A characterization of Jordan algebras using solid lines

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Axial algebras and groups related to them

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Background

Background

Primitive axial algebras of Jordan type η were introduced in 2015 by Hall, Rehren and Shpectorov [HRS15].

$\star$	1	0	η
1	1		η
0		0	η
η	η	η	$0, 1$

Background

Primitive axial algebras of Jordan type $\eta \neq \frac{1}{2}$ are Matsuo algebras [HRS15].

$\eta \neq \frac{1}{2}$	$\eta = \frac{1}{2}$
Matsuo algebras	Matsuo algebras Jordan algebras ?

2022: Talk by Shpectorov on *solid lines*.

2024: Paper on arXiv by Gorshkov, Shpectorov, Staroletov [GSS24].

Today: Solid lines and Jordan algebras.

Assumptions

- $\mathbb{F}$ is an algebraically closed field with $\operatorname{char} \mathbb{F} \neq 2$.
- (A, X) is a primitive axial algebra of Jordan type $\frac{1}{2}$.
- A admits a unique Frobenius form $(,)$ such that $(a, a) = 1$ for every primitive axis $a \in A$ [HSS18].

1 Background

Definition.
We call a 2-generated subalgebra $\langle\langle a, b\rangle\rangle$ of (A, X) solid if every primitive idempotent in $\langle\langle a, b\rangle\rangle$ is an axis of A.

Theorem ([GSS24]).
If $\operatorname{char}\mathbb{F} = 0$ and $(a, b) \neq \frac{1}{4}$, then $\langle\langle a, b\rangle\rangle$ is solid.

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Theorem ([GSS24]).
If $(a, b) \neq \frac{1}{4}, 0, 1$ then $\langle\langle a, b\rangle\rangle$ is solid.

→ What is the link with Jordan and Matsuo algebras?

$o(\tau_a \tau_b) = 3$

2-generated subalgebras

2-generated subalgebras

Let $J(\phi)$, $\phi \in \mathbb{F}$, be the algebra over $\mathbb{F}$ with basis x, y, σ and multiplication given by:

	x	y	σ
x	x	$\frac{1}{2}(x+y)+\sigma$	$\frac{1}{2}(\phi-1)x$
y	$\frac{1}{2}(x+y)+\sigma$	y	$\frac{1}{2}(\phi-1)y$
σ	$\frac{1}{2}(\phi-1)x$	$\frac{1}{2}(\phi-1)y$	$\frac{1}{2}(\phi-1)\sigma$

Then $J(\phi)$ is a Jordan algebra and x and y are primitive axes.

Theorem ([HRS15]).
Let $A = \langle\langle a, b\rangle\rangle$ be a primitive axial algebra of Jordan type half, where a and b are primitive axes. Let $\phi = (a, b)$. Then there is a unique epimorphism from $J(\phi)$ onto A sending x and y onto a and b, respectively.

2-generated subalgebras

Theorem ([HRS15]).

$J(\phi)$ is simple except in the following two cases:

1. $J(0)$ has a unique minimal ideal, with quotient $2B = \mathbb{F} \oplus \mathbb{F}$;
2. $J(1)$ has two quotients, the 1-dimensional point and a 2-dimensional algebra $\overline{J(1)}$.

2-generated subalgebras

$u = 1 \quad e^2 = f^2 = 0$

Toric lines

Subalgebras $\langle\langle a, b\rangle\rangle$ with $(a, b) \neq 0, 1$. These are spanned by elements e, u, f with $a = e + \frac{1}{2}u + f$ and $b = \mu e + \frac{1}{2}u + \mu^{-1}f$ for some $\mu \in \mathbb{F}^\times$. Let V be the variety of primitive idempotents in $\langle\langle a, b\rangle\rangle$.

In [GSS24], the following was shown:

$$V = \left\{ a_\lambda := \lambda e + \tfrac{1}{2}u + \lambda^{-1}f \mid \lambda \in \mathbb{F}^\times \right\}$$

$$a_\lambda^{\tau_{a_\mu}} = a_{\lambda^{-1}\mu^2}$$

Flat lines

Subalgebras $\langle\langle a, b\rangle\rangle$ with $(a, b) = 0$. These are spanned by elements $a, b, v = ab$. Let V be the variety of primitive idempotents in $\langle\langle a, b\rangle\rangle$.

In [GSS24], the following was shown:

$$V = \{a_\lambda := a + \lambda v, b_\lambda := b + \lambda v \mid \lambda \in \mathbb{F}\}$$

$$a_\lambda^{\tau_{a_\mu}} = a_{2\mu-\lambda}, b_\lambda^{\tau_{a_\mu}} = b_{-2-2\mu-\lambda}$$

$$V = \{a, b\}^{\langle \tau_c \mid c \in \langle\langle a, b\rangle\rangle \rangle}$$

2-generated subalgebras

Baric lines

Subalgebras $\langle\langle a, b\rangle\rangle$ with $(a, b) = 1$. These are spanned by elements $a, v = 2(ab - a), v^2$. Let V be the variety of idempotents in $\langle\langle a, b\rangle\rangle$.

In [GSS24], the following was shown:

$$V = \left\{a_\lambda := a + \lambda v + \lambda^2 v^2 \mid \lambda \in \mathbb{F}\right\}$$

$$a_\lambda^{\tau_{a_\mu}} = a_{2\mu-\lambda}$$

Two important polynomials

Two important polynomials

Lemma.
Let (A, X) be a primitive axial algebra of Jordan type half over a field $\mathbb{F}$, and $a, b \in X$. the subalgebra $\langle\langle a, b\rangle\rangle$ is solid if and only if

$$Q_x(c) = c(cx) - \frac{1}{2}(cx + (c, x)c)$$

and

$$P_{x,y}(c) = 4(cx)(cy) - (c, y)cx - (c, x)cy - (cy)x - (cx)y - (c, xy)c + c(xy)$$

are zero for all primitive idempotents $c \in \langle\langle a, b\rangle\rangle$ and $x, y \in A$.

Two important polynomials

Write $x^{\tau_c} = x + 4(c, x)c - 4cx$. Then

$$x^{\tau_c} y^{\tau_c} - (xy)^{\tau_c} = 4P_{x,y}(c) - 16Q_x(c) - 16Q_y(c).$$

If c is an axis, then for all $x, y \in A$, we have $Q_x(c) = 0$ so $P_{x,y}(c) = 0$.

Conversely, if $P_{x,y}(c) = Q_x(c) = 0$ for all $x, y \in A$, then is an automorphism of A, and $\{a, b\}^{\langle \tau_c | c \rangle} = V$.

Most lines are solid

Most lines are solid

↳ 0, 1

Note that $P^u_{x,y}(\lambda) = P_{x,y}(\overline{a_\lambda})$ and $Q^u_x(\lambda) = P_{x,y}(\overline{a_\lambda})$ are always polynomial maps, for all $x, y \in A$. Suppose that $\langle\langle a, b\rangle\rangle \ncong J(1), \mathbb{F} \oplus \mathbb{F}$.

↳ deg(a λ) = 1

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- $\deg_\lambda(a_\lambda) = 1$, so $\deg_\lambda(P^u_{x,y}), \deg_\lambda(Q^u_x) \leq 2$.

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- $\deg_\lambda(a_\lambda) = 1$, so $\deg_\lambda(P^u_{x,y}), \deg_\lambda(Q^u_x) \leq 2$.
- $\langle\langle a, b\rangle\rangle$ contains $a, a^{\tau_b}, a^{\tau_b\tau_a}$, so $P^u_{x,y}(\lambda), Q^u_x(\lambda)$ have at least 3 roots.

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- $\deg P^u_{x,y}, \deg Q^u_x \leq 2$, so $P^u_{x,y} = Q^u_x = 0$.

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- $\deg_\lambda(a_\lambda) = 1$, so $\deg_\lambda(P^u_{x,y}), \deg_\lambda(Q^u_x) \leq 2$.
- $\langle\langle a, b\rangle\rangle$ contains $a, a^{\tau_b}, a^{\tau_b \tau_a}$, so $P^u_{x,y}(\lambda), Q^u_x(\lambda)$ have at least 3 roots.
- $\deg P^u_{x,y}, \deg Q^u_x \leq 2$, so $P^u_{x,y} = Q^u_x = 0$.
- For flat lines, we also have that $P_{x,y}(b_\lambda) = Q_x(b_\lambda) = 0$.

Most lines are solid

Note that $P^u_{x,y}(\lambda) = P_{x,y}(a_\lambda)$ and $Q^u_x(\lambda) = P_{x,y}(a_\lambda)$ are always polynomial maps, for all $x, y \in A$. Suppose that $\langle\langle a, b\rangle\rangle \ncong J(1), \mathbb{F} \oplus \mathbb{F}$.

- $\deg_\lambda(a_\lambda) = 1$, so $\deg_\lambda(P^u_{x,y}), \deg_\lambda(Q^u_x) \leq 2$.
- $\langle\langle a, b\rangle\rangle$ contains $a, a^{\tau_b}, a^{\tau_b\tau_a}$, so $P^u_{x,y}(\lambda), Q^u_x(\lambda)$ have at least 3 roots.
- $\deg P^u_{x,y}, \deg Q^u_x \leq 2$, so $P^u_{x,y} = Q^u_x = 0$.
- For flat lines, we also have that $P_{x,y}(b_\lambda) = Q_x(b_\lambda) = 0$.

If $\langle\langle a, b\rangle\rangle \cong J(1)$ and $\operatorname{char}\mathbb{F} \neq 3$, then $\langle\langle a, b\rangle\rangle$ is also solid.

Most lines are solid

Theorem.
Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with $\operatorname{char} \mathbb{F} \neq 2$. Denote by $(\cdot, \cdot)$ the unique Frobenius form on A. Given $a, b \in X$, the subalgebra $\langle\langle a, b \rangle\rangle$ is solid whenever $(a, b) \neq \frac{1}{4}$ or $\langle\langle a, b \rangle\rangle$ is not 3-dimensional.

J(1) is solid

Solid lines and derivations

Let R be a commutative associative $\mathbb{F}$-algebra. If $Q^u_x, P^u_{x,y} \in A[\lambda]$ are zero polynomials, then this is still true over $A_R := A \otimes_{\mathbb{F}} R$.

Let R be a commutative associative $\mathbb{F}$-algebra. If $Q_x^u, P_{x,y}^u \in A[\lambda]$ are zero polynomials, then this is still true over $A_R := A \otimes_{\mathbb{F}} R$.

Dual numbers

The algebra of the dual numbers is $\mathbb{F}[\varepsilon] := \{a + \varepsilon b \mid a, b \in \mathbb{F}\}$ with $\varepsilon^2 = 0$.

If $Q_x^u(\varepsilon) = P_{x,y}^u(\varepsilon) = 0$ for all $x, y \in A$, then τ_{a_ε} is automorphism of $A_{\mathbb{F}[\varepsilon]}$.

Associators/Inner derivations

Given $a, b, x \in A$, define

$$D_{a,b}(x) = a(bx) - b(ax).$$

$$x^{\tau_a \tau_{a_\varepsilon}} = x + \mu\varepsilon D_{a,b}(x) \text{ for a } \mu \in \mathbb{F}, \text{ for all } x \in A.$$

Corollary.

If $\langle\langle a, b\rangle\rangle$ is solid, then $D_{a,b}$ is a derivation.

Solid lines and derivations

Proposition.
If $D_{a,b}$ is a derivation, then $\langle\langle a, b\rangle\rangle$ is solid.

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Assume $\operatorname{char}\mathbb{F} = 3$, $\langle\langle a, b\rangle\rangle$ baric.

- $\chi = \tau_a\tau_{a_\varepsilon}$ is automorphism of $A_{\mathbb{F}[\varepsilon]}$
- $a_\lambda^\chi = a_{\lambda-4\varepsilon}$ is an axis for $\lambda \in \mathbb{F}_3$.
- $P^u_{x,y}(\lambda - 4\varepsilon) = Q^u_x(\lambda - 4\varepsilon) = 0$ implies that λ is a root of multiplicity 2.
- $\deg P^u_{x,y}$, $\deg_\lambda Q^u_x \leq 4$, so $P^u_{x,y} = Q^u_x = 0$.

Jordan algebras

6 Jordan algebras

Definition.
A non-associative commutative ring A with $2 \in A^\times$ is *almost Jordan* if $D_{x,y}$ is a derivation for all $x, y \in A$.

Marshall Osborn studied almost Jordan algebras in [Osb65].

Theorem ([Osb65]).
Let A be an almost Jordan ring and e a ~~primitive~~ idempotent.
Then A is Jordan if and only if $A_0(e)$ is.

and $A_1(e)$

Jordan algebras

Definition.
A non-associative commutative ring A with $2 \in A^\times$ is *almost Jordan* if $D_{x,y}$ is a derivation for all $x, y \in A$.

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Theorem.
Let A be an almost Jordan ring *linearly spanned by idempotents*. Then A is Jordan.

Main result

Theorem.
A primitive axial algebra of Jordan type half (A, X) of Jordan type half over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$ such that for all $a, b \in X$ the line $\langle\langle a, b\rangle\rangle$ is solid, is Jordan.

Further work

Further work

- More structure theory of Jordan type half axial algebras (joint with Sergey Shpectorov)
- Non-Jordan, non-Matsuo Jordan type half axial algebras.

References I

Thank you for listening!

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[Osb65] **J. Marshall Osborn.** Commutative algebras satisfying an identity of degree four. *Proc. Amer. Math. Soc.*, 16:1114–1120, 1965.