Representations of algebraic groups with an axial structure

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Axial algebras and groups related to them

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Motivation

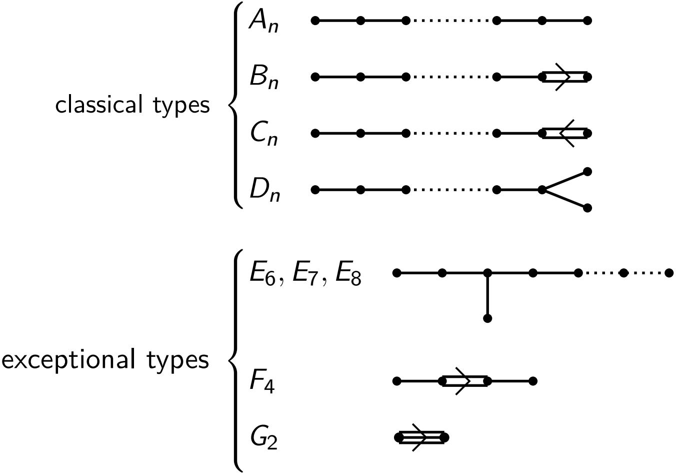
A quick reminder on linear algebraic groups

Linear algebraic groups $\approx$ matrix groups defined by polynomial equations over a field k.

$\mathrm{char}(k) \gg 0$ (i.p., $\mathrm{char}(k) \neq 2, 3$).

Motivation

A quick reminder on linear algebraic groups



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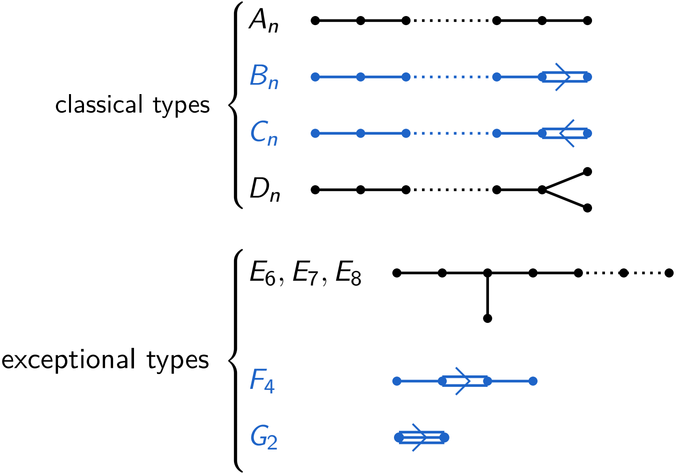

Type G_2: Automorphism group of *octonion algebra*

Type F_4: Automorphism group of *Albert algebra*

Motivation

Proposition ([GG15]).
On the second smallest irrep V of type E_8 (with dimension 3785) there exists a unique equivariant algebra product $\diamond$ together with a unique invariant Frobenius form τ, i.e.

$$\tau(a \diamond b, c) = \tau(a, b \diamond c) \quad \forall a, b, c \in V.$$

Two constructions

The idea

V: 2nd smallest irrep of E_8

Construction starts from the associated Lie algebra $\mathfrak{g}$.

$$A := k \oplus V \hookrightarrow \mathrm{S}^2\mathfrak{g} = \mathfrak{g}^{\otimes 2}/(a \otimes b - b \otimes a \mid \forall a, b \in \mathfrak{g})$$

Two constructions

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Two different constructions

De Medts - Van Couwenberghe [DMVC21]	Chayet - Garibaldi [CG21]

Two constructions

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Two different constructions

De Medts - Van Couwenberghe [DMVC21]	→ Chayet - Garibaldi [CG21]

2 Two constructions

The Chayet-Garibaldi construction

$$\begin{array}{ccc} A & \twoheadleftarrow & \mathrm{S}^2\mathfrak{g} \\ \cong\uparrow & & \downarrow P \\ \mathrm{Im}(P) & \subset & \mathrm{End}(\mathfrak{g}) \end{array}$$

$$\begin{array}{ccc} \mathrm{S}^2\mathfrak{g} \otimes \mathrm{S}^2\mathfrak{g} & \xrightarrow{\diamond} & \mathrm{S}^2\mathfrak{g} \\ P\otimes P \downarrow & & \downarrow P \\ A \otimes A & \longrightarrow & A \end{array}$$

$\rightarrow$ Explicit formulas for $P, \diamond$ in terms of Killing form K and Lie bracket $[\cdot, \cdot]$

$\rightarrow$ Frobenius form τ is just the trace map!

The formulas (you can ignore this slide)

$h^\vee$: Dual Coxeter number

$$\begin{aligned} P\colon\ \mathrm{S}^2\,\mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ ab &\mapsto \tfrac{1}{2}\big(h^\vee\,\mathrm{ad}(a)\circ\mathrm{ad}(b) + h^\vee\,\mathrm{ad}(a)\circ\mathrm{ad}(b) \\ &\qquad + K(b,\_)a + K(a,\_)b\big), \end{aligned}$$

$$\begin{aligned} P(ab)\diamond P(cd) := \tfrac{h^\vee}{2}\big(&P(a,(\mathrm{ad}\,c\bullet\mathrm{ad}\,d)b) + P((\mathrm{ad}\,c\bullet\mathrm{ad}\,d)a,b) \\ &+ P(c,(\mathrm{ad}\,a\bullet\mathrm{ad}\,b)d) + P((\mathrm{ad}\,a\bullet\mathrm{ad}\,b)c,d) \\ &+ P([a,c][b,d]) + P([a,d][b,c])\big) \\ +\tfrac{1}{4}\big(&K(a,c)P(b,d) + K(a,d)P(b,c) \\ &+ K(b,c)P(ad) + K(b,d)P(ac)\big). \end{aligned}$$

Theorem ([CG21], D. 2022+).
Given an (almost) simple algebraic group G, one can construct unital algebra $A(G)$ s.t.

- $A(G)$ is simple,
- $\tau(a \diamond b, c) = \tau(a, b \diamond c)$ for all $a, b, c \in A(G)$,
- For types G_2, F_4 and E_8, $\mathrm{Aut}(A(G))$ is adjoint group of type G_2, F_4 and E_8 (resp.)

If $[h, h] = 0$ then $P(h^2) \diamond P(h^2) = K(h, h)P(h^2) \implies$ idempotents!

Two constructions

Two different constructions

→ De Medts - Van Couwenberghe [DMVC21]

Chayet - Garibaldi [CG21]

Two constructions

gebra

Theorem ([DMVC21]).
Given a *split* simple algebraic group G over $\mathbb{C}$ of *simply laced type*, one can construct an *axial decomposition algebra* $A(G)$.

Two constructions

gebra

Theorem ([DMVC21]).
Given a *split* simple algebraic group G over $\mathbb{C}$ of *simply laced type*, one can construct an *axial decomposition algebra* $A(G)$.

$\circledast$	1	0	$\frac{4}{3}c_1 - \frac{1}{6}$	$(\frac{1}{2})_1$	$(\frac{1}{2})_2$	c_1
1	1	$\emptyset$	$\frac{4}{3}c_1 - \frac{1}{6}$	$(\frac{1}{2})_1$	$(\frac{1}{2})_2$	c_1
0	$\emptyset$	0	$\frac{4}{3}c_1 - \frac{1}{6}$	$(\frac{1}{2})_1$	$(\frac{1}{2})_2$	c_1
$\frac{4}{3}c_1 - \frac{1}{6}$	$\frac{4}{3}c_1 - \frac{1}{6}$	$\frac{4}{3}c_1 - \frac{1}{6}$	$1, 0, \frac{4}{3}c_1 - \frac{1}{6}$	$(\frac{1}{2})_1$	$(\frac{1}{2})_2, c_1$	$(\frac{1}{2})_2, c_1$
$(\frac{1}{2})_1$	$(\frac{1}{2})_1$	$(\frac{1}{2})_1$	$(\frac{1}{2})_1$	$1, 0, \frac{4}{3}c_1 - \frac{1}{6}, (\frac{1}{2})_1$	$(\frac{1}{2})_2, c_1$	$(\frac{1}{2})_2, c_1$
$(\frac{1}{2})_2$	$(\frac{1}{2})_2$	$(\frac{1}{2})_2$	$(\frac{1}{2})_2, c_1$	$(\frac{1}{2})_2, c_1$	$1, 0, \frac{4}{3}c_1 - \frac{1}{6}, (\frac{1}{2})_1$	$\frac{4}{3}c_1 - \frac{1}{6}, (\frac{1}{2})_1$
c_1	c_1	c_1	$(\frac{1}{2})_2, c_1$	$(\frac{1}{2})_2, c_1$	$\frac{4}{3}c_1 - \frac{1}{6}, (\frac{1}{2})_1$	$1, 0, \frac{4}{3}c_1 - \frac{1}{6}, (\frac{1}{2})_1$

Fusion laws for E_8 with parameter c_1.

$\rightarrow$ related to idempotents $m_h = P(h^2)/K(h,h)$.

Axes with non-trivial grading $\to$ idempotents in trivial component of grading.

Cartan grading

Split simple algebraic group G of rank n acts on algebra A as automorphisms $\implies$ Algebra A admits grading by $\mathbb{Z}^n$.

Theorem (Classificiation of Lie algebras).

Over $\mathbb{C}$, for any Lie algebra $\mathfrak{g}$ there exists a root system Φ of rank n s.t.

$$L = L_0 \oplus \bigoplus_{\alpha\in\Phi} L_\alpha,$$

with $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$ and $L_\alpha = \langle e_\alpha \rangle$.

$$\begin{aligned} A(G)_0 &= \langle P(e_\alpha e_{-\alpha}), P(h_1 h_2) \mid \alpha \in \Phi, h_1, h_2 \in L_0 \rangle \\ &\cong W \oplus \mathrm{S}^2 L_0 \text{ (as vector spaces).} \end{aligned}$$

$$A(G)_0 = \langle P(e_\alpha e_{-\alpha}), P(h_1 h_2) \mid \alpha \in \Phi, h_1, h_2 \in L_0 \rangle$$
$$\cong W \oplus \mathrm{S}^2 L_0 \text{ (as vector spaces)}.$$

If α is a long root then

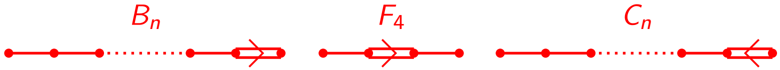
$$2P(e_\alpha e_{-\alpha}) = P([e_\alpha, e_{-\alpha}]^2)$$

$$\Phi = A_n, D_n \text{ or } E_n \implies A(G)_0 \cong \mathrm{S}^2 L_0.$$

Cartan gradings in Chayet-Garibaldi algebras

The trivial component

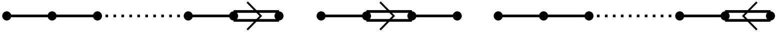
If α is a short root then



$u_\alpha = \lambda_2(4S(e_\alpha e_{-\alpha}) - S([e_\alpha, e_{-\alpha}]^2))$ is an idempotent.

If α is a short root then

B_n $\quad F_4$ $\quad C_n$

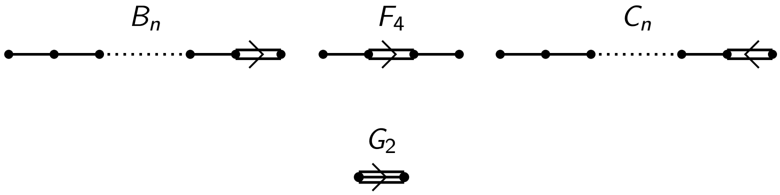


$$u_\alpha = \lambda_2(4S(e_\alpha e_{-\alpha}) - S([e_\alpha, e_{-\alpha}]^2)) \text{ is an idempotent.}$$

G_2

$$u_\alpha = \lambda_3(6S(e_\alpha e_{-\alpha}) - S([e_\alpha, e_{-\alpha}]^2)) \text{ is an idempotent.}$$

$$u_\alpha = u_{-\alpha}, \quad u_{\alpha\pm\beta} = u_\alpha \text{ for any long root } \beta.$$



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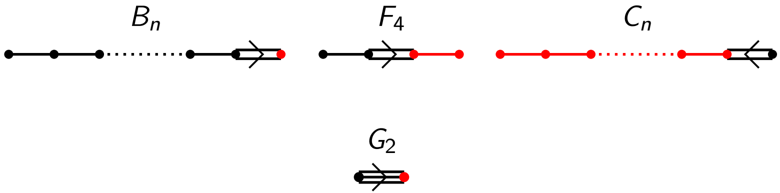
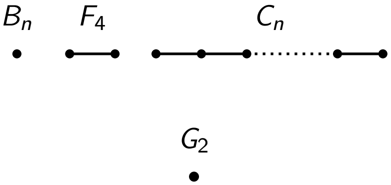


Cartan gradings in Chayet-Garibaldi algebras

Relations on the $u'_\alpha s$

$$u_\alpha = u_{-\alpha}, \quad u_{\alpha\pm\beta} = u_\alpha \text{ for any long root } \beta.$$

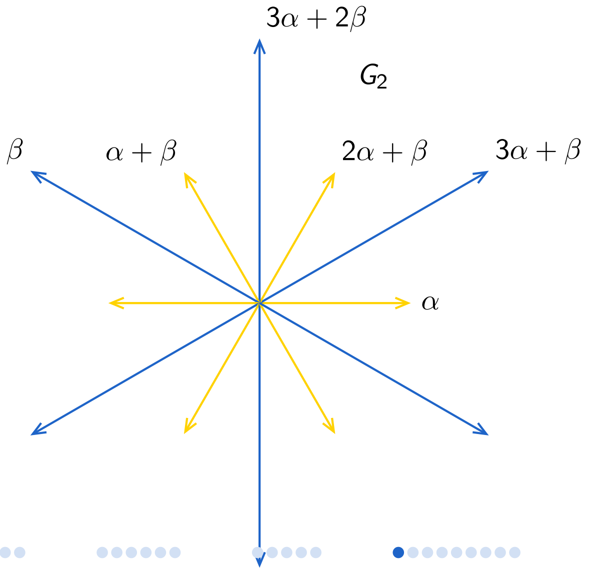


Computations show the u_α's span a Matsuo algebra of type A_k!
(rank determined as above)

$$\Phi = B_n, C_n, G_2 \text{ or } F_4$$
$$\Longrightarrow$$
$$A(G)_0 \cong \underbrace{M_{1/4}(A_k)}_{\text{Matsuo algebra of Jordan type } \frac{1}{4}} \oplus \underbrace{\mathrm{S}^2 L_0}_{\text{Jordan algebra related to Killing form}}.$$

Axes:

Resulting fusion laws



For root $\gamma \in \Phi$ write $h_\gamma := [e_\gamma, e_{-\gamma}]$.

0	1	$\frac{1}{2}$	$\frac{1}{3}$
$P(S^2 L_0)$	u_α	$P(e_{\pm\alpha} h_\alpha)$	$P(e_{\pm\alpha} h_1)$, $h_\alpha \perp_K h_1 \in L_0$
$P(e_{\pm\beta} L_0)$		$P(e_{\pm(\alpha+\beta)} h_{\alpha+\beta})$	$P(e_{\pm(\alpha+\beta)} h_1)$, $h_{\alpha+\beta} \perp_K h_1 \in L_0$
$P(e_{\pm(3\alpha+\beta)} L_0)$		$P(e_{\pm(2\alpha+\beta)} h_{2\alpha+\beta})$	$P(e_{\pm(2\alpha+\beta)} h_1)$, $h_{2\alpha+\beta} \perp_K h_1 \in L_0$
$P(e_{\pm(3\alpha+2\beta)} L_0)$			$e_{\pm\alpha} e_{\pm(3\alpha+2\beta)}$
			$e_{\pm(\alpha+\beta)} e_{\pm(3\alpha+\beta)}$
			$e_{\pm(2\alpha+\beta)} e_{\pm\beta}$

Proposition.
The fusion law for type G_2 with respect to the primitive idempotent u_α is

$*$	0	1	$\frac{1}{2}$	$\frac{1}{3}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}, \frac{1}{3}\}$	$\{\frac{1}{2}, \frac{1}{3}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{3}\}$
$\frac{1}{2}$	$\{\frac{1}{2}, \frac{1}{3}\}$	$\{\frac{1}{2}\}$	$\{1, 0, \frac{1}{3}\}$	$\{\frac{1}{2}, 0\}$
$\frac{1}{3}$	$\{\frac{1}{2}, \frac{1}{3}\}$	$\{\frac{1}{3}\}$	$\{\frac{1}{2}, 0\}$	$\{1, 0, \frac{1}{3}\}$

Resulting fusion laws

Short roots: $\{\alpha, \alpha+\beta, -(2\alpha+\beta)\} \sqcup \{-\alpha, -(\alpha+\beta), 2\alpha+\beta\}$
orbits under $\langle \sigma_\beta, \sigma_{3\alpha+\beta}, \sigma_{3\alpha+2\beta} \rangle$.

0	1	$\frac{1}{2}+$	$\frac{1}{3}+$
$P(S^2 L_0)$ $P(e_{\pm\beta} L_0)$ $P(e_{\pm(3\alpha+\beta)} L_0)$ $P(e_{\pm(3\alpha+2\beta)} L_0)$	u_α	$P(e_\alpha h_\alpha)$ $P(e_{\alpha+\beta} h_{\alpha+\beta})$ $P(e_{-(2\alpha+\beta)} h_{2\alpha+\beta})$	$P(e_\alpha h_1),\ h_\alpha \perp_K h_1 \in L_0$ $P(e_{\alpha+\beta} h_1),\ h_{\alpha+\beta} \perp_K h_1 \in L_0$ $P(e_{-(2\alpha+\beta)} h_1),\ h_{2\alpha+\beta} \perp_K h_1 \in L_0$ $e_\alpha e_{\pm(3\alpha+2\beta)}$ $e_{(\alpha+\beta)} e_{\pm(3\alpha+\beta)}$ $e_{-(2\alpha+\beta)} e_{\pm\beta}$
		$\frac{1}{2}-$	$\frac{1}{3}-$
		$P(e_{-\alpha} h_\alpha)$ $P(e_{-(\alpha+\beta)} h_{\alpha+\beta})$ $P(e_{2\alpha+\beta} h_{2\alpha+\beta})$	$P(e_{-\alpha} h_1),\ h_\alpha \perp_K h_1 \in L_0$ $P(e_{-(\alpha+\beta)} h_1),\ h_{\alpha+\beta} \perp_K h_1 \in L_0$ $P(e_{2\alpha+\beta} h_1),\ h_{2\alpha+\beta} \perp_K h_1 \in L_0$ $e_{-\alpha} e_{\pm(3\alpha+2\beta)}$ $e_{-(\alpha+\beta)} e_{\pm(3\alpha+\beta)}$ $e_{2\alpha+\beta} e_{\pm\beta}$

Resulting fusion laws

Proposition.
The decomposed fusion law for type G_2 with respect to the primitive idempotent u_α is

$*$	0	1	$\frac{1}{2}+$	$\frac{1}{3}+$	$\frac{1}{2}-$	$\frac{1}{3}-$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}+, \frac{1}{3}+\}$	$\{\frac{1}{2}+, \frac{1}{3}+\}$	$\{\frac{1}{2}-, \frac{1}{3}-\}$	$\{\frac{1}{2}-, \frac{1}{3}-\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}+\}$	$\{\frac{1}{3}+\}$	$\{\frac{1}{2}-\}$	$\{\frac{1}{3}-\}$
$\frac{1}{2}+$	$\{\frac{1}{2}+, \frac{1}{3}+\}$	$\{\frac{1}{2}+\}$	$\{\frac{1}{3}-\}$	$\{\frac{1}{2}-\}$	$\{0, 1\}$	$\{0\}$
$\frac{1}{3}+$	$\{\frac{1}{2}+, \frac{1}{3}+\}$	$\{\frac{1}{3}+\}$	$\{\frac{1}{2}-\}$	$\{\frac{1}{3}-\}$	$\{0\}$	$\{1, 0\}$
$\frac{1}{2}-$	$\{\frac{1}{2}-, \frac{1}{3}-\}$	$\{\frac{1}{2}-\}$	$\{0, 1\}$	$\{0\}$	$\{\frac{1}{3}+\}$	$\{\frac{1}{2}+\}$
$\frac{1}{3}-$	$\{\frac{1}{2}-, \frac{1}{3}-\}$	$\{\frac{1}{3}-\}$	$\{0\}$	$\{1, 0\}$	$\{\frac{1}{2}+\}$	$\{\frac{1}{3}+\}$

Resulting fusion laws

Proposition.
The fusion law for type B_n with respect to the primitive idempotent u_α is

$*$	0	1	$\frac{1}{2}$	$\frac{1}{4}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{2}, \frac{1}{4}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{4}\}$
$\frac{1}{2}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{2}\}$	$\{1, 0\}$	$\{0\}$
$\frac{1}{4}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{0\}$	$\{1, 0\}$

Resulting fusion laws

Conjecture.
The fusion law for type C_n, F_4 with respect to the primitive idempotent u_α is

$*$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{32}\}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{2}, \frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{2}$	$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{32}\}$	$\{\frac{1}{2}\}$	$\{1, \frac{1}{4}, \frac{1}{32}, 0\}$	$\{\frac{1}{2}, \frac{1}{32}, 0\}$	$\{\frac{1}{2}, \frac{1}{4}, 0\}$
$\frac{1}{4}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{2}, \frac{1}{32}, 0\}$	$\{1, 0\}$	$\{\frac{1}{2}, \frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{2}, \frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{2}, \frac{1}{4}, 0\}$	$\{\frac{1}{2}, \frac{1}{32}\}$	$\{1, \frac{1}{4}, 0\}$

Problem
Verified up to C_6, sufficient if known for C_8.

Type C_n

$*$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2},\frac{1}{4},\frac{1}{32}\}$	$\{\frac{1}{2},\frac{1}{4}\}$	$\{\frac{1}{2},\frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{2}$	$\{\frac{1}{2},\frac{1}{4},\frac{1}{32}\}$	$\{\frac{1}{2}\}$	$\{1,\frac{1}{4},\frac{1}{32},0\}$	$\{\frac{1}{2},\frac{1}{32},0\}$	$\{\frac{1}{2},\frac{1}{4},0\}$
$\frac{1}{4}$	$\{\frac{1}{2},\frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{2},\frac{1}{32},0\}$	$\{1,0\}$	$\{\frac{1}{2},\frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{2},\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{2},\frac{1}{4},0\}$	$\{\frac{1}{2},\frac{1}{32}\}$	$\{1,\frac{1}{4},0\}$

Type C_n

$*$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2},\frac{1}{4},\frac{1}{32}\}$	$\{\frac{1}{2},\frac{1}{4}\}$	$\{\frac{1}{2},\frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{2}$	$\{\frac{1}{2},\frac{1}{4},\frac{1}{32}\}$	$\{\frac{1}{2}\}$	$\{1,\frac{1}{4},\frac{1}{32},0\}$	$\{\frac{1}{2},\frac{1}{32},0\}$	$\{\frac{1}{2},\frac{1}{4},0\}$
$\frac{1}{4}$	$\{\frac{1}{2},\frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{2},\frac{1}{32},0\}$	$\{1,0\}$	$\{\frac{1}{2},\frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{2},\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{2},\frac{1}{4},0\}$	$\{\frac{1}{2},\frac{1}{32}\}$	$\{1,\frac{1}{4},0\}$

Type C_n

$*$	0	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{32}\}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{2}, \frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{2}$	$\{\frac{1}{2}, \frac{1}{4}, \frac{1}{32}\}$	$\{\frac{1}{2}\}$	$\{1, \frac{1}{4}, \frac{1}{32}, 0\}$	$\{\frac{1}{2}, \frac{1}{32}, 0\}$	$\{\frac{1}{2}, \frac{1}{4}, 0\}$
$\frac{1}{4}$	$\{\frac{1}{2}, \frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{2}, \frac{1}{32}, 0\}$	$\{1, 0\}$	$\{\frac{1}{2}, \frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{2}, \frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{2}, \frac{1}{4}, 0\}$	$\{\frac{1}{2}, \frac{1}{32}\}$	$\{1, \frac{1}{4}, 0\}$

Remove $\frac{1}{2}$:

$*$	0	1	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{4}$	$\{\frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{1, 0\}$	$\{\frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{1, \frac{1}{4}, 0\}$

Type C_n

Remove $\frac{1}{2}$:

$*$	0	1	$\frac{1}{4}$	$\frac{1}{32}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{4}\}$	$\{\frac{1}{32}\}$
$\frac{1}{4}$	$\{\frac{1}{4}\}$	$\{\frac{1}{4}\}$	$\{1, 0\}$	$\{\frac{1}{32}\}$
$\frac{1}{32}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{\frac{1}{32}\}$	$\{1, \frac{1}{4}, 0\}$

Remove $\frac{1}{4}$, $\frac{1}{32}$:

$*$	0	1	$\frac{1}{2}$
0	$\{0\}$	$\emptyset$	$\{\frac{1}{2}\}$
1	$\emptyset$	$\{1\}$	$\{\frac{1}{2}\}$
$\frac{1}{2}$	$\{\frac{1}{2}\}$	$\{\frac{1}{2}\}$	$\{1, 0\}$

5

Further questions

1. Can we decompose the fusion law for C_n, F_4?
2. Can we use other gradings to find different idempotents?
   e.g. Cayley Dickson
3. Idempotents u_α exist for compact real form of the groups; does the Norton inequality hold in these algebras?
4. Is there a natural explanation for this fusion law?

References I

Thank you for listening!

[CG21] **Maurice Chayet and Skip Garibaldi.** A class of continuous non-associative algebras arising from algebraic groups including e8. *Forum Math. Sigma*, 9:Paper No. e6, 2021.

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