SUPREMUM PRESERVING UPPER PROBABILITIES

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Abstract. We study the relation between possibility measures and the theory of imprecise probabilities, and argue that possibility measures have an important part in this theory. It is shown that a possibility measure is a coherent upper probability if and only if it is normal. A detailed comparison is given between the possibilistic and natural extension of an upper probability, both in the general case and for upper probabilities defined on a class of nested sets. We prove in particular that a possibility measure is the restriction to events of the natural extension of a special kind of upper probability, defined on a class of nested sets. We show that possibilistic extension can be interpreted in terms of natural extension. We also prove that when either the upper or the lower cumulative distribution function of a random quantity is specified, possibility measures very naturally emerge as the corresponding natural extensions. Next, we go from upper probabilities to upper previsions. We show that if a coherent upper prevision defined on the convex cone of all non-negative gambles is supremum preserving, then it must take the form of a Shilkret integral associated with a possibility measure. But at the same time, we show that such a supremum preserving upper prevision is never coherent unless it is the vacuous upper prevision with respect to a non-empty subset of the universe of discourse.

1. Introduction

Supremum preserving set functions have been studied in the literature under a number of different guises and names, and in a diversity of contexts. For one thing, they are join-morphisms and therefore play an important part in order (or lattice) theory [1, 4]. They appear in a modified form in Shackle's logic of surprise [23], and they were studied in a measure-theoretic context by Shilkret [25], who also pointed out that such set functions frequently appear in many measure-theoretic areas, but had not yet been studied in a systematic manner. To give but one example, the distance from the subsets of a metric space to a fixed subset leads naturally to possibility measures. They are also special limit cases in Shafer's theory of belief functions [24]. In the context of fuzzy set theory, special attention was drawn to these set functions by Zadeh [33]. He called them *possibility measures* because, in his view, they model a graded notion of possibility. He advanced the thesis that they are a mathematical representation of the information conveyed by typical statements in natural language. For recent discussions of this interpretation, we refer to [6, 29, 30], where it is shown that Zadeh's claims are not unreasonable when considered in the context of the behavioural theory of imprecise probabilities.

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In this paper, we investigate how supremum preserving set functions — or possibility measures, adopting Zadeh's name for the sake of continuity — fit into the behavioural theory of imprecise probabilities, as it was formulated by Walley [28]. We show that they have an important role in the theory, and arise naturally in a number of interesting situations. Here, contrary to the discussions in [6, 29, 30], we are less concerned with matters of interpretation, and the stress is on the formal mathematical aspects of possibility measures. To a certain extent, we also generalise a number of results by Dubois and Prade [20] from a finitary towards a more general context. A related discussion, dealing with natural extension in the context of random sets, can be found in [10]. The discussion in [30] is an interesting application of the mathematical results derived here.

The paper is organised as follows. Section 2 is a brief survey of the preliminary material, necessary for understanding the rest of the paper. In particular, we give a concise introduction to possibility measures and to the theory of imprecise probabilities. The coherence of possibility measures interpreted as upper probabilities, and their natural extension to larger domains are studied in Section 3. We also discuss the role which the Choquet integral [3, 16] has in this extension.

Given an upper probability defined on a class of events, it may be asked whether this set function can be extended to a normal possibility measure defined for all events. In Section 4, we discuss this so-called *possibilistic extension* problem in its full generality, and compare possibilistic with natural extension. In particular, we show that if possibilistic extension is possible there is an interesting link between the two extension methods. In Sections 5 and 6, both extension methods are studied for upper probabilities defined on special classes of events where possibilistic extension is especially interesting, namely collections of nested sets. Our main conclusion is that for upper probabilities defined on such chains of sets, natural and possibilistic extension coincide under a number of additional (continuity) conditions. Since, as we see in Section 7, upper probabilities on classes of nested sets appear naturally in a number of situations, this indicates that supremum preserving upper probabilities are not without importance for the theory of imprecise probabilities.

On the other hand, it is argued in Section 8 that supremum preserving upper *previsions* are less promising in this respect, because they are only coherent in trivial cases. We also lay bare the connection between such previsions and the so-called Shilkret integral [25].

We have relegated the proofs of theorems and propositions, together with some additional discussion, to the Appendix.

2. Preliminary definitions and notations

In what follows, we consider a non-empty set Ω, called the *possibility space* or the *universe of discourse*. Note that Ω may be infinite. It can be interpreted as the set of the mutually exclusive possible states of the world that are relevant to a given problem.

2.1. Gambles and events.

A real-valued mapping X on Ω is called a *gamble* on Ω if it is bounded, i.e. if $\sup[X] = \sup\{X(\omega)\colon \omega \in \Omega\}$ and $\inf[X] = \inf\{X(\omega)\colon \omega \in \Omega\}$ are finite real numbers. The set of the gambles on Ω is denoted by $\mathcal{L}(\Omega)$. It is a linear space under the pointwise addition of gambles and the scalar multiplication of gambles with real numbers. If X and Y are gambles on Ω, we write $X \leq Y$ if $(\forall \omega \in \Omega)(X(\omega) \leq Y(\omega))$. Also, a *constant gamble* will be denoted by the unique value it assumes. Gambles can be interpreted as uncertain rewards in some linear utility [28]: if a subject is uncertain about which ω will actually obtain among a set Ω of possible alternatives, he will also be uncertain about the real value (or utility) $X(\omega)$.

There is a special class of gambles that assume only values in $\{0, 1\}$. If X is such a mapping, then clearly it is the *indicator* (or characteristic function) of the subset $A = \{\omega \in \Omega \colon X(\omega) = 1\}$ of Ω, and it is also denoted as I_A. A subset of Ω will be called an *event*, and the set of events will be denoted by $\wp(\Omega)$. In what follows, we shall often identify an event A with its characteristic function I_A. In such cases it should be clear from the context whether A denotes an event (a set) or a gamble (an indicator).

With a real-valued function X on Ω and a real number x, we may associate a number of special events: $\{X > x\} = \{\omega \in \Omega \colon X(\omega) > x\}$ is the *strict cut set*, $\{X \leq x\} = \{\omega \in \Omega \colon X(\omega) \leq x\}$ the *dual cut set* and $\{X < x\} = \{\omega \in \Omega \colon X(\omega) < x\}$ the *strict dual cut set* of X at level x.

We shall make frequent use of special classes of events, called ample fields [11, 31, 32]. An *ample field* $\mathcal{R}$ on Ω is a class of subsets of Ω that is closed under arbitrary unions and complementation. It is therefore also closed under arbitrary intersections. For any ω in Ω, the *atom* $[\,\omega\,]_{\mathcal{R}}$ of $\mathcal{R}$ containing ω is defined as $[\,\omega\,]_{\mathcal{R}} = \bigcap\{A \in \mathcal{R} \colon \omega \in A\}$. The atoms of $\mathcal{R}$ make up a partition of Ω. Interestingly, for any subset A of Ω we have that $A \in \mathcal{R}$ if and only if $A = \bigcup_{\omega \in A}[\,\omega\,]_{\mathcal{R}}$. Any element of $\mathcal{R}$ is called *$\mathcal{R}$-measurable*. A gamble X on Ω is called *$\mathcal{R}$-measurable* if its cut sets (or dual cut sets, ...) are $\mathcal{R}$-measurable, or equivalently, if it is constant on the atoms of $\mathcal{R}$.

2.2. **Possibility and necessity measures.** A *possibility measure* Π on $(\Omega, \mathcal{R})$ is a *complete join-morphism* [4] between the complete lattices $(\mathcal{R}, \subseteq)$ and $([0, 1], \leq)$. Equivalently, for any family $(A_j \colon j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j \in J} A_j) = \bigvee_{j \in J} \Pi(A_j)$, where $\vee$ is the supremum[1] on the chain $([0, 1], \leq)$. This definition implies that $\Pi(\emptyset) = 0$. Π is called *normal* if $\Pi(\Omega) = 1$. A *distribution* for Π is a $\mathcal{R}$-measurable $\Omega - [0, 1]$-mapping π for which for any A in $\mathcal{R}$: $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$. Clearly, such a distribution is unique and completely determined by $\pi(\omega) = \Pi([\,\omega\,]_{\mathcal{R}})$, $\omega \in \Omega$. Conversely, a possibility measure is uniquely determined by its distribution. When Π is defined on all subsets of Ω, i.e. $\mathcal{R} = \wp(\Omega)$, we have as a special case that $\pi(\omega) = \Pi(\{\omega\})$ for all $\omega \in \Omega$.

Dually, a *necessity measure* N on $(\Omega, \mathcal{R})$ is a *complete meet-morphism* between the complete lattices $(\mathcal{R}, \subseteq)$ and $([0, 1], \leq)$. In other words, for any family $(A_j \colon j \in J)$ of elements of $\mathcal{R}$, $\mathrm{N}(\bigcap_{j \in J} A_j) = \bigwedge_{j \in J} \mathrm{N}(A_j)$, where $\wedge$ is the infimum on $([0, 1], \leq)$. Note that by definition $\mathrm{N}(\Omega) = 1$. N is called *normal* if $\mathrm{N}(\emptyset) = 0$. A *distribution* for N is a $\mathcal{R}$-measurable $\Omega - [0, 1]$-mapping ν such that for any A in $\mathcal{R}$: $\mathrm{N}(A) = \bigwedge_{\omega \in \mathrm{co}A} \nu(\omega)$. Such a distribution is unique and determined by $\nu(\omega) = \mathrm{N}(\mathrm{co}[\,\omega\,]_{\mathcal{R}})$, $\omega \in \Omega$. Conversely, a necessity measure is uniquely determined by its distribution. For more information about possibility and necessity measures, we refer to [7, 8, 9, 14, 19, 33]. A good recent survey article is [21].

[1] We distinguish between the supremum sup on $\mathbb{R}$ and the supremum $\vee$ on $[0, 1]$ for the following reason: for any non-empty subset A of $[0, 1]$, we have $\sup A = \bigvee A$, but $\bigvee \emptyset = 0$ whereas $\sup \emptyset = -\infty$. A similar (dual) remark can be made for infimum.

2.3. **Upper and lower previsions.** We continue with a number of notions from the behavioural theory of imprecise probabilities. More details can be found in Walley's book on the subject [28]. An *upper prevision* $\overline{P}$ can be formally defined as a real-valued function on a class of gambles $\mathcal{K} \subseteq \mathcal{L}(\Omega)$. In order to identify its universe of discourse and domain, we also denote $\overline{P}$ as $(\Omega, \mathcal{K}, \overline{P})$. The *conjugate lower prevision* $\underline{P}$ is defined on $-\mathcal{K} = \{-X \colon X \in \mathcal{K}\}$ by $\underline{P}(X) = -\overline{P}(-X)$, $X \in -\mathcal{K}$. When the domain of an upper (lower) prevision is a class of events — indicators —, it will also be called an upper (lower) *probability*.

Upper and lower previsions and probabilities can be given the following behavioural interpretation[2]. A subject's upper prevision $\overline{P}(X)$ for the gamble X is the lowest real number r such that he will be disposed to sell X for any price $x > r$. Similarly, his lower prevision $\underline{P}(X)$ for X is the highest real number s such that he will be disposed to buy X for any price $y < s$. Informally, $\overline{P}(X)$ is the subject's *infimum selling price* and $\underline{P}(X)$ is his *supremum buying price* for the gamble X.

Since upper and lower previsions have this interpretation in terms of behaviour, they are subject to criteria of rationality. The basic rationality requirement is that if a subject is disposed to accept a finite number of gambles $X_1, \ldots, X_n$, he should also be disposed to accept a non-negative linear combination $\sum_{k=1}^{n} \lambda_k X_k$ of them (here $\lambda_1, \ldots, \lambda_n$ are non-negative real numbers). This basic requirement leads to the following rationality criteria which will be imposed on upper and lower previsions.

An upper prevision $(\Omega, \mathcal{K}, \overline{P})$ is said to *avoid sure loss* if for any $X_1, \ldots, X_n$ in $\mathcal{K}$, $n \geq 1$, $\sup[\sum_{k=1}^{n} G(X_k)] \geq 0$. In this expression, we have used the notation $G(X) = \overline{P}(X) - X$. It follows from the definition of $\overline{P}(X)$ that the subject will be disposed to accept the gamble $G(X) + \epsilon$ for any $\epsilon > 0$. To see why this condition is a significant consistency criterion, let us assume that it fails to hold. Then there are $n \geq 1$, $X_1, \ldots, X_n$ in $\mathcal{K}$ and $\epsilon > 0$ such that the sum of the acceptable gambles $G(X_1) + \epsilon, \ldots, G(X_n) + \epsilon$ is strictly negative: $\sum_{k=1}^{n} [G(X_k) + \epsilon] < -\epsilon$. This sum should be acceptable, so the subject can be induced to accept a gamble which is certain to produce an overall loss! This can only be avoided by imposing the above condition.

$(\Omega, \mathcal{K}, \overline{P})$ is called *coherent* if for any $m \geq 0$, $n \geq 0$ and $X_o, X_1, \ldots, X_n$ in $\mathcal{K}$, $\sup[\sum_{k=1}^{n} G(X_k) - mG(X_o)] \geq 0$. In that case, $\overline{P}$ obviously also avoids sure loss (let $m = 0$). To justify this condition, assume that it fails for some $m > 0$. Then there are $n \geq 0$, $X_o, X_1, \ldots, X_n$ in $\mathcal{K}$ and $\epsilon > 0$ such that

$$\sum_{k=1}^{n} [G(X_k) + \epsilon] \leq m[(\overline{P}(X_o) - \epsilon) - X_o].$$

Since the left-hand side is acceptable, the right-hand side will be acceptable too, as it represents a gamble with a uniformly higher net reward. This means that the subject should be effectively disposed to sell X_o at a price $\overline{P}(X_o) - \epsilon$, which is strictly lower than his infimum selling price $\overline{P}(X_o)$: in specifying $\overline{P}(X_o)$ the subject did not take into account the implications of his other upper prevision assessments.

[2]The interpretation is minimal, because it is allowed that upper and lower previsions have other interpretations besides a behavioural one; how they are obtained is practically irrelevant, provided that they can (also) be interpreted as infimum selling prices and supremum buying prices. Similarly, upper and lower previsions need not be maximally precise. It is not excluded that a more careful analysis of the available information may lead to a more precise model.

This produces a kind of logical inconsistency, which is not as bad as incurring a sure loss, but should nevertheless be avoided and/or corrected.

Finally, a lower prevision is said to avoid sure loss if its conjugate upper prevision does, and is called coherent if its conjugate upper prevision is.

We shall often use the following consequences of avoiding sure loss and coherence. If $(\Omega,\mathcal{K},\overline{P})$ avoids sure loss, then $\emptyset\in\mathcal{K}\Rightarrow\overline{P}(\emptyset)\geq 0$ and $\Omega\in\mathcal{K}\Rightarrow\overline{P}(\Omega)\geq 1$. If it is coherent, we find that $\emptyset\in\mathcal{K}\Rightarrow\overline{P}(\emptyset)=0$ and $\Omega\in\mathcal{K}\Rightarrow\overline{P}(\Omega)=1$. Coherent upper previsions also have the following interesting properties, which will be useful further on. If $\overline{P}$ is a coherent upper prevision defined on an arbitrary set $\mathcal{K}$ of gambles, it satisfies:

1. $\inf[X]\leq\overline{P}(X)\leq\sup[X]$;
2. if $X\leq Y$ then $\overline{P}(X)\leq\overline{P}(Y)$ [monotonicity];
3. if $X+Y\in\mathcal{K}$ then $\overline{P}(X+Y)\leq\overline{P}(X)+\overline{P}(Y)$ [sub-additivity];
4. if $\lambda>0$ and $\lambda X\in\mathcal{K}$ then $\overline{P}(\lambda X)=\lambda\overline{P}(X)$ [positive homogeneity];
5. if $-X\in\mathcal{K}$ then $\overline{P}(X)+\overline{P}(-X)\geq 0$, or equivalently, $\underline{P}(X)\leq\overline{P}(X)$;

for all X and Y in $\mathcal{K}$. The following properties of a coherent upper probability $\overline{P}$ on a set $\mathcal{S}$ of events follow at once:

1. $0\leq\overline{P}(A)\leq 1$;
2. if $A\subseteq B$ then $\overline{P}(A)\leq\overline{P}(B)$ [monotonicity];
3. if $\mathrm{co}A\in\mathcal{S}$ then $\overline{P}(A)+\overline{P}(\mathrm{co}A)\geq 1$, or equivalently, $\underline{P}(A)\leq\overline{P}(A)$;

for any A and B in $\mathcal{S}$. A much more detailed discussion of the consequences of coherence can be found in [28].

2.4. **Linear previsions.** A real-valued function P on a class of gambles $\mathcal{K}$ is called a *linear prevision* on $\mathcal{K}$ if for any $X_1,\dots,X_n$, $Y_1,\dots,Y_m$ in $\mathcal{K}$, where $n\geq 0$ and $m\geq 0$, $\sup[\sum_{k=1}^{n}G(X_k)-\sum_{\ell=1}^{m}G(Y_\ell)]\geq 0$, where $G(X)=P(X)-X$. A linear prevision is coherent both when interpreted as an upper and as a lower prevision. Moreover, an upper prevision $(\Omega,\mathcal{K},\overline{P})$ with $\mathcal{K}=-\mathcal{K}$ is a linear prevision on $\mathcal{K}$ if and only if it avoids sure loss and is *self-conjugate*, i.e. $\overline{P}(X)=-\overline{P}(-X)=\underline{P}(X)$. Also, a linear prevision P on $\mathcal{L}(\Omega)$ is a positive ($X\geq 0\Rightarrow P(X)\geq 0$) linear functional with unit norm ($P(1)=1$) on the linear space $\mathcal{L}(\Omega)$. Its restriction to an arbitrary class of gambles $\mathcal{K}$ is a linear prevision on $\mathcal{K}$. In particular, its restriction to $\wp(\Omega)$ is a finitely additive probability. Conversely, if $(\Omega,\mathcal{K},P)$ is a linear prevision, then it is the restriction to $\mathcal{K}$ of some linear prevision defined on $\mathcal{L}(\Omega)$. The linear previsions are the precise probability models, and they are previsions or 'fair prices' in the sense of de Finetti [15].

We denote the set of all linear previsions on $\mathcal{L}(\Omega)$ by $\mathcal{P}(\Omega)$. With an upper prevision $(\Omega,\mathcal{K},\overline{P})$, we may associate a set $\mathcal{M}(\overline{P})$ of *dominated linear previsions* as follows:

$$\mathcal{M}(\overline{P})=\{P\in\mathcal{P}(\Omega)\colon(\forall X\in\mathcal{K})(P(X)\leq\overline{P}(X))\}.$$

Walley [28, Theorem 3.3.3] has shown that $(\Omega,\mathcal{K},\overline{P})$ avoids sure loss if and only if $\mathcal{M}(\overline{P})\neq\emptyset$, and that $(\Omega,\mathcal{K},\overline{P})$ is coherent if and only if it is the upper envelope of $\mathcal{M}(\overline{P})$, i.e. if for all $X\in\mathcal{K}$, $\overline{P}(X)=\sup\{P(X)\colon P\in\mathcal{M}(\overline{P})\}$.

2.5. **Natural extension.** If the upper prevision $(\Omega,\mathcal{K},\overline{P})$ avoids sure loss, its *natural extension* $(\Omega,\mathcal{L}(\Omega),\overline{E})$ is the greatest coherent upper prevision on $\mathcal{L}(\Omega)$ that is dominated by $\overline{P}$ on $\mathcal{K}$ [28, Theorem 3.1.2]. It is then given by $\overline{E}(X)=\sup\{P(X)\colon P\in\mathcal{M}(\overline{P})\}$, for $X\in\mathcal{L}(\Omega)$. If moreover $(\Omega,\mathcal{K},\overline{P})$ is coherent, $\overline{E}$ is the greatest coherent upper prevision on $\mathcal{L}(\Omega)$ that coincides with $\overline{P}$ on $\mathcal{K}$. From a behavioural point of view, the process of natural extension is very important, because it is *least committal*. In other words, it enables us to extend upper previsions to a larger domain, *taking only coherence into account*. Any coherent upper prevision $\overline{P}'$ on $\mathcal{L}(\Omega)$ that extends a coherent $\overline{P}$ on $\mathcal{K}$ has behavioural implications that are at least as strong as the ones in the natural extension $\overline{E}$: $\overline{P}'$ implies a willingness to sell gambles for prices that are lower than or equal to the prices implicit in $\overline{E}$.

For the natural extension $\underline{E}$ of the conjugate lower prevision $(\Omega, -\mathcal{K}, \underline{P})$, which is the smallest coherent lower prevision on $\mathcal{L}(\Omega)$ that dominates $\underline{P}$ on its domain $-\mathcal{K}$, we have that $\underline{E}(X) = \inf\{P(X) \colon P \in \mathcal{M}(\overline{P})\}$, $X \in \mathcal{L}(\Omega)$.

This discussion of upper and lower previsions, coherence and natural extension is necessarily very limited. Only very little attention has been paid to the behavioural interpretation of these notions. For a much more detailed exposition of the behavioural theory of imprecise probabilities, we refer to Walley's book [28]. A reasonable knowledge of the material covered there is very helpful for a proper understanding of much of what follows.

2.6. **Set functions.** We call *set function* a mapping μ from a collection of events $\mathcal{S}$ to the unit interval $[0, 1]$. μ is called *monotone* if for any A and B in $\mathcal{S}$, $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$. A coherent upper probability is in particular a monotone set function.

μ is called *sub-modular* if for any A and B in $\mathcal{S}$ such that $A \cap B \in \mathcal{S}$ and $A \cup B \in \mathcal{S}$, $\mu(A \cap B) + \mu(A \cup B) \leq \mu(A) + \mu(B)$, and *modular* if the equality holds in the above condition.

If $\mathcal{S}$ is an algebra (or field of sets), then μ is called *2-alternating* [3, 27, 28] if $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and μ is sub-modular. In particular, it will be shown in Section 3 that a normal possibility measure is a 2-alternating set function.

Given a monotone set function μ, its *outer set function*[3] $\mu^* \colon \wp(\Omega) \to [0, 1]$ can be defined as follows: $\mu^*(\emptyset) = 0$ and if A is a non-empty subset of Ω, then $\mu^*(A) = \bigwedge\{\mu(B) \colon B \in \mathcal{S} \text{ and } A \subseteq B\}$. Providing that $\emptyset \in \mathcal{S} \Rightarrow \mu(\emptyset) = 0$ holds, μ^* is a monotone extension of μ to $\wp(\Omega)$. Indeed, if we ignore its behaviour at $\emptyset$, it is the greatest such extension.

3. The coherence of possibility measures

In this section, we investigate when possibility measures, interpreted as upper probabilities, avoid sure loss, and when they are coherent. We also look at the natural extension of possibility measures to larger domains of events and of gambles.

First of all, consider a possibility measure Π defined on an ample field $\mathcal{R}$ of subsets of Ω. We denote its distribution by π. We can interpret the set function Π as an upper probability. The *conjugate lower probability* N, defined by $\mathrm{N}(A) = 1 - \Pi(\mathrm{co}A)$, $A \in \mathcal{R}$, is a necessity measure, with distribution $\nu = 1 - \pi$. In what follows, we shall mainly be concerned with possibility measures. Of course, any result given for possibility measures interpreted as upper probabilities has its dual (or conjugate) counterpart for necessity measures interpreted as lower probabilities.

It turns out that coherence is guaranteed if a subject specifies his beliefs in the form of a *normal* possibility measure on an ample field.

[3]Outer set functions are mostly defined for set functions that are zero at the empty set, see for instance [16]. We define $\mu^*(\emptyset) = 0$ here for the sake of convenience (see also Theorem 8).

Theorem 1. *Let $\mathcal{R}$ be an ample field on Ω, and let Π be a possibility measure on $(\Omega, \mathcal{R})$. The following statements are equivalent:*

1. Π *is normal;*
2. *the upper probability* $(\Omega, \mathcal{R}, \Pi)$ *avoids sure loss;*
3. $(\Omega, \mathcal{R}, \Pi)$ *is a coherent upper probability.*

This also implies that a normal possibility measure Π on $(\Omega, \mathcal{R})$ is an upper envelope of a class of linear previsions (or finitely additive probabilities). Indeed, if we consider the set $\mathcal{M}(\Pi) = \{P \in \mathcal{P}(\Omega)\colon (\forall A \in \mathcal{R})(P(A) \leq \Pi(A))\}$, then Π is the upper envelope of $\mathcal{M}(\Pi)$ and its conjugate N the lower envelope of $\mathcal{M}(\Pi)$, i.e. for any A in $\mathcal{R}$:

$$\Pi(A) = \sup\{P(A)\colon P \in \mathcal{M}(\Pi)\} \text{ and } \mathrm{N}(A) = \inf\{P(A)\colon P \in \mathcal{M}(\Pi)\}.$$

Note by the way that $\mathcal{M}(\mathrm{N}) = \{P \in \mathcal{P}(\Omega)\colon (\forall A \in \mathcal{R})(\mathrm{N}(A) \leq P(A))\} = \mathcal{M}(\Pi)$. If we denote by Π also the natural extension of Π to the set $\mathcal{L}(\Omega)$, and by N the natural extension of N, we know furthermore that for any $X \in \mathcal{L}(\Omega)$:

$$\Pi(X) = \sup\{P(X)\colon P \in \mathcal{M}(\Pi)\} \text{ and } \mathrm{N}(X) = \inf\{P(X)\colon P \in \mathcal{M}(\Pi)\}.$$

In the rest of this section, we present a number of formulas which facilitate the calculation of these natural extensions. Let us first concentrate on the natural extension of normal Π and N to the set of the $\mathcal{R}$-measurable gambles on Ω. Recall that a gamble X on Ω is called *$\mathcal{R}$-measurable* if $\{X \leq x\} \in \mathcal{R}$ for every real x, or equivalently, if X is constant on the atoms of $\mathcal{R}$. For any $\mathcal{R}$-measurable gamble X, we may therefore define the *lower* and *upper cumulative distribution functions* of X under Π and N as follows:

$$\begin{aligned}
\underline{F}_X(x) &= \mathrm{N}(\{X \leq x\}) = 1 - \bigvee\{\pi(\omega)\colon X(\omega) > x\} \\
\overline{F}_X(x) &= \Pi(\{X \leq x\}) = 1 - \bigwedge\{\nu(\omega)\colon X(\omega) \leq x\}.
\end{aligned}$$

Since the coherent upper probability $(\Omega, \mathcal{R}, \Pi)$ is 2-alternating (see the proof of Theorem 1 in the Appendix), we may use a result by Walley concerning the natural extension of 2-alternating upper probabilities [28, Section 3.2.4], which states that the natural extensions of Π and N to an $\mathcal{R}$-measurable gamble X are given by the following *Choquet integrals*:

$$\begin{aligned}
\Pi(X) &= \int_{-\infty}^{+\infty} x \mathrm{d}\underline{F}_X(x) = \inf[X] + \int_{\inf[X]}^{\sup[X]} 1 - \underline{F}_X(x) \mathrm{d}x \\
&= \inf[X] + \int_{\inf[X]}^{\sup[X]} \bigvee\{\pi(\omega)\colon X(\omega) > x\} \mathrm{d}x \\
&= \inf[X] + \int_{\inf[X]}^{\sup[X]} \bigvee\{\pi(\omega)\colon X(\omega) \geq x\} \mathrm{d}x \qquad (1)
\end{aligned}$$

and

$$\begin{aligned}\mathrm{N}(X) &= \int_{-\infty}^{+\infty} x\mathrm{d}\overline{F}_X(x) = \inf[X] + \int_{\inf[X]}^{\sup[X]} 1 - \overline{F}_X(x)\mathrm{d}x \\ &= \inf[X] + \int_{\inf[X]}^{\sup[X]} \bigwedge\{\nu(\omega)\colon X(\omega) \leq x\}\mathrm{d}x \\ &= \inf[X] + \int_{\inf[X]}^{\sup[X]} \bigwedge\{\nu(\omega)\colon X(\omega) < x\}\mathrm{d}x, \qquad (2)\end{aligned}$$

where the last equalities in (1) and (2) hold because the integrands considered may only differ in their points of discontinuity, which are at most countable in number. Note the order-duality between (1) and (2).

Moreover, since the set of $\mathcal{R}$-measurable gambles is a linear space containing all constant gambles, we deduce from [28, Theorem 3.1.4] that for the natural extensions of Π and N to $\mathcal{L}(\Omega)$:

$$\Pi(X) = \inf\{\Pi(Y)\colon X \leq Y \text{ and } Y \text{ is } \mathcal{R}\text{-measurable}\} = \Pi(X^{\uparrow})$$
$$\mathrm{N}(X) = \sup\{\mathrm{N}(Y)\colon Y \leq X \text{ and } Y \text{ is } \mathcal{R}\text{-measurable}\} = \mathrm{N}(X^{\downarrow})$$

for any gamble X on Ω. In this expression, we have used the notations $X^{\uparrow}(\omega) = \sup_{\varpi\in[\,\omega\,]_{\mathcal{R}}} X(\varpi)$ and $X^{\downarrow}(\omega) = \inf_{\varpi\in[\,\omega\,]_{\mathcal{R}}} X(\varpi)$, $\omega \in \Omega$. Clearly, $X^{\uparrow}$ and $X^{\downarrow}$ are $\mathcal{R}$-measurable for any gamble X. In particular, we find[4] for any event A that $\Pi(A) = \Pi(A^{\uparrow})$ and $\mathrm{N}(A) = \mathrm{N}(A^{\downarrow})$, where $A^{\uparrow} = \bigcup_{\omega\in A}[\,\omega\,]_{\mathcal{R}}$, and $A^{\downarrow} = \bigcap_{\omega\in \mathrm{co}A} \mathrm{co}[\,\omega\,]_{\mathcal{R}}$.

Observation 1. *The natural extension of a normal possibility (necessity) measure on $(\Omega, \mathcal{R})$ to any ample field which includes $\mathcal{R}$ is still a normal possibility (necessity) measure, with the same distribution.*

This also means that natural extension essentially preserves possibility measures and their distributions in the process of refining a possibility space.

In the following sections we generalise this result by answering the following two questions. Consider an upper probability $(\Omega, \mathcal{S}, \overline{P})$ on a class of events $\mathcal{S}$ which is not necessarily an ample field. Is it possible to extend this upper probability to a normal possibility measure on an ample field which includes $\mathcal{S}$? And if so, what is the relation between this so-called *possibilistic extension* and the natural extension of $(\Omega, \mathcal{S}, \overline{P})$?

4. Natural and possibilistic extension

Let us consider an arbitrary non-empty and not necessarily finite set S, a so-called *multi-valued mapping* $\Gamma\colon S \to \wp(\Omega)$, and a mapping $\xi\colon S \to [0,1]$ assuming values in the real unit interval. If we assume that

$$(\forall (s,t) \in S^2)(\Gamma(s) = \Gamma(t) \Rightarrow \xi(s) = \xi(t)) \qquad \text{(N0)}$$

then ξ may serve as an *upper probability assignment*[5] on the values of Γ. In other words, we may consider the upper probability $(\Omega, \Gamma(S), \overline{P}_{\Gamma})$ defined by

$$\overline{P}_{\Gamma}(\Gamma(s)) = \xi(s), \quad s \in S,$$

[4] This in fact states that the natural extension of Π to $\wp(\Omega)$ is its outer set function.

[5] Recall from Section 2.3 that a coherent upper *probability* can only assume values in the unit interval.

where we have used the common notation $\Gamma(S) = \{\Gamma(s)\colon s \in S\}$ for the direct image of the set S under the mapping Γ.

First of all, if $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ avoids sure loss[6], we may consider the natural extension $(\Omega, \mathcal{L}(\Omega), \overline{E}_\Gamma)$ of this upper probability to the set of all gambles. Recall that it is the greatest coherent upper prevision that is dominated by $\overline{P}_\Gamma$ on its domain $\Gamma(S)$. Moreover, if $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a coherent upper probability, then $\overline{E}_\Gamma$ and $\overline{P}_\Gamma$ coincide on $\Gamma(S)$.

On the other hand, since we are interested in the relation between natural extension and possibility measures, it is fairly natural to look for possibility measures that are dominated by, or equal to, the upper probability $\overline{P}_\Gamma$ on its domain $\Gamma(S)$. To put it more precisely, we want to find out whether there is a possibility measure Π, defined on some ample field $\mathcal{R}$ of subsets of Ω such that $\Pi \circ \Gamma \leq \xi$, i.e.

$$(\forall s \in S)(\Pi(\Gamma(s)) \leq \xi(s)), \tag{3}$$

or more stringently

$$(\forall s \in S)(\Pi(\Gamma(s)) = \xi(s)), \tag{4}$$

which can be rewritten as $\Pi \circ \Gamma = \xi$. The requirements $\Pi \circ \Gamma \leq \xi$ and $\Pi \circ \Gamma = \xi$ presuppose that $\Gamma(S) = \{\Gamma(s)\colon s \in S\} \subseteq \mathcal{R}$ or equivalently $\mathcal{R}_\Gamma \subseteq \mathcal{R}$, where $\mathcal{R}_\Gamma$ is the ample field generated by $\Gamma(S)$, i.e. the smallest ample field for which the sets $\Gamma(s)$, $s \in S$, are all measurable.

Solving (4) amounts to extending $\overline{P}_\Gamma$ to a possibility measure, and (4) is therefore called a *possibilistic extension problem*. We summarise what is known about this problem in the following definition and theorem, due to Wang [32]. Generalisations of these results in a more general context have been given by Boyen *et al.* [2]. We want to mention in passing that the distribution π^g in Theorem 3 is also the basic tool in the construction of a semantics for the possibilistic logic of Dubois *et al.* [17]. In possibilistic logic, the inequalities (3) are written in the equivalent form $\mathrm{N}(\mathrm{co}\Gamma(s)) \geq 1 - \xi(s)$.

Definition 2. The upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is called P-consistent if for any $s \in S$ and $T \subseteq S$,

$$\Gamma(s) \subseteq \bigcup_{t \in T} \Gamma(t) \Rightarrow \xi(s) \leq \bigvee_{t \in T} \xi(t).$$

Theorem 3. *Let $\mathcal{R}$ be an ample field which includes $\Gamma(S)$, i.e. $\mathcal{R}_\Gamma \subseteq \mathcal{R}$. The greatest solution of the system of inequalities* (3) *is the possibility measure Π^g on $(\Omega, \mathcal{R})$ with distribution π^g, given by*

$$\pi^g(\omega) = \bigwedge_{s \in S, \omega \in \Gamma(s)} \xi(s) = \bigwedge_{A \in \Gamma(S), \omega \in A} \overline{P}_\Gamma(A).$$

Moreover, the following statements are equivalent:

1. *the system of equations* (4) *has a solution;*
2. Π^g *is the greatest solution of* (4)*;*
3. $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ *is P-consistent.*

[6] If an upper probability does not avoid sure loss, then its natural extension assumes the value $-\infty$ everywhere [28, Definition 3.1.1].

It should be noted that the distribution π^g is constant on the atoms of $\mathcal{R}_\Gamma$ and therefore also constant of the atoms of $\mathcal{R} \supseteq \mathcal{R}_\Gamma$. It is the same for every such choice of $\mathcal{R}$.

We are primarily interested in possibility measures that avoid sure loss, and are therefore coherent, as upper probabilities. This leads to the question whether (3) and (4) admit normal solutions. For obvious reasons, we call the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ *supremum-normalisable* if

$$\Pi^g(\Omega) = \bigvee_{\omega\in\Omega} \bigwedge_{s\in S, \omega\in\Gamma(s)} \xi(s) = \bigvee_{\omega\in\Omega} \bigwedge_{A\in\Gamma(S), \omega\in A} \overline{P}_\Gamma(A) = 1.$$

Theorem 4. *Let $\mathcal{R}$ be an ample field which includes $\Gamma(S)$, i.e. $\mathcal{R}_\Gamma \subseteq \mathcal{R}$. The system of inequalities* (3) *admits normal solutions if and only if the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is supremum-normalisable. The system of equations* (4) *admits normal solutions if and only if the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is P-consistent and supremum-normalisable.*

We are therefore mainly interested in P-consistent and supremum-normalisable upper probabilities. It should be noted that *these conditions imply the coherence of* $(\Omega, \Gamma(S), \overline{P}_\Gamma)$, because they assure that there is a normal and therefore coherent possibility measure that coincides with $\overline{P}_\Gamma$ on its domain $\Gamma(S)$.

Observation 2. *A P-consistent and supremum-normalisable upper probability is coherent.*

Note that $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is always supremum-normalisable if $\Gamma(S)$ does not cover Ω. Indeed, if $\bigcup \Gamma(S) \subset \Omega$, there is an ω in Ω such that $(\forall s \in S)(\omega \notin \Gamma(s))$, whence $\pi^g(\omega) = 1$, and therefore $\Pi^g(\Omega) = 1$. On the other hand, if $\Gamma(S)$ covers Ω a necessary condition[7] for supremum-normalisability is that for any $T \subseteq S$, $\bigcup \Gamma(T) = \Omega \Rightarrow \bigvee_{t\in T} \xi(t) = 1$. If ξ assumes only a finite number of values, a necessary and sufficient condition for supremum-normalisability is that there is an ω in Ω such that $(\forall s \in S)(\omega \in \Gamma(s) \Rightarrow \xi(s) = 1)$, or equivalently, $(\forall A \in \Gamma(S))(\omega \in A \Rightarrow \overline{P}_\Gamma(A) = 1)$. Both P-consistency and supremum-normalisability are obviously rather strong requirements.

Let us now consider the set of linear previsions that are dominated by a possibility measure Π on its domain $\mathcal{R}$:

$$\mathcal{M}(\Pi) = \{P \in \mathcal{P}(\Omega) \colon (\forall A \in \mathcal{R})(P(A) \leq \Pi(A))\}.$$

We know from Theorem 1 that $\mathcal{M}(\Pi) \neq \emptyset$ if and only if Π is normal. In general, there are the following relations between the sets $\mathcal{M}(\Pi)$ and $\mathcal{M}(\overline{P}_\Gamma)$ for solutions Π of the system of inequalities (3).

Proposition 5. *Let $\mathcal{R}$ be an ample field on Ω such that $\Gamma(S) \subseteq \mathcal{R}$, or equivalently, $\mathcal{R}_\Gamma \subseteq \mathcal{R}$. Let Π be a possibility measure on $(\Omega, \mathcal{R})$. If Π is a solution of the system of inequalities* (3) *then $\mathcal{M}(\Pi) \subseteq \mathcal{M}(\overline{P}_\Gamma)$. Moreover, if Π is normal, then Π is a solution of* (3) *if and only if $\mathcal{M}(\Pi) \subseteq \mathcal{M}(\Pi^g) \subseteq \mathcal{M}(\overline{P}_\Gamma)$.*

We may conclude from this discussion that in a behavioural context, possibilistic extension is only of interest when the original upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is P-consistent and supremum-normalisable. In that case, possibilistic extension Π^g is

[7] In possibilistic logic, the failure of this condition relates to a so-called inconsistent possibilistic knowledge base: a set of constraints $\{\mathrm{N}(A_i) \geq \alpha_i \colon i = 1, \ldots, n\}$ where $\bigcap_{i=1}^n A_i = \emptyset$ and $\min_{i=1}^n \alpha_i > 0$.

generally more precise and therefore more committal than natural extension $\overline{E}_\Gamma$: for all $A \subseteq \Omega$, $\Pi^g(A) \leq \overline{E}_\Gamma(A)$. But it turns out that if $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ can be extended to a normal possibility measure, there is an interesting relation between the greatest such possibilistic extension Π^g and the natural extension $\overline{E}_\Gamma$. Another interesting connection between the two types of extension will be discussed in Section 7.3.

Theorem 6. *Let $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ be a P-consistent and supremum-normalisable upper probability. Then for any ω in Ω, $\overline{E}_\Gamma(\{\omega\}) = \overline{E}_\Gamma([\,\omega\,]_{\mathcal{R}_\Gamma}) = \pi^g(\omega)$.*

The existence of this special relationship between natural and possibilistic extension can make us wonder in what special cases natural and possibilistic extension coincide, or equivalently, for which kind of upper probabilities the natural extension is a possibility measure. Such upper probabilities could be called *possibilistic* in nature.

In the following sections, we take a closer look at the relation between possibilistic and natural extension in an important special case, namely where Γ is increasing, so that $\Gamma(S)$ is a chain of sets.

5. Natural extension for nested sets

Consider the following special case of the possibilistic extension problem described in the previous section. We assume that S is totally ordered by a relation $\leq$, or more specifically, that $(S, \leq)$ is a bounded[8] chain. The top of this structure is denoted by 1_S, its bottom by 0_S. We also assume that the mapping $\Gamma\colon S \to \wp(\Omega)$ is increasing in the following sense:

$$(\forall (s,t) \in S^2)(s \leq t \Rightarrow \Gamma(s) \subseteq \Gamma(t)). \tag{N1}$$

This guarantees that $(\Gamma(S), \subseteq)$ is a chain of subsets of Ω, or in other words, a collection of *nested sets*[9]. On the mapping ξ from S to $[0,1]$ we impose the following condition, which expresses that ξ should follow the behaviour of Γ:

$$(\forall (s,t) \in S^2)(\Gamma(s) \subseteq \Gamma(t) \Rightarrow \xi(s) \leq \xi(t)). \tag{N2}$$

If (N1) holds, (N2) is equivalent to the following conditions, which mean that ξ should be increasing, and should be constant wherever Γ is:

$$(\forall (s,t) \in S^2)(s \leq t \Rightarrow \xi(s) \leq \xi(t)) \text{ and } (\forall (s,t) \in S^2)(\Gamma(s) = \Gamma(t) \Rightarrow \xi(s) = \xi(t)).$$

The last condition is precisely (N0), so we may interpret ξ as an assignment of upper probability to the nested sets $\Gamma(s)$, $s \in S$. In this section, we determine the *natural extension* $(\Omega, \mathcal{L}(\Omega), \overline{E}_\Gamma)$ of the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$. Of course, we want $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ to avoid sure loss. From the definition, it follows that a necessary condition for avoiding sure loss is that

$$\Omega \in \Gamma(S) \Rightarrow \overline{P}_\Gamma(\Omega) = 1. \tag{N3}$$

For coherence it is moreover necessary that

$$\emptyset \in \Gamma(S) \Rightarrow \overline{P}_\Gamma(\emptyset) = 0. \tag{N4}$$

[8]Boundedness is not an essential requirement, since any chain can be trivially extended to a bounded one.

[9]In general, nested sets can be described by a multi-valued mapping Γ that is either increasing or decreasing. The case of decreasing Γ can be formally converted to the increasing case, by considering the dual order relation $\leq^d = \geq$ on S. This trick is exploited in Section 7.7.

We presently show that these conditions are also sufficient. Note that by (N1) and (N2), condition (N3) is equivalent to $\Gamma(1_S) = \Omega \Rightarrow \xi(1_S) = 1$, and condition (N4) is equivalent to $\Gamma(0_S) = \emptyset \Rightarrow \xi(0_S) = 0$.

Proposition 7. *Let $(S, \leq)$ be a bounded chain and assume that Γ and ξ satisfy* (N1) *and* (N2)*. Then the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a linear prevision (finitely additive probability) if and only if it satisfies* (N3) *and* (N4)*.*

As a corollary, we find that (N3) and (N4) are necessary and sufficient for the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ to be coherent. Similarly, it follows quite easily that (N3) is a criterion for $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ to avoid sure loss[10]. We must therefore impose (N3) in order to discuss natural extension.

Theorem 8. *Let $(S, \leq)$ be a bounded chain. Assume that Γ and ξ satisfy* (N1)*,* (N2) *and* (N3)*. The natural extension $(\Omega, \mathcal{L}(\Omega), \overline{E}_\Gamma)$ of the upper probability $\overline{P}_\Gamma$ coincides on $\wp(\Omega)$ with the outer set function $\overline{P}^*_\Gamma$ of $\overline{P}_\Gamma$, i.e. $\overline{E}_\Gamma(\emptyset) = 0$ and for any non-empty $A \subseteq \Omega$:*

$$\overline{E}_\Gamma(A) = \overline{P}^*_\Gamma(A) = \bigwedge\{\xi(s) \colon s \in S \text{ and } A \subseteq \Gamma(s)\}.$$

$\overline{E}_\Gamma$ coincides with $\overline{P}_\Gamma$ on $\Gamma(S) \setminus \{\emptyset\}$. Moreover, $\overline{E}_\Gamma$ is 2-alternating on $\wp(\Omega)$. $\overline{E}_\Gamma$ is even maxitive: $\overline{E}_\Gamma(A \cup B) = \max\{\overline{E}_\Gamma(A), \overline{E}_\Gamma(B)\}$ for all subsets A and B of Ω. For any X in $\mathcal{L}(\Omega)$,

$$\overline{E}_\Gamma(X) = \inf[X] + \int_{\inf[X]}^{\sup[X]} \bigwedge\{\xi(s) \colon s \in S \text{ and } \{X > x\} \subseteq \Gamma(s)\} \mathrm{d}x.$$

Finally, if (N4) *holds, then $\overline{E}_\Gamma$ coincides with $\overline{P}_\Gamma$ on $\Gamma(S)$.*

If (N3) holds, we see that $\overline{E}_\Gamma$ *coincides* with $\overline{P}_\Gamma$ on every element of $\Gamma(S)$, except possibly on $\emptyset$. If $\overline{P}_\Gamma$ is incoherent, the incoherence is localised in $\emptyset$ because $\overline{P}_\Gamma(\emptyset) > 0$, and this is corrected by natural extension since $\overline{E}_\Gamma(\emptyset) = 0$.

6. The possibilistic extension problem for nested sets

In the previous section, we have identified the natural extension $(\Omega, \mathcal{L}(\Omega), \overline{E}_\Gamma)$ of the upper probability $\overline{P}_\Gamma$ on the chain of sets $\Gamma(S)$, where $(S, \leq)$ is a bounded chain and Γ and ξ satisfy the conditions (N1), (N2) and (N3). We already know from Theorem 8 that $\overline{E}_\Gamma$ is maxitive on $\wp(\Omega)$. Let us now find out under what conditions the restriction of this natural extension to $\wp(\Omega)$ is a normal possibility measure. This will clearly be the case if and only if for any non-empty A in $\wp(\Omega)$, $\overline{E}_\Gamma(A) = \bigvee_{\omega \in A} \overline{E}_\Gamma(\{\omega\})$, which is equivalent to

$$\bigwedge_{s \in S, A \subseteq \Gamma(s)} \xi(s) \leq \bigvee_{\omega \in A} \bigwedge_{s \in S, \omega \in \Gamma(s)} \xi(s). \tag{5}$$

A necessary condition is of course that $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ should be extendable to a normal possibility measure, i.e. that $\overline{P}_\Gamma$ should be P-consistent and supremum-normalisable. Condition (5) is rather forbidding from the point of view of interpretation. Moreover, verifying whether given Γ and ξ satisfy this condition will in general not be an easy matter.

In this section, we look at two special cases, in which condition (5) takes a decidedly simpler form, and which are at the same time sufficiently general to cover

[10]To see this, use the same trick as in the proof of Theorem 13.1 (Appendix A4).

a number of interesting and practical situations. The first restriction we impose, is that the (bounded) chain $(S, \leq)$ should be *complete*, i.e. that for any subset T of S, its infimum $\inf T$ and its supremum $\sup T$ exist. Using the multi-valued mapping Γ, we may then define the $\Omega - S$-mappings α and β as follows:

$$\alpha(\omega) = \inf\{s \in S \colon \omega \in \Gamma(s)\} \text{ and } \beta(\omega) = \sup\{s \in S \colon \omega \notin \Gamma(s)\}, \quad \omega \in \Omega.$$

For any ω in Ω, $\beta(\omega) \leq \alpha(\omega)$, and there is no t in S such that $\beta(\omega) < t < \alpha(\omega)$. In other words, $\alpha(\omega)$ either covers[11] or equals $\beta(\omega)$. For any s in S,

$$\{\omega \in \Omega \colon \beta(\omega) < s\} \subseteq \Gamma(s) \subseteq \{\omega \in \Omega \colon \alpha(\omega) \leq s\}.$$

Note finally that if $\alpha(\omega)$ covers $\beta(\omega)$, then $\omega \in \Gamma(\alpha(\omega))$ and $\omega \notin \Gamma(\beta(\omega))$, and in the expression above both inclusions are actually equalities. On the other hand, if $\alpha(\omega) = \beta(\omega)$, then obviously ω belongs to both $\Gamma(\alpha(\omega))$ and $\Gamma(\beta(\omega))$, or to none of them.

This discussion leads us to consider the following two special cases, which will be dealt with in separate subsections.

6.1. **A family of dual cut sets.** We assume that $(S, \leq)$ is a complete chain, and that Γ and ξ satisfy conditions (N1) and (N2), implying (N0). We also require that Γ should satisfy the additional condition

$$(\forall \omega \in \Omega)(\omega \in \Gamma(\alpha(\omega))). \tag{N5}$$

Conditions (N1) and (N5) together are equivalent to the following continuity condition:

$$\Gamma(\inf T) = \bigcap_{t \in T} \Gamma(t), \quad T \subseteq S. \tag{6}$$

In particular, this implies that $\Gamma(1_S) = \Omega$, and that $\Gamma(S)$ is closed under arbitrary intersections, i.e. $\Gamma(S)$ constitutes a topped intersection structure, or closure system [4]. The following result tells us that there is an interesting relation between the $\Omega - S$-mapping α and the sets $\Gamma(s)$, $s \in S$. It ensures that for any s in S, $\Gamma(s) = \{\omega \in \Omega \colon \alpha(\omega) \leq s\}$, or in other words, that the $\Gamma(s)$ are generalised dual cut sets for the mapping α.

Proposition 9. *Let $(S, \leq)$ be a complete chain. Assume that Γ satisfies* (N1) *and* (N5)*. For any s in S and ω in Ω, $\omega \in \Gamma(s) \Leftrightarrow \alpha(\omega) \leq s$.*

The question we eventually want to answer is under what conditions the restriction to $\wp(\Omega)$ of the natural extension $\overline{E}_\Gamma$ of the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a possibility measure. As a first step, we investigate whether $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is P-consistent, i.e. whether there is a possibility measure Π on $(\Omega, \mathcal{R}_\Gamma)$, such that $\Pi \circ \Gamma = \xi$. The requirement (N2) imposed on ξ is a necessary condition for the existence of such a possibility measure. Since any possibility measure is zero at the empty set, (N4) is also a necessary condition for possibilistic extendability. On the other hand, if $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is P-consistent, it follows from $\Gamma(1_S) = \Omega$ that Π will be normal if and only if $\xi(1_S) = 1$, or in other words, that $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is supremum-normalisable if and only if $\xi(1_S) = 1$, that is, if (N3) holds.

[11] If the complete chain $(S, \leq)$ is order-dense, i.e. none of its elements covers or is covered by any other element, then the mappings α and β must be equal. This is the case for the real unit interval $([0, 1], \leq)$.

We now formulate a necessary and sufficient condition that ξ must satisfy besides (N2) in order that the set function $\overline{P}_\Gamma$ assuming the values $\xi(s)$ on $\Gamma(s)$, $s \in S$, would be extendable to a possibility measure on $(\Omega, \mathcal{R}_\Gamma)$. This is an immediate consequence of Theorem 3. However, in order to illustrate the line of reasoning behind that theorem, we give a more direct proof for this result in the Appendix. Remark that (7) implies (N4).

Theorem 10. *Let $(S, \leq)$ be a complete chain. Assume that Γ and ξ satisfy* (N1), (N2) *and* (N5). *Then there is a possibility measure Π on $(\Omega, \mathcal{R}_\Gamma)$ that is a solution of $\Pi \circ \Gamma = \xi$ if and only if*

$$\xi(s) = \bigvee_{\omega \in \Omega, \alpha(\omega) \leq s} \xi(\alpha(\omega)), \quad s \in S. \tag{7}$$

In that case, the greatest such possibility measure has distribution $\xi \circ \alpha$. Moreover, any solution Π will be normal if and only if $\xi(1_S) = 1$, that is, if (N3) *holds.*

It is shown in the Appendix (Example 20) that conditions (N1), (N2), (N4) and (N5) are in general not sufficient for (7) to hold, or in other words, that condition (7) is not redundant.

We are now ready to take a closer look at the natural extension $\overline{E}_\Gamma$ of $\overline{P}_\Gamma$. Since we want $\overline{P}_\Gamma$ to avoid sure loss, we impose at least (N3), which implies that $\xi(1_S) = 1$. It turns out that natural extension is now very easy to calculate. Indeed (see also the proof of Theorem 11 below), for any non-empty subset A of Ω, $\overline{E}_\Gamma(A) = \xi(\sup \alpha(A))$, which, by the way, is also the value that $\overline{P}_\Gamma$ assigns to the closure $C_\Gamma(A)$ of A in the closure system $\Gamma(S)$: $C_\Gamma(A) = \bigcap\{\Gamma(s) \colon A \subseteq \Gamma(s)\} = \Gamma(\sup \alpha(A))$.

Theorem 11. *Let $(S, \leq)$ be a complete chain. Assume that Γ and ξ satisfy* (N1), (N2) *and* (N5).

1. *If* (N3) *holds, the restriction to $\wp(\Omega)$ of the natural extension $\overline{E}_\Gamma$ of $\overline{P}_\Gamma$ is a possibility measure on $(\Omega, \wp(\Omega))$ if and only if*[12]

$$\bigvee \xi(\alpha(A)) = \xi(\sup \alpha(A)), \quad \emptyset \subset A \subseteq \Omega. \tag{8}$$

2. *If* (N3) *and* (N4) *hold, then $\overline{E}_\Gamma$ is the greatest solution of the equation $\Pi \circ \Gamma = \xi$ if and only if ξ satisfies* (8).

In both cases, the distribution of $\overline{E}_\Gamma$ is given by $\xi \circ \alpha$.

6.2. **A family of strict dual cut sets.** Let us now turn to the second special case. We again assume that $(S, \leq)$ is a complete chain, and that Γ and ξ satisfy conditions (N1) and (N2), implying (N0). Now, we require that Γ should satisfy the additional condition

$$(\forall \omega \in \Omega)(\omega \notin \Gamma(\beta(\omega))), \tag{N6}$$

[12]This condition can be stated less densely as $\bigvee_{\omega \in A} \xi(\alpha(\omega)) = \xi(\sup\{\alpha(\omega) \colon \omega \in A\})$. Actually, it follows from Proposition 9 and Proposition 18 in the Appendix that α is constant on the atoms of $\mathcal{R}_\Gamma$, so (8) is equivalent to $(\forall A \in \mathcal{R}_\Gamma \setminus \{\emptyset\})(\bigvee \xi(\alpha(A)) = \xi(\sup \alpha(A)))$, which also implies that $\overline{E}_\Gamma$ is a possibility measure on $(\Omega, \wp(\Omega))$ if and only if its restriction to $\mathcal{R}$ is a possibility measure on $(\Omega, \mathcal{R})$, where $\mathcal{R}$ is any ample field on Ω with $\mathcal{R}_\Gamma \subseteq \mathcal{R}$.

rather than (N5). Conditions (N1) and (N6) are together equivalent to the continuity requirement

$$\Gamma(\sup T) = \bigcup_{t \in T} \Gamma(t), \quad T \subseteq S, \tag{9}$$

which implies that $\Gamma(0_S) = \emptyset$, and that $\Gamma(S)$ is closed under arbitrary unions, and therefore constitutes a topped dual intersection structure, or dual closure system [4]. This condition is necessary and sufficient for $(\Gamma(s) \colon s \in S)$ to be a family of strict dual cut sets of some $\Omega - S$-mapping. Indeed, we have for ω in Ω and s in S that $\omega \in \Gamma(s) \Leftrightarrow \beta(\omega) < s$.

Here too, we want to find out under what conditions the natural extension $\overline{E}_\Gamma$ of the upper probability $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a possibility measure when restricted to $\wp(\Omega)$. As a first step, we again want to investigate whether $\overline{P}_\Gamma$ can be extended to a possibility measure on $(\Omega, \mathcal{R}_\Gamma)$, i.e. whether $\overline{P}_\Gamma$ is P-consistent. As before, we do not require at this point that (N3) should hold. Since $\Gamma(0_S) = \emptyset$ and any possibility measure is zero at $\emptyset$, it will be necessary that $\xi(0_S) = 0$, i.e. (N4) should hold. Note that (N4) is implied by (10).

Theorem 12. *Let* $(S, \leq)$ *be a complete chain. Assume that* Γ *and* ξ *satisfy* (N1), (N2) *and* (N6)*. Then there is a possibility measure* Π *on* $(\Omega, \mathcal{R}_\Gamma)$ *that is a solution of* $\Pi \circ \Gamma = \xi$ *if and only if*

$$\xi(\sup T) = \bigvee_{t \in T} \xi(t), \quad T \subseteq S. \tag{10}$$

In that case, the greatest such possibility measure Π^g *has distribution* $\pi^g(\omega) = \bigwedge_{s \in S, \beta(\omega) < s} \xi(s)$, $\omega \in \Omega$.

To conclude this section, we take a closer look at the natural extension $\overline{E}_\Gamma$ of $(\Omega, \Gamma(S), \overline{P}_\Gamma)$. Of course, we must now impose (N3) in order to make sure that $\overline{P}$ avoids sure loss.

Theorem 13. *Let* $(S, \leq)$ *be a complete chain. Assume that* Γ *and* ξ *satisfy* (N1), (N2) *and* (N6).

1. *If* (N3) *holds, the restriction of the natural extension* $\overline{E}_\Gamma$ *of* $\overline{P}_\Gamma$ *to* $\wp(\Omega)$ *is a normal possibility measure if and only if* ξ *satisfies:*
$$\xi(\sup T) = \bigvee_{t \in T} \xi(t), \quad \emptyset \subset T \subseteq \{s \in S \colon \emptyset \subset \Gamma(s)\}.$$
2. *If* (N3) *and* (N4) *hold, the restriction of the natural extension* $\overline{E}_\Gamma$ *of* $\overline{P}_\Gamma$ *to* $\wp(\Omega)$ *is the greatest solution of* $\Pi \circ \Gamma = \xi$ *if and only if* ξ *satisfies* (10)*, i.e. if* $\overline{P}_\Gamma$ *is P-consistent.*

In both cases, $\overline{E}_\Gamma$ *will then have distribution* $\pi^g(\omega) = \bigwedge_{s \in S, \beta(\omega) < s} \xi(s)$, $\omega \in \Omega$.

It appears that the conditions in Theorems 12 and 13 are simpler and more elegant than the ones in Theorems 10 and 11. This will also become apparent in Sections 7.3, 7.6 and 7.7.

7. Important special cases and applications

Let us apply the general results obtained in the previous sections in a number of important special cases. This will in particular serve to illustrate the relevance of possibility measures to the theory of imprecise probabilities.

7.1. **Recovering possibility measures.** Consider a normal possibility measure Π, defined on an ample field $\mathcal{R}$ of subsets of Ω. We denote its distribution by π. We first turn our attention to the family $(\{\pi \leq x\} : x \in [0,1])$ of dual cut sets of π. Let us consider the complete chain $(S, \leq) = ([0,1], \leq)$, and define the $[0,1] - \wp(\Omega)$-mapping Γ and the $[0,1] - [0,1]$-mapping ξ by $\Gamma(x) = \{\pi \leq x\}$ and $\xi(x) = \Pi(\{\pi \leq x\}) = \bigvee_{\pi(\omega) \leq x} \pi(\omega)$, $x \in [0,1]$. These mappings — or the corresponding upper probability $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ — satisfy conditions (N1), (N2), (N3), (N4) and (N5), which means that we may apply the results of Section 6.1. For any ω in Ω, we find that $\alpha(\omega) = \bigwedge\{x \in [0,1] : \omega \in \{\pi \leq x\}\} = \pi(\omega)$ and $\xi(\alpha(\omega)) = \pi(\omega)$. Therefore, $\bigvee_{\alpha(\omega) \leq x} \xi(\alpha(\omega)) = \bigvee_{\pi(\omega) \leq x} \pi(\omega) = \xi(x)$, $x \in [0,1]$, which tells us that (7) holds. From Theorem 10 we deduce that Π is the greatest (normal) possibility measure on $(X, \mathcal{R})$ that coincides with $\overline{P}_\Gamma$ on $\Gamma([0,1])$. Since (8) also holds, Theorem 11 assures us that the restriction of the natural extension $\overline{E}_\Gamma$ to $\wp(\Omega)$ is a normal possibility measure that coincides on $\mathcal{R}$ with Π — actually, it is the outer set function of Π.

Observation 3. *Any normal possibility measure is the natural extension (to its domain) of its restriction to the dual cut sets of its distribution.*

Next, we consider the family of *strict dual cuts sets* $(\{\pi < x\} : x \in [0,1])$ of the distribution π. As before, we may consider an upper probability $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ associated with the mappings $\Gamma : [0,1] \to \wp(\Omega)$ and $\xi : [0,1] \to [0,1]$, where now, for any x in $[0,1]$, $\Gamma(x) = \{\pi < x\}$ and $\xi(x) = \Pi(\{\pi < x\}) = \bigvee_{\pi(\omega) < x} \pi(\omega)$. It is verified that (N1), (N2), (N3), (N4) and (N6) hold, which brings us to the context of Section 6.2. Note that $\beta = \pi$ and that (10) holds. From Theorem 12 we deduce that Π is the greatest (normal) possibility measure on $(X, \mathcal{R})$ that coincides with $\overline{P}_\Gamma$ on $\Gamma([0,1])$, and Theorem 13 states that the restriction of the natural extension $\overline{E}_\Gamma$ of $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ to $\wp(\Omega)$ is a normal possibility measure that coincides on $\mathcal{R}$ with Π — as before, it is the outer set function of Π.

Observation 4. *Any normal possibility measure is the natural extension (to its domain) of its restriction to the strict dual cut sets of its distribution.*

7.2. **Assessments on a finite number of nested sets.** Assume that we have a monotone upper probability $\overline{P}$ defined on a *finite* number n of nested proper subsets $A_1, \ldots, A_n$ of Ω. It follows from Proposition 7 that such an upper probability is always coherent (indeed, it is a finitely additive probability!). We are interested in whether its natural extension is a possibility measure. Consider the mappings Γ and ξ for the finite and therefore complete chain $(S, \leq) = (\{0, 1, \ldots, n, n{+}1\}, \leq)$, defined by $\Gamma(0) = \emptyset$, $\Gamma(n+1) = \Omega$ and $\Gamma(k) = A_k$, $k = 1, \ldots, n$; $\xi(0) = 0$, $\xi(n+1) = 1$ and $\xi(k) = \overline{P}(A_k)$, $k = 1, \ldots, n$. Obviously, (N1), (N2), (N3) and (N4) are verified, and $\overline{P}$ and $\overline{P}_\Gamma$ have the same natural extension $\overline{E}_\Gamma$. Moreover, since S is finite, both (N5) and (N6) hold, which means that the results of both Sections 6.1 and 6.2 apply. Using either, it follows that the restriction of the natural extension to $\wp(\Omega)$ is a normal possibility measure, with distribution:

$$\pi^g(\omega) = \min_{k=1}^{n} \max(\xi(k), I_{\mathrm{co}\Gamma(k)}(\omega)) = \min_{k=1}^{n} \max(\overline{P}(A_k), I_{\mathrm{co}A_k}(\omega)), \quad \omega \in \Omega.$$

Our discussion therefore includes and at the same time generalises the finite version, treated by Dubois and Prade [20].

Observation 5. *Given a monotone upper probability on a finite number of nested proper events, its natural extension is a possibility measure.*

In the Appendix, it is proven by means of a counterexample (see Example 20) that this no longer necessarily holds when the number of sets is (countably) infinite; but the natural extension will nevertheless still be maxitive.

7.3. **Interpretation of the greatest solution of a possibilistic extension problem.** The discussion in Section 6.2 also provides an interesting interpretation for the greatest solution of any possibilistic extension problem, and allows us to interpret possibilistic extension in terms of natural extension. To see this, consider a P-consistent upper probability $(\Omega, \mathcal{S}, \overline{P})$, defined on a collection of subsets $\mathcal{S}$ of Ω that is not necessarily nested. With this upper probability, we may associate a possibility measure Π^g, with distribution π^g given by

$$\pi^g(\omega) = \bigwedge_{A \in \mathcal{S}, \omega \in A} \overline{P}(A), \quad \omega \in \Omega.$$

Recall that, since $\overline{P}$ is P-consistent, Π^g is the greatest possibility measure that coincides with $\overline{P}$ on $\mathcal{S}$.

We use the upper probability $(\Omega, \mathcal{S}, \overline{P})$ to construct in a fairly natural way a new upper probability on nested sets, which will be called the *nested approximation* of the original upper probability. Let $(S, \leq)$ be the complete chain $([0,1], \leq)$ and define for any x in $[0,1]$,

$$\Gamma(x) = \bigcup \{A\colon A \in \mathcal{S} \text{ and } \overline{P}(A) < x\}$$

and

$$\xi(x) = \bigvee \{\overline{P}(A)\colon A \in \mathcal{S} \text{ and } \overline{P}(A) < x\}.$$

Then Γ and ξ satisfy (N1), (N2), (N4) and (N6).[13] Moreover, (10) holds — meaning that the nested approximation $\overline{P}_\Gamma$ is P-consistent as well — which by Theorem 12 implies that $\overline{P}_\Gamma$ is extendable to a possibility measure. The distribution π_Γ of the greatest such possibility measure Π_Γ is given by $\pi_\Gamma(\omega) = \bigwedge_{x \in [0,1], \omega \in \Gamma(x)} \xi(x)$, $\omega \in \Omega$. Interestingly, $\pi_\Gamma = \pi^g$, and therefore $\Pi_\Gamma = \Pi^g$. A proof of these equalities is given in the Appendix.

It is also proven there that if $(\Omega, \mathcal{S}, \overline{P})$ is moreover supremum-normalisable, then Γ and ξ also satisfy (N3), which by Theorem 13 implies that the restriction of the natural extension $\overline{E}_\Gamma$ of $\overline{P}_\Gamma$ to $\wp(\Omega)$ is precisely the possibility measure $\Pi_\Gamma = \Pi^g$.

Observation 6. *If an upper probability is P-consistent, then its greatest possibilistic extension coincides with the greatest possibilistic extension of its nested approximation. Moreover, if an upper probability is P-consistent and supremum-normalisable, then its greatest possibilistic extension is the natural extension of its nested approximation.*

7.4. **Product possibility measures.** Consider the universes Ω_1 and Ω_2, provided with the respective ample fields $\mathcal{R}_1$ and $\mathcal{R}_2$. Also consider the normal possibility measures Π_1 on $(\Omega_1, \mathcal{R}_1)$ with distribution π_1, and Π_2 on $(\Omega_2, \mathcal{R}_2)$ with distribution π_2. We want to use these upper probabilities to construct a new coherent upper probability on the product $\Omega_1 \times \Omega_2$.

Consider the sets $\mathcal{M}_1 = \{A \times \Omega_2\colon A \in \mathcal{R}_1\}$, $\mathcal{M}_2 = \{\Omega_1 \times B\colon B \in \mathcal{R}_2\}$ and $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. We denote by $\mathcal{R}_1 \times \mathcal{R}_2$ the smallest ample field which includes

[13] Actually, (N2) holds because $\overline{P}$ is P-consistent.

$\mathcal{M}$. Since $\Pi_1(\Omega_1) = \Pi_2(\Omega_2) = 1$, we may define in a fairly natural way an upper probability $(\Omega_1 \times \Omega_2, \mathcal{M}, \overline{P})$ as follows. For any A in $\mathcal{R}_1$ and $B \in \mathcal{R}_2$:

$$\overline{P}(A \times \Omega_2) = \Pi_1(A) \text{ and } \overline{P}(\Omega_1 \times B) = \Pi_2(B),$$

i.e. Π_1 and Π_2 are the marginals of $\overline{P}$. Note that $\mathcal{R}_1 \times \mathcal{R}_2$ is the smallest ample field to which the set mapping $\overline{P}$ can be extended. It is proven in the Appendix that $(\Omega_1 \times \Omega_2, \mathcal{M}, \overline{P})$ is P-consistent and supremum-normalisable. From the discussion in Section 4, we know that the greatest possibility measure Π^g which coincides with $\overline{P}$ on $\mathcal{M}$ has distribution

$$\pi^g(\omega_1, \omega_2) = \bigwedge_{A \in \mathcal{M}, (\omega_1, \omega_2) \in A} \overline{P}(A) = \pi_1(\omega_1) \wedge \pi_2(\omega_2), \quad (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2.$$

Consequently, for $A_1 \in \mathcal{R}_1$ and $A_2 \in \mathcal{R}_2$, $\Pi^g(A_1 \times A_2) = \Pi_1(A_1) \wedge \Pi_2(A_2)$. Π^g is also denoted as $\Pi_1 \times_\wedge \Pi_2$ and is called the ($\wedge$-)*product possibility measure* of Π_1 and Π_2 [7, 33].

Observation 7. *A product possibility measure is the greatest possibilistic extension of its marginals.*

Walley mentions that the natural extension of $(\Omega_1 \times \Omega_2, \mathcal{M}, \overline{P})$ coincides with Π^g on the algebra generated by $\mathcal{M}$, i.e. the set $\{A \times B \colon A \in \mathcal{R}_1 \text{ and } B \in \mathcal{R}_2\}$ [29, Section 6]. This is stronger than what Theorem 6 tells us in this case, namely that the natural extension and Π^g coincide on the atoms of $\mathcal{R}_1 \times \mathcal{R}_2$, which take the general form $[\,(\omega_1, \omega_2)\,]_{\mathcal{R}_1 \times \mathcal{R}_2} = [\,\omega_1\,]_{\mathcal{R}_1} \times [\,\omega_2\,]_{\mathcal{R}_2}$.

It was suggested by one of the referees that Observation 7 was also made by Dubois, Moral and Prade in [18], when they justified the $\wedge$-rule for combining two possibility distributions as the one that yields 'the *most* specific knowledge that is a consequence of both distributions'. Not so: what we are saying is something different: the $\wedge$-*product* of two *marginal* possibility distributions is the *least* specific joint distribution which has the given distributions as its marginals.

7.5. **Specifying the cumulative distribution function of a random quantity.** Consider a real-valued function Q defined on a universe Ω. We call Q a *real random quantity*, Ω its *basic space* and $\mathbb{R}$ its *sample space*. We use the notation $\mathbb{A} = \{Q(\omega) \colon \omega \in \Omega\}$. For the sake of simplicity, we assume that Q is bounded, i.e. $\inf[Q] = \inf \mathbb{A} \in \mathbb{R}$ and $\sup[Q] = \sup \mathbb{A} \in \mathbb{R}$. We want to stress, however, that in this and the following subsections similar conclusions can be reached without the assumption of boundedness.

We also consider a $\mathbb{R} - \mathbb{R}$-mapping F, and we ask under what conditions F can be interpreted as a *cumulative distribution function* of the random variable Q. In other words, when is there a linear prevision (or finitely additive probability) P such that

$$P(\{\omega \in \Omega \colon Q(\omega) \leq x\}) = F(x), \quad x \in \mathbb{R}? \tag{11}$$

It is a consequence of the properties of a linear prevision (finitely additive probability) that the following are necessary conditions.

$$(\forall(x,y)\in\mathbb{R}^2)\,((\forall a\in\mathbb{A})(a\le x\Leftrightarrow a\le y)\Rightarrow F(x)=F(y)) \tag{DF1}$$

$$(\forall(x,y)\in\mathbb{R}^2)(x\le y\Rightarrow F(x)\le F(y)) \tag{DF2}$$

$$(\forall x\in\mathbb{R})\,((\forall a\in\mathbb{A})(a>x)\Rightarrow F(x)=0) \tag{DF3}$$

$$(\forall x\in\mathbb{R})\,((\forall a\in\mathbb{A})(a\le x)\Rightarrow F(x)=1) \tag{DF4}$$

Conversely, assume that these conditions hold. Let $S_Q=[\inf\mathbb{A},\sup\mathbb{A}]$ and consider the mappings $\Gamma_Q\colon S_Q\to\wp(\Omega)$ and $\xi_Q\colon S_Q\to[0,1]$, defined as

$$\Gamma_Q(x)=\{\omega\colon Q(\omega)\le x\}\text{ and }\xi_Q(x)=F(x),\quad x\in S_Q.$$

$(S_Q,\le)$ is a complete chain with top $1_{S_Q}=\sup\mathbb{A}$ and bottom $0_{S_Q}=\inf\mathbb{A}$. It follows from (DF1) and (DF2) that Γ_Q and ξ_Q satisfy (N1) and (N2). From (DF3) and (DF4) we deduce that they also satisfy (N3) and (N4). Proposition 7 then guarantees that $(\Omega,\Gamma_Q(S_Q),\overline{P}_{\Gamma_Q})$ is a linear prevision, and is therefore extendable to a linear prevision P on $\mathcal{L}(\Omega)$, or on $\wp(\Omega)$, so that (11) holds.

Observation 8. *Conditions* (DF1)–(DF4) *are necessary and sufficient for a function $F\colon\mathbb{R}\to\mathbb{R}$ to be interpretable as a cumulative distribution function for the random quantity Q.*

7.6. **Specifying the upper cumulative distribution function of a random quantity.** Let us now investigate the case that F is interpreted as an *upper cumulative distribution function* of the bounded random quantity Q [28, Section 4.4.6]. In other words, we want to find out under what conditions there is a coherent upper prevision (or probability) $\overline{P}$, defined on a sufficiently large domain, such that

$$\overline{P}(\{\omega\in\Omega\colon Q(\omega)\le x\})=F(x),\quad x\in\mathbb{R}.$$

Using a similar argument as in the previous subsection, we find that this is the case if and only if $(\Omega,\Gamma_Q(S_Q),\overline{P}_{\Gamma_Q})$ is a coherent upper probability, i.e. if (DF1)–(DF4) hold. For the corresponding $\Omega-S_Q$-mapping α_Q we find in that case that

$$\alpha_Q(\omega)=\inf\{x\in S_Q\colon\omega\in\Gamma_Q(x)\}=\inf\{x\in S_Q\colon Q(\omega)\le x\}=Q(\omega),$$

and consequently $\Gamma_Q(\alpha_Q(\omega))=\{\varpi\in\Omega\colon Q(\varpi)\le Q(\omega)\}\ni\omega$, $\omega\in\Omega$, which tells us that (N5) holds. We may therefore apply the results of Section 6.1.

In particular, using Theorem 11 we find that the restriction to $\wp(\Omega)$ of the natural extension $\overline{E}_\Gamma$ of $\overline{P}_\Gamma$ is a possibility measure if and only if

$$\bigvee_{a\in A}F(a)=F(\sup A),\quad\emptyset\subset A\subseteq\mathbb{A}.$$

Denote by $\mathbb{A}^{li}$ the set of the *left-isolated* points of $\mathbb{A}$ in the metric topology on $\mathbb{R}$, i.e. those points a of $\mathbb{A}$ for which there is a left-neighbourhood $]a-\epsilon,a]$ of a such that $\mathbb{A}\cap\,]a-\epsilon,a]=\{a\}$. It is shown in the Appendix that if (DF1)–(DF4) hold, the above condition is equivalent to

$$F\text{ is left-continuous in }\mathbb{R}\setminus\mathbb{A}^{li}. \tag{DF5}$$

In that case, the distribution π^g of this possibility measure is given by $\pi^g(\omega)=F(Q(\omega))$, $\omega\in\Omega$.

Observation 9. *Suppose that we specify information about the values a random quantity may assume in the form of an upper cumulative distribution function. Then the natural extension of this information is of a possibilistic nature if and only if F is left-continuous in any point which is not a left-isolated point of the set of values of the random quantity.*

7.7. **Specifying the lower cumulative distribution function of a random quantity.** To end this section, we consider the case that F is interpreted as a *lower cumulative distribution function* of the bounded random quantity Q [28, Section 4.4.6]. In other words, we want to investigate under what conditions there is a coherent lower prevision (or lower probability) $\underline{P}$ defined on an appropriate domain, such that $\underline{P}(\{\omega \in \Omega \colon Q(\omega) \leq x\}) = F(x)$, $x \in \mathbb{R}$. This is of course equivalent to the existence of a conjugate coherent upper prevision (or upper probability) $\overline{P}$ such that

$$\overline{P}(\{\omega \in \Omega \colon Q(\omega) > x\}) = 1 - F(x), \quad x \in \mathbb{R}.$$

Similarly as in Section 7.5, it follows from the properties of a coherent upper prevision that (DF1)–(DF4) are necessary conditions. In order to prove that they are also sufficient, consider the complete chain $(S_Q, \leq_Q) = ([\inf \mathbb{A}, \sup \mathbb{A}], \geq)$ with top $1_{S_Q} = \inf \mathbb{A}$ and bottom $0_{S_Q} = \sup \mathbb{A}$. If we define the mappings $\Gamma_Q \colon S_Q \to \wp(\Omega)$ and $\xi_Q \colon S_Q \to [0,1]$ as follows

$$\Gamma_Q(x) = \{\omega \colon Q(\omega) > x\} \text{ and } \xi_Q(x) = 1 - F(x), \quad x \in S_Q,$$

then it follows from (DF1) and (DF2) that (N1) and (N2) are satisfied. Moreover, (DF3) and (DF4) imply that (N3) and (N4) hold. Using Proposition 7 and the fact that any linear prevision can be coherently extended to an upper prevision on $\mathcal{L}(\Omega)$, we may conclude that the conditions (DF1)–(DF4) are also sufficient.

Moreover, for any $\omega \in \Omega$,

$$\beta_Q(\omega) = \sup_Q\{x \in S_Q \colon \omega \notin \Gamma_Q(x)\} = \inf\{x \in S_Q \colon Q(\omega) \leq x\} = Q(\omega),$$

and also $\omega \notin \Gamma_Q(\beta_Q(\omega))$. Since this tells us that (N6) holds, we may apply the results of Section 6.2. In particular, Theorem 13 tells us (after some manipulation) that the restriction to $\wp(\Omega)$ of the natural extension $\overline{E}_{\Gamma_Q}$ of $\overline{P}_{\Gamma_Q}$ is a possibility measure if and only if for any subset T of S_Q, $\xi_Q(\sup_Q T) = \bigvee_{t \in T} \xi_Q(t)$, or equivalently, $F(\inf T) = \bigwedge_{t \in T} F(t)$. Taking into account (DF1)–(DF4), this is equivalent to

$$F \text{ is right-continuous} \tag{DF6}$$

In that case, the distribution π^g of this possibility measure is given by

$$\pi^g(\omega) = 1 - \lim_{t \to Q(\omega)^-} F(t) = 1 - F(Q(\omega)^-), \quad \omega \in \Omega.$$

Observation 10. *Suppose that we specify information about the values a random quantity may assume in the form of a lower cumulative distribution function. Then the natural extension of this information is of a possibilistic nature if and only if F is right-continuous.*

8. Supremum preserving upper previsions

We have seen that the normal possibility measures constitute an important and interesting subclass of the coherent upper probabilities. In this section, we investigate whether anything similar can be said for supremum preserving *upper previsions*.

For a start, it is fairly natural when considering supremum preservation to restrict ourselves to gambles which are uniformly non-negative. We therefore consider the convex cone $\mathcal{C} = \{X \in \mathcal{L}(\Omega)\colon X \geq 0\}$ of all non-negative gambles on Ω. The supremum $\sup \mathcal{D}$ of any subset $\mathcal{D}$ of $\mathcal{C}$ is defined pointwise: $(\sup \mathcal{D})(\omega) = \operatorname{Sup}_{X \in \mathcal{D}} X(\omega)$, $\omega \in \Omega$, where Sup denotes the supremum[14] on the chain $(\mathbb{R}^+, \leq)$ of the nonnegative real numbers. Obviously, $\sup \mathcal{D}$ is again a member of $\mathcal{C}$, provided that it is still bounded. Also, if $\mathcal{D}$ is empty, it follows that $\sup \mathcal{D} = 0$ belongs to $\mathcal{C}$.

We call an upper prevision $(\Omega, \mathcal{C}, \overline{P})$ *supremum preserving* if for any subset $\mathcal{D}$ of $\mathcal{C}$ such that $\sup \mathcal{D} \in \mathcal{C}$, $\overline{P}(\sup \mathcal{D}) = \operatorname{Sup}_{X \in \mathcal{D}} \overline{P}(X)$. Obviously, if the restriction to $\wp(\Omega)$ of a supremum preserving upper prevision is coherent, it is a normal possibility measure. The following result is even stronger.

Proposition 14. *Let $(\Omega, \mathcal{C}, \overline{P})$ be a coherent upper prevision. Then for any X in $\mathcal{C}$, $\overline{P}(X) \geq \operatorname{Sup}_{\omega \in \Omega} X(\omega)\overline{P}(\{\omega\})$. Moreover, $(\Omega, \mathcal{C}, \overline{P})$ is supremum preserving if and only if*

$$\overline{P}(X) = \operatorname*{Sup}_{\omega \in \Omega} X(\omega)\overline{P}(\{\omega\}), \quad X \in \mathcal{C}. \tag{12}$$

This result, and in particular (12), tells us that if a coherent prevision $(\Omega, \mathcal{C}, \overline{P})$ is supremum preserving, then its restriction to $\wp(\Omega)$ is the possibility measure with distribution π given by $\pi(\omega) = \overline{P}(\{\omega\})$, $\omega \in \Omega$. Moreover, $\overline{P}$ must take the form (12) of a Shilkret[15] integral [25] associated with this possibility measure.

Of course, at this point we have no guarantee that upper previsions of this type can be coherent. Let us therefore study upper previsions $(\Omega, \mathcal{C}, \overline{P}_\pi)$ of the form $\overline{P}_\pi(X) = \operatorname{Sup}_{\omega \in \Omega} X(\omega)\pi(\omega)$, $X \in \mathcal{C}$, where π can be any $\Omega - [0,1]$-mapping. More in particular, we want to find out whether such upper previsions are necessarily coherent. For any $A \subseteq \Omega$, $\overline{P}_\pi(A) = \bigvee_{\omega \in A} \pi(\omega)$, which means that the restriction of $\overline{P}_\pi$ to events is a possibility measure with distribution π, and is, by Theorem 1, coherent if and only if π is *supremum-normal*, i.e. $\bigvee\{\pi(\omega)\colon \omega \in \Omega\} = 1$. The supremum-normality of π is therefore a necessary condition for the coherence of $(\Omega, \mathcal{C}, \overline{P}_\pi)$.

Walley [28, Theorem 2.5.5] has shown that any upper prevision $(\Omega, \mathcal{C}, \overline{P})$ is coherent if and only if it satisfies the following three conditions:

$$(\forall X \in \mathcal{C})(\forall \lambda > 0)(\overline{P}(\lambda X) = \lambda \overline{P}(X)) \tag{CC1}$$

$$(\forall (X, Y) \in \mathcal{C}^2)(\overline{P}(X + Y) \leq \overline{P}(X) + \overline{P}(Y)) \tag{CC2}$$

$$(\forall (X, Y) \in \mathcal{C}^2)(\forall \mu \in \mathbb{R})(X \geq Y + \mu \Rightarrow \overline{P}(X) \geq \overline{P}(Y) + \mu) \tag{CC3}$$

$\overline{P}_\pi$ always satisfies (CC1) and (CC2), so we only need to verify whether $\overline{P}_\pi$ satisfies (CC3). There is a special case in which $(\Omega, \mathcal{C}, \overline{P}_\pi)$ is always coherent, as the following proposition tells us.

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[14] For any non-empty subset of $\mathbb{R}^+$, $\operatorname{Sup} A = \sup A$, but $\operatorname{Sup} \emptyset = 0$ whereas $\sup \emptyset = -\infty$.

[15] Actually, in a different context Shilkret proved a somewhat stronger result, because he only used the preservation of countable suprema, and imposed conditions which are weaker than coherence.

Proposition 15. *Let π be a $\Omega - [0,1]$-mapping that is supremum-normal. If π can only assume the values 0 and 1, then $(\Omega, \mathcal{C}, \overline{P}_\pi)$ is coherent, and equals on its domain $\mathcal{C}$ the natural extension of the possibility measure Π with distribution π.*

It turns out that the previous proposition identifies the *only* case in which $\overline{P}_\pi$ is coherent.

Proposition 16. *Let π be a supremum-normal $\Omega - [0,1]$-mapping. If there is an ω_o in Ω such that $0 < \pi(\omega_o) < 1$, then $(\Omega, \mathcal{C}, \overline{P}_\pi)$ is not coherent.*

Observation 11. *A supremum preserving upper prevision $\overline{P}$ on $\mathcal{C}$ — or equivalently a Shilkret integral associated with a (normal) possibility measure — is never coherent, unless the corresponding distribution assumes only the values 0 and 1, in which case $\overline{P}$ is the vacuous conditional prevision with respect to a non-empty subset A of Ω:*

$$\overline{P}(X) = \mathrm{Sup}\{X(\omega) \colon \omega \in A\}, \quad X \in \mathcal{C}.$$

Moreover, the natural extension of a normal possibility measure is never supremum preserving on $\mathcal{C}$, unless its distribution is $0-1$-valued.

A Shilkret integral is a special case of a more general type of integral, namely Sugeno's fuzzy integral [13, 26], where the min operator may be generalised into a more general binary operator $\star$, such as product (see for instance [13]). When this integral is associated with a possibility measure, it allows us to define an upper prevision on $\mathcal{C}$. It will be supremum preserving if the operator $\star$ is left-continuous, but never coherent, except possibly when the normal possibility measure only assumes the values 0 and 1.

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Appendix

In this appendix, we have collected the proofs of the original results, and some additional discussion.

A.1 Proof of the result in Section 3.

Proof of Theorem 1. It is obvious from the definition of coherence and avoiding sure loss that the third statement implies the second, and that the second implies the first. It therefore remains to be shown the first statement implies the third. Assume that $\Pi(\Omega) = 1$. We know that $\Pi(\emptyset) = 0$. Moreover, we have for any A and B in $\mathcal{R}$, since Π is monotone and $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$,

$$\Pi(A \cap B) + \Pi(A \cup B) \leq \min(\Pi(A), \Pi(B)) + \max(\Pi(A), \Pi(B)) = \Pi(A) + \Pi(B),$$

which means that the upper probability $(\Omega, \mathcal{R}, \Pi)$ is 2-alternating. Walley has shown that this ensures that Π is coherent [27, Section 6]. □

A.2 Proofs of the results in Section 4.

Proof of Theorem 4. Immediately from Theorem 3 and the obvious fact that (3) admits normal solutions if and only if Π^g is normal and (4) admits normal solutions if and only if Π^g is a normal solution. □

Proof of Proposition 5. To prove the first statement, assume that Π is a solution of (3). If $\mathcal{M}(\Pi) = \emptyset$, the proof is immediate. Let us therefore assume that $\mathcal{M}(\Pi)$ is non-empty and let P be any element of $\mathcal{M}(\Pi)$. Consider any A in $\Gamma(S)$, then since $\Gamma(S) \subseteq \mathcal{R}$, $P(A) \leq \Pi(A)$. Moreover, since Π solves (3), $\Pi(A) \leq \overline{P}_\Gamma(A)$. We may therefore conclude that $P \in \mathcal{M}(\overline{P}_\Gamma)$. To prove the second statement, assume that Π is normal, whence $\mathcal{M}(\Pi) \neq \emptyset$. If Π is a solution of (3), then the first statement, together with Theorem 3 implies that $\mathcal{M}(\Pi) \subseteq \mathcal{M}(\Pi^g) \subseteq \mathcal{M}(\overline{P}_\Gamma)$. It therefore remains to be proven that $\mathcal{M}(\Pi) \subseteq \mathcal{M}(\overline{P}_\Gamma)$ implies that Π is a solution of (3). *Ex absurdo*, assume that $\mathcal{M}(\Pi) \subseteq \mathcal{M}(\overline{P}_\Gamma)$ and that there is an A in $\Gamma(S)$ such that $\Pi(A) > \overline{P}_\Gamma(A)$. Since Π is normal, it is a coherent upper probability, and from Walley's lower envelope theorem [28, Theorem 3.3.3] we infer that there is a P in $\mathcal{M}(\Pi)$ such that $P(A) = \Pi(A) > \overline{P}_\Gamma(A)$, whence $P \notin \mathcal{M}(\overline{P}_\Gamma)$, a contradiction. □

Proof of Theorem 6. Since it follows from the assumptions that $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is coherent and therefore avoids sure loss, we may invoke Walley's natural extension theorem [28, Theorem 3.4.1] and Proposition 5 to obtain $\mathcal{M}(\Pi^g) \subseteq \mathcal{M}(\overline{P}_\Gamma) = \mathcal{M}(\overline{E}_\Gamma)$ and therefore also, for any X in $\mathcal{L}(\Omega)$: $\overline{E}_{\Pi^g}(X) = \sup\{P(X) \colon P \in \mathcal{M}(\Pi^g)\} \leq \sup\{P(X) \colon P \in \mathcal{M}(\overline{P}_\Gamma)\} = \overline{E}_\Gamma(X)$, where $\overline{E}_{\Pi^g}$ is the natural extension of Π^g to $\mathcal{L}(\Omega)$. Consequently, using the results and notations from the previous section, we find for any ω in Ω, since Π^g is by assumption normal and therefore coherent: $\overline{E}_{\Pi^g}(\{\omega\}) = \Pi^g(\{\omega\}^\uparrow) = \Pi^g([\omega]_{\mathcal{R}_\Gamma}) = \pi^g(\omega)$ and $\overline{E}_{\Pi^g}(\{\omega\}) \leq \overline{E}_\Gamma(\{\omega\})$. On the other hand, it follows from the coherence of $\overline{E}_\Gamma$ that $\overline{E}_\Gamma(\{\omega\}) \leq \overline{E}_\Gamma([\omega]_{\mathcal{R}_\Gamma}) = \overline{E}_\Gamma(\bigcap\{A \in \mathcal{R}_\Gamma \colon \omega \in A\}) \leq \bigwedge_{A \in \mathcal{R}_\Gamma, \omega \in A} \overline{E}_\Gamma(A) \leq \bigwedge_{A \in \Gamma(S), \omega \in A} \overline{P}_\Gamma(A) = \pi^g(\omega)$. □

A.3 Proofs of the results in Section 5.

Proof of Proposition 7. If $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a linear prevision, it is coherent, which implies that (N3) and (N4) hold. Conversely, assume that (N3) and (N4) are verified. Consider the monotone set function μ defined on the chain of sets $\mathcal{S} = \Gamma(S) \cup \{\emptyset, \Omega\}$ by $\mu(\emptyset) = 0$, $\mu(\Omega) = 1$ and $\mu(A) = \overline{P}_\Gamma(A)$, $A \in \Gamma(S)$. Denneberg [16, Proposition 2.10] has shown that μ has a unique extension α to the algebra $\mathcal{A}$ generated by $\mathcal{S}$ that is modular, i.e. $(\forall (A, B) \in \mathcal{A}^2)(\alpha(A \cup B) + \alpha(A \cap B) = \alpha(A) + \alpha(B))$. From a theorem by Walley [28, Theorem 2.8.9], we deduce that $(\Omega, \mathcal{A}, \alpha)$ is a linear prevision. Since $\overline{P}_\Gamma$ is clearly the restriction of α to $\Gamma(S)$, it follows that $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ is a linear prevision. Alternatively, this proposition is an immediate consequence of a basic extension theorem for linear maps; see [22, Section 11.10]. □

Proof of Theorem 8. Since (N3) holds, $(\Omega, \Gamma(S), \overline{P}_\Gamma)$ avoids sure loss, and the natural extension $(\Omega, \mathcal{L}(\Omega), \overline{E}_\Gamma)$ is the greatest coherent upper prevision that is dominated by $\overline{P}_\Gamma$ on $\Gamma(S)$. It is obvious that $\overline{E}_\Gamma(\emptyset) = 0 = \overline{P}^*_\Gamma(\emptyset)$. Let A be a non-empty subset of Ω. For any $s \in S$ such that $A \subseteq \Gamma(s)$, we have that

$\overline{E}_\Gamma(A) \leq \overline{E}_\Gamma(\Gamma(s)) \leq \xi(s)$, since $\overline{E}_\Gamma$ is coherent and therefore monotone, and is dominated by $\overline{P}_\Gamma$ on $\Gamma(S)$. Consequently, $\overline{E}_\Gamma(A) \leq \overline{P}^*_\Gamma(A)$. If there is no $s \in S$ such that $A \subseteq \Gamma(s)$, then still $\overline{E}_\Gamma(A) \leq 1 = \overline{P}^*_\Gamma(A)$, since $\overline{E}_\Gamma$ is coherent. $\overline{E}_\Gamma$ is therefore dominated by $\overline{P}^*_\Gamma$ on $\wp(\Omega)$. Moreover, $\overline{P}^*_\Gamma$ is dominated by $\overline{P}_\Gamma$ on $\Gamma(S)$. Actually, it coincides with $\overline{P}_\Gamma$ except possibly on $\emptyset$. If we can show that the upper probability $(\Omega, \wp(\Omega), \overline{P}^*_\Gamma)$ is coherent, then it must be dominated by and therefore equal to the greatest coherent upper probability $\overline{E}_\Gamma$ that is dominated by $\overline{P}_\Gamma$ on $\Gamma(S)$. Consider non-empty A and B in $\wp(\Omega)$. First assume that there are s and t in S such that $A \subseteq \Gamma(s)$ and $B \subseteq \Gamma(t)$. Then by (N1), $A \cap B \subseteq \Gamma(s) \cap \Gamma(t) = \Gamma(\min(s,t)) \in \Gamma(S)$ and $A \cup B \subseteq \Gamma(s) \cup \Gamma(t) = \Gamma(\max(s,t)) \in \Gamma(S)$. Therefore, $\overline{P}^*_\Gamma(A \cap B) + \overline{P}^*_\Gamma(A \cup B) \leq \xi(\min(s,t)) + \xi(\max(s,t)) = \xi(s) + \xi(t)$, whence

$$\overline{P}^*_\Gamma(A \cap B) + \overline{P}^*_\Gamma(A \cup B) \leq \bigwedge_{s \in S, A \subseteq \Gamma(s)} \xi(s) + \bigwedge_{t \in S, B \subseteq \Gamma(t)} \xi(t) = \overline{P}^*_\Gamma(A) + \overline{P}^*_\Gamma(B),$$

and this inequality remains trivially valid when A or B are not included in any element of $\Gamma(S)$, or when A and B may be empty. Since by (N3) it also follows that $\overline{P}^*_\Gamma(\Omega) = 1$, we may conclude that $\overline{P}^*_\Gamma$ is 2-alternating on $\wp(\Omega)$, and therefore a coherent upper probability on $\wp(\Omega)$ [27, Section 6]. This proves that $\overline{E}_\Gamma$ coincides on $\wp(\Omega)$ with $\overline{P}^*_\Gamma$, and is therefore also 2-alternating. In order to prove that $\overline{E}_\Gamma(A \cup B) = \max\{\overline{E}_\Gamma(A), \overline{E}_\Gamma(B)\}$, we only need to prove that $\overline{E}_\Gamma(A \cup B) \leq \max\{\overline{E}_\Gamma(A), \overline{E}_\Gamma(B)\}$, since $\overline{E}_\Gamma$ is monotone. Assume *ex absurdo* that $\overline{E}_\Gamma(A \cup B) > \max\{\overline{E}_\Gamma(A), \overline{E}_\Gamma(B)\}$. This means that there are s and t in S such that $A \subseteq \Gamma(s)$, $B \subseteq \Gamma(t)$, $\xi(s) < \overline{E}_\Gamma(A \cup B)$ and $\xi(t) < \overline{E}_\Gamma(A \cup B)$. Consequently $\xi(\max\{s,t\}) < \overline{E}_\Gamma(A \cup B)$. But on the other hand $A \cup B \subseteq \Gamma(s) \cup \Gamma(t) = \Gamma(\max\{s,t\})$, so it follows that $\overline{E}_\Gamma(A \cup B) = \bigwedge\{\xi(s') \colon A \cup B \subseteq \Gamma(s')\} \leq \xi(\max\{s,t\}))$, a contradiction. This proves that $\overline{E}_\Gamma$ is maxitive. Finally, as in Section 3, we may conclude from the 2-alternating behaviour of $\overline{E}_\Gamma$ on $\wp(\Omega)$ and [28, Section 3.2.4] that for any X in $\mathcal{L}(\Omega)$, $\overline{E}_\Gamma(X)$ is given by the Choquet integral:

$$\overline{E}_\Gamma(X) = \inf[X] + \int_{\inf[X]}^{\sup[X]} \overline{E}_\Gamma(\{X > x\}) \mathrm{d}x.$$

The rest of the proof is now trivial. □

A.4 Proofs of the results in Section 6.

Proof of Proposition 9. If $\omega \in \Gamma(s)$ then clearly $\alpha(\omega) = \inf\{t \in S \colon \omega \in \Gamma(t)\} \leq s$, by (N1). Conversely, if $\alpha(\omega) \leq s$, then it follows from the properties (N1) and (N5) of Γ that $\omega \in \Gamma(\alpha(\omega)) \subseteq \Gamma(s)$. □

Proof of Theorem 10. Consider an arbitrary possibility measure Π on $(\Omega, \mathcal{R}_\Gamma)$. If we denote its distribution by ρ, we find that $\Pi \circ \Gamma$ coincides with ξ on S if and only if

$$\xi(s) = \bigvee_{\omega \in \Omega} (I_{\Gamma(s)}(\omega) \wedge \rho(\omega)), \quad s \in S, \tag{13}$$

where $I_{\Gamma(s)}$ is the indicator of the event $\Gamma(s)$. In other words, there is a possibility measure Π such that $\Pi \circ \Gamma = \xi$ if and only if the so-called *system of relational equations* (13) in ρ has a solution. It is a standard result in relational equation theory [5] that (13) has a solution if and only if the $\Omega-[0,1]$-mapping π^g, defined as

$$\pi^g(\omega)=\bigwedge_{s\in S}\mathcal{I}_\wedge(I_{\Gamma(s)}(\omega),\xi(s)),\quad \omega\in\Omega,$$

is a solution. In that case, π^g is also the greatest solution. In this expression $\mathcal{I}_\wedge$ is the residual operator associated with the minimum operator $\wedge$ on $[0,1]$ [1, 12], given by, for any μ and λ in $[0,1]$:

$$\mathcal{I}_\wedge(\mu,\lambda)=\bigvee\{\beta\in[0,1]\colon \mu\wedge\beta\leq\lambda\}=\begin{cases}1 & ; \quad \mu\leq\lambda\\ \lambda & ; \quad \mu>\lambda.\end{cases}$$

Clearly, $\mathcal{I}_\wedge(0,\lambda)=1$ and $\mathcal{I}_\wedge(1,\lambda)=\lambda$, whence, also taking into account (N2) and Proposition 9, $\pi^g(\omega)=\bigwedge_{s\in S,\alpha(\omega)\leq s}\xi(s)=\xi(\alpha(\omega))$. Substitution of π^g in Eq. (13) then tells us that (7) is a necessary and sufficient condition for the existence of a possibility measure Π on $(\Omega,\mathcal{R}_\Gamma)$ satisfying $\Pi\circ\Gamma=\xi$. If this condition is satisfied, then clearly $\pi^g=\xi\circ\alpha$ is the distribution of the greatest solution of $\Pi\circ\Gamma=\xi$. Finally, since $\Gamma(1_S)=\Omega$, any solution Π is indeed normal if and only if $\xi(1_S)=1$. □

Proof of Theorem 11. We begin with the first statement. It follows from the assumptions that $(\Omega,\Gamma(S),\overline{P}_\Gamma)$ avoids sure loss, and that we may calculate $\overline{E}_\Gamma$ using the results of Theorem 8. For any ω in Ω, also using Proposition 9, $\overline{E}_\Gamma(\{\omega\})=\bigwedge_{s\in S,\omega\in\Gamma(s)}\xi(s)=\bigwedge_{s\in S,\alpha(\omega)\leq s}\xi(s)=\xi(\alpha(\omega))$. Therefore, $\overline{E}_\Gamma$ is a possibility measure on $(\Omega,\wp(\Omega))$ if and only if for any non-empty A in $\wp(\Omega)$, $\overline{E}_\Gamma(A)=\bigvee_{\omega\in A}\overline{E}_\Gamma(\{\omega\})=\bigvee\xi(\alpha(A))$. Since, by Proposition 9, $A\subseteq\Gamma(s)\Leftrightarrow\sup_{\omega\in A}\alpha(\omega)\leq s$, it is obvious that $\overline{E}_\Gamma(A)=\xi(\sup_{\omega\in A}\alpha(\omega))=\xi(\sup\alpha(A))$. This proves the first statement. To prove the second statement, it now clearly suffices to assume that (8) holds, and to show that $\overline{E}_\Gamma$ is the greatest solution of $\Pi\circ\Gamma=\xi$. It follows from the first part of this theorem that $\overline{E}_\Gamma$ is a possibility measure. Moreover, by Theorem 8, $\overline{E}_\Gamma$ coincides with $\overline{P}_\Gamma$ on $\Gamma(S)$, so $\overline{E}_\Gamma\circ\Gamma=\xi$. The rest of the proof is now obvious. □

Proof of Theorem 12. By Theorem 3 it only needs to be proven that (10) is equivalent to the P-consistency of $\overline{P}_\Gamma$. Assume that $\overline{P}_\Gamma$ is P-consistent and consider a subset T of S. Since ξ is increasing, we already have that $\xi(\sup T)\geq\bigvee_{t\in T}\xi(t)$. By (N1) and (N6), $\Gamma(\sup T)\subseteq\bigcup_{t\in T}\Gamma(t)$, and P-consistency therefore implies that $\xi(\sup T)\leq\bigvee_{t\in T}\xi(t)$. Conversely, assume that (10) holds. Consider arbitrary s in S and $T\subseteq S$, and assume that $\Gamma(s)\subseteq\bigcup_{t\in T}\Gamma(t)=\Gamma(\sup T)$. Define for any t in S, $\widetilde{t}=\sup\{r\in S\colon\Gamma(r)=\Gamma(t)\}$. Then clearly $\Gamma(\widetilde{t})=\bigcup\{\Gamma(r)\colon\Gamma(r)=\Gamma(t)\}=\Gamma(t)$, whence by (N2), $\xi(\widetilde{t})=\xi(t)$. $\Gamma(s)\subseteq\Gamma(\sup T)$ is clearly equivalent to $s\leq\widetilde{\sup T}$. It now follows from the assumptions that $\xi(s)\leq\xi(\widetilde{\sup T})=\xi(\sup T)=\bigvee_{t\in T}\xi(t)$. Therefore, $\overline{P}_\Gamma$ is P-consistent. The rest of the proof is now trivial, taking into account Theorem 3. □

In order to prove the following theorem, we need an auxiliary result. Its proof is immediate and therefore omitted.

Lemma 17. *Let* $(S,\leq)$ *be a complete chain. Assume that* Γ *and* ξ *satisfy* (N1), (N2) *and* (N6). *Then for any subset* A *of* Ω, *the following statements are equivalent:*

1. $A \not\subseteq \Gamma(\sup_{\omega\in A}\beta(\omega))$;
2. $(\exists\varpi\in A)(\forall\omega\in A)(\beta(\varpi)\geq\beta(\omega))$;
3. $(\forall s\in S)(A\subseteq\Gamma(s)\Leftrightarrow s>\sup_{\omega\in A}\beta(\omega))$.

Proof of Theorem 13. We begin with the second statement. It follows from the assumptions (see Theorem 8) that $(\Omega,\Gamma(S),\overline{P}_\Gamma)$ is coherent and therefore coincides on its domain with its natural extension $\overline{E}_\Gamma$. We have shown in Theorem 12 that the P-consistency of $(\Omega,\Gamma(S),\overline{P}_\Gamma)$ is equivalent to (10). We first show that the restriction of $\overline{E}_\Gamma$ to $\wp(\Omega)$ is a possibility measure if and only if (10) holds. By Theorem 3, if the restriction of $\overline{E}_\Gamma$ to $\wp(\Omega)$, and therefore also to $\mathcal{R}_\Gamma$, is a possibility measure, then its restriction $\overline{P}_\Gamma$ to $\Gamma(S)$ must be P-consistent. Conversely, assume that ξ satisfies (10). From Theorem 12 we deduce that the greatest possibilistic extension Π^g of $\overline{P}_\Gamma$ to $\wp(\Omega)$ has distribution $\pi^g(\omega)=\bigwedge_{\omega\in\Gamma(s)}\xi(s)$, $\omega\in\Omega$. Since it follows from the assumptions that $\overline{E}_\Gamma$ is the greatest coherent extension of $\overline{P}_\Gamma$ to $\mathcal{L}(\Omega)$, it is obvious that the restriction of $\overline{E}_\Gamma$ to $\wp(\Omega)$ is a possibility measure if and only if it is equal to Π^g, i.e. if for any non-empty subset A of Ω, also taking into account Theorem 8,

$$\bigwedge_{s\in S,A\subseteq\Gamma(s)}\xi(s)=\bigvee_{\omega\in A}\bigwedge_{t\in S,\beta(\omega)<t}\xi(t). \tag{14}$$

Taking into account (10), we find that

$$\bigwedge_{s\in S,A\subseteq\Gamma(s)}\xi(s)\geq\bigvee_{\omega\in A}\bigwedge_{t\in S,\beta(\omega)<t}\xi(t)\geq\xi(\sup_{\omega\in A}\beta(\omega)).$$

If therefore also $\xi(\sup_{\omega\in A}\beta(\omega))\geq\bigwedge_{s\in S,A\subseteq\Gamma(s)}\xi(s)$, it follows that (14) holds for A. Assume on the other hand that $\xi(\sup_{\omega\in A}\beta(\omega))<\bigwedge_{s\in S,A\subseteq\Gamma(s)}\xi(s)$, whence clearly $A\not\subseteq\Gamma(\sup_{\omega\in A}\beta(\omega))$. It then follows from Lemma 17 that $(\exists\varpi\in A)(\forall\omega\in A)(\beta(\varpi)\geq\beta(\omega))$, whence

$$\bigvee_{\omega\in A}\bigwedge_{t\in S,\beta(\omega)<t}\xi(t)=\bigwedge_{t\in S,\beta(\varpi)<t}\xi(t)=\bigwedge_{t\in S,\sup_{\omega\in A}\beta(\omega)<t}\xi(t)=\bigwedge_{t\in S,A\subseteq\Gamma(t)}\xi(t),$$

which tells us that (14) also holds for A in this case. The rest of the proof of the second statement is now obvious.

Let us continue with the first statement. Define $S^o=\{s\in S\colon\emptyset\subset\Gamma(s)\}\subset S$, and consider the $S-[0,1]$-mapping ξ^o which coincides with ξ on S^o and assumes the value $\xi^o(s)=0$ whenever $\Gamma(s)=\emptyset$. If we also define the upper probability $(\Omega,\Gamma(S),\overline{P}^o_\Gamma)$ by $\overline{P}^o_\Gamma(\Gamma(s))=\xi^o(s)$, $s\in S$, then $\overline{P}_\Gamma$ and $\overline{P}^o_\Gamma$ coincide, except possibly on $\emptyset$. They are completely identical if and only if Γ and ξ satisfy (N4), i.e. if $\xi(s)=0$ whenever $\Gamma(s)=\emptyset$. Γ and ξ^o satisfy (N1)–(N4) and (N6), so we may apply the results of the second part of this theorem to the upper probability $(\Omega,\Gamma(S),\overline{P}^o_\Gamma)$. Since moreover this upper probability has the same natural extension $\overline{E}_\Gamma$ as $(\Omega,\Gamma(S),\overline{P}_\Gamma)$ — use Theorem 8 —, the course of reasoning in the proof of the second statement allows us to conclude that $\overline{E}_\Gamma$ is a possibility measure if and only if $\xi^o(\sup T)=\bigvee_{t\in T}\xi^o(t)$, $T\subseteq S$, which is equivalent to $\xi(\sup T)=\bigvee_{t\in T}\xi(t)$, $\emptyset\subset T\subseteq S^o$. □

A.5 Additional discussion for Section 6.1. Assume that the assumptions made in Section 6.1 hold. We take a closer look at condition (7). We also intend to show that conditions (N1), (N2), (N4) and (N5) are in general not sufficient for (7) to hold.

In order to characterise the atoms of the ample field $\mathcal{R}_\Gamma$, it will be convenient to introduce a new class of subsets of Ω. For any s in S:

$$\Delta_\alpha(s) = \Gamma(s) \setminus \bigcup_{t<s} \Gamma(t). \tag{15}$$

For any $s \in S$ and $\omega \in \Omega$, $\omega \in \Delta_\alpha(s) \Leftrightarrow \alpha(\omega) = s$. The sets $\Delta_\alpha(s)$, $s \in S$, satisfy the following disjointness property: if s and t are different elements of S, then $\Delta_\alpha(s) \cap \Delta_\alpha(t) = \emptyset$. Indeed, we may assume without loss of generality that $t < s$. By definition, $\Delta_\alpha(t) \subseteq \Gamma(t)$ and $\Delta_\alpha(s) \cap \Gamma(t) = \emptyset$, which proves our assertion. Moreover, it is obvious that the set $\Delta_\alpha(S)$ covers Ω. Therefore, the set $\{\Delta_\alpha(s) \colon \Delta_\alpha(s) \neq \emptyset\} = \{\Delta_\alpha(\alpha(\omega)) \colon \omega \in \Omega\}$ is a partition of Ω. Since the elements of this partition are all $\mathcal{R}_\Gamma$-measurable, and, moreover, any element of $\Gamma(S)$ is a union of elements of this partition, we are led to the following conclusion.

Proposition 18. *Let $(S, \leq)$ be a complete chain. Assume that Γ satisfies* (N1) *and* (N5). *The atoms of the ample field $\mathcal{R}_\Gamma$ are given by $[\omega]_{\mathcal{R}_\Gamma} = \Delta_\alpha(\alpha(\omega))$, $\omega \in \Omega$.*

We have not required that the mapping Γ should be invertible. Since Γ is increasing, this means that it may be *constant* on an order-convex[16] subset of S. The smallest elements of these subsets play an important part in the study of condition (7). In order to characterise them, it is helpful to consider the following definitions. For any s in S:

$$\widehat{s} = \inf\{t \in S \colon \Gamma(t) = \Gamma(s)\}.$$

Also, $\widehat{S} = \{\widehat{s} \colon s \in S\}$. The following result tells us that $\widehat{s}$ is the smallest element of the order-convex subset of S on which Γ assumes the value $\Gamma(s)$. It also identifies an important subset of $\widehat{S}$.

Proposition 19. *Let $(S, \leq)$ be a complete chain and assume that Γ and ξ satisfy* (N1), (N2) *and* (N5). *For any s and t in S, we have that $\widehat{s} \leq s$, $\widehat{\widehat{s}} = \widehat{s}$ and $s \leq t \Rightarrow \widehat{s} \leq \widehat{t}$. Furthermore, $\Gamma(\widehat{s}) = \Gamma(s)$ and therefore also $\xi(\widehat{s}) = \xi(s)$. Finally, $\alpha(\Omega) \subseteq \widehat{S}$.*

Proof. Consider s in S. The first assertion follows immediately from the definition of $\widehat{s}$. It follows from (6) that $\Gamma(\widehat{s}) = \bigcap\{\Gamma(t) \colon \Gamma(t) = \Gamma(s)\} = \Gamma(s)$. By (N2), we find that $\xi(\widehat{s}) = \xi(s)$. From $\Gamma(s) = \Gamma(\widehat{s})$ we also deduce that $\widehat{\widehat{s}} = \widehat{s}$. Next, consider t in S such that $s \leq t$. We prove that $\widehat{s} \leq \widehat{t}$. It follows from (N1) that $\Gamma(s) \subseteq \Gamma(t)$. If $\Gamma(s) = \Gamma(t)$, clearly $\widehat{s} = \widehat{t}$. If $\Gamma(s) \subset \Gamma(t)$, we find that $\Gamma(\widehat{s}) \subset \Gamma(\widehat{t})$, whence $\widehat{s} < \widehat{t}$. Finally, to prove that $\alpha(\Omega) \subseteq \widehat{S}$, consider any ω in Ω. Proposition 9 tells us that $\omega \in \Delta_\alpha(\alpha(\omega))$. This implies that for any t in S with $t < \alpha(\omega)$, $\Gamma(t) \subset \Gamma(\alpha(\omega))$, whence $\widehat{\alpha(\omega)} = \alpha(\omega)$. $\square$

[16]A subset I of S is order-convex if for any two elements s and t of I, $\{r \in S \colon s \leq r \leq t\} \subseteq I$ [4].

Since, by Propositions 9 and 19 for any s in S and any ω in Ω, $\xi(s) = \xi(\widehat{s})$, and $\alpha(\omega) \leq s \Leftrightarrow \alpha(\omega) \leq \widehat{s}$, we find that (7) is equivalent to

$$\xi(\widehat{s}) = \bigvee_{\omega\in\Omega,\alpha(\omega)\leq\widehat{s}} \xi(\alpha(\omega)), \quad \widehat{s} \in \widehat{S}. \tag{16}$$

Furthermore, it is obvious that for any $\widehat{s}$ in $\alpha(\Omega)$ the above condition is trivially satisfied, which means that in order to verify whether condition (7) holds, our only concern lies in the elements of $\widehat{S} \setminus \alpha(\Omega)$, i.e. those elements s of S for which $s = \widehat{s}$ and $\Delta_\alpha(s) = \emptyset$. Remark that for any s in S:

$$s = \widehat{s} \Leftrightarrow (\forall t < s)(\exists\omega \in \Omega)(t < \alpha(\omega) \leq s) \tag{17}$$

and, taking into account (15),

$$\Delta_\alpha(s) = \emptyset \Leftrightarrow (\forall\omega \in \Omega)(\alpha(\omega) \leq s \Rightarrow \alpha(\omega) < s). \tag{18}$$

Interestingly, if we consider any $\widehat{s}$ in $\widehat{S}$, (17) tells us that there is a net of elements of $\alpha(\Omega)$ which order-converges[17] to $\widehat{s}$, whence $\alpha(\Omega) \subseteq \widehat{S} \subseteq \mathrm{Cl}(\alpha(\Omega))$, where Cl is the closure operator in the order topology on the complete chain $(S, \leq)$. This also implies that

$$\widehat{s} = \sup_{\omega\in\Omega,\alpha(\omega)\leq s} \alpha(\omega), \quad s \in S. \tag{19}$$

We may use the results derived above to show that in general the conditions (N1), (N2), (N4) and (N5) are *not sufficient* to ensure the existence of a possibility measure Π on $(\Omega, \mathcal{R}_\Gamma)$ such that $\Pi \circ \Gamma = \xi$. We do this by giving a counterexample.

Example 20. Let $\Omega = \mathbb{N}^*$, the set of the strictly positive natural numbers, let $S = [0, 1]$ and define Γ as follows:

$$\begin{aligned}\Gamma(x) &= \{n \in \mathbb{N}^* \colon (n-1)/n \leq x\} \\ &= \begin{cases} \{1, \ldots, M\} & ; \quad M \leq \dfrac{1}{1-x} < M+1 \text{ and } x \neq 1 \\ \mathbb{N}^* & ; \quad x = 1. \end{cases}\end{aligned}$$

For any n in $\mathbb{N}^*$, $\alpha(n) = (n-1)/n$, and therefore $n \in \Gamma(\alpha(n))$. This means that Γ satisfies conditions (N1) and (N5). Moreover, $\widehat{[0,1]} = \{\alpha(n) \colon n \in \mathbb{N}^*\} \cup \{1\}$, with $1 \notin \alpha(\mathbb{N}^*)$, and therefore $\alpha(\mathbb{N}^*) \subset \widehat{[0,1]}$. If we define ξ as follows:

$$\xi(x) = \begin{cases} 0 & ; \quad x < 1 \\ \epsilon & ; \quad x = 1, \end{cases}$$

where $0 < \epsilon \leq 1$, then conditions (N2) and (N4) are also satisfied, but clearly $\xi(1) = \epsilon > 0 = \bigvee\{\xi(\alpha(n)) \colon \alpha(n) \leq 1\}$, violating (7). Therefore, by Theorem 10, $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ is not extendable to a possibility measure. Finally, by choosing $\epsilon = 1$ we can make sure that (N3) holds, but still (7) and *a fortiori* (8) will not hold, and therefore, by Theorem 11, the restriction to $\wp(\Omega)$ of the natural extension of $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ will not be a possibility measure. This can also be verified directly. For the natural extension (if $\epsilon = 1$) we have that $\overline{E}_\Gamma(A)$ equals zero if A is finite, and one if A is infinite, and is therefore maxitive. But $\pi^g(n) = \inf_{n\in\Gamma(x)} \xi(x) = 0$ for all $n \in \mathbb{N}^*$, so any possibility measure which extends $\overline{P}_\Gamma$ must be identically

[17] For details about order-convergence and the order topology on partially ordered sets, we refer to [1].

zero, which contradicts $\overline{P}_\Gamma(\mathbb{N}^*) = \epsilon$. This example is in no way pathological: the sets $\Gamma(x)$ merely converge to $\Gamma(1) = \mathbb{N}^*$ as $x \to 1$, and 'continuity' condition (7) merely expresses that $\xi(x)$ should consequently converge to $\xi(1)$.

It was shown in Section 7.2 that the natural extension of a monotone upper probability on a finite number of nested proper events is always a possibility measure. From the above example, we deduce that this does not necessarily hold for an upper probability on a (countably) infinite number of nested sets.

A.6 Proofs of the results mentioned in Section 7.3. Assume that the assumptions made in Section 7.3 hold. We use the notations introduced there. We know that the nested approximation $(\Omega, \Gamma([0,1]), \overline{P}_\Gamma)$ can be extended to a possibility measure. First of all, we show that the distribution π_Γ of the greatest such possibility measure Π_Γ is equal to π^g, which is the distribution of the greatest possibility measure Π^g that coincides with $\overline{P}$ on $\mathcal{S}$.

If we write $\mathcal{S}_\omega = \{A \in \mathcal{S} \colon \omega \in A\}$, it is obvious that for any ω in Ω and x in $[0,1]$, $\omega \in \Gamma(x) \Leftrightarrow (\exists A \in \mathcal{S}_\omega)(\overline{P}(A) < x)$. On the one hand, this tells us that $\beta = \pi^g$, and on the other hand, we find that

$$\pi_\Gamma(\omega) = \bigwedge_{\omega \in \Gamma(x)} \xi(x) = \bigwedge_{A \in \mathcal{S}_\omega} \bigwedge_{\overline{P}(A) < x} \bigvee_{\overline{P}(B) < x} \overline{P}(B) = \bigwedge_{A \in \mathcal{S}_\omega} \psi(A),$$

where we have written $\psi(A) = \bigwedge_{\overline{P}(A) < x} \bigvee_{\overline{P}(B) < x} \overline{P}(B)$. We prove that $\psi(A) = \overline{P}(A)$, $A \in \mathcal{S}$. If $\overline{P}(A) = 1$ then obviously also $\psi(A) = 1$, since $\bigwedge \emptyset = 1$. Let us therefore assume that $\overline{P}(A) < 1$. For any $x \in [0,1]$ such that $\overline{P}(A) < x$ it is obvious that $\overline{P}(A) \leq \bigvee_{\overline{P}(B) < x} \overline{P}(B)$, whence $\overline{P}(A) \leq \psi(A)$. Conversely, note that $\psi(A) \leq \bigwedge_{\overline{P}(A) < x} x = \overline{P}(A)$. This proves that $\pi_\Gamma = \pi^g$, and therefore also $\Pi_\Gamma = \Pi^g$.

Next, we prove that if $(\Omega, \mathcal{S}, \overline{P})$ is supremum-normalisable, then (N3) holds. We need to verify the condition $\Gamma(1) = \Omega \Rightarrow \xi(1) = 1$. Assume that $(\Omega, \mathcal{S}, \overline{P})$ is supremum-normalisable. It follows from $\Gamma(1) = \Omega$ that for any ω in Ω there is an A_ω in $\mathcal{S}$ such that $\omega \in A_\omega$ and $\overline{P}(A_\omega) < 1$. Consequently, $\bigwedge_{A \in \mathcal{S}, \omega \in A} \overline{P}(A) \leq \overline{P}(A_\omega)$ and from the supremum-normalisability of $(\Omega, \mathcal{S}, \overline{P})$ we also deduce that $1 = \bigvee_{\omega \in \Omega} \bigwedge_{A \in \mathcal{S}, \omega \in A} \overline{P}(A) \leq \bigvee_{\omega \in \Omega} \overline{P}(A_\omega) \leq \bigvee_{A \in \mathcal{S}, \overline{P}(A) < 1} \overline{P}(A) \leq 1$, whence $\xi(1) = 1$.

A.7 Proofs of the results mentioned in Section 7.4. We use the notations introduced in Section 7.4. Let us first prove that $(\Omega_1 \times \Omega_2, \mathcal{M}, \overline{P})$ is P-consistent. Consider A in $\mathcal{M}$ and a family $(A_j \colon j \in J)$ of elements of $\mathcal{M}$. We may assume without loss of generality that $A \in \mathcal{M}_1$, i.e. $A = A' \times \Omega_2$, $A' \in \mathcal{R}_1$. We can always divide J into two disjoint subsets J_1 and J_2 such that $A_j \in \mathcal{M}_1$, $j \in J_1$ and $A_j \in \mathcal{M}_2$, $j \in J_2$. Now, from $A \subseteq \bigcup_{j \in J} A_j$, we deduce that $A' \subseteq \bigcup_{j \in J_1} A'_j$ or that $\bigcup_{j \in J_2} A'_j = \Omega_2$, where, with obvious notations, $A_j = A'_j \times \Omega_2$, $j \in J_1$ and $A_j = \Omega_1 \times A'_j$, $j \in J_2$. Either way, $\sup_{j \in J} \overline{P}(A_j) = (\bigvee_{j \in J_1} \Pi_1(A'_j)) \vee (\bigvee_{j \in J_2} \Pi_2(A'_j)) \geq \Pi_1(A') = \overline{P}(A)$, and we conclude that $\overline{P}$ is P-consistent. That $\overline{P}$ is supremum-normalisable follows at once from $\Pi^g(\Omega_1 \times \Omega_2) = \Pi_1(\Omega_1) \wedge \Pi_2(\Omega_2) = 1$.

A.8 Proof of the result mentioned in Section 7.6. We refer to the notations established in Sections 7.5 and 7.6. We assume that (DF1) and (DF2) hold and want to prove that the condition

$$\bigvee_{a \in A} F(a) = F(\sup A), \quad \emptyset \subset A \subseteq \mathbb{A} \tag{20}$$

is equivalent to the left-continuity of F in $\mathbb{R}\setminus\mathbb{A}^{li}$. It is fairly obvious that (20) is implied by (DF2) and the left-continuity of F in $\mathbb{R}\setminus\mathbb{A}^{li}$. In order to prove the converse, we introduce the following notation: $<y>_{\mathbb{A}} = \{z \in \mathbb{R} \colon (\forall a \in \mathbb{A})(a \leq y \Leftrightarrow a \leq z)\}$, $y \in \mathbb{R}$. We also need the following lemma.

Lemma 21. *Let y and z be elements of $\mathbb{R}$.*

1. $y \in <y>_{\mathbb{A}}$.
2. $<y>_{\mathbb{A}} = <z>_{\mathbb{A}} \Leftrightarrow y \in <z>_{\mathbb{A}} \Leftrightarrow z \in <y>_{\mathbb{A}}$.
3. *If $y < z$ then $y \in <z>_{\mathbb{A}} \Leftrightarrow \mathbb{A} \cap\,]y, z] = \emptyset$.*
4. $y \in <z>_{\mathbb{A}} \Rightarrow F(y) = F(z)$.

Proof. The proof of the first two statements is immediate. Statement 4 is a reformulation of (DF1). Let us prove the third statement. Let $y < z$. Assume that $\mathbb{A} \cap\,]y, z] = \emptyset$ and consider a in $\mathbb{A}$. If $a \leq y$ then obviously also $a \leq z$. If $a > y$ then it follows from the assumption that $a > z$. This implies by definition that $y \in <z>_{\mathbb{A}}$. Conversely, assume that there is an a_o in $\mathbb{A}$ such that $y < a_o \leq z$. $y \in <z>_{\mathbb{A}}$ would in particular imply that $a_o \leq z \Leftrightarrow a_o \leq y$, a contradiction. □

Let us now assume that (20) holds. Consider an arbitrary x in $\mathbb{R}\setminus\mathbb{A}^{li}$. There are two possibilities. Either $(\exists \delta > 0)(\mathbb{A} \cap\,]x - \delta, x] = \emptyset)$, whence using Lemma 21.3–4, $F(x - \delta) = F(x)$. Taking into account (DF2), this implies that F is constant in a left-neighbourhood of x, and therefore left-continuous in x. Since x cannot be a left-isolated element of $\mathbb{A}$, the only other possibility is that $(\forall \delta > 0)(\exists a_\delta \in \mathbb{A})(x - \delta < a_\delta < x)$. This implies that $\sup_{\delta > 0} a_\delta = x$, whence $F(x) = \bigvee_{\delta > 0} F(a_\delta)$, using (20). Also using (DF2), we conclude that for any $\epsilon > 0$ there is a left-neighbourhood $]a_\delta, x]$ of x such that for any $y \in\,]a_\delta, x]$, $|F(x) - F(y)| < \epsilon$. F is therefore left-continuous in x.

A.9 Proofs of the results in Section 8.

Proof of Proposition 14. Consider an arbitrary X in $\mathcal{C}$. Since X is non-negative, we have $X = \mathrm{Sup}_{a \geq 0}\, a I_{X^{-1}(\{a\})}$. Since $\overline{P}$ is coherent, and therefore also monotone, $\overline{P}(X) \geq \mathrm{Sup}_{a \geq 0}\, \overline{P}(a I_{X^{-1}(\{a\})}) = \mathrm{Sup}_{a \geq 0}\, a \overline{P}(X^{-1}(\{a\}))$. Moreover, we may write that $I_{X^{-1}(\{a\})} = \mathrm{Sup}_{X(\omega) = a}\, I_{\{\omega\}}$, whence, again taking into account the coherence of $\overline{P}$, $\overline{P}(X^{-1}\{a\}) \geq \mathrm{Sup}_{X(\omega) = a}\, \overline{P}(\{\omega\})$. This leads to

$$\overline{P}(X) \geq \operatorname*{Sup}_{a \geq 0} a \operatorname*{Sup}_{X(\omega) = a} \overline{P}(\{\omega\}) = \operatorname*{Sup}_{a \geq 0} \operatorname*{Sup}_{X(\omega) = a} X(\omega) \overline{P}(\{\omega\}) = \operatorname*{Sup}_{\omega \in \Omega} X(\omega) \overline{P}(\{\omega\}).$$

To complete the proof, note that if (12) holds then the associativity of supremum ensures that $\overline{P}$ is supremum preserving on $\mathcal{C}$. Conversely, assume that $\overline{P}$ is supremum preserving on $\mathcal{C}$, then in the above argument '$\geq$' may be consistently replaced by '$=$'. □

Proof of Proposition 15. Let $A = \{\omega \in \Omega \colon \pi(\omega) = 1\}$. The supremum-normality of π implies that $A \neq \emptyset$. For any X, Y in $\mathcal{C}$ and any real μ such that $X \geq Y + \mu$, we find

$$\overline{P}_\pi(X) = \operatorname*{Sup}_{\omega \in A} X(\omega) \geq \operatorname*{Sup}_{\omega \in A} (Y(\omega) + \mu) = \operatorname*{Sup}_{\omega \in A} Y(\omega) + \mu = \overline{P}_\pi(Y) + \mu,$$

which proves that $\overline{P}_\pi$ satisfies (CC3) besides (CC1) and (CC2), and is therefore coherent on its domain $\mathcal{C}$. Moreover, let Π also denote the natural extension of Π. It is verified using (1) that for any X in $\mathcal{C}$, $\Pi(X) = \text{Sup}\{x \in \mathbb{R}^+ : A \cap \{X > x\} \neq \emptyset\} = \text{Sup}\{X(\omega) : \omega \in A\} = \text{Sup}_{\omega \in \Omega} X(\omega)\pi(\omega)$, which completes the proof. □

Proof of Proposition 16. It only needs to be proven that $\overline{P}_\pi$ does not satisfy (CC3), or in other words, that there are X and Y in $\mathcal{C}$, and μ in $\mathbb{R}$, such that $X \geq Y + \mu$ and $\overline{P}_\pi(X) < \overline{P}_\pi(Y) + \mu$. Choose $Y = 1 - \pi \geq 0$, $\mu = 1$ and $X = Y + 1 \geq 0$. If we introduce, for notational simplicity, $A = \{\pi(\omega) : \omega \in \Omega\}$, $f : [0,1] \to \mathbb{R} : x \mapsto 2x - x^2$ and $g : [0,1] \to \mathbb{R} : x \mapsto x - x^2 + 1$, then it is easy to see that $\overline{P}_\pi(X) = \text{Sup}_{a \in A} f(a)$ and $\overline{P}_\pi(Y) + 1 = \text{Sup}_{a \in A} g(a)$. Since $(\forall x \in [0,1])(f(x) \leq 1)$, we find that $\overline{P}_\pi(X) \leq 1$. On the other hand, it follows from the assumptions that there is an $a_o = \pi(\omega_o) \in A$ such that $0 < a_o < 1$, whence $g(a_o) > 1$ and therefore also $\overline{P}_\pi(Y) + 1 > 1$. □

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