Solid lines in axial algebras of Jordan type $\frac{1}{2}$ and derivations

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Background

Primitive axial algebras of Jordan type η: introduced in 2015 by Hall, Rehren and Shpectorov [HRS15].

Definition ([HRS15]).
Let $0, 1 \neq \eta \in \mathbb{F}$. We consider commutative $\mathbb{F}$-algebras generated by idempotents $a \in A$ (called *axes*) such that

1. $A = A_1(a) \oplus A_0(a) \oplus A_\eta(a)$.
2. $A_0(a)$ is a subalgebra of A and $A_1(a) = \langle a \rangle$.
3. For all $\delta, \epsilon \in \{\pm\}$,

$$A_\delta(a) A_\epsilon(a) \subseteq A_{\delta\epsilon}(a),$$

where $A_+(a) = A_1(a) \oplus A_0(a)$ and $A_-(a) = A_\eta(a)$.

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Jordan algebras are of Jordan type $\frac{1}{2}$ by Peirce decomposition!

Another example

Matsuo algebras: algebras constructed combinatorially from 3-transposition groups in the context of vertex operator algebras

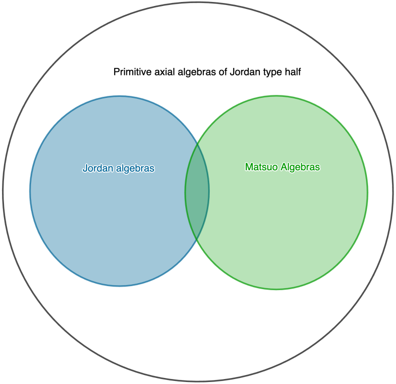
Definition (Matsuo algebra).
G a group with $D \subseteq G$ a G-stable, generating subset of elements of order 2. Then $M_\eta(G, D) = \mathbb{F}D$ with for $a, b \in D$

$$a \cdot b := \begin{cases} a & \text{if } a = b, \\ 0 & \text{if } o(ab) = 2, \\ \frac{\eta}{2}(a + b - b^a) & \text{if } o(ab) = 3. \end{cases}$$

Primitive axial algebras of Jordan type $\eta \neq \frac{1}{2}$ are quotients of Matsuo algebras [HRS15].

Background

When $\eta = \frac{1}{2}$: Dichotomy between continuous and discrete setting!



Assumptions

- $\mathbb{F}$ is an algebraically closed field with $\operatorname{char}\mathbb{F} \neq 2$.
- A is a primitive axial algebra of Jordan type $\frac{1}{2}$.
- A admits a unique associative bilinear form $(,)$ such that $(a, a) = 1$ for every primitive axis $a \in A$ [HSS18].

In this case, we have a formula

$$x^{\tau_a} = x + 4(a, x)a - 4ax \text{ for all } x \in A.$$

Solid lines

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Definition ([GSS24]).
We call a 2-generated subalgebra $\langle\langle a, b\rangle\rangle$ of (A, X) solid if every primitive idempotent in $\langle\langle a, b\rangle\rangle$ is an axis of A.

Theorem ([GSS24]).
If $(a, b) \neq \frac{1}{4}$,0,1, then $\langle\langle a, b\rangle\rangle$ is solid.

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What is the link with Jordan and Matsuo algebras?

Lemma.
The order of $\tau_a\tau_b$ is 3 if and only if $(a, b) = \frac{1}{4}$.

Two important polynomials

Lemma.
Let (A, X) be a primitive axial algebra of Jordan type half over a field $\mathbb{F}$, and $a, b \in X$. the subalgebra $\langle\langle a, b\rangle\rangle$ is solid if and only if

$$Q_x(c) = c(cx) - \frac{1}{2}(cx + (c, x)c) \qquad \text{(Eigenvalues)}$$

and

$$\begin{aligned} P_{x,y} = 4(cx)(cy) - (c, y)cx - (c, x)cy \\ - (cy)x - (cx)y - (c, xy)c + c(xy) \end{aligned} \qquad \text{(Fusion)}$$

are zero for all primitive idempotents $c \in \langle\langle a, b\rangle\rangle$ and $x, y \in A$.

(Recall $x^{\tau_a} = x + 4(a, x)a - 4ax$ for all $x \in A$.)

Most lines are solid

Theorem.
Let (A, X) be a primitive axial algebra of Jordan type $\eta = \frac{1}{2}$ over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2$. Denote by $(\cdot, \cdot)$ the unique Frobenius form on A. Given $a, b \in X$, the subalgebra $\langle\langle a, b\rangle\rangle$ is solid whenever $(a, b) \neq \frac{1}{4}$ or $\langle\langle a, b\rangle\rangle$ is not 3-dimensional.

Proof idea

- For most 2-generated subalgebras: there is a (vector-valued) rational map $f : \mathbb{F} \to \langle\langle a, b \rangle\rangle$ that surjects onto a connected component of idempotents, with numerator of low degree.
- $(Q_x \circ f)$,$(P_{x,y} \circ f)$ are rational maps of low degree, and axes are zeroes for these maps.
- Whenever $(a, b) \neq \frac{1}{4}$ there are more axes than the degree of the numerator.

Solid lines and derivations

Associators/Inner derivations

Given $a, b, x \in A$, define

$$D_{a,b}(x) = a(bx) - b(ax).$$

over $\mathbb{F}[t]/(t^2) = \mathbb{F}[\varepsilon]$, there exists $a_\varepsilon \in \langle\langle a, b\rangle\rangle \otimes_{\mathbb{F}} \mathbb{F}[\varepsilon]$ with

$$x^{\tau_a \tau_{a_\varepsilon}} = x + \mu\varepsilon D_{a,b}(x) \text{ for a } \mu \in \mathbb{F} \setminus \{0\}, \text{ for all } x \in A.$$

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Corollary.
$\langle\langle a, b\rangle\rangle$ is solid if and only if $D_{a,b}$ is a derivation.

Jordan algebras

Definition.
A non-associative commutative ring A with $2 \in A^\times$ is *almost Jordan* if $D_{x,y}$ is a derivation for all $x, y \in A$.

Marshall Osborn studied almost Jordan algebras in [Osb65].

Jordan algebras

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A non-associative commutative ring A with $2 \in A^\times$ is *almost Jordan* if $D_{x,y}$ is a derivation for all $x, y \in A$.

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Equivalently, A is almost Jordan if

$$2((yx)x)x + yx^3 = 3(yx^2)x \text{ for all } x, y \in A.$$

Theorem.
Let A be an almost Jordan ring *generated by idempotents*. Then A is Jordan.

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Sketch of proof if $\operatorname{char}\mathbb{F} \neq 3$ *and* A *simple [CG21].*

Idempotents in almost Jordan algebras are *axes*, so A is primitive axial algebra of Jordan type with associative non-degenerate form $(,)$. Then

$$3L_xL_{x^2} = (3L_{x^2}L_x)^\top = (2(L_x)^3 + L_{x^3})^\top = 2(L_x)^3 + L_{x^3} = 3L_{x^2}L_x$$

for $x \in A$, so

$$3x(x^2y) = 3x^2(xy) \text{ for all } x, y \in A.$$

Main result

Theorem.
A primitive axial algebra of Jordan type half (A, X) of Jordan type half over a field $\mathbb{F}$ with $\operatorname{char}\mathbb{F} \neq 2, 3$ such that for all $a, b \in X$ the line $\langle\langle a, b\rangle\rangle$ is solid, is Jordan.

References I

Thank you for listening!

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