Markovian Imprecise Jump Processes: Foundations, Algorithms and Applications

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Corrigendum

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E.1 Issues due to Lemma 5.24

Unfortunately, Lemma 5.24_{234} is incorrect – and I thank Arne Decadt for pointing this out. That said, we can fix this statement in such a way that all – non-intermediary – results in the dissertation still hold, although in some cases we need to slightly alter the statement. Let us start by stating and proving the replacement for Lemma 5.24_{234}. Here and in the remainder, we indicate changes as follows: this is new, this replacesthis is replaced and this is deleted.

Lemma 5.24. *Consider time points s, r in $\mathbb{R}_{\geq 0}$ such that $s < r$ and a grid $v = (t_0, \ldots, t_n)$ over $[s, r]$ with $n \geq 2$. Then*

$$\eta_{(s,r)} + \sum_{k=1}^{n-1} \mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}} \leq \eta_v \leq = \eta_{(s,r)} + 2 \sum_{k=1}^{n-1} \mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}}.$$

Proof. Crucial to our proof is the following observation. Let $w = (s_0, \ldots, s_m)$ be any grid over $[s, r]$, and fix some time point t in $]s_{m-1}, s_m[$. Then for all ω in Ω,

$$\eta_{w \cup (t)}(\omega) = \begin{cases} \eta_w(\omega) + 2 & \text{if } \omega(s_{m-1}) \neq \omega(t) \neq \omega(s_{m\,n}) \text{ and } \omega(s_{m-1}) = \omega(s_m), \\ \eta_w(\omega) + 1 & \text{if } \omega(s_{m-1}) \neq \omega(t) \neq \omega(s_{m\,n}) \text{ and } \omega(s_{m-1}) \neq \omega(s_m), \\ \eta_w(\omega) & \text{otherwise.} \end{cases}$$

Hence,

$$\eta_w + \mathbb{1}_{\{X_{s_{m-1}} \neq X_t \neq X_{s_m}\}} \leq \eta_{w \cup (t)} \leq = \eta_w + 2\mathbb{1}_{\{X_{s_{m-1}} \neq X_t \neq X_{s_m}\}}. \tag{E.1}$$

Fix some ω in Ω, and let $v_0 := (s, r)$. Furthermore, for all k in $\{1, \ldots, n-1\}$, we let $v_k := (t_0, t_1, \ldots, t_k, t_n)$; note that $v_{n-1} = v$. Then it follows from Eq. (E.1) that for all k in $\{1, \ldots, n-1\}$,

$$\eta_{v_{k-1}} + \mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}} \leq \eta_{v_k} \leq = \eta_{v_{k-1}} + 2\mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}}.$$

We repeatedly apply the second inequalitypreceding equality, to yield

$$\eta_v = \eta_{v_{n-1}} \leq = \eta_{v_{n-2}} + 2\mathbb{1}_{\{X_{t_{n-2}} \neq X_{t_{n-1}} \neq X_r\}} = \cdots \leq = \eta_{(s,r)} + 2\sum_{k=1}^{n-1} \mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}};.$$

similarly, by repeated use of the first inequality we find that

$$\eta_v = \eta_{v_{n-1}} \geq \eta_{v_{n-2}} + \mathbb{1}_{\{X_{t_{n-2}} \neq X_{t_{n-1}} \neq X_r\}} \geq \cdots \geq \eta_{(s,r)} + \sum_{k=1}^{n-1} \mathbb{1}_{\{X_{t_{k-1}} \neq X_{t_k} \neq X_r\}}.$$

□

Lemma 5.24$_{234}$ is used in the proof of Lemma 5.23$_{234}$ and Proposition 6.2$_{275}$; we need to change the statement and proof of the former and the proof of the latter accordingly. First let us fix Lemma 5.23$_{234}$.

Lemma 5.23. *Consider time points s and r in $\mathbb{R}_{\geq 0}$ such that $s \leq r$, and two grids v and w over $[s,r]$ such that w refines v – that is, $w \supseteq v$. Then for all ω in Ω, there is some k_ω in $\mathbb{Z}_{\geq 0}$ such that*

$$\eta_w(\omega) = \eta_v(\omega) + 2k_\omega;$$

consequently, $\eta_w \geq \eta_v$.

Proof. The statement is clearly trivial in case $[s,r]$ is a degenerate interval, so we assume without loss of generality that $s < r$. Enumerate the time points in v as $(t_0, \ldots, t_n)$, and note that $n \geq 1$ because $s < r$. For all ℓ in $\{1, \ldots, n\}$, we let w_ℓ be the sequence of time points that consists of those time points in w that belong to $[t_{\ell-1}, t_\ell]$; because w refines v, w_ℓ is a grid over $[t_{\ell-1}, t_\ell]$. It follows from repeated application of Lemma 5.22$_{234}$ that

$$\eta_v = \sum_{\ell=1}^{n} \eta_{(t_{\ell-1}, t_\ell)} \quad \text{and} \quad \eta_w = \sum_{\ell=1}^{n} \eta_{w_\ell}. \tag{E.2}$$

Fix some ω in Ω. Then it follows from Lemma 5.24$_{234}$ that for all ℓ in $\{1, \ldots, n\}$, $\eta_{w_\ell} \geq \eta_{(t_{\ell-1}, t_\ell)}$.there is a non-negative integer $k_{\omega,\ell}$ such that

$$\eta_{w_\ell}(\omega) = \eta_{(t_{\ell-1}, t_\ell)}(\omega) + 2k_{\omega,\ell}.$$

It follows immediately from this and Eq. (E.2) that $\eta_w \geq \eta_v$.

$$\eta_w(\omega) = \sum_{\ell=1}^{n} \eta_{w_\ell}(\omega) = \sum_{\ell=1}^{n} \left(\eta_{(t_{\ell-1}, t_\ell)}(\omega) + 2k_{\omega,\ell}\right) = \eta_v(\omega) + \sum_{\ell=1}^{n} 2k_{\omega,\ell} = \eta_v(\omega) + 2k_\omega,$$

where we let $k_\omega := \sum_{\ell=1}^{n} k_{\omega,\ell}$. □

Lemma 5.23$_{234}$ is used in the proof of Theorem 5.26$_{236}$, Theorem 5.27$_{236}$, Proposition 6.2$_{275}$ and Lemma 6.8$_{279}$. Of these proofs, the only one that uses the (incorrect) equality is that of Lemma 6.8$_{279}$. We will get to this in Section E.1.1$_{\text{IV}}$ further on.

Second, we fix the proof of Proposition 6.2$_{275}$.

Proposition 6.2. *Consider a jump process P that has uniformly bounded rate, with rate bound λ. Fix a state history $\{X_u = x_u\}$ in $\mathscr{H}$, time points s, r in $\mathbb{R}_{\geq 0}$ such that $\max u \leq s < r$ and a grid $v = (t_0, \ldots, t_n)$ over $[s,r]$. Then $\eta_{[s,r]} - \eta_v$ is a non-negative $\mathscr{F}_u$-over variable, and*

$$\begin{aligned} E_P^{\mathrm{D}}(\eta_{[s,r]} - \eta_v \mid X_u = x_u) &= E_P^{\mathrm{D}}(\eta_{[s,r]} \mid X_u = x_u) - E_P(\eta_v \mid X_u = x_u) \\ &\leq \frac{1}{4}\Delta(v)(r-s)\lambda^2. \end{aligned}$$

Proof. For every ℓ in $\mathbb{N}$ and k in $\{1,\dots,n\}$, we let $v_{\ell,k}$ be the grid over $[t_{k-1},t_k]$ that divides this subinterval in 2^ℓ subintervals of equal length. That is, for all ℓ in $\mathbb{N}$ and k in $\{1,\dots,n\}$, we let $v_{\ell,k} := \big(t_{\ell,k,0},\dots,t_{\ell,k,2^\ell}\big)$ where for all i in $\{0,\dots,2^\ell\}$,

$$t_{\ell,k,i} := t_{k-1} + (t_k - t_{k-1})\frac{i}{2^\ell}.$$

Next, for all ℓ in $\mathbb{N}$, we let v_ℓ be the (ordered) union of $v_{\ell,1}$, …, $v_{\ell,n}$; this way, v_ℓ is a grid over $[s,r]$ with $\Delta(v_\ell) = \Delta(v)2^{-\ell}$ such that $v \subseteq v_\ell \subseteq v_{\ell+1}$. Recall from Lemma 5.21$_{234}$ that η_v and, for all ℓ in $\mathbb{N}$, η_{v_ℓ} are $\mathscr{F}_u$-simple variables. Therefore, it follows immediately from Lemma 2.39$_{36}$ that for all ℓ in $\mathbb{N}$, $(\eta_{v_\ell} - \eta_v)$ is an $\mathscr{F}_u$-simple variable. Furthermore, for all ℓ in $\mathbb{N}$, it follows immediately from Lemma 5.23$_{234}$ that $\eta_{v_{\ell+1}} \geq \eta_{v_\ell} \geq \eta_v$ because $v_{\ell+1} \supseteq v_\ell \supseteq v$ by construction. Thus, we have shown that $(\eta_{v_\ell} - \eta_v)_{\ell\in\mathbb{N}}$ is a non-decreasing sequence of non-negative $\mathscr{F}_u$-simple variables; that this sequence converges point-wise to $\eta_{[s,r]} - \eta_v$ follows immediately from Theorem 5.26$_{236}$. Hence, $\eta_{[s,r]} - \eta_v$ is a non-negative $\mathscr{F}_u$-over variable, and it follows from (DE1)$_{225}$, (DE3)$_{335}$ and Theorem 5.10$_{226}$ that

$$E_P^{\mathrm{D}}(\eta_{[s,r]} - \eta_v \,|\, X_u = x_u) = \lim_{\ell\to+\infty} E_P(\eta_{v_\ell} - \eta_v \,|\, X_u = x_u). \tag{E.3}$$

In order to verify the inequality of the statement, we investigate the expectations on the right-hand side of the preceding equality. To this end, we fix any ℓ in $\mathbb{N}$. It follows from (repeated application of) Lemma 5.22$_{234}$ that

$$\eta_{v_\ell} - \eta_v = \sum_{k=1}^n \eta_{v_{\ell,k}} - \sum_{k=1}^n \eta_{(t_{k-1},t_k)} = \sum_{k=1}^n \big(\eta_{v_{\ell,k}} - \eta_{(t_{k-1},t_k)}\big). \tag{E.4}$$

Recall from Lemma 5.24$_{234}$ that, for all k in $\{1,\dots,n\}$,

$$\eta_{v_{\ell,k}} \leq = \eta_{(t_{k-1},t_k)} + 2\sum_{i=1}^{2^\ell-1} \mathbb{I}_{\{X_{t_{\ell,k,i-1}} \neq X_{t_{\ell,k,i}} \neq X_{t_{\ell,k,2^\ell}}\}}.$$

It follows immediately from this inequality and Eq. (E.4) thatWe substitute the preceding equality in Eqn. (E.4), to yield

$$\eta_{v_\ell} - \eta_v \leq = 2\sum_{k=1}^n \sum_{i=1}^{2^\ell-1} \mathbb{I}_{\{X_{t_{\ell,k,i-1}} \neq X_{t_{\ell,k,i}} \neq X_{t_{\ell,k,2^\ell}}\}};$$

from this inequality and (DE6)$_{226}$, it follows that

$$\begin{aligned} E_P(\eta_{v_\ell} - \eta_v \,|\, X_u = x_u) \leq = E_P\left(2\sum_{k=1}^n \sum_{i=1}^{2^\ell-1} \mathbb{I}_{\{X_{t_{\ell,k,i-1}} \neq X_{t_{\ell,k,i}} \neq X_{t_{\ell,k,2^\ell}}\}} \,\middle|\, X_u = x_u\right) \\ = 2\sum_{k=1}^n \sum_{i=1}^{2^\ell-1} P\big(X_{t_{\ell,k,i-1}} \neq X_{t_{\ell,k,i}} \neq X_{t_{\ell,k,2^\ell}} \,\big|\, X_u = x_u\big), \end{aligned}$$

where for the second equality we used Eqn (2.19)$_{36}$. We replace the probabilities on the right-hand side of the equality by the upper bound in Lemma 6.53$_{322}$, to yield

$$\begin{aligned}
E_P(\eta_{v_\ell} - \eta_v \,|\, X_u = x_u) &\le 2 \sum_{k=1}^{n} \sum_{i=1}^{2^\ell - 1} \frac{1}{4} \big(t_{\ell,k,i} - t_{\ell,k,i-1}\big)\big(t_{\ell,k,2^\ell} - t_{\ell,k,i}\big)\lambda^2 \\
&= 2 \sum_{k=1}^{n} \sum_{i=1}^{2^\ell - 1} \frac{1}{4} \frac{t_k - t_{k-1}}{2^\ell} \frac{(t_k - t_{k-1})(2^\ell - i)}{2^\ell} \lambda^2 \\
&= \frac{1}{2}\lambda^2 \sum_{k=1}^{n} (t_k - t_{k-1})^2 \frac{1}{2^\ell} \sum_{i=1}^{2^\ell - 1} \frac{2^\ell - i}{2^\ell},
\end{aligned}$$

where the two equalities follow after some straightforward manipulations. Because

$$\sum_{i=1}^{2^\ell - 1} \frac{2^\ell - i}{2^\ell} = \frac{1}{2^\ell} \sum_{i=1}^{2^\ell - 1} (2^\ell - i) = \frac{1}{2^\ell} \sum_{i=1}^{2^\ell - 1} i = \frac{1}{2^\ell} \frac{(2^\ell - 1)2^\ell}{2} = \frac{2^\ell - 1}{2},$$

it follows from this inequality that

$$\begin{aligned}
E_P(\eta_{v_\ell} - \eta_v \,|\, X_u = x_u) &\le \frac{1}{2}\lambda^2 \sum_{k=1}^{n} (t_k - t_{k-1})^2 \frac{1}{2^\ell} \frac{2^\ell - 1}{2} \\
&= \frac{1}{4}\lambda^2 \frac{2^\ell - 1}{2^\ell} \sum_{k=1}^{n} (t_k - t_{k-1})^2 \\
&\le \frac{1}{4}\Delta(v)(r - s)\lambda^2 \frac{2^\ell - 1}{2^\ell},
\end{aligned}$$

where for the last inequality we used that $(t_k - t_{k-1}) \le \Delta(v)$ for all k in $\{1, \ldots, n\}$ and that $\sum_{k=1}^{n} (t_k - t_{k-1}) = (r - s)$.

It follows from the preceding inequality and Eq. (E.3)$_{\mathrm{III}}$ that

$$E_P^{\mathrm{D}}(\eta_{[s,r]} - \eta_v \,|\, X_u = x_u) \le \lim_{\ell \to +\infty} \frac{1}{4}\Delta(v)(r - s)\lambda^2 \frac{2^\ell - 1}{2^\ell} = \frac{1}{4}\Delta(v)(r - s)\lambda^2,$$

establishing the inequality in the statement. Furthermore, because $\eta_{[s,r]}$ and $\eta_{[s,r]} - \eta_v$ are non-negative $\mathscr{F}_u$-over variables – see Theorem 5.26$_{236}$ for the former – and because η_v is an $\mathscr{F}_u$-simple variable (and hence bounded), it follows from (DE1)$_{225}$, (DE2)$_{225}$, (DE3)$_{225}$ and (DE5)$_{225}$ that

$$\begin{aligned}
E_P^{\mathrm{D}}(\eta_{[s,r]} - \eta_v \,|\, X_u = x_u) &= E_P^{\mathrm{D}}(\eta_{[s,r]} \,|\, X_u = x_u) - E_P^{\mathrm{D}}(\eta_v \,|\, X_u = x_u) \\
&= E_P^{\mathrm{D}}(\eta_{[s,r]} \,|\, X_u = x_u) - E_P(\eta_v \,|\, X_u = x_u),
\end{aligned}$$

and this proves the equality in the statement. □

E.1.1 Fixing Lemma 6.8 and its dependencies

Next, let us fix the statement and proof of Lemma 6.8$_{279}$.

Lemma 6.8. *Consider some time points s and r in $\mathbb{R}_{\geq 0}$ such that $s < r$. Then for any grid v over $[s,r]$,*

$$\frac{1}{2}(\eta_{[s,r]} - \eta_{v}) \geq \mathbb{I}_{A} \qquad \textit{with } A := \left\{\omega \in \Omega \colon \eta_{v}(\omega) < \eta_{[s,r]}(\omega)\right\}.$$

Proof. Let $(v_\ell)_{\ell\in\mathbb{N}}$ be the sequence of grids as constructed in the proof of Proposition 6.2$_{275}$ – see Appendix 6.A$_{321}$. Furthermore, for all ℓ in $\mathbb{N}$, we let

$$A_\ell := \left\{\omega \in \Omega \colon \eta_{v}(\omega) < \eta_{v_\ell}(\omega)\right\}.$$

Fix some ℓ in $\mathbb{N}$. Then because $v_\ell \supseteq v$ by construction, it follows from Lemma 5.23$_{234}$ that $\eta_{v_\ell} \geq \eta_{v}$, and therefore for any ω in Ω, $\eta_{v_\ell}(\omega) - \eta_{v}(\omega)$ is either equal to 0 or greater than or equal to 2. Consequently,

$$\mathbb{I}_{A_\ell} = (\eta_{v_\ell} - \eta_{v}) \wedge 1 \leq \frac{1}{2}(\eta_{v_\ell} - \eta_{v}). \tag{E.5}$$

Recall from the proof of Proposition 6.2$_{275}$ that the sequence $(\eta_{v_\ell} - \eta_{v})_{\ell\in\mathbb{N}}$ converges point-wise to $\eta_{[s,r]} - \eta_{v}$. It follows immediately from this and Eq. (E.5) that $\mathbb{I}_{A_\ell}$ converges point-wise to $\mathbb{I}_{A}$, and that

$$\mathbb{I}_{A} = \operatorname*{p\text{-}w\,lim}_{\ell\to+\infty} \mathbb{I}_{A_\ell} \leq \operatorname*{p\text{-}w\,lim}_{\ell\to+\infty} \frac{1}{2}(\eta_{v_\ell} - \eta_{v}) = \frac{1}{2}(\eta_{[s,r]} - \eta_{v}),$$

as required. □

Lemma 6.8$_{279}$ is used in the proof of Lemma 6.11$_{282}$ and Lemma 6.20$_{292}$. Changing the proof of these results is trivial, so here we will only give the changed statement.

Lemma 6.11. *Consider subsets S, G of $\mathscr{X}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that $s \leq r$. Then for any grid v over $[s,r]$,*

$$\left|h^{S,G}_{[s,r]} - h^{S,G}_{v}\right| \leq \frac{1}{2}(\eta_{[s,r]} - \eta_{v}).$$

Lemma 6.20. *Consider a subset G of $\mathscr{X}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that $s \leq r$. Then for any grid v over $[s,r]$,*

$$\left|\tau^{G}_{[s,r]} - \tau^{G}_{v}\right| \leq \Delta(v) + \frac{1}{2}(r-s)(\eta_{[s,r]} - \eta_{v}).$$

Changes due to the corrected statement of Lemma 6.11

In turn, Lemma 6.11$_{282}$ is used in two follow-up results. It is used in the proof of Lemma 6.12$_{283}$, and the trivial required change in the proof does not change the statement of this intermediary result. It is also used in the proof of Proposition 6.13$_{283}$, and the correction to Lemma 6.11$_{282}$ leads to the following corrected statement of Proposition 6.13$_{283}$.

Proposition 6.13. *Consider a non-empty and bounded set $\mathcal{Q}$ of rate operators, and an imprecise jump process $\mathscr{P}$ such that $\mathscr{P} \subseteq \mathbb{P}_{\mathcal{Q}}$. Fix some subsets S, G of $\mathscr{X}$, a state history $\{X_u = x_u\}$ in $\mathscr{H}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that* $\max u \leq s \leq r$. *Then for any grid v over $[s, r]$,*

$$\left| \underline{P}^{\mathrm{D}}_{\mathscr{P}}\big(H^{S,G}_{[s,r]} \,\big|\, X_u = x_u\big) - \underline{P}_{\mathscr{P}}\big(H^{S,G}_{v} \,\big|\, X_u = x_u\big) \right| \leq \frac{1}{4}\,\frac{1}{8} \Delta(v)(r-s)\|\mathcal{Q}\|^2_{\mathrm{op}}$$

and

$$\left| \overline{P}^{\mathrm{D}}_{\mathscr{P}}\big(H^{S,G}_{[s,r]} \,\big|\, X_u = x_u\big) - \overline{P}_{\mathscr{P}}\big(H^{S,G}_{v} \,\big|\, X_u = x_u\big) \right| \leq \frac{1}{4}\,\frac{1}{8} \Delta(v)(r-s)\|\mathcal{Q}\|^2_{\mathrm{op}}.$$

In particular, this holds for $\mathscr{P} = \mathbb{P}^{\mathrm{HM}}_{\mathscr{M},\mathcal{Q}}$, $\mathscr{P} = \mathbb{P}^{\mathrm{M}}_{\mathscr{M},\mathcal{Q}}$ and $\mathscr{P} = \mathbb{P}_{\mathscr{M},\mathcal{Q}}$, with $\mathscr{M}$ a non-empty set of initial mass functions.

Proposition 6.13$_{283}$ is subsequently used in the proof of Theorem 6.46$_{314}$, and this leads to the following change in its statement.

Theorem 6.46. *Consider a non-empty set $\mathscr{M}$ of initial mass functions, a non-empty and bounded set $\mathcal{Q}$ of rate operators that has separately specified rows and an imprecise jump process $\mathscr{P}$ such that $\mathbb{P}^{\mathrm{M}}_{\mathscr{M},\mathcal{Q}} \subseteq \mathscr{P} \subseteq \mathbb{P}_{\mathscr{M},\mathcal{Q}}$. Fix subsets S, G of $\mathscr{X}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that $s < r$. For all n in $\mathbb{N}$, we let $\Delta_n := {}^{(r-s)}/_{n}$ and let $\tilde{f}_{n,0}$ be the gamble on $\mathscr{X}$ that is defined by the initial condition $\tilde{f}_{n,n} := \mathbb{I}_G$ and, for all k in $\{0, \ldots n-1\}$, by the recursive relation*

$$\tilde{f}_{n,k} := \mathbb{I}_G + \mathbb{I}_{S\setminus G}\big(I + \Delta_n \underline{Q}_{\mathcal{Q}}\big)\tilde{f}_{n,k+1}. \tag{E.6}$$

Then for all x in $\mathscr{X}$ and n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}_{\mathcal{Q}}\|_{\mathrm{op}} \leq 2n$,

$$\left| \underline{P}^{\mathrm{D}}_{\mathscr{P}}\big(H^{S,G}_{[s,r]} \,\big|\, X_s = x\big) - \tilde{f}_{n,0}(x) \right| \leq \frac{1}{2}\,\frac{3}{8} \frac{(r-s)^2}{n} \|\underline{Q}_{\mathcal{Q}}\|^2_{\mathrm{op}},$$

and therefore

$$\underline{P}^{\mathrm{D}}_{\mathscr{P}}\big(H^{S,G}_{[s,r]} \,\big|\, X_s = x\big) = \lim_{n\to+\infty} \tilde{f}_{n,0}(x).$$

The same holds for $\overline{P}^{\mathrm{D}}_{\mathscr{P}}$ if in Eq. (E.6) *we replace $\underline{Q}_{\mathcal{Q}}$ by $\overline{Q}_{\mathcal{Q}}$.*

Proof. Let $\underline{Q} := \underline{Q}_{\mathcal{Q}}$. Because every rate operator Q in $\mathcal{Q}$ dominates $\underline{Q}$, it follows immediately from (LR7)$_{111}$ that $\|\mathcal{Q}\|_{\mathrm{op}} \leq \|\underline{Q}\|_{\mathrm{op}}$.

Fix some n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}\|_{\mathrm{op}} \leq 2n$, and let v be the grid over $[s,r]$ with n subintervals of length Δ_n – that is, we let $v := (s, s+\Delta_n, \ldots, s+n\Delta_n)$. Then by Proposition 6.13$_{283}$,

$$\begin{aligned} \left| \underline{P}^{\mathrm{D}}_{\mathscr{P}}\big(H^{S,G}_{[s,r]} \,\big|\, X_s = x\big) - \underline{E}_{\mathscr{P}}\big(h^{S,G}_{v} \,\big|\, X_s = x\big) \right| &\leq \frac{1}{4}\,\frac{1}{8} \Delta(v)(r-s)\|\mathcal{Q}\|^2_{\mathrm{op}} \\ &\leq \frac{1}{4}\,\frac{1}{8} \frac{(r-s)^2}{n} \|\underline{Q}\|^2_{\mathrm{op}}, \end{aligned} \tag{E.7}$$

where for the second inequality we used that $\Delta(v) = {}^{(r-s)}/_{n}$ and that $\|\mathcal{Q}\|_{\mathrm{op}} \leq \|\underline{Q}\|_{\mathrm{op}}$.

Recall from Lemma 6.10$_{281}$ that $h_v^{S,G}$ has a sum-product representation over v:

$$h_v^{S,G} = \sum_{k=0}^{n} g_k(X_{s+k\Delta_n}) \prod_{\ell=0}^{k-1} h_\ell(X_{s+\ell\Delta_n}),$$

with $g_k \coloneqq \mathbb{I}_G$ for all k in $\{0,\dots,n\}$ and $h_\ell \coloneqq \mathbb{I}_{S\setminus G}$ for all ℓ in $\{0,\dots,n-1\}$. For this reason, it follows from Theorem 4.9$_{166}$ that

$$\underline{E}_{\mathscr{P}}\big(h_v^{S,G} \,\big|\, X_s = x\big) = f_{n,0}(x), \tag{E.8}$$

where $f_{n,0}$ is the gamble on $\mathscr{X}$ that is defined by the initial condition $f_{n,n} \coloneqq g_n = \mathbb{I}_G$ and, for all k in $\{0,\dots,n-1\}$, by the recursive relation

$$f_{n,k} \coloneqq \mathbb{I}_G + \mathbb{I}_{S\setminus G} e^{\Delta_n \underline{Q}} f_{n,k+1}.$$

Furthermore, it follows from Lemma 6.43$_{311}$ that

$$\big|f_{n,0}(x) - \tilde{f}_{n,0}(x)\big| \leq \big\|f_{n,0} - \tilde{f}_{n,0}\big\|_{\mathrm{op}} \leq \frac{1}{2}\|\underline{Q}\|_{\mathrm{op}}^2 \frac{(r-s)^2}{n^2} \sum_{k=1}^{n} \|\tilde{f}_{n,k}\|_{\mathrm{c}}. \tag{E.9}$$

We now claim that for all k in $\{1,\dots,n\}$, $\min \tilde{f}_{n,k} \geq 0$ and $\max \tilde{f}_{n,k} \leq 1$, and therefore $\|\tilde{f}_{n,k}\|_{\mathrm{c}} \leq \frac{1}{2}$. Our proof will be one by induction. For the base case $k = n$, this is obvious because $\tilde{f}_{n,n} = \mathbb{I}_G$ by definition. For the inductive step, we fix some k in $\{1,\dots,n-1\}$ and assume that $\min \tilde{f}_{n,k+1} \geq 0$ and $\max \tilde{f}_{n,k+1} \leq 1$. Because $\Delta_n \|\underline{Q}\|_{\mathrm{op}} \leq 2$, $(I + \Delta_n \underline{Q})$ is a lower transition operator due to Lemma 3.72$_{112}$. Hence, it follows from the induction hypothesis and (LT4)$_{108}$ that

$$0 \leq \min \tilde{f}_{n,k+1} \leq (I + \Delta_n \underline{Q}) \tilde{f}_{n,k+1} \leq \max \tilde{f}_{n,k+1} \leq 1.$$

For this reason, and because $\tilde{f}_{n,k} = \mathbb{I}_G + \mathbb{I}_{S\setminus G}(I + \Delta_n \underline{Q})\tilde{f}_{n,k+1}$ by definition, we see that $\min \tilde{f}_{n,k} \geq 0$ and $\max \tilde{f}_{n,k} \leq 1$, as required.

Because $\|\tilde{f}_{n,k}\|_{\mathrm{c}} \leq 1/2$ for all k in $\{1,\dots,n\}$, it follows from Eq. (E.9) that

$$\big|f_{n,0}(x) - \tilde{f}_{n,0}(x)\big| \leq \frac{1}{4}\|\underline{Q}\|_{\mathrm{op}}^2 \frac{(r-s)^2}{n}. \tag{E.10}$$

Finally, it follows from Eqs. (E.7)$_{\mathrm{VI}}$, (E.8) and (E.10) and the triangle inequality that

$$\begin{aligned} \Big|\underline{P}_{\mathscr{P}}^{\mathrm{D}}\big(H_{[s,r]}^{S,G} \,\big|\, X_s = x\big) - \tilde{f}_{n,0}(x_s)\Big| &\leq \frac{1}{4}\,\frac{1}{8}\,\frac{(r-s)^2}{n}\|\underline{Q}\|_{\mathrm{op}}^2 + \frac{1}{4}\|\underline{Q}\|_{\mathrm{op}}^2 \frac{(r-s)^2}{n} \\ &= \frac{1}{2}\,\frac{3}{8}\,\frac{(r-s)^2}{n}\|\underline{Q}\|_{\mathrm{op}}^2. \end{aligned} \tag{E.11}$$

Because Eq. (E.11) holds for all n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}\|_{\mathrm{op}} \leq 2n$ and because the right-hand side of the inequality vanishes as n recedes to $+\infty$, we have proven the limit statement for $\underline{P}_{\mathscr{P}}^{\mathrm{D}}$.

The statement for $\overline{P}_{\mathscr{P}}^{\mathrm{D}}$ essentially follows from conjugacy. More precisely, the argument is almost exactly the same as the argument in the first part of this proof. We do need a couple of extra steps though. First, we use that

$$\overline{E}_{\mathscr{P}}\big(h_v^{S,G} \,\big|\, X_s = x\big) = -\underline{E}_{\mathscr{P}}\big(-h_v^{S,G} \,\big|\, X_s = x\big),$$

for any grid v over $[s,r]$. Second, we use that $-h_v^{S,G}$ also has a sum-product representation over v: by Lemma 4.7$_{165}$,

$$-h_v^{S,G} = \sum_{k=0}^{n} [-\mathbb{I}_G]\big(X_{s+k\Delta_n}\big) \prod_{\ell=0}^{k-1} \mathbb{I}_{S\setminus G}\big(X_{s+\ell\Delta_n}\big)$$

Third, we again use Lemmas 4.9$_{166}$ and 6.43$_{311}$, but this time to approximate $\underline{E}_{\mathscr{P}}\big(-h_v^{S,G} \,\big|\, X_s = x\big)$ instead of $\underline{E}_{\mathscr{P}}\big(h_v^{S,G} \,\big|\, X_s = x\big)$. This way, we find that

$$\big|\underline{E}_{\mathscr{P}}(-h_v^{S,G} \,|\, X_s = x) - \check{f}_{n,0}(x)\big| \leq \frac{1}{2}\|\underline{Q}\|_{\mathrm{op}}^2 \frac{(r-s)^2}{n^2} \sum_{k=1}^{n} \|\check{f}_{n,k}\|_{\mathrm{c}},$$

where $\check{f}_{n,0}(x)$ is recursively defined by the initial condition $\check{f}_{n,n} := -\mathbb{I}_G$ and, for all k in $\{0,\dots,n-1\}$, by the recursive relation

$$\check{f}_{n,k} := -\mathbb{I}_G + \mathbb{I}_{S\setminus G}\big(I + \Delta_n \underline{Q}\big)\check{f}_{n,k+1}.$$

Obviously, $\tilde{f}_{n,n} = -\check{f}_{n,n}$. Furthermore, it is easy to verify that, for all k in $\{0,\dots,n-1\}$, $\|\check{f}_{n,k}\|_{\mathrm{c}} \leq 1/2$ and that, by conjugacy,

$$\tilde{f}_{n,k} = \mathbb{I}_G + \mathbb{I}_{S\setminus G}\big(I + \Delta_n \overline{Q}\big)\tilde{f}_{n,k} = -\Big(-\mathbb{I}_G + \mathbb{I}_{S\setminus G}\big(I + \Delta_n \underline{Q}\big)(-\tilde{f}_{n,k+1})\Big) = -\check{f}_{n,k}.$$

Therefore, and because $\overline{E}_{\mathscr{P}}\big(h_v^{S,G} \,\big|\, X_s = x\big) = -\underline{E}_{\mathscr{P}}\big(-h_v^{S,G} \,\big|\, X_s = x\big)$,

$$\big|\overline{E}_{\mathscr{P}}(h_v^{S,G} \,|\, X_s = x) - \tilde{f}_{n,0}(x)\big| \leq \frac{1}{4}\|\underline{Q}\|_{\mathrm{op}}^2 \frac{(r-s)^2}{n}.$$

The remainder of the proof is again similar to the first part of the proof. □

Changes due to the corrected statement of Lemma 6.20

Lemma 6.20$_{292}$ is also used in two follow-up results. It is used in the proof of Lemma 6.21$_{293}$, and the trivial required change in the proof does not change the statement of this intermediary result. It is also used in the proof of Proposition 6.22$_{293}$, and the correction to Lemma 6.20$_{292}$ induces the following obvious change to Proposition 6.22$_{293}$.

Proposition 6.22. *Consider a non-empty and bounded set $\mathscr{Q}$ of rate operators, and an imprecise jump process $\mathscr{P}$ such that $\mathscr{P} \subseteq \mathbb{P}_{\mathscr{Q}}$, and let $\lambda := \|\mathscr{Q}\|_{\mathrm{op}}$. Fix some subset G of $\mathscr{X}$, a state history $\{X_u = x_u\}$ in $\mathscr{H}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that $\max u \leq s \leq r$. Then for any grid v over $[s,r]$,*

$$\big|\underline{E}_{\mathscr{P}}^{\mathrm{D}}\big(\tau_{[s,r]}^{G} \,\big|\, X_u = x_u\big) - \underline{E}_{\mathscr{P}}\big(\tau_v^{G} \,\big|\, X_u = x_u\big)\big| \leq \Delta(v) + \frac{1}{4}\frac{1}{8}\Delta(v)(r-s)^2\|\mathscr{Q}\|_{\mathrm{op}}^2$$

and

$$\big|\overline{E}_{\mathscr{P}}^{\mathrm{D}}\big(\tau_{[s,r]}^{G} \,\big|\, X_u = x_u\big) - \overline{E}_{\mathscr{P}}\big(\tau_v^{G} \,\big|\, X_u = x_u\big)\big| \leq \Delta(v) + \frac{1}{4}\frac{1}{8}\Delta(v)(r-s)^2\|\mathscr{Q}\|_{\mathrm{op}}^2.$$

In particular, this holds for $\mathscr{P} = \mathbb{P}_{\mathscr{M},\mathscr{Q}}^{\mathrm{HM}}$, $\mathscr{P} = \mathbb{P}_{\mathscr{M},\mathscr{Q}}^{\mathrm{M}}$ and $\mathscr{P} = \mathbb{P}_{\mathscr{M},\mathscr{Q}}$, with $\mathscr{M}$ a non-empty set of initial mass functions.

Proposition 6.22$_{293}$ is subsequently used in the proof of Theorem 6.48$_{316}$, and this leads to the following change in its statement.

Theorem 6.48. *Consider a non-empty set $\mathscr{M}$ of initial mass functions, a non-empty and bounded set $\mathscr{Q}$ of rate operators that has separately specified rows and an imprecise jump process $\mathscr{P}$ such that $\mathbb{P}^{\mathrm{M}}_{\mathscr{M},\mathscr{Q}} \subseteq \mathscr{P} \subseteq \mathbb{P}_{\mathscr{M},\mathscr{Q}}$. Fix some subset G of $\mathscr{X}$ and time points s, r in $\mathbb{R}_{\geq 0}$ such that $s < r$. For all n in $\mathbb{N}$, we let $\Delta_n := {}^{(r-s)}/_{n}$ and let $\tilde{f}_{n,0}$ be the gamble on $\mathscr{X}$ that is defined by the initial condition $\tilde{f}_{n,n} := \Delta_n$ and, for all k in $\{0, \ldots n-1\}$, by the recursive relation*

$$\tilde{f}_{n,k} := \begin{cases} \Delta_n + \mathbb{I}_{G^c}\big(I + \Delta_n \underline{Q}_{\mathscr{Q}}\big)\tilde{f}_{n,k+1} & \text{if } k \geq 1, \\ s + \mathbb{I}_{G^c}\big(I + \Delta_n \underline{Q}_{\mathscr{Q}}\big)\tilde{f}_{n,k+1} & \text{if } k = 0. \end{cases} \tag{E.12}$$

Then for all x in $\mathscr{X}$ and all n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}_{\mathscr{Q}}\|_{\mathrm{op}} \leq 2$,

$$\Big|\underline{E}^{\mathrm{D}}_{\mathscr{P}}\big(\tau^{G}_{[s,r]} \,\big|\, X_s = x\big) - \tilde{f}_{n,0}(x)\Big| \leq \frac{r-s}{n} + \frac{1}{8}\frac{(r-s)^3}{n}\frac{{\color{orange}3}{\color{lightgray}2}n+1}{n}\|\underline{Q}_{\mathscr{Q}}\|^2_{\mathrm{op}},$$

and therefore

$$\underline{E}^{\mathrm{D}}_{\mathscr{P}}\big(\tau^{G}_{[s,r]} \,\big|\, X_s = x\big) = \lim_{n \to +\infty} \tilde{f}_{n,0}(x).$$

The same holds for $\overline{E}^{\mathrm{D}}_{\mathscr{P}}$ if in Eq. (E.12) we replace $\underline{Q}_{\mathscr{Q}}$ by $\overline{Q}_{\mathscr{Q}}$.

Proof. Let $\underline{Q} := \underline{Q}_{\mathscr{Q}}$. Because every rate operator Q in $\mathscr{Q}$ dominates $\underline{Q}$, it follows immediately from (LR7)$_{111}$ that $\|\mathscr{Q}\|_{\mathrm{op}} \leq \|\underline{Q}\|_{\mathrm{op}}$.

Fix some n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}\|_{\mathrm{op}} \leq 2n$, and let v be the grid over $[s, r]$ with n subintervals of length Δ_n – that is, we let $v := (s, s+\Delta_n, \ldots, s+n\Delta_n)$. Then by Proposition 6.22$_{293}$,

$$\begin{aligned} \Big|\underline{E}^{\mathrm{D}}_{\mathscr{P}}\big(\tau^{G}_{[s,r]} \,\big|\, X_s = x\big) - \underline{E}_{\mathscr{P}}\big(\tau^{G}_{v} \,\big|\, X_s = x\big)\Big| &\leq \Delta(v) + {\color{orange}\frac{1}{4}}{\color{lightgray}\frac{1}{8}}\Delta(v)(r-s)^2\|\mathscr{Q}\|^2_{\mathrm{op}} \\ &\leq \frac{r-s}{n} + {\color{orange}\frac{1}{4}}{\color{lightgray}\frac{1}{8}}\frac{(r-s)^3}{n}\|\underline{Q}\|^2_{\mathrm{op}}, \end{aligned} \tag{E.13}$$

where for the second inequality we used that $\Delta(v) = {}^{(r-s)}/_{n}$ and that $\|\mathscr{Q}\|_{\mathrm{op}} \leq \|\underline{Q}\|_{\mathrm{op}}$.

Recall from Lemma 6.18$_{290}$ that τ^G_v has a sum-product representation over v:

$$\tau^G_v = \sum_{k=0}^{n} g_k(X_{s+k\Delta}) \prod_{\ell=0}^{k-1} h_\ell(X_{s+\ell\Delta}),$$

with $g_0 := s$ and, for all k in $\{1, \ldots, n\}$, $g_k := \Delta_n$ and $h_{k-1} := \mathbb{I}_{G^c}$. For this reason, it follows from Theorem 4.9$_{166}$ that

$$\underline{E}_{\mathscr{P}}\big(\tau^G_v \,\big|\, X_s = x\big) = f_{n,0}(x), \tag{E.14}$$

where $f_{n,0}$ is the gamble on $\mathscr{X}$ that is defined by the initial condition $f_{n,n} := g_n$ and, for all k in $\{0, \ldots, n-1\}$, by the recursive relation

$$f_{n,k} := g_k + h_{k-1}e^{\Delta_n \underline{Q}} f_{n,k+1}.$$

Furthermore, it follows from Lemma 6.43_{311} that

$$\left|f_{n,0}(x)-\tilde{f}_{n,0}(x)\right| \leq \left\|f_{n,0}-\tilde{f}_{n,0}\right\|_{\mathrm{op}} \leq \frac{1}{2}\|\underline{Q}\|_{\mathrm{op}}^{2}\frac{(r-s)^{2}}{n^{2}}\sum_{k=1}^{n}\|\tilde{f}_{n,k}\|_{\mathrm{c}}. \tag{E.15}$$

We now claim that for all k in $\{1,\dots,n\}$, $\min\tilde{f}_{n,k}\geq 0$ and $\max\tilde{f}_{n,k}\leq(n-k+1)\Delta_n$, and therefore $\|\tilde{f}_{n,k}\|_{\mathrm{c}}\leq\frac{(n-k+1)\Delta_n}{2}$. Our proof will be one by induction. For the base case $k=n$, this is obvious because $\tilde{f}_{n,n}=\Delta_n$ by definition. For the inductive step, we fix some k in $\{1,\dots,n-1\}$ and assume that $\min\tilde{f}_{n,k+1}\geq 0$ and $\max\tilde{f}_{n,k+1}\leq(n-k)\Delta_n$. Because $\Delta_n\|\underline{Q}\|_{\mathrm{op}}\leq 2$, $(I+\Delta_n\underline{Q})$ is a lower transition operator due to Lemma 3.72_{112}. Hence, it follows from the induction hypothesis and $(\mathrm{LT4})_{108}$ that

$$0\leq\min\tilde{f}_{n,k+1}\leq(I+\Delta_n\underline{Q})\tilde{f}_{n,k+1}\leq\max\tilde{f}_{n,k+1}\leq(n-k)\Delta_n.$$

For this reason, and because $\tilde{f}_{n,k}=\Delta_n+\mathbb{I}_{G^{\mathrm{c}}}(I+\Delta_n\underline{Q})\tilde{f}_{n,k+1}$ by definition, we see that $\min\tilde{f}_{n,k}\geq 0$ and $\max\tilde{f}_{n,k}\leq(n-k+1)\Delta_n$, as required.

Because $\|\tilde{f}_{n,k}\|_{\mathrm{c}}\leq\frac{(n-k+1)\Delta_n}{2}$ for all k in $\{1,\dots,n\}$, it follows from Eq. (E.15) that

$$\begin{aligned}\left|f_{n,0}(x)-\tilde{f}_{n,0}(x)\right| &\leq \frac{1}{4}\|\underline{Q}\|_{\mathrm{op}}^{2}\frac{(r-s)^{3}}{n^{3}}\sum_{k=1}^{n}(n-k+1)\\ &= \frac{1}{4}\|\underline{Q}\|_{\mathrm{op}}^{2}\frac{(r-s)^{3}}{n^{3}}\frac{n(n+1)}{2}\\ &= \frac{1}{8}\|\underline{Q}\|_{\mathrm{op}}^{2}\frac{(r-s)^{3}}{n}\frac{n+1}{n},\end{aligned} \tag{E.16}$$

where for the inequality we also used that $\Delta_n={(r-s)}/{n}$. Finally, it follows from Eqs. $(\mathrm{E.13})_{\mathrm{IX}}$, $(\mathrm{E.14})_{\mathrm{IX}}$ and (E.16) and the triangle inequality that

$$\begin{aligned}\left|\underline{E}_{\mathscr{P}}^{\mathrm{D}}(\tau_{[s,r]}^{G}\,|\,X_s=x)-\tilde{f}_{n,0}(x)\right| &\leq \frac{r-s}{n}+\frac{1}{4}\frac{1}{8}\frac{(r-s)^{3}}{n}\|\underline{Q}\|_{\mathrm{op}}^{2}+\frac{1}{8}\frac{(r-s)^{3}}{n}\frac{n+1}{n}\|\underline{Q}\|_{\mathrm{op}}^{2}\\ &= \frac{r-s}{n}+\frac{1}{8}\frac{(r-s)^{3}}{n}\,32\,\frac{n+1}{n}\|\underline{Q}\|_{\mathrm{op}}^{2}.\end{aligned} \tag{E.17}$$

Because Eq. (E.17) holds for all n in $\mathbb{N}$ such that $(r-s)\|\underline{Q}\|_{\mathrm{op}}\leq 2n$ and because the right-hand side of the inequality vanishes as n recedes to $+\infty$, we have proven the limit statement for $\underline{E}_{\mathscr{P}}^{\mathrm{D}}$.

The statement for $\overline{E}_{\mathscr{P}}^{\mathrm{D}}$ essentially follows from conjugacy; as in the proof of Theorem 6.46_{314}, we need some obvious extra/different steps. □