colon \mathbb{S} \times \mathbb{N} \to \mathcal{P}_{\mathrm{fin}}(\mathcal{M}_{\mathrm{rat}}(\mathscr{X}))$ such that $d_{\mathrm{H}}\big(\varphi_\ell(s), \mathrm{CH}\big(q(s, N)\big)\big) \le 2^{-N}$ for all $s \in \mathbb{S}$ and $N \in \mathbb{N}$. Hence, if we let φ'_ℓ be the recursive rational forecasting system defined by $\varphi'_\ell(s) = \mathrm{CH}\big(q(s, N_\ell)\big)$ for all $s \in \mathbb{S}$, then clearly $d_{\mathrm{H}}\big(\varphi_\ell(s), \varphi'_\ell(s)\big) \le 2^{-N_\ell} < \epsilon_\ell$ for all $s \in \mathbb{S}$. By recalling that $\max f - f \in \mathscr{L}_1(\mathscr{X})$ if $f \in \mathscr{L}_1(\mathscr{X})$, it follows from Corollary 7.9[48] and conjugacy that

$$\begin{aligned}\Big|\underline{E}_{\varphi_\ell(s)}(f) - \underline{E}_{\varphi'_\ell(s)}(f)\Big| &= \Big|\overline{E}_{\varphi'_\ell(s)}(-f) - \overline{E}_{\varphi_\ell(s)}(-f)\Big| \\ &\overset{\mathrm{C4}_{20}}{=} \Big|\overline{E}_{\varphi'_\ell(s)}\big(\max f - f\big) - \overline{E}_{\varphi_\ell(s)}\big(\max f - f\big)\Big| \\ &< \epsilon_\ell \text{ for all } s \in \mathbb{S} \text{ and } f \in \mathscr{L}_1(\mathscr{X}).\end{aligned} \tag{19.3}$$

Since $\{{}^{\ell}m_1,\dots,{}^{\ell}m_{i_\ell}\}\subseteq\mathcal{M}_{\mathrm{rat}}(\mathscr{X})$ is a finite set of rational probability mass functions and since φ'_ℓ is a recursive rational forecasting system, it follows from Lemma 19.22[191] that there is a recursive rational map $q_\ell\colon\{1,\dots,i_\ell\}\times\mathbb{S}\to\mathbb{Q}$ such that

$$\left|d\big({}^{\ell}m_i,\varphi'_\ell(s)\big)-q_\ell(i,s)\right|<\epsilon_\ell \text{ for all } 1\leq i\leq i_\ell \text{ and } s\in\mathbb{S}. \tag{19.4}$$

Let the natural map $q'_\ell\colon\mathbb{S}\to\{1,\dots,i_\ell\}$ be defined by

$$q'_\ell(s):=\min\big\{1\leq i\leq i_\ell\colon q_\ell(i,s)>7\epsilon_\ell\big\} \text{ for all } s\in\mathbb{S}. \tag{19.5}$$

Observe that q'_ℓ is (well-)defined because it holds for every $s\in\mathbb{S}$ that

$$\begin{aligned}
&\max_{1\leq i\leq i_\ell} q_\ell(i,s)\\
&\qquad\geq\max_{1\leq i\leq i_\ell} d\Big({}^{\ell}m_i,\varphi'_\ell(s)\Big)-\epsilon_\ell\\
&\qquad\geq d_{\mathrm{H}}\Big(\mathrm{CH}\Big(\Big\{{}^{\ell}m_1,\dots,{}^{\ell}m_{i_\ell}\Big\}\Big),\varphi'_\ell(s)\Big)-\epsilon_\ell\\
&\qquad\geq d_{\mathrm{H}}\big(C,\varphi_\ell(s)\big)-d_{\mathrm{H}}\Big(C,\mathrm{CH}\Big(\Big\{{}^{\ell}m_1,\dots,{}^{\ell}m_{i_\ell}\Big\}\Big)\Big)-d_{\mathrm{H}}\big(\varphi'_\ell(s),\varphi_\ell(s)\big)-\epsilon_\ell\\
&\qquad> d_{\mathrm{H}}\big(C,\varphi_\ell(s)\big)-3\epsilon_\ell\\
&\qquad\geq\frac{\mathrm{dia}(C)-\mathrm{dia}\big(\varphi_\ell(s)\big)}{2}-3\epsilon_\ell\overset{\text{Eq. (19.2)}}{>}\frac{20\epsilon_\ell}{2}-3\epsilon_\ell=7\epsilon_\ell,
\end{aligned}$$

where the first inequality holds by Eq. (19.4), where the second inequality holds by Lemma 19.19[190], where the third inequality holds by the triangle inequality for the Hausdorff distance, and where the fifth inequality holds by Lemma 19.20[190]; this implies that the set $\{1\leq i\leq i_{\ell_s}\colon q_{\ell_s}(i,s)>7\epsilon_{\ell_s}\}$ is non-empty. Moreover, since the rational map q_ℓ is recursive, the natural map q'_ℓ is recursive as well.

With this set-up phase completed, we're now ready to use the map λ, the rational ϵ_ℓ, the rational probability mass functions $\{{}^{\ell}m_1,\dots,{}^{\ell}m_{i_\ell}\}$, the probability mass functions $\{{}^{\ell}m'_1,\dots,{}^{\ell}m'_{i_\ell}\}\subseteq C$, the recursive rational forecasting system φ'_ℓ, the recursive rational map q_ℓ and the recursive natural map q'_ℓ we have just determined in the construction above for any $\ell\in\mathbb{N}$, to define a precise forecasting system φ_C as follows: let $\ell_s:=\lambda(|s|)$ for all $s\in\mathbb{S}$ and

$$\varphi_C(s):={}^{\ell_s}m'_{q'_{\ell_s}(s)} \text{ for all } s\in\mathbb{S}. \tag{19.6}$$

Observe that φ_C is (well-)defined because $q'_{\ell_s}(s)\in\{1,\dots,i_{\ell_s}\}$ for every $s\in\mathbb{S}$.

Let's consider any path $\omega\in\Omega$ that's R-random for this precise forecasting system φ_C; we know from Corollary 9.3[56] and Proposition 11.10(ii)[96] that there's at least one such path. For any $l\in\mathbb{N}$, we know that ${}^{\ell}m'_i\in C$ for all $1\leq i\leq i_\ell$, so it follows from our construction that $\varphi_C(s)\in C$ for all $s\in\mathbb{S}$. Since ω is assumed to be R-random for φ_C, it follows from Propositions 9.5[56] and 11.10(iv)[96] that ω is also R-random for C.

Let's now fix any $\ell_o\in\mathbb{N}$. Since ℓ_o is chosen arbitrarily, we'll be done if we can show that the path ω isn't wCH-random for φ_{ℓ_o}, because then, since φ_{ℓ_o} is computable, it's immediate from Corollary 12.4[105] that ω isn't R-random for φ_{ℓ_o}. This is what we now set out to do.

Let the selection process $S^{\ell_o} \in \mathcal{S}$ be defined by

$$S^{\ell_o}(s) := \begin{cases} 1 & \text{if } \ell_s = \ell_o \\ 0 & \text{otherwise} \end{cases} \text{ for all } s \in \mathbb{S}. \tag{19.7}$$

This selection process is recursive and temporal, because λ is recursive and $\ell_s = \lambda(|s|)$ for all $s \in \mathbb{S}$; it is moreover total, because λ maps infinitely many naturals to ℓ_o. Since every probability mass function $m \in \mathcal{M}(\mathscr{X})$ is almost computable, since $\{{}^{\ell_o}m'_1, \dots, {}^{\ell_o}m'_{i_{\ell_o}}\}$ is a finite set of probability mass functions, since q'_{ℓ_o} is a recursive natural map and since $\varphi_C(s) = {}^{\ell_o}m'_{q'_{\ell_o}(s)}$ for all $s \in \mathbb{S}$ with $S^{\ell_o}(s) = 1$, this implies that the forecasting system φ_C is almost computable for S^{ℓ_o}.

For every $1 \leq i \leq i_{\ell_o}$, let the selection process $S_i^{\ell_o} \in \mathcal{S}$ be defined by

$$S_i^{\ell_o}(s) := \begin{cases} 1 & \text{if } S^{\ell_o}(s) = 1 \text{ and } q'_{\ell_o}(s) = i \\ 0 & \text{otherwise} \end{cases} \text{ for all } s \in \mathbb{S}; \tag{19.8}$$

this selection process is recursive because S^{ℓ_o} and q'_{ℓ_o} are recursive. Since $q'_{\ell_o}(s) \in \{1, \dots, i_{\ell_o}\}$ for all $s \in \mathbb{S}$, it's immediate that

$$S^{\ell_o}(s) = \sum_{1 \leq i \leq i_{\ell_o}} S_i^{\ell_o}(s) \text{ for all } s \in \mathbb{S}. \tag{19.9}$$

By invoking Lemma 19.21$_{190}$, we infer that there's a finite set of rational gambles $\{f_1, \dots, f_{j_{\ell_o}}\} \subseteq \mathscr{L}_1(\mathscr{X}) \cap \mathscr{L}_{\mathrm{rat}}(\mathscr{X})$ such that

$$d(m, C) \leq \max_{1 \leq j \leq j_{\ell_o}} \big(\underline{E}_C(f_j) - E_m(f_j)\big) + \epsilon_{\ell_o} \text{ for all } m \in \mathcal{M}(\mathscr{X}) \text{ and } C \in \mathscr{C}(\mathscr{X}),$$

and hence, it follows for every $1 \leq i \leq i_{\ell_o}$ and for all $s \in \mathbb{S}$ that

$$\begin{aligned} S_i^{\ell_o}(s) = 1 \Rightarrow q'_{\ell_o}(s) = i & \\ \Rightarrow q_{\ell_o}(i, s) > 7\epsilon_{\ell_o} & \\ \Rightarrow d\big({}^{\ell_o}m_i, \varphi'_{\ell_o}(s)\big) > 6\epsilon_{\ell_o} & \\ \Rightarrow \max_{1 \leq j \leq j_{\ell_o}} \Big(\underline{E}_{\varphi'_{\ell_o}(s)}(f_j) - E_{{}^{\ell_o}m_i}(f_j)\Big) \geq d\Big({}^{\ell_o}m_i, \varphi'_{\ell_o}(s)\Big) - \epsilon_{\ell_o} > 5\epsilon_{\ell_o}, & \end{aligned} \tag{19.10}$$

where the second implication holds by Eq. (19.5)$_{\curvearrowleft}$, and where the third implication holds by Eq. (19.4)$_{\curvearrowleft}$. With any $1 \leq i \leq i_{\ell_o}$ and $1 \leq j \leq j_{\ell_o}$, we associate the selection process $S_{i,j}^{\ell_o} \in \mathcal{S}$ defined for all $s \in \mathbb{S}$ by

$$S_{i,j}^{\ell_o}(s) := \begin{cases} 1 & \text{if } S_i^{\ell_o}(s) = 1 \text{ and } \underline{E}_{\varphi'_{\ell_o}(s)}(f_j) - E_{{}^{\ell_o}m_i}(f_j) > 5\epsilon_{\ell_o} \\ 0 & \text{otherwise,} \end{cases}$$

which is recursive because $S_i^{\ell_o}$ is a recursive selection process, and because the strict inequality is decidable for all $s \in \mathbb{S}$ by Lemma 7.1$_{43}$ and conjugacy since ${}^{\ell_o}m_i$ is a rational probability mass function, f_j is a rational gamble, ϵ_{ℓ_o} is a rational constant and φ'_{ℓ_o} is a recursive rational forecasting system. By Eq. (19.8)↶, it holds for all $s \in \mathbb{S}$ that

$$S^{\ell_o}_{i,j}(s) = \begin{cases} 1 & \text{if } S^{\ell_o}(s) = 1,\ q'_{\ell_o}(s) = i \text{ and } \underline{E}_{\varphi'_{\ell_o}(s)}(f_j) - E_{^{\ell_o}m_i}(f_j) > 5\epsilon_{\ell_o} \\ 0 & \text{otherwise.} \end{cases} \tag{19.11}$$

Moreover, by construction, for all $s \in \mathbb{S}$ with $S^{\ell_o}_{i,j}(s) = 1$,

$$\begin{aligned} E_{^{\ell_o}m'_i}(f_j) &\leq E_{^{\ell_o}m_i}(f_j) + \epsilon_{\ell_o} \\ &< \underline{E}_{\varphi'_{\ell_o}(s)}(f_j) - 4\epsilon_{\ell_o} \\ &\overset{\text{Eq. (19.3)}_{183}}{<} \underline{E}_{\varphi_{\ell_o}(s)}(f_j) - 3\epsilon_{\ell_o}, \end{aligned} \tag{19.12}$$

where the first inequality is immediate from Lemma 11.9[95] because $d({}^{\ell_o}m'_i, {}^{\ell_o}m_i) < \epsilon_{\ell_o}$. Obviously, for every $1 \leq i \leq i_{\ell_o}$ and $1 \leq j \leq j_{\ell_o}$, $S^{\ell_o}_{i,j}(s) \leq S^{\ell_o}(s)$ for all $s \in \mathbb{S}$. It's immediate from Eq. (19.10)↶ that

$$S^{\ell_o}_i(s) \leq \sum_{1 \leq j \leq j_{\ell_o}} S^{\ell_o}_{i,j}(s) \text{ for all } 1 \leq i \leq i_{\ell_o} \text{ and } s \in \mathbb{S},$$

and it's therefore immediate from Eq. (19.9)↶ that

$$S^{\ell_o}(s) \leq \sum_{1 \leq i \leq i_{\ell_o}, 1 \leq j \leq j_{\ell_o}} S^{\ell_o}_{i,j}(s) \text{ for all } s \in \mathbb{S}.$$

Hence, we can fix some $1 \leq i \leq i_{\ell_o}$ and $1 \leq j \leq j_{\ell_o}$ such that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\ell_o}_{i,j}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} > 0,$$

because otherwise

$$\begin{aligned} 1 &= \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} \\ &\leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} \sum_{1 \leq i \leq i_{\ell_o}, 1 \leq j \leq j_{\ell_o}} S^{\ell_o}_{i,j}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} \\ &\leq \sum_{1 \leq i \leq i_{\ell_o}, 1 \leq j \leq j_{\ell_o}} \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\ell_o}_{i,j}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} = 0. \end{aligned}$$

Let $\delta_{\ell_o} > 0$ be such that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\ell_o}_{i,j}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} > \delta_{\ell_o};$$

this implies that $S^{\ell_o}_{i,j}$ accepts ω, because otherwise, since the total selection process S^{ℓ_o} obviously accepts ω,

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\ell_o}_{i,j}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} = 0.$$

Consequently, there's an infinite subset $\mathcal{N}_{\ell_o} \subseteq \mathbb{N}_0$ such that

$$\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) > 0 \text{ and } \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} > \delta_{\ell_o} \text{ for all } n \in \mathcal{N}_{\ell_o}. \tag{19.13}$$

While the recursive selection process $S_{i,j}^{\ell_o}$ accepts ω, it isn't guaranteed that it's total. We'll 'slightly' change this selection process in order for it to be total. To this end, consider the recursive total selection process $S^{\ell_o,\wedge} \in \mathcal{S}$ defined by

$$S^{\ell_o,\wedge}(s) := \begin{cases} 1 & \text{if } S^{\ell_o}(s) = 1 \text{ and } \sqrt{\sum_{k=0}^{|s|-1} S^{\ell_o}(s_{1:k})} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \text{ for all } s \in \mathbb{S}.$$

By construction, $S^{\ell_o,\wedge}(s) \le S^{\ell_o}(s)$ for all $s \in \mathbb{S}$ and

$$\begin{aligned} \frac{\sum_{k=0}^{|s|-1} S^{\ell_o,\wedge}(s_{1:k})}{\sum_{k=0}^{|s|-1} S^{\ell_o}(s_{1:k})} &= \frac{\left\lfloor \sqrt{\sum_{k=0}^{|s|-1} S^{\ell_o,\wedge}(s_{1:k})} \right\rfloor}{\sum_{k=0}^{|s|-1} S^{\ell_o}(s_{1:k})} \\ &\le \frac{1}{\sqrt{\sum_{k=0}^{|s|-1} S^{\ell_o}(s_{1:k})}} \text{ for all } s \in \mathbb{S} \text{ with } \sum_{k=0}^{|s|-1} S^{\ell_o}(s_{1:k}) > 0. \end{aligned} \tag{19.14}$$

Let the recursive total selection process $S_{i,j}^{\ell_o,\wedge} \in \mathcal{S}$ be defined by

$$S_{i,j}^{\ell_o,\wedge}(s) := \max\left\{S_{i,j}^{\ell_o}(s), S^{\ell_o,\wedge}(s)\right\} \text{ for all } s \in \mathbb{S}. \tag{19.15}$$

By construction,

$$S_{i,j}^{\ell_o,\wedge}(s) \le S_{i,j}^{\ell_o}(s) + S^{\ell_o,\wedge}(s) \text{ for all } s \in \mathbb{S}. \tag{19.16}$$

Also by construction, $S_{i,j}^{\ell_o,\wedge}(s) \le S^{\ell_o}(s)$ for all $s \in \mathbb{S}$, and hence, by recalling that φ_C is almost computable for the selection process S^{ℓ_o}, φ_C is almost computable as well for the recursive total selection process $S_{i,j}^{\ell_o,\wedge}$. Since $\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) > 0$ for all $n \in \mathcal{N}_{\ell_o}$, it's obvious that $\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k}) > 0$ and $\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k}) > 0$ as well for all $n \in \mathcal{N}_{\ell_o}$. Consequently, for all $n \in \mathcal{N}_{\ell_o}$:

$$\begin{aligned} \frac{\sum_{k=0}^{n-1} S^{\ell_o,\wedge}(\omega_{1:k})}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} &\le \frac{\sum_{k=0}^{n-1} S^{\ell_o,\wedge}(\omega_{1:k})}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k})} \\ &= \frac{\sum_{k=0}^{n-1} S^{\ell_o,\wedge}(\omega_{1:k})}{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})} \frac{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k})} \\ &\le \frac{1}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}}, \end{aligned} \tag{19.17}$$

where the last inequality holds by Eqs. (19.13) and (19.14). Since ω is R-random for φ_C, since φ_C is almost computable for the recursive total selection process $S_{i,j}^{\ell_o,\wedge}$, and since the recursive *total* selection process $S_{i,j}^{\ell_o,\wedge}$ obviously accepts ω, it follows from Corollary 11.21$_{101}$ [for $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$] and from Definition 12.1$_{102}$ and Proposition 11.3$_{89}$ [for $\mathrm{R} \in \{\mathrm{CH}, \mathrm{wCH}\}$] that

$$\liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\left[f_j(\omega_{k+1}) - E_{\varphi_C(\omega_{1:k})}(f_j)\right]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \geq 0 \qquad \text{[by letting } f \text{ equal } f_j]$$

and

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\left[f_j(\omega_{k+1}) - E_{\varphi_C(\omega_{1:k})}(f_j)\right]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \leq 0, \qquad \text{[by letting } f \text{ equal } -f_j]$$

which implies that

$$\lim_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\left[f_j(\omega_{k+1}) - E_{\varphi_C(\omega_{1:k})}(f_j)\right]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} = 0. \tag{19.18}$$

Meanwhile, for all $n \in \mathcal{N}_{\ell_o}$,

$$\begin{aligned}
&\frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k}) E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \\
&\overset{\text{Eq. (19.16)}\curvearrowleft}{\leq} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\sum_{k=0}^{n-1} S^{\ell_o,\wedge}(\omega_{1:k}) E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \\
&\overset{\text{C1}_{20},\text{C5}_{20}}{\leq} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\sum_{k=0}^{n-1} S^{\ell_o,\wedge}(\omega_{1:k}) \max|f_j|}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \\
&\overset{\text{Eq. (19.17)}\curvearrowleft}{\leq} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\max|f_j|}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
&\overset{\text{Eq. (19.11)}_{186}}{=} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) E_{\ell_o m_i'}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\max|f_j|}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
&\overset{\text{Eq. (19.12)}_{186}}{\leq} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k})\left[\underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j) - 3\epsilon_{\ell_o}\right]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\max|f_j|}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
&= \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) \underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \\
&\qquad - \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) 3\epsilon_{\ell_o}}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\max|f_j|}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
&\overset{\text{Eq. (19.16)}\curvearrowleft}{\leq} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k}) \underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\sum_{k=0}^{n-1}\Big[S_{i,j}^{\ell_o,\wedge}(\omega_{1:k}) - S^{\ell_o,\wedge}(\omega_{1:k})\Big]3\epsilon_{\ell_o}}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + \frac{\max|f_j|}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
\overset{\text{Eq. (19.17)}_{187}}{\leq}\; & \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o}(\omega_{1:k})\underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} - 3\epsilon_{\ell_o} + \frac{\max|f_j| + 3\epsilon_{\ell_o}}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} \\
\overset{\text{Eq. (19.15)}_{187}}{\leq}\; & \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} - 3\epsilon_{\ell_o} + \frac{\max|f_j| + 3\epsilon_{\ell_o}}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}},
\end{aligned}
$$

where the first and the seventh inequalities make use of the fact that $\underline{E}_{C'}(f) \overset{\text{C1}_{20}}{\geq} 0$ for all gambles $f \geq 0$ and $C' \in \mathscr{C}(\mathscr{X})$, and where the first equality also makes use of Eqs. (19.6)$_{184}$ and (19.7)$_{185}$. Since $\lim_{n\to\infty}\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k}) = \infty$, there's a natural $M_{\ell_o} \in \mathbb{N}$ such that

$$
\frac{\max|f_j| + 3\epsilon_{\ell_o}}{\delta_{\ell_o}\sqrt{\sum_{k=0}^{n-1} S^{\ell_o}(\omega_{1:k})}} < \epsilon_{\ell_o} \text{ for all } n \geq M_{\ell_o}.
$$

This implies for all $n \in \mathcal{N}_{\ell_o}$ with $n \geq M_{\ell_o}$—which is still an infinite number of naturals—that

$$
\frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})E_{\varphi_C(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + 2\epsilon_{\ell_o} < \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})}.
$$

By Eq. (19.18)$_{\curvearrowleft}$, it holds that there's a natural $O_{\ell_o} \in \mathbb{N}$ such that

$$
\epsilon_{\ell_o} > \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\Big[f_j(\omega_{k+1}) - E_{\varphi_C(\omega_{1:k})}(f_j)\Big]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \text{ for all } n \geq O_{\ell_o},
$$

and hence, it holds for all $n \in \mathcal{N}_{\ell_o}$ with $n \geq \max\{M_{\ell_o}, O_{\ell_o}\}$—which is still an infinite number of naturals—that

$$
\begin{aligned}
\epsilon_{\ell_o} &> \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\Big[f_j(\omega_{k+1}) - E_{\varphi_C(\omega_{1:k})}(f_j)\Big]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} \\
&> \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\Big[f_j(\omega_{k+1}) - \underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)\Big]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})} + 2\epsilon_{\ell_o}.
\end{aligned}
$$

Consequently,

$$
-\epsilon_{\ell_o} \geq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})\Big[f_j(\omega_{k+1}) - \underline{E}_{\varphi_{\ell_o}(\omega_{1:k})}(f_j)\Big]}{\sum_{k=0}^{n-1} S_{i,j}^{\ell_o,\wedge}(\omega_{1:k})},
$$

which implies by Definition 12.1$_{102}$ and Proposition 11.3$_{89}$ that ω cannot be wCH-random for φ_{ℓ_o}. □

Lemma 19.19. *Consider any non-empty finite set of probability mass functions* $\{m_1,\dots,m_n\} \subseteq \mathcal{M}(\mathcal{X})$ *and any credal set* $C \in \mathscr{C}(\mathcal{X})$. *Then,* $d_H(CH(\{m_1,\dots,m_n\}),C) \leq \max_{m\in\{m_1,\dots,m_n\}} d(m,C)$.

Proof. Since $\overline{E}_{\mathrm{CH}(\{m_1,\dots,m_n\})}(f) = \max_{m\in\{m_1,\dots,m_n\}} E_m(f)$ for all $f \in \mathscr{L}(\mathcal{X})$, it follows from Lemma 7.8₄₈ that

$$\begin{aligned}
d_{\mathrm{H}}(\mathrm{CH}(\{m_1,\dots,m_n\}),C) &= \max_{f\in\mathscr{L}_1(\mathcal{X})} \left|\overline{E}_{\mathrm{CH}(\{m_1,\dots,m_n\})}(f) - \overline{E}_C(f)\right| \\
&= \max_{f\in\mathscr{L}_1(\mathcal{X})} \left|\max_{m\in\{m_1,\dots,m_n\}} E_m(f) - \overline{E}_C(f)\right| \\
&\leq \max_{m\in\{m_1,\dots,m_n\}} \max_{f\in\mathscr{L}_1(\mathcal{X})} \left|E_m(f) - \overline{E}_C(f)\right| \\
&= \max_{m\in\{m_1,\dots,m_n\}} d(m,C). \qquad \square
\end{aligned}$$

Lemma 19.20. *Consider any two credal sets* $C, C' \in \mathscr{C}(\mathcal{X})$. *Then,* $|dia(C) - dia(C')| \leq 2d_H(C,C')$.

Proof. Fix any two probability mass functions $m_1, m_2 \in C$ such that $\|m_1 - m_2\|_{\mathrm{tv}} = \mathrm{dia}(C)$; it's immediate from the definition of the diameter on p. 182 that this is always possible. Since $m_1, m_2 \in C$, it's immediate from the definition of the Hausdorff distance on p. 18 that there are probability mass function $m_1', m_2' \in C'$ such that $\|m_1 - m_1'\|_{\mathrm{tv}} \leq d_{\mathrm{H}}(C,C')$ and $\|m_2 - m_2'\|_{\mathrm{tv}} \leq d_{\mathrm{H}}(C,C')$. Since $m_1', m_2' \in C'$, we also have that $\|m_1' - m_2'\|_{\mathrm{tv}} \leq \mathrm{dia}(C')$. By applying the triangle inequality for the total variation norm, we get that

$$\begin{aligned}
\mathrm{dia}(C) &= \|m_1 - m_2\|_{\mathrm{tv}} \\
&\leq \|m_1 - m_1'\|_{\mathrm{tv}} + \|m_1' - m_2'\|_{\mathrm{tv}} + \|m_2' - m_2\|_{\mathrm{tv}} \\
&\leq d_{\mathrm{H}}(C,C') + \mathrm{dia}(C') + d_{\mathrm{H}}(C,C') \\
&= \mathrm{dia}(C') + 2d_{\mathrm{H}}(C,C')
\end{aligned}$$

By reversing the roles of C and C' in the above argument, we also have that $\mathrm{dia}(C') \leq \mathrm{dia}(C) + 2d_{\mathrm{H}}(C,C')$, and hence, $|\mathrm{dia}(C) - \mathrm{dia}(C')| \leq 2d_{\mathrm{H}}(C,C')$. □

Lemma 19.21. *Consider any real* $\epsilon > 0$. *Then there is a finite set of rational gambles* $\{f_1,\dots,f_N\} \subseteq \mathscr{L}_1(\mathcal{X}) \cap \mathscr{L}_{\mathrm{rat}}(\mathcal{X})$ *such that* $\left|d(m,C) - \max_{1\leq i\leq N}(\underline{E}_C(f_i) - E_m(f_i))\right| < \epsilon$ *for any probability mass function* $m \in \mathcal{M}(\mathcal{X})$ *and any credal set* $C \in \mathscr{C}(\mathcal{X})$.

Proof. Since $\mathscr{L}_1(\mathcal{X})$ is closed and bounded, and since the rational gambles are a dense subset of the real gambles, there is a finite set of rational gambles $\{f_1,\dots,f_N\} \subseteq \mathscr{L}_1(\mathcal{X}) \cap \mathscr{L}_{\mathrm{rat}}(\mathcal{X})$ such that for every $f \in \mathscr{L}_1(\mathcal{X})$ there's some $f' \in \{f_1,\dots,f_N\}$ for which $|f - f'| \leq \epsilon/2$. Consequently, for any probability mass function $m \in \mathcal{M}(\mathcal{X})$ and any credal set $C \in \mathscr{C}(\mathcal{X})$, it follows from Lemma 11.9₉₅ that

$$d(m,C) = \max_{f\in\mathscr{L}_1(\mathcal{X})} \left(\underline{E}_C(f) - E_m(f)\right)$$

$$\overset{\text{C5}_{20}}{\leq} \max_{1\leq i\leq N} \big(\underline{E}_C(f_i + \epsilon/2) - E_m(f_i - \epsilon/2)\big)$$
$$\overset{\text{C4}_{20}}{\leq} \max_{1\leq i\leq N} \big(\underline{E}_C(f_i) - E_m(f_i)\big) + \epsilon$$

and

$$d(m, C) = \max_{f\in\mathscr{L}_1(\mathscr{X})} \big(\underline{E}_C(f) - E_m(f)\big)$$
$$\overset{\text{C5}_{20}}{\geq} \max_{1\leq i\leq N} \big(\underline{E}_C(f_i - \epsilon/2) - E_m(f_i + \epsilon/2)\big)$$
$$\overset{\text{C4}_{20}}{\geq} \max_{1\leq i\leq N} \big(\underline{E}_C(f_i) - E_m(f_i)\big) - \epsilon,$$

which implies that

$$\left| d(m, C) - \max_{1\leq i\leq N} \big(\underline{E}_C(f_i) - E_m(f_i)\big) \right| \leq \epsilon. \qquad \square$$

Lemma 19.22. *Consider any real $\epsilon > 0$. Then there is a single algorithm that, when provided with a rational probability mass function $m_{\text{rat}} \in \mathcal{M}_{\text{rat}}(\mathscr{X})$ and a code for a recursive rational credal set $C_{\text{rat}} \in \mathscr{C}_{\text{rat}}(\mathscr{X})$, outputs a rational $q \in \mathbb{Q}$ such that $|d(m_{\text{rat}}, C_{\text{rat}}) - q| < \epsilon$.*

Proof. By Lemma 19.21$_\curvearrowleft$, we can fix a finite set of rational gambles $\{f_1, \dots, f_N\} \subseteq \mathscr{L}_1(\mathscr{X}) \cap \mathscr{L}_{\text{rat}}(\mathscr{X})$ such that $\big|d(m, C) - \max_{1\leq i\leq N}\big(\underline{E}_C(f_i) - E_m(f_i)\big)\big| < \epsilon$ for any probability mass function $m \in \mathcal{M}(\mathscr{X})$ and any credal set $C \in \mathscr{C}(\mathscr{X})$. Let $q := \max_{1\leq i\leq N}\big(\underline{E}_{C_{\text{rat}}}(f_i) - E_{m_{\text{rat}}}(f_i)\big)$. By Lemma 7.1$_{43}$ and conjugacy, it's immediate that there's a single algorithm that outputs q upon the input of m_{rat} and a code for C_{rat}. By construction, $|d(m_{\text{rat}}, C_{\text{rat}}) - q| < \epsilon$. $\square$

When we restrict our attention to non-degenerate computable credal sets, then the above theorem also applies to our test-theoretic notions of randomness.

Corollary 19.23. *Consider any* R $\in$ {ML, S} *and any non-degenerate computable credal set $C \in \mathscr{C}(\mathscr{X})$. Then there's a path $\omega \in \Omega$ that's* R*-test-random for the credal set C, but that's never* R*-test-random for any computable forecasting system $\varphi \in \Phi(\mathscr{X})$ whose highest imprecision is smaller than that of C, in the specific sense that $\sup_{s\in\mathbb{S}} dia\big(\varphi(s)\big) < dia(C)$.*

Proof. By Theorem 19.1$_{183}$, consider a path $\omega \in \Omega$ that's R-random for C but not R-random for any computable forecasting system $\varphi \in \Phi(\mathscr{X})$ whose highest imprecision is smaller than that of C. Since C is non-degenerate and computable by assumption, it follows from Propositions 14.4$_{121}$ and 15.6$_{137}$ that ω is R-test-random for C. Consider any computable forecasting system $\varphi \in \Phi(\mathscr{X})$ whose highest imprecision is smaller than that of C. Since ω isn't R-random for φ by assumption, it follows from Propositions 14.2$_{120}$ and 15.2$_{136}$ that ω isn't R-test-random for φ. $\square$

We repeat that (a less general version of) Theorem 19.1$_{183}$ led De Cooman & De Bock [36] to claim that R-randomness is inherently imprecise, because, for all credal sets $C \in \mathscr{C}(\mathscr{X})$ with $\mathrm{dia}(C) > 0$, the randomness of the paths ω in this theorem can only be captured by an imprecise forecasting system. In the next section, we'll show and explain that the assumption that φ is computable is crucial for this claim: indeed, Theorem 20.1↷ further on implies that for every credal set $C \in \mathscr{C}(\mathscr{X})$ there's a non-computable precise forecasting system $\varphi_{\mathrm{pr}} \in C$—so with $\sup_{s \in \mathbb{S}} \mathrm{dia}(\varphi_{\mathrm{pr}}(s)) = 0$—such that ω is R-random for C if and only if it's R-random for φ_{pr}. Hence, for this particular φ_{pr}, there's no path $\omega \in \Omega$ that is R-random for C but not for φ_{pr}.

20 Forecasting systems are sets of precise forecasting systems

At the start of Chapter ⚀$_{13}$, in Section 5.3$_{17}$, we mentioned that a credal set can be seen as a (closed convex) set of probability mass functions. Similarly, as explained in Section 6.2$_{24}$, a forecasting system $\varphi \in \Phi(\mathscr{X})$ can be seen as a set $\{\varphi_{\mathrm{pr}} \colon \varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X}) \text{ and } \varphi_{\mathrm{pr}} \in \varphi\}$ of compatible precise forecasting systems. This point of view continued to make sense in Section 6.4$_{32}$, where we explained that the corresponding upper expectation $\overline{E}^{\varphi}(\bullet)$ coincides with the upper envelope $\sup_{\varphi_{\mathrm{pr}} \in \varphi} E^{\varphi_{\mathrm{pr}}}(\bullet)$ of all global expectations $E^{\varphi_{\mathrm{pr}}}$ that are determined by compatible precise forecasting systems $\varphi_{\mathrm{pr}} \in \varphi$, and this on all global gambles that are measurable with respect to $\mathscr{B}(\Omega)$ [70, Theorem 13]. In this section, we'll explain and show that this kind of interpretation continues to make sense in an algorithmic randomness context as well: a path is random for a forecasting system if and only if it's random for at least one compatible precise forecasting system.

We start by having a look at the measure-theoretic approach to randomness. In Martin-Löf's seminal paper [30], he didn't only introduce measure-theoretic tests to test the agreement between a path and a computable probability measure, he also devised a measure-theoretic test that tests whether a path is Bernoulli random, that is, whether it's (uniformly) random with respect to the set *Ber* of all Bernoulli probability measures. It turns out that a path $\omega \in \Omega$ is Bernoulli random if and only if it's (uniformly) random for some Bernoulli probability measure $\mu \in \mathit{Ber}$ [6, Theorem 4.12]. As is immediate from the introduction to this chapter, this feature continues to hold for Levin's more general notion of *uniform randomness* which—as we recall—tests the correspondence between a path $\omega \in \Omega$ and a so-called effectively compact class of measures $\mathcal{C} \subseteq \mathcal{M}(\Omega)$ [4, 6]: a path $\omega \in \Omega$ is uniformly random for $\mathcal{C}$ if and only if it's uniformly random with respect to at least one member of the considered class $\mathcal{C}$.

An analogous property will turn out to hold for the imprecise-probabilistic martingale-theoretic randomness notions that we introduced in Chapter ⚁$_{49}$: a path $\omega \in \Omega$ is martingale-theoretically random for a(n imprecise-probabilistic) forecasting system if and only if it's martingale-theoretically random for at least one compatible precise forecasting system. Interestingly, this property readily follows from the even more general result below: for every forecasting system $\varphi \in \Phi(\mathscr{X})$ there's a compatible precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ such that a path $\omega \in \Omega$ is martingale-theoretically random for φ if and only if it's martingale-theoretically random for φ_{pr}. We originally proved this property in the restricted setting of binary state spaces [43, 44] using an argument that is not straightforwardly generalisable to arbitrary but finite state spaces. Here, we provide a more direct and shorter argument (for proving Theorem 20.1), which applies to arbitrary but finite state spaces and is based on a diagonal argument put forward by Alexander Shen. Our original argument for binary state spaces can be found in Section 20.1196; we include both because we deem it insightful to provide a different road that leads to Rome and because we find it interesting that the argument for binary state spaces provides us with a(n even) more explicit construction for φ_{pr}.

Theorem 20.1. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ *and any forecasting system* $\varphi \in \Phi(\mathscr{X})$*. Then there's a precise forecasting system* $\varphi_{\mathrm{pr}} \in \varphi$ *such that* $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(\varphi)$.

In the proof below, as you'll notice, we make use of a similar argument as in Example 14.15125.

Proof. We'll construct the precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ iteratively. Let $(F_i)_{i\in\mathbb{N}}$ be an enumeration (not necessarily recursive) of all *implementable* processes in $\mathscr{F}_{\mathrm{R}}$; this is always possible by Lemma 7.646 and Proposition 8.150. We start by considering F_1. If $\overline{E}_{\varphi(s)}(F_1(s\,\cdot)) > F_1(s)$ for an infinite number of $s \in \mathbb{S}$, then we fix some $s_1 \in \mathbb{S}$ such that $\overline{E}_{\varphi(s_1)}(F_1(s_1\,\cdot)) > F_1(s_1)$. Let $\varphi_{\mathrm{pr}}(s_1)$ be equal to some probability mass function $m_1 \in \varphi(s_1)$ for which $E_{m_1}(F_1(s_1\,\cdot)) > F_1(s_1)$ [it's easy to infer from Eq. (5.7)19 that this is always possible]. Otherwise, we let $s_1 \coloneqq \square$. We continue by considering F_2. If $\overline{E}_{\varphi(s)}(F_2(s\,\cdot)) > F_2(s)$ for an infinite number of $s \in \mathbb{S}$, then we (can) fix some $s_2 \in \mathbb{S}$ such that $|s_2| > |s_1| + 1$ and $\overline{E}_{\varphi(s)}(F_2(s_2\,\cdot)) > F_2(s_2)$, and we let $\varphi_{\mathrm{pr}}(s_2)$ be equal to some probability mass function $m_2 \in \varphi(s_2)$ for which $E_{m_2}(F_2(s_2\,\cdot)) > F_2(s_2)$. Otherwise, we let $s_2 \coloneqq s_1$. We continue by considering F_3. If $\overline{E}_{\varphi(s)}(F_3(s\,\cdot)) > F_3(s)$ for an infinite number of $s \in \mathbb{S}$, then we (can) fix some $s_3 \in \mathbb{S}$ such that $|s_3| > |s_2| + 1$ and $\overline{E}_{\varphi(s)}(F_3(s_3\,\cdot)) > F_3(s_3)$, and we let $\varphi_{\mathrm{pr}}(s_3)$ be equal to some probability mass function $m_3 \in \varphi(s_3)$ for which $E_{m_3}(F_3(s_3\,\cdot)) > F_3(s_3)$. Otherwise, we let $s_3 \coloneqq s_2$. Repeat this procedure *ad infinitum* and let $\varphi_{\mathrm{pr}}(s)$ be equal to some probability mass function $m \in \varphi(s)$ in all situations $s \in \mathbb{S}$ that weren't assigned a probability mass function yet. In this way, we obtain a precise forecasting system that's compatible with φ.

We continue by proving that $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(\varphi)$. $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) \subseteq \Omega_{\mathrm{R}}(\varphi)$ is obvious by Proposition 9.556. To prove that $\Omega_{\mathrm{R}}(\varphi) \subseteq \Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}})$, consider any path $\omega \in \Omega_{\mathrm{R}}(\varphi)$ and any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi_{\mathrm{pr}})$. If there's an infinite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi(s)}(T(s\,\cdot)) > T(s)$, then by construction there's some $t \in \mathbb{S}$ such that $E_{\varphi_{\mathrm{pr}}(t)}(T(t\,\cdot)) > T(t)$, contradicting the fact that T is a test supermartingale for φ_{pr}. This implies that there's only a finite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi(s)}(T(s\,\cdot)) > T(s)$. Consequently, by Lemma 10.468 [with $\varphi \to \varphi_{\mathrm{pr}}$ and $\varphi' \to \varphi$], T remains (computably) bounded on ω. $\square$

By Theorem 19.1$_{183}$, in the case of stationary imprecise-probabilistic forecasting systems, the precise forecasting systems φ_{pr} in Theorem 20.1$_{\curvearrowleft}$ are necessarily *non-computable*, as well as *non-stationary*.

Corollary 20.2. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$*, any credal set* $C \in \mathscr{C}(\mathscr{X})$ *with dia*$(C) > 0$*, and any precise forecasting system* $\varphi_{\mathrm{pr}} \in C$*. If* $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(C)$*, then* φ_{pr} *must be non-computable and non-stationary.*

Proof. Since $\mathrm{dia}(C) > 0$ by assumption, and since $\sup_{s \in \mathbb{S}} \mathrm{dia}\big(\varphi_{\mathrm{pr}}(s)\big) = 0 < \mathrm{dia}(C)$ due to the precision of φ_{pr}, it follows from Theorem 19.1$_{183}$ that φ must be non-computable, because, otherwise, this theorem would guarantee that there's some path $\omega \in \Omega$ that's R-random for C but not R-random for φ_{pr}.

It remains to prove that the precise forecasting system φ_{pr} can't be stationary either. Indeed, assume *ex absurdo* that $\varphi_{\mathrm{pr}}(s) = m \in \mathcal{M}(\mathscr{X})$ for all $s \in \mathbb{S}$. Since $\mathrm{dia}(C) > 0$, there's some $m' \in C$ such that $m \neq m'$. Consider any path $\omega \in \Omega$ that's R-random for m' [this is always possible by Corollary 9.3$_{56}$]. By Proposition 9.5$_{56}$, $\omega \in \Omega_{\mathrm{R}}(C)$. Since $\omega \in \Omega_{\mathrm{R}}(m')$, it follows from Corollary 12.4$_{105}$ and Proposition 11.7$_{93}$ [with $S = 1$] that

$$0 = \lim_{n\to\infty} d\left(\frac{\sum_{k=0}^{n-1} \mathbb{I}_{\omega_{k+1}}}{n}, m'\right) = \lim_{n\to\infty} \left\| \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}_{\omega_{k+1}} - m' \right\|_{\mathrm{tv}},$$

and hence,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}_{\omega_{k+1}}(x) = m'(x) \text{ for all } x \in \mathscr{X}.$$

This implies that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{I}_{\omega_{k+1}}(x) \neq m(x)$ for some $x \in \mathscr{X}$, which in its turn implies that ω cannot be wCH-random for m, and therefore, by Corollary 12.4$_{105}$, cannot be R-random for m. We conclude that $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) \neq \Omega_{\mathrm{R}}(C)$. □

We repeat that, in this martingale-theoretic setting, Theorem 20.1$_{\curvearrowleft}$ complements Theorem 19.1$_{183}$ by laying bare the importance of the computability assumption on the precise forecasting systems. As we've already highlighted in the discussion before Theorem 20.1$_{\curvearrowleft}$, Theorem 20.1$_{\curvearrowleft}$ is also interesting from a measure-theoretic randomness perspective, since it readily leads to corollaries that are reminiscent of existing measure-theoretic randomness results. The next result shows that our martingale-theoretic notions of randomness satisfy a property that is similar to the one for uniform randomness that we mentioned at the start of this section: a path $\omega \in \Omega$ is random for a(n imprecise-probabilistic) forecasting system if and only if it's random for at least one compatible precise forecasting system.

Corollary 20.3. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$ *and any forecasting system* $\varphi \in \Phi(\mathscr{X})$*. Then a path* $\omega \in \Omega$ *is* R*-random for* φ *if and only if it's* R*-random for at least one compatible precise forecasting system* $\varphi_{\mathrm{pr}} \in \varphi$*.*

Proof. By Proposition 9.5$_{56}$, the 'if' part is straightforward, so we proceed to the 'only if' part. Assume that ω is R-random for the forecasting system $\varphi \in \Phi(\mathscr{X})$. Then it follows from Theorem 20.1₁₉₃ that there's some precise compatible forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ for which ω is R-random. □

Another measure-theoretic result that we can now show has a martingale-theoretic counterpart, is the existence of a so-called *neutral* measure for which all paths $\omega \in \Omega$ are random. In a measure-theoretic context, this is true for uniform randomness [6, Theorem 6.2]. We here obtain a similar result for any of the four martingale-theoretic notions of randomness that we consider.

Corollary 20.4. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$. *Then there's a precise—but necessarily non-stationary and non-computable—forecasting system* $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ *for which all paths* $\omega \in \Omega$ *are* R*-random.*

Proof. From Theorem 20.1$_{193}$, it immediately follows that there's some precise forecasting system $\varphi_{\mathrm{pr}} \in \Phi(\mathscr{X})$ such that $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(C_{\mathrm{v}})$, and hence, by Proposition 9.4$_{56}$, $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega$. Corollary 20.2$_{\curvearrowleft}$ then implies that φ_{pr} is necessarily non-stationary and non-computable. □

We find this result to be particularly intriguing. Proposition 9.4$_{56}$ guarantees that every path $\omega \in \Omega$ is random for the vacuous forecasting system. Since all precise forecasting systems are compatible with the vacuous forecasting system, Corollary 20.3$_{\curvearrowleft}$ then tells us that this amounts to every path $\omega \in \Omega$ being random for at least one precise forecasting system; this statement can also be proven by realising that every path $\omega \in \Omega$ is random for the temporal (precise) forecasting system that assigns at every time instant $n \in \mathbb{N}_0$ all mass to the next state ω_{n+1} in ω [see also Proposition 21.1$_{199}$ further on]. The result above strengthens this, by showing that there's in fact one single precise forecasting system all paths are random for.

Where does this discussion leave us? Theorem 20.1$_{193}$ and Corollary 20.2$_{\curvearrowleft}$ show that the randomness of a path $\omega \in \Omega$ with respect to a credal set $C \in \mathscr{C}(\mathscr{X})$ with $\mathrm{dia}(C) > 0$, be it computable or not, can be equivalently described by a precise forecasting system that's then necessarily *non-computable* and non-stationary. This furthermore implied, as we have seen in Corollary 20.3$_{\curvearrowleft}$, that randomness with respect to a stationary forecasting system can be equivalently described in terms of the compatible precise forecasting systems. It may therefore seem that, on purely theoretical grounds, stationary imprecise forecasting systems aren't needed in the study of algorithmic randomness. However, if we want to maintain the claim that randomness is inherently imprecise, Corollaries 20.2$_{\curvearrowleft}$ and 20.4 tell us we need only explain why we believe that non-computable forecasting systems are non-satisfactory, and even fairly useless. We'll come to that in Section 21$_{199}$, where we argue why the computability assumption on the forecasting system is justified on practical grounds. But before getting to that, we now devote ourselves to providing an alternative proof for Theorem 20.1$_{193}$ in the specific setting of binary state spaces.

20.1 An alternative proof of the main result for binary state spaces

In this section, we'll provide an alternative proof for Theorem 20.1$_{193}$ when restricting our attention to the binary state space $\mathscr{X} = \{a, b\}$; in the remainder of this section, as we mentioned on p. 26, a forecasting system $\varphi\colon \mathbb{S} \to \mathscr{I}$ will then be seen as map that specifies for every situation $s \in \mathbb{S}$ an interval forecast $\varphi(s) \in \mathscr{I}$ that's associated with the outcome $X_{|s|+1} = a$. We'll do so by proving a slightly different statement: for every forecasting system $\varphi \in \Phi(\mathscr{X})$ and $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$, we'll (more) explicitly provide and construct a compatible precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$ such that $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(\varphi)$. In this construction, we'll make use of paths that are $\mathcal{S}^\infty$-random for an interval forecast $I \subseteq (0,1)$, with $\mathcal{S}^\infty$ some well-chosen countable set of selection processes $\mathcal{S}^\infty$. By Definition 11.2$_{89}$ and Proposition 11.3$_{89}$, a path $\omega \in \Omega$ is $\mathcal{S}^\infty$-random for an interval forecast $I \in \mathscr{I}$ if and only if for any gamble $f \in \mathscr{L}(\mathscr{X})$ and any selection process $S \in \mathcal{S}^\infty(\omega)$ that accepts ω:

$$\underline{E}_I(f) \leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) f(\omega_{k+1})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S(\omega_{1:k}) f(\omega_{k+1})}{\sum_{k=0}^{n-1} S(\omega_{1:k})} \leq \overline{E}_I(f). \tag{20.5}$$

In order to use such $\mathcal{S}^\infty$-random paths in the construction of an appropriate precise forecasting system, we of course need to be sure that for every interval forecast I and every countable set of selection processes $\mathcal{S}^\infty$, there's at least one $\mathcal{S}^\infty$-random path for I. This is guaranteed by Proposition 11.10(ii)$_{96}$.

We'll now use such $\mathcal{S}^\infty$-random paths $\omega \in \Omega$ to craft the special precise forecasting systems we have been talking about. To this end, fix any forecasting system φ and any path $\varpi \in \Omega$, and consider the associated *compatible* precise forecasting system $\varphi^{\varpi} \in \Phi_{\mathrm{pr}}(\mathscr{X})$, defined by

$$\varphi^{\varpi}(s) := \begin{cases} \overline{\varphi}(s) & \text{if } \varpi_{|s|+1} = a \\ \underline{\varphi}(s) & \text{if } \varpi_{|s|+1} = b \end{cases} \quad \text{for all } s \in \mathbb{S}. \tag{20.6}$$

In the main result of this section, Theorem 20.10$_{\curvearrowright}$ below, we'll in particular use paths ϖ that are $\mathcal{S}^\infty$-random for an interval forecast $I \subseteq (0,1)$, where the countable set of selection processes $\mathcal{S}^\infty$ is of a special type.

To define these sets $\mathcal{S}^\infty$, we start by associating with every real process F and every *precise* forecasting system $\varphi_{\mathrm{pr}} \in \Phi_{\mathrm{pr}}(\mathscr{X})$ the *temporal* selection process $S_F^{\varphi_{\mathrm{pr}}}$, defined for all $n \in \mathbb{N}_0$ by

$$S_F^{\varphi_{\mathrm{pr}}}(n) := \begin{cases} 1 & \text{if } E_{\varphi_{\mathrm{pr}}(s)}(\Delta F(s)) > 0 \text{ for some } s \in \mathbb{S} \text{ with } |s| = n \\ 0 & \text{if } E_{\varphi_{\mathrm{pr}}(s)}(\Delta F(s)) \leq 0 \text{ for all } s \in \mathbb{S} \text{ with } |s| = n. \end{cases} \tag{20.7}$$

We use these temporal selection processes to associate with every countable set $\mathscr{F}$ of real processes and every forecasting system $\varphi \in \Phi(\mathscr{X})$ the clearly countable set $\mathcal{S}^{\varphi}_{\mathscr{F}}$ of temporal selection processes, defined by

$$\mathcal{S}^{\varphi}_{\mathscr{F}} \coloneqq \left\{ S^{\varphi_{\mathrm{pr}}}_{F} \colon F \in \mathscr{F} \text{ and } \varphi_{\mathrm{pr}} \in \{\underline{\varphi}, \overline{\varphi}\} \right\}. \tag{20.8}$$

Since $\mathcal{S}^{\varphi}_{\mathscr{F}}$ is countable, Proposition 11.10(ii)$_{96}$ guarantees that there's at least one path that's $\mathcal{S}^{\varphi}_{\mathscr{F}}$-random for a given interval forecast $I \subseteq (0,1)$.

In this construction of the sets $\mathcal{S}^{\varphi}_{\mathscr{F}}$, the specific countable sets $\mathscr{F}$ of real processes that we'll consider, are the sets $\mathscr{F}_{\mathrm{S}}$, $\mathscr{F}_{\mathrm{C}}$, $\mathscr{F}_{\mathrm{wML}}$ and $\mathscr{F}_{\mathrm{ML}}$ introduced in Section 8$_{50}$. If we recall that $\mathscr{F}_{\mathrm{S}} = \mathscr{F}_{\mathrm{C}} \subseteq \mathscr{F}_{\mathrm{wML}} \subseteq \mathscr{F}_{\mathrm{ML}}$, Eq. (20.8) tells us that

$$\mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{S}}} = \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{C}}} \subseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{wML}}} \subseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{ML}}} \text{ for all } \varphi \in \Phi(\mathscr{X}). \tag{20.9}$$

We're now ready to move on to the main result of this section, where we use the special countable sets of selection processes $\mathcal{S}^{\varphi}_{\mathscr{F}}$ and the special forecasting systems φ^{ϖ} to provide an alternative proof for Theorem 20.1$_{193}$.

Theorem 20.10. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$*, any forecasting system* $\varphi \in \Phi(\mathscr{X})$*, any interval forecast* $I \subseteq (0,1)$*, any countable set of selection processes* $\mathcal{S}^{\infty} \supseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{R}}}$*, and any path* $\varpi \in \Omega$ *that's* $\mathcal{S}^{\infty}$*-random for* I*. Then a path* $\omega \in \Omega$ *is* R*-random for* φ *if and only if it's* R*-random for* φ^{ϖ}*, that is,* $\Omega_{\mathrm{R}}(\varphi) = \Omega_{\mathrm{R}}(\varphi^{\varpi})$*.*

Proof. We begin with the direct implication. Assume that $\omega \in \Omega$ is R-random for φ^{ϖ}. Since $\varphi^{\varpi} \in \varphi$, it follows from Proposition 9.5$_{56}$ that ω is also R-random for φ.

To prove the converse implication, assume that $\omega \in \Omega$ is R-random for φ. Taking into account Definitions 8.5$_{52}$ and 8.6$_{54}$, in order to prove that ω is R-random for φ^{ϖ}, we consider any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\varpi})$ and prove that it isn't unbounded on ω when $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}\}$, and that it isn't computably unbounded on ω when $\mathrm{R} = \mathrm{S}$.

To this end, consider the two temporal selection processes $S^{\underline{\varphi}}_{T}$ and $S^{\overline{\varphi}}_{T}$ as defined by Eq. (20.7)$_{\curvearrowleft}$. We'll take a closer look at the temporal selection process $S^{\underline{\varphi}}_{T}$ and prove that there's only a finite number of non-negative integers $n \in \mathbb{N}_0$ for which $S^{\underline{\varphi}}_{T}(n) = 1$. To this end, assume *ex absurdo* that there's an infinite number of them, and therefore that $\lim_{n\to\infty} \sum_{k=0}^{n-1} S^{\underline{\varphi}}_{T}(k) = \infty$. Consider any $k \in \mathbb{N}_0$ such that $S^{\underline{\varphi}}_{T}(k) = 1$, then it follows from Eq. (20.7)$_{\curvearrowleft}$ that there's some $s \in \mathbb{S}$ with $|s| = k$ such that $E_{\underline{\varphi}(s)}(\Delta T(s)) > 0$. Since $E_{\varphi^{\varpi}(s)}(\Delta T(s)) \leq 0$ (because ΔT is a supermartingale for φ^{ϖ}), this implies that necessarily $\varphi^{\varpi}(s) = \overline{\varphi}(s) \neq \underline{\varphi}(s)$ and therefore, since $|s| = k$, we infer from Eq. (20.6)$_{\curvearrowleft}$ that $\varpi_{k+1} = a$. Since this is true for each of the infinitely many $k \in \mathbb{N}_0$ such that $S^{\underline{\varphi}}_{T}(k) = 1$, it follows that

$$\limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\underline{\varphi}}_{T}(k) \mathbb{I}_a(\varpi_{k+1})}{\sum_{k=0}^{n-1} S^{\underline{\varphi}}_{T}(k)} = \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S^{\underline{\varphi}}_{T}(k)}{\sum_{k=0}^{n-1} S^{\underline{\varphi}}_{T}(k)} = 1. \tag{20.11}$$

Since $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\varpi})$, and therefore also $T \in \mathscr{F}_{\mathrm{R}}$, we can infer from Eq. (20.8)↶ that $S_T^{\varphi} \in \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{R}}} \subseteq \mathcal{S}^{\infty}$. Consequently, since ϖ is $\mathcal{S}^{\infty}$-random for I by assumption and since S_T^{φ} accepts ω [because $\lim_{n\to\infty} \sum_{k=0}^{n-1} S_T^{\varphi}(k) = \infty$], it follows from Eq. (20.5)$_{196}$ [with $f \to \mathbb{I}_a$] that

$$\max I \geq \limsup_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_T^{\varphi}(k)\mathbb{I}_a(\varpi_{k+1})}{\sum_{k=0}^{n-1} S_T^{\varphi}(k)} \overset{\text{Eq. (20.11)↶}}{=} 1,$$

contradicting the assumption that $I \subseteq (0,1)$. We conclude that, indeed, there's only a finite number of non-negative integers $n \in \mathbb{N}_0$ for which $S_T^{\underline{\varphi}}(n) = 1$.

In a completely similar manner, it can be shown that there's only a finite number of non-negative integers $n \in \mathbb{N}_0$ for which $S_T^{\overline{\varphi}}(n) = 1$. Indeed, assume *ex absurdo* that there's an infinite number of them. Then, by adopting a similar argument, it follows that $\lim_{n\to\infty} \sum_{k=0}^{n-1} S_T^{\overline{\varphi}}(k) = \infty$, that $S_T^{\overline{\varphi}} \in \mathcal{S}^{\overline{\varphi}}_{\mathscr{F}_{\mathrm{R}}} \subseteq \mathcal{S}^{\infty}$, and that for all $k \in \mathbb{N}_0$, if $S_T^{\overline{\varphi}}(k) = 1$, then $\varpi_{k+1} = b$. That being so, it follows from Eq. (20.5)$_{196}$ [with $f \to \mathbb{I}_a$], since ϖ is $\mathcal{S}^{\infty}$-random for I by assumption and since $S_T^{\overline{\varphi}}$ accepts ω, that

$$\min I \leq \liminf_{n\to\infty} \frac{\sum_{k=0}^{n-1} S_T^{\overline{\varphi}}(k)\mathbb{I}_a(\varpi_{k+1})}{\sum_{k=0}^{n-1} S_T^{\overline{\varphi}}(k)} = 0,$$

contradicting the assumption that $I \subseteq (0,1)$.

Since there's only a finite number of non-negative integers $n \in \mathbb{N}_0$ for which $S_T^{\underline{\varphi}}(n) = 1$ or $S_T^{\overline{\varphi}}(n) = 1$, and since for each such n, there's only a finite number of situations $s \in \mathbb{S}$ such that $|s| = n$, it follows from Eq. (20.7)$_{196}$ that there's only a finite number of situations $s \in \mathbb{S}$ for which $E_{\underline{\varphi}(s)}(\Delta T(s)) > 0$ or $E_{\overline{\varphi}(s)}(\Delta T(s)) > 0$. Hence, there's only a finite number of situations $s \in \mathbb{S}$ for which $\overline{E}_{\varphi(s)}(\Delta T(s)) = \max\{E_{\underline{\varphi}(s)}(\Delta T(s)), E_{\overline{\varphi}(s)}(\Delta T(s))\} > 0$. By invoking Lemma 10.4$_{68}$ [with $\varphi \to \varphi^{\varpi}$ and $\varphi' \to \varphi$], we conclude that T remains (computably) bounded on ω. □

According to Theorem 20.10↶, for every choice of R in {ML, wML, C, S}, there's some path $\varpi \in \Omega$ such that the R-random paths for the forecasting system φ and for the precise forecasting system φ^{ϖ} coincide. Interestingly, there's also a single path $\varpi \in \Omega$ that does this job for all four notions of randomness that we consider here. Basically, this is true because the weaker the notion of randomness, the weaker the conditions on ϖ that are required in Theorem 20.10↶, in the sense that the minimally required countable set of selection processes $\mathcal{S}^{\infty}$ becomes smaller.

Corollary 20.12. *Consider any forecasting system φ, any interval forecast $I \subseteq (0,1)$, any countable set of selection processes $\mathcal{S}^{\infty} \supseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{ML}}}$, and any path $\varpi \in \Omega$ that's $\mathcal{S}^{\infty}$-random for I. Then, for any* R ∈ {ML, wML, C, S}, *a path $\omega \in \Omega$ is* R*-random for φ if and only if it's* R*-random for φ^{ϖ}.*

Proof. Since $\mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{S}}} = \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{C}}} \subseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{wML}}} \subseteq \mathcal{S}^{\varphi}_{\mathscr{F}_{\mathrm{ML}}}$ by Eq. (20.9)↶, this follows readily from Theorem 20.10↶. □

21 Theoretical and practical necessity of credal sets in statistics

Let's now zoom out and move away from the technicalities in the previous sections, in order to better understand the implications of Theorem 20.1~193~ and its Corollary 20.3~194~. In trying to come to a better understanding, we have found it useful to look at these results from the point of view of statistics, whose aim it is to learn an uncertainty model from data. Regarding the data, we'll consider a finite sequence $\omega_{1:n}$ and assume that it's an initial segment of an idealised (and unobserved) path ω that is (ML-, wML-, C- or S-)random for some forecasting system; there are clearly a multitude of forecasting systems for which this is the case. Under this assumption, we'll examine what forecasting systems—that make the path ω random—can be learned from the finite initial segment $\omega_{1:n}$. Notice that, whilst doing so, we have changed our point of view: instead of focusing on the paths that are random for a forecasting system $\varphi \in \Phi(\mathscr{X})$, as we have done before, we have a look at the forecasting systems that make a path $\omega \in \Omega$ random. Even though it's commonly assumed that the uncertainty model φ to be estimated or identified from the data $\omega_{1:n}$ is precise, we thus question the assumption that a path's randomness should always be described by a precise forecasting system $\varphi \in \Phi_{\mathrm{pr}}(\mathscr{X})$. So, in the discussion below, we want to remain open about that possibility, and see what can be said if we don't assume *a priori* that the sequence ω is necessarily random for a *precise* forecasting system.

From Proposition 9.4~56~, we know that there's at least one candidate (stationary) credal set that makes ω random: all paths are random for the vacuous credal set C_{v}. Meanwhile, it isn't guaranteed that there's a stationary precise probability mass function m that makes ω random; by Theorem 19.1~183~, for every credal set $C \in \mathscr{C}(\mathscr{X})$ with $\mathrm{dia}(C) > 0$, there's a path $\omega \in \Omega$ such that ω is random for C, but not for any probability mass function $m \in \mathcal{M}(\mathscr{X})$ (for which $\mathrm{dia}(m) = 0$). Hence, generally speaking, imprecision is needed if we insist on a *stationary* uncertainty model to describe a path's randomness. If we also allow for non-stationary uncertainty models however, then Theorem 20.1~193~ shows that we could replace every forecasting system φ that makes ω random by a non-stationary precise forecasting system $\varphi_{\mathrm{pr}} \in \varphi$. In fact, there's an even more (theoretically) straightforward way to associate a non-stationary precise forecasting system with a path ω: the temporal precise forecasting system φ^{ω} defined by

$$\varphi^{\omega}(n)(x) := \begin{cases} 1 & \text{if } \omega_{n+1} = x \\ 0 & \text{otherwise} \end{cases} \text{ for all } n \in \mathbb{N}_0 \text{ and } x \in \mathscr{X},$$

which assigns probability 1 to the actual next value, and hence, makes a perfect prediction.

Proposition 21.1. *Consider any* $\mathrm{R} \in \{\mathrm{ML}, \mathrm{wML}, \mathrm{C}, \mathrm{S}\}$*, then any path* $\omega \in \Omega$ *is* R*-random for the precise forecasting system* φ^{ω}*.*

Proof. Consider any test supermartingale $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\omega})$. Since T is a supermartingale for φ^{ω}, it holds for any $n \in \mathbb{N}_0$ that

$$0 \geq E_{\varphi^{\omega}(\omega_{1:n})}(\Delta T(\omega_{1:n})) = \Delta T(\omega_{1:n})(\omega_{n+1}),$$

and therefore,

$$T(\omega_{1:n}) = T(\Box) + \sum_{k=0}^{n-1} \Delta T(\omega_{1:k})(\omega_{k+1}) \leq T(\Box) = 1.$$

Consequently, all test supermartingales $T \in \overline{\mathbb{T}}_{\mathrm{R}}(\varphi^{\omega})$ are bounded above by 1 along ω. It therefore holds [see Definitions 8.5$_{52}$ and 8.6$_{54}$] that ω is R-random for φ^{ω}. □

Hence, if ω is random for a forecasting system φ, then it's definitely random for at least one (and at least two if $\varphi_{\mathrm{pr}} \neq \varphi^{\omega}$) precise models. We won't risk getting bogged down into a discussion on what uncertainty models are best associated with a path ω; that would require a chapter on its own. But we do want to point out that the uncertainty models that correspond with ω typically don't contain the same information; that is, they don't share the same set of random paths. Interestingly, however, as we know from Theorem 20.1$_{193}$, φ and φ_{pr} do have the same set of random paths and are, in that sense, equally expressive. On that ground, theoretically, one might argue that the imprecision in φ isn't needed.

We believe that this story changes when moving to more practical grounds though. If we're given an initial finite segment $\omega_{1:n}$ of a path $\omega \in \Omega$ and want to learn a forecasting system φ for which ω is random, we'll have to do so by adopting a finite algorithm that, given the data $\omega_{1:n}$, outputs (a code for) a forecasting system φ' whose set of random paths is then believed to contain ω. A candidate for φ' could be the precise forecasting system φ^{ω} that's generated by ω itself. However, it's unfeasible to output a code for this forecasting system, or to even approximate it, as it basically requires us to know the entire path ω itself.

Another candidate for φ' could be the precise forecasting system φ_{pr} [for which $\Omega_{\mathrm{R}}(\varphi_{\mathrm{pr}}) = \Omega_{\mathrm{R}}(\varphi)$]. However, if φ is stationary and has non-zero diameter, it seems impossible to output a code for φ_{pr} or to even approximate it because it's then non-computable and non-stationary by Corollary 20.2$_{194}$. At the same time, learning a stationary forecasting system φ—which is as expressive as φ_{pr}—seems a much less daunting, and practically more feasible, task, especially if φ is computable.

In summary, it's one thing to associate precise uncertainty models with a path ω that isn't random for any precise stationary probability mass function, but it's another thing to actually learn them. When it comes to the latter, computable stationary imprecise forecasting systems seem more promising than non-computable non-stationary precise ones.

Conclusions

In the introduction to this dissertation, we asked a few questions:

When do you consider a binary sequence to be generated by flipping a fair coin?

> *Or put differently, (when) would you say that a sequence agrees with probability* $1/2$*, where* $1/2$ *is the probability for the coin landing heads?*

What sequences do (and don't) you deem random?

> *What happens when we allow for imprecise probability models in the field of algorithmic randomness?*

In particular, how do we allow for imprecise uncertainty models in several classical randomness definitions, and how do the corresponding generalisations shine new light on our understanding of random sequences?

We also indicated how we would answer these questions:

> *We'll allow for imprecise uncertainty models in various frequentist, test- and martingale-theoretic notions of randomness. We'll argue that these definitions are natural since (i) they coincide with the classical definitions when considering precise (computable) forecasting systems, and since (ii) they have similar properties as the classical precise-probabilistic definitions. In particular, we'll study how all definitions relate to each other, and these relationships will be reminiscent of the classical precise-probabilistic relations. Moreover, we're able to ask and address some questions for which imprecise probabilities are pivotal. For instance, should or could the randomness of a sequence always be defined with respect to a precise uncertainty model? And how do imprecise probability models change our understanding of random sequences?*

After reading this dissertation, we can give more detail to these answers (and provide answers to the additional questions).

In summary, we've generalised several randomness notions by allowing for forecasting systems, which associate a possibly different credal set with every situation. In particular, under the standard approach to randomness, we've generalised the martingale-theoretic notions of (weak) Martin-Löf, computable and Schnorr randomness in Chapter ⚁$_{49}$, and the frequentist notions of (weak) Church stochasticity in Chapter ⚂$_{85}$. As we've explained, (i) all these randomness notions obviously coincide with their precise-probabilistic counterparts when restricting our attention to precise (computable) forecasting systems, and (ii) they have properties that generalise and extend the precise-probabilistic ones; to give but a few examples, the set of R-random paths $\Omega_{\mathrm{R}}(\varphi)$ is almost sure for φ [Propositions 9.2$_{55}$ and 11.10(i)$_{96}$], the R-randomness of a path with respect to a computable (non-degenerate) forecasting system φ only depends on the forecasts that are specified along the path [Propositions 9.6$_{57}$, Proposition 9.18$_{65}$ and 11.23$_{101}$], the betting strategies used to define computable and Schnorr randomness can be assumed to be rational-valued and recursive [Propositions 10.16$_{78}$ and 10.22$_{82}$], these randomness notions relate to each other in the exact same way as their precise-probabilistic counterparts do when restricting attention to (almost) computable forecasting systems [Corollary 12.4$_{105}$], and (weak) Church stochasticity has a martingale-theoretic characterisation [Proposition 12.7$_{107}$].

This leads us to say that we've not just succeeded in allowing for imprecise probability models in several algorithmic randomness notions, but that we've succeeded in doing so in a very natural way. That our approach is natural is reconfirmed when adopting a (standard) test-theoretic approach to randomness. In Chapter ⚃$_{111}$, we introduced two test-theoretic notions of randomness: Martin-Löf and Schnorr test randomness [Definitions 13.1$_{114}$, 13.4$_{115}$ and 13.14$_{119}$]; we explained that these test-theoretic randomness notions coincide with their precise-probabilistic counterparts when restricting our attention to precise (computable) forecasting systems [Propositions 13.7$_{116}$ and 13.10$_{118}$], and showed that they coincide with the respective martingale-theoretic definitions when restricting our attention to non-degenerate computable forecasting systems [Theorems 14.1$_{120}$ and 15.1$_{136}$], thereby generalising earlier results by Schnorr [1] and Levin [4]. Moreover, as is also true for the precise-probabilistic case, for every computable forecasting system φ there is a universal test such that a path is Martin-Löf test random for φ if and only if it passes this single test [Proposition 14.18$_{127}$]. Last, as yet another argument in favour of our approach, we showed that our imprecise-probabilistic notion of Martin-Löf test randomness coincides with Levin's imprecise-probabilistic notion of uniform randomness when considering computable forecasting systems [Theorem 14.34$_{134}$].

In Chapter ⚄$_{143}$, we answered the question '*What sequences do we deem random?*' yet again. Only, this time we did away with the standard approach to algorithmic randomness and adopted a so-called prequential approach. Instead of defining what it means for a path to be random for a (rational) forecasting system, we defined the randomness of a path only with respect to the (rational) forecasts that are specified along the path. We did this adopting both a martingale- and a test-theoretic approach, and called the respective randomness notions game-randomness and test-randomness [Definitions 17.6$_{153}$ and 18.3$_{169}$]. We succeeded in showing that both prequential randomness notions coincide, without having to impose any computability conditions on the forecasts [Theorem 18.6$_{171}$]. Moreover, when restricting our attention to non-degenerate recursive rational forecasting systems, we showed that the prequential randomness notions coincide with Martin-Löf (test) randomness [Theorem 17.24$_{161}$ and Corollary 17.25$_{161}$]. Last, we showed that our prequential randomness notions satisfy similar properties as our standard notion of Martin-Löf randomness: there is a universal test [Corollary 18.13$_{176}$], the frequency of the outcomes along a random (prequential) path is bounded by the forecasts along this path [Theorem 17.31$_{166}$], etc.

In Chapter ⚅$_{179}$, we gave an answer to the questions '*Should or could the randomness of a sequence always be defined with respect to a precise uncertainty model? And how do imprecise probability models change our understanding of random sequences?*', and we drew special attention to the importance of computability assumptions on the forecasting systems in answering these questions. On the one hand, when restricting our attention to computable forecasting systems, we showed the existence of paths that are random for a credal set, but not for any computable precise forecasting system [Theorem 19.1$_{183}$ and Corollary 19.23$_{191}$]. On the other hand, when letting go of the computability assumption on the precise forecasting systems, we showed that for every forecasting system there is a compatible precise forecasting system that has the exact same set of random paths [Theorem 20.1$_{193}$]. Both results seem to contradict each other, but the contradiction is easily resolved by looking at the role that computability plays; the answer to the above question thus ultimately depends on your view of the computability matter. In Section 21$_{199}$, we argued why computable forecasting systems are to be favoured from the (practical) vantage point of statistics whose aim it is to learn an uncertainty model from data, and thereby provided an argument in favour of using imprecise uncertainty models.

Future work

In accordance with tradition, we conclude this dissertation with a discussion of some possible avenues for future research. We'll do so by walking one more time through this dissertation, and gathering a number of research questions that spring from our exposition.

We introduced imprecise-probabilistic forecasting systems in Chapter ⚀$_{13}$, and used them in Chapters ⚁$_{49}$ and ⚂$_{85}$ to define a number of martingale-theoretic and frequentist randomness notions. We wonder whether we can allow in these randomness notions for uncertainty models that are even more general, such as choice functions [93].

In Section 9_{54} of Chapter ⚁$_{49}$ and in Sections 11.3_{95} and 12.2_{103} of Chapter ⚂$_{85}$ we looked into a number of properties that these randomness notions have. There is however a precise-probabilistic property that we find particularly interesting, and for which we haven't provided an imprecise-probabilistic analogue. Building upon the work by Vladimir Vovk [94, 95], we'd like to find out if there's some path-dependent 'distance' between any two computable (imprecise) forecasting systems $\varphi, \varphi' \in \Phi(\mathscr{X})$ such that, if a path $\omega \in \Omega$ is random for φ, then it will be random for φ' if (and only if) the distance between both forecasting systems remains bounded on ω; for this to be possible, we may need to impose additional properties on the forecasting systems alongside computability, such as non-degeneracy.

In Chapter ⚃$_{111}$, we allowed for imprecise-probabilistic forecasting systems in a test-theoretic approach to Martin-Löf and Schnorr randomness. In an overall program articulated by Downey et al. [32, 96], which aims to 'calibrate' several results in algorithmic randomness, the authors succeeded in equipping computable randomness with a test-theoretic characterisation by devising so-called computably graded tests. We wonder whether we can generalise these tests to our imprecise-probabilistic setting such that a path $\omega \in \Omega$ is computably random for a computable (non-degenerate) forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if ω is computably test random for φ.

In our work in Chapters ⚁$_{49}$ and ⚃$_{111}$, we focused on extending several martingale- and test-theoretic randomness definitions of randomness to deal with credal sets. In the precise-probabilistic setting, there are also other approaches to defining the classical notions of Martin-Löf, computable and Schnorr randomness, besides the martingale- and test-theoretic ones: via Kolmogorov complexity [1, 30, 32, 83, 84], order-preserving transformations of the event tree associated with a sequence of outcomes [1], or specific limit laws (such as Lévy's zero-one law) [85, 86], and so on. It remains to be investigated whether our credal set extensions can also be arrived at via such alternative routes.

In Chapter ⚄$_{143}$, we allowed for rational credal sets in a prequential approach to Martin-Löf (test) randomness; a first obvious thing to do then is to come up with prequential versions of our other imprecise-probabilistic randomness notions. In future work, we also intend to come closer to Vovk and Shen's work [9], by allowing for arbitrary real credal sets and adopting a more involved notion of lower semicomputability that allows for real maps $r\colon \mathcal{D}' \to \mathbb{R}$ whose domain $\mathcal{D}'$ can be uncountable, such as the set $(\mathscr{C}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$. We suspect that, in this continuous setting, the conditions required for obtaining results analogous to the ones in Section 17.4 will turn out to be different; for one thing, we expect the computability (recursiveness) requirement on the forecasting systems in Proposition 17.29_{164} to drop, which would then yield an arguably more natural monotonicity property.

We are also led to wonder whether a precise-probabilistic interpretation can be given to our prequential imprecise-probabilistic randomness notions. In the standard setting, we've shown [Theorem 20.1$_{193}$] that a path $\omega \in \Omega$ is Martin-Löf random for a forecasting system $\varphi \in \Phi(\mathscr{X})$ if and only if it's random for some compatible *precise* (but typically non-computable) forecasting system $\varphi_{\mathrm{p}} \in \varphi$, in the sense that $\varphi_{\mathrm{p}}(s) \in \varphi(s)$ for all $s \in \mathbb{S}$. In a prequential context, then, could an infinite sequence of credal sets $\zeta = (C_1, \ldots, C_n, \ldots) \in \mathscr{C}_{\mathrm{rat}}(\mathscr{X})^{\mathbb{N}}$ be interpreted as bounds on—then so-called *compatible*—precise forecasts, and could it be concluded that a prequential path $(\zeta, \omega) \in (\mathscr{C}_{\mathrm{rat}}(\mathscr{X}) \times \mathscr{X})^{\mathbb{N}}$ is game-random if and only if at least one corresponding prequential path with compatible precise forecasts is?

As explained, Theorem 19.1$_{183}$ in Chapter ⚅$_{179}$ shows that both our martingale-theoretic and frequentist notions of randomness are inherently imprecise, with Theorem 20.1$_{193}$—which only considers martingale-theoretic randomness notions—laying bare the importance of the computability assumption on the forecasting systems for this claim. Consequently, we ask ourselves whether Theorem 20.1$_{193}$ can also allow for our frequentist notions of randomness. If the answer turned out to be negative, this would possibly provide a different and imprecise-probabilistic way to differentiate between randomness notions, which would complement the precise-probabilistic comparison [28, 29, 78, 79, 97]; while we believe that the work in this dissertation doesn't (yet) add anything to this discussion, we believe that our work in Refs. [47, 48, 49] for the binary setting can, because it seems that ML-randomness cannot associate with every binary path a smallest probability interval it's (almost) ML-random for, whereas the other randomness notions do.

Finally, as a continuation of our discussion in Section 21$_{199}$ of Chapter ⚅$_{179}$, we believe that our research can function as a point of departure for developing completely new types of imprecise learning methods. That is, we would like to create and implement novel algorithms that, given a finite sequence of data out of some infinite sequence, can estimate a (most informative/smallest) credal set for which this infinite sequence is random. In this way, we would obtain statistical methods that are reliable in the sense that they do not insist anymore on associating a single precise probability mass function with a(n in)finite sequence. This assumption of precision is for example, as was already mentioned in the Introduction, not defensible in situations where relative frequencies do not converge.

Index

Index of symbols

This index groups the symbols in three categories—Latin, Greek and other symbols—, orders them alphabetically, and provides each one of them with both a short explanation and a pointer to the page where it's introduced.

Latin alphabet

Other symbols

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