Monte Carlo Estimation for Imprecise Probabilities: Basic Properties

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Problem Statement

We estimate lower expectations $\underline{\mathrm{E}}^{\mathscr{P}}(f) = \inf_{t\in T} \mathrm{E}^{P_t}(f)$ for some set of probability measures $\mathscr{P} = \{P_t : t \in T\}$ and do this using the technique of Monte Carlo. In the classical case, Monte Carlo says that for large sample sizes n we have $\mathrm{E}^P(f) \approx \sum_{k=1}^n f(X_k^P) = \hat{\mathrm{E}}^P(f)$. We consider imprecise extensions of such estimators. In this poster we will use an example where $f(x) = \cos(2x)$, $T = [-2,2]$, $P_t \sim N(t,1)$. For this example it can be calculated exactly that $\underline{\mathrm{E}}^{\mathscr{P}}(f) = -\mathrm{e}^{-2} = \mathrm{E}^{P_{\frac{\pi}{2}}}(f) = \mathrm{E}^{P_{-\frac{\pi}{2}}}(f)$.

Naive Method

A straightforward attempt at an estimator is to take a representative finite subset T' of T and do independent classical Monte Carlo simulations for every one of them and take their minimum.

$$\hat{\underline{\mathrm{E}}}^{\mathscr{P}}(f) = \min_{t\in T'} \hat{\mathrm{E}}^{P_t}(f)$$

Let m be the number of elements in T'. For $m > 1$, we can choose T' equidistant from -2 to 2.

Transform Method

We consider estimators of the form

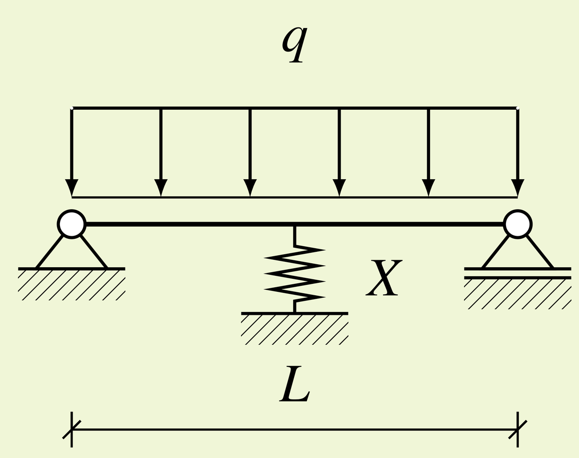
$$\hat{\underline{\mathrm{E}}}_n^{\mathscr{P}}(f) = \inf_{t\in T} \frac{1}{n}\sum_{k=1}^n f_t(X_k^P),$$

for chosen functions f_t for which $\mathrm{E}^P(f_t) = \mathrm{E}^{P_t}(f)$.

We give two examples of such functions:

1. Inverse transform sampling: P is the uniform distribution on $(0,1)$. Consider the quantile function $F_{P_t}^{\dagger}$ (the pseudo-inverse of the cdf). Now if $f_t = f \circ F_{P_t}^{\dagger}$ on $(0,1)$, then we have the desired property.
2. Importance sampling: Suppose the probability measures P and P_t have densities p and p_t respectively for every $t \in T$, and for every $t \in T$: $\operatorname{supp} p \supseteq \operatorname{supp} p_t$. If $f_t = f \cdot \frac{p_t}{p}$ on $\operatorname{supp} p$, then we have the desired property.

Example from Literature



Thomas Fetz and Michael Oberguggenberger used importance sampling to estimate the upper failure probability of a beam on a spring of unknown stiffness X. They assume X distributed normally with with mean and standard deviation (μ,σ) in $[\underline{\mu},\overline{\mu}] \times [\underline{\sigma},\overline{\sigma}]$.

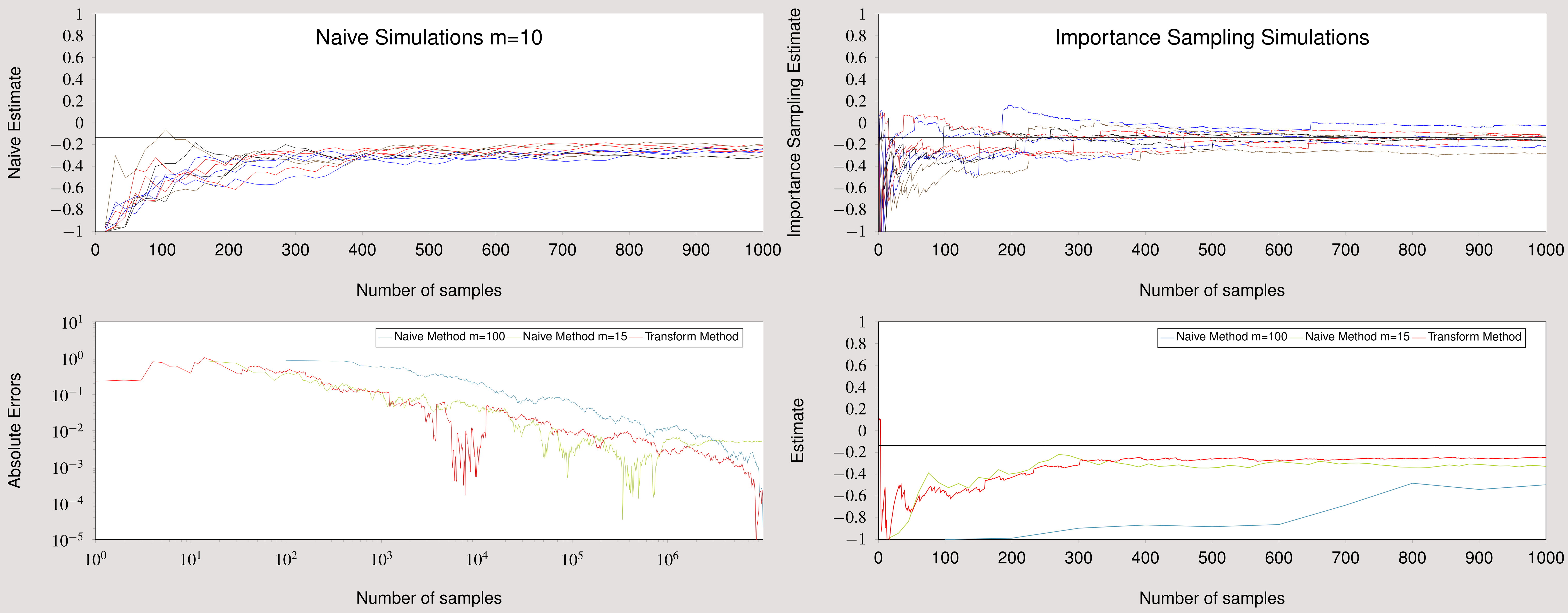
The objective can be rephrased in our context of lower expectations as

$$\overline{P}(g(X) \leqslant 0) = 1 - \underline{\mathrm{E}}^{\mathscr{P}}(\mathbb{I}_{g(X)>0}).$$

Our method proves that their estimator is consistent.

Example Simulations

For the importance sampling $P \sim N(0,1)$.



Bias

1. The bias is negative and the absolute bias decreases with n:

$$\mathrm{E}\Big(\hat{\underline{\mathrm{E}}}_{n-1}^{\mathscr{P}}(f)\Big) \leqslant \mathrm{E}\Big(\hat{\underline{\mathrm{E}}}_{n}^{\mathscr{P}}(f)\Big) \leqslant \underline{\mathrm{E}}^{\mathscr{P}}(f).$$

2. Observation: the absolute bias increases with the size of T.

In the paper we prove slightly more general results for estimators that are not measurable.

Consistency

Intuitively, consistency can be guaranteed when T is 'small' enough and if f_t is 'smooth' enough. In the paper we have conditions for the general setting, but here we will – for brevity and simplicity – only discuss the case of importance sampling. In the following we assume that $\mathrm{E}^P(F) < +\infty$ and for some theorems p_t are required to have an extension to values of t outside of T. For the exact theorems, we refer to the paper.

T is finite	$\mathbb{R}^n \supset T$ compact $p_t(x)$ cont. diff. in (x,t) $\mathrm{E}^P(\sup_{t\in T}\lvert p_t\rvert) < +\infty$	$\mathbb{R}^n \supset T$ bounded $\lVert\nabla_t p_t(x)\rVert < F(x)$	$\mathbb{R}^n \supset T$ has a finite ε-cover for all $\varepsilon > 0$ $\lvert p_t(x) - p_s(x)\rvert \leqslant d(s,t)F(x)$

← easier but more restrictive — more general but more complex →

Example of No Consistency

We will look at an importance sampling estimator with central density $p = \frac{1}{2}\mathbb{I}_{[0,1]}$ and a countable set of densities

$$\left\{\sum_{\ell=1}^{2k} a_\ell \mathbb{I}_{\left[\frac{\ell-1}{k},\frac{\ell}{k}\right]} : k \in \mathbb{N} \text{ and } (a_1,\dots,a_{2k}) \text{ is a binary sequence with the same number of zeros and ones}\right\}.$$

For a sample of size n it is always possible to choose a binary sequence of at most size $2n$, such that the corresponding density is zero on the sampled values.