Non-associative Frobenius algebras for simple algebraic groups

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July 19, 2022

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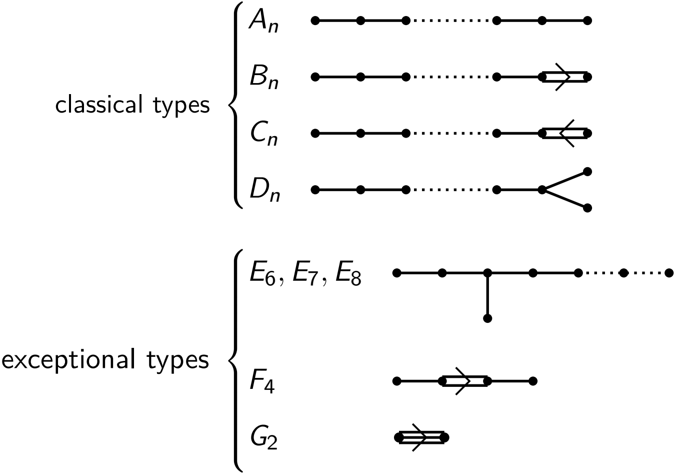
A quick reminder on linear algebraic groups

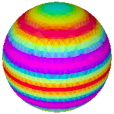
Linear algebraic groups $\approx$ matrix groups defined by polynomial equations over a field k.

$\mathrm{char}(k) \gg 0$ (i.p., $\mathrm{char}(k) \neq 2, 3$).

A quick reminder on linear algebraic groups



SO(n): stabiliser of non-degenerate symmetric bilinear form



Analogous:

Type G_2: Automorphism group of octonion algebra $(\mathbb{O}, \cdot, \langle \cdot, \cdot \rangle)$

Type F_4: Automorphism group of Albert algebra $(\mathcal{H}_3(\mathbb{O}), \bullet, \langle \cdot, \cdot \rangle)$

Motivation

Proposition ([GG15]).
On the second smallest representation V of type E_8 (with dimension 3785) there exists a unique equivariant algebra product $\diamond$ together with a unique invariant Frobenius form τ, i.e.

$$\tau(a \diamond b, c) = \tau(a, b \diamond c) \quad \forall a, b, c \in V.$$

A construction

V: 2nd smallest representation of E_8

Construction starts from the associated Lie algebra $\mathfrak{g}$.

$$A := k \oplus V \hookrightarrow \mathrm{S}^2\mathfrak{g} = \mathfrak{g}^{\otimes 2}/(a \otimes b - b \otimes a \mid \forall a, b \in \mathfrak{g})$$

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Two different constructions

De Medts-Van Couwenberghe [DMVC21]	Chayet-Garibaldi [CG21]

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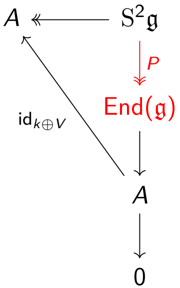
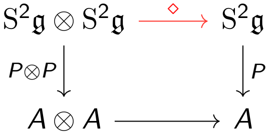
Two different constructions

De Medts-Van Couwenberghe [DMVC21]	$\rightarrow$ Chayet-Garibaldi [CG21]

The Chayet-Garibaldi construction





$\rightarrow$ Explicit formulas for $P, \diamond$ in terms of Killing form and Lie bracket

$\rightarrow$ Frobenius form τ is just the trace map!

Theorem ([CG21]).
Given an (almost) simple algebraic group G, one can construct algebra $A(G)$ s.t.

- $A(G)$ is simple,
- $\tau(a \diamond b, c) = \tau(a, b \diamond c)$ for all $a, b, c \in A(G)$,
- For types F_4 and E_8, $\mathrm{Aut}(A(G))$ is adjoint group of type F_4 and E_8 (resp.)

Issue: construction starts from Lie algebra

For types G_2, F_4, E_6, E_7, Chayet-Garibaldi described embedding $\sigma\colon A(G) \hookrightarrow \mathrm{End}(W)$, with W the smallest representation.

$$W = \begin{cases} \text{traceless octonions, for type } G_2, \\ \text{traceless albert elements, for type } F_4, \end{cases}$$

$$\begin{array}{ccc} A(G) & \overset{\sigma}{\hookrightarrow} & \mathrm{End}(W) \\ & & \cup \\ & & \sigma(A(G)) \end{array}$$

$$\begin{array}{ccc} A(G) \otimes A(G) & \xrightarrow{\ \diamond\ } & A(G) \\ \downarrow & & \downarrow \\ \sigma(A(G)) \otimes \sigma(A(G)) & \xrightarrow{\ \star\ } & \sigma(A(G)) \end{array}$$

Types G_2 and F_4

$$A(G) \xhookrightarrow{\sigma} \mathrm{End}(W) \supset \sigma(A(G))$$

$$\begin{array}{ccc} A(G) \otimes A(G) & \xrightarrow{\diamond} & A(G) \\ \downarrow & & \downarrow \\ \sigma(A(G)) \otimes \sigma(A(G)) & \xrightarrow{\star} & \sigma(A(G)) \end{array}$$

Theorem (D., 2022+).
For G_2, F_4, there are explicit formulas for $\star, \sigma(A(G))$ in terms of the algebra product and the bilinear form on $\mathbb{O}, \mathcal{H}_3(\mathbb{O})$.

Corollary (D., 2022+).
For type G_2, $\mathrm{Aut}(A(G))$ is adjoint group of type G_2.

References

[CG21] **Maurice Chayet and Skip Garibaldi.** A class of continuous non-associative algebras arising from algebraic groups including e8. *Forum Math. Sigma*, 9:Paper No. e6, 2021.

[DMVC21] **Tom De Medts and Michiel Van Couwenberghe.** Non-associative frobenius algebras for simply laced chevalley groups. *Trans. Amer. Math. Soc.*, 374:8715–8774, 2021.

[GG15] **Skip Garibaldi and Robert M. Guralnick.** Simple groups stabilizing polynomials. *Forum of Mathematics, Pi*, 3:e3, 2015.