AMPLE FIELDS

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In this paper, we study the notion of an ample or complete field, a special case of the well-known fields and σ-fields of sets. These collections of sets serve as candidates for the domains of possibility and necessity measures, and are therefore important for the further development of a general fuzzy set and possibility theory. The existence of a one-one relationship between ample fields and atomic complete Boolean lattices is proven. Furthermore, the concept of measurability of general fuzzy sets w.r.t. ample fields is explored.

1. Ample Fields and Their Atoms

Let us consider a mapping h from a universe (i.e., a non-empty set) X into the real unit interval $[0,1]$. Such a mapping will be called a $([0,1],\leq)$-fuzzy set (or Zadeh fuzzy set) on X [11]. We also consider the $\mathcal{P}(X)-[0,1]$ mappings Π and N, defined as follows: for an arbitrary subset A of X

$$\Pi(A)=\sup_{x\in A}h(x) \qquad \text{and} \qquad \mathrm{N}(A)=\inf_{x\in \mathrm{co}A}(1-h(x)).$$

Remark also that

$$\Pi(A)=1-\mathrm{N}(\mathrm{co}A). \tag{1.1}$$

Π is called the possibility measure on $\mathcal{P}(X)$ [12] and N the necessity measure on $\mathcal{P}(X)$ [4] based on h. It is easily verified that Π is a *complete join-morphism*[1] between the complete lattices $(\mathcal{P}(X),\subseteq)$ and $([0,1],\leq)$, whereas N is a *complete meet-morphism* between $(\mathcal{P}(X),\subseteq)$ and $([0,1],\leq)$. These characterizations can also serve as an alternative way of defining possibility and necessity measures, without explicit reference to fuzzy sets. They also

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[1]Note that by a complete join-morphism we mean a mapping that preserves *arbitrary* suprema, and therefore also the supremum of the empty set, which means that it preserves the smallest element.

show the way towards a generalization of these notions: a $(L, \leq)$-possibility measure on $\mathcal{L}$ can in principle be defined as a complete join-morphism between a structure $(\mathcal{L}, \subseteq)$ and a complete lattice $(L, \leq)$. Here, $\mathcal{L}$ is a collection of subsets of X, closed under arbitrary unions (i.e., a dual Moore family, see for instance [3]). Dually, a $(L, \leq)$-necessity measure on $\mathcal{M}$ can in principle be defined as a complete meet-morphism between a structure $(\mathcal{M}, \subseteq)$ and a complete lattice $(L, \leq)$. Here, $\mathcal{M}$ is a collection of subsets of X, closed under arbitrary intersections (i.e., a Moore family). But, if we want to study possibility and necessity measures having the same domain, possibly linked by relationships that are suitable generalizations of (1.1), we immediately arrive at ample fields as the appropriate domains for these measures.

Definition 1.1. *Let X be a universe. An ample field $\mathcal{R}$ on X is a set of subsets of X that is closed under complementation and arbitrary unions. An ample field on X is also called a complete field on X.*

In the sequel, we shall use the notation X for an arbitrary universe and $\mathcal{R}$ for an arbitrary ample field on X. From definition 1.1 it is obvious that an ample field $\mathcal{R}$ on a universe X contains the set X and the empty set $\emptyset$, and is also closed under arbitrary intersections. This implies that the structure $(\mathcal{R}, \subseteq)$ is a complete Boloney sublattice of $(\mathcal{P}(X), \subseteq)$. Furthermore, any ample field is in particular a field and a σ-field.

A fairly comprehensive study of ample fields (as possible domains of generalized possibility measures) was given by Wang in 1982 [10], although it cannot be said that the subject was at that time entirely new (see for instance [8]). In this paper, we shall first study a few general lattice-theoretic and topological aspects of ample fields. In this section, we briefly discuss the notion of an atom of an ample field. We also discuss the interesting fact that an ample field can be considered as a very special topology on X. In the next section, the relationship between ample fields and atomic complete Boloney lattices is explored in detail. We also prove a representation theorem for atomic complete Boloney lattices that is a generalization of the well-known representation theorem for finite Boloney lattices. Finally, in section 3, we go into the measurability of fuzzy sets w.r.t. ample fields. The study of the latter notion is a necessary step towards a measure- and integral-theoretic account of possibility and necessity theory (for more detail, see [2]).

An important characteristic of an ample field is that it can be constructed using elementary building blocks, henceforth called atoms, and defined as follows.

Definition 1.2 [10]**.** *Let x be an element of X. The atom of $\mathcal{R}$ containing x is defined as*

$$[\, x \,]_{\mathcal{R}} \stackrel{\text{def}}{=} \bigcap \{\, A \mid A \in \mathcal{R} \text{ and } x \in A \,\}.$$

Furthermore, a subset A of X is called an atom of $\mathcal{R}$ if $(\exists x \in X)(A = [\, x \,]_{\mathcal{R}})$.

The justification of the name '*atom of $\mathcal{R}$ containing x*' is given in the following theorem. A proof can be found in [10].

Theorem 1.1 [10]**.** *Let x and y be elements of X and let A be a subset of X. Then the following propositions hold.*

(1) $x \in [\,x\,]_{\mathcal{R}}$ *and* $[\,x\,]_{\mathcal{R}} \in \mathcal{R}$.
(2) A *is an atom of* $\mathcal{R}$ *if and only if* $(\forall z \in A)([\,z\,]_{\mathcal{R}} = A)$.
(3) A *is an atom of* $\mathcal{R}$ *if and only if* $A \in \mathcal{R} \setminus \{\emptyset\}$ *and* A *is indivisible in* $\mathcal{R}$, *i.e.,* $(\forall B \in \mathcal{R})(B \cap A = \emptyset$ *or* $\mathrm{co}B \cap A = \emptyset)$.
(4) $A \in \mathcal{R} \Leftrightarrow A = \bigcup_{x \in A} [\,x\,]_{\mathcal{R}}$.
(5) *The set of the atoms of* $\mathcal{R}$ *is a partition of* X. *We shall denote this set by* $X_{\mathcal{R}}$, *i.e.,* $X_{\mathcal{R}} \stackrel{\mathrm{def}}{=} \{\, [\,x\,]_{\mathcal{R}} \mid x \in X \,\}$.

We can interpret $\mathcal{R}$ as a topology (i.e., a set of open sets) on X. Since $\mathcal{R}$ is closed under complementation, all the sets in $\mathcal{R}$ are at the same time open and closed, or *clopen*. Furthermore, the set of atoms $X_{\mathcal{R}}$ is a basis for the topology $\mathcal{R}$ on X, since theorem 1.1(4) guarantees that every open set—i.e., each element of $\mathcal{R}$— is a union of elements of $X_{\mathcal{R}}$.

We continue this topological discussion with the introduction of special mappings, the topological interpretation of which will soon become clear.

Definition 1.3.

(1) *The* $\mathcal{P}(X) - \mathcal{P}(X)$ *mapping* $\mathfrak{p}_{\mathcal{R}}$ *is defined by*

$$(\forall A \in \mathcal{P}(X))(\mathfrak{p}_{\mathcal{R}}(A) \stackrel{\mathrm{def}}{=} \bigcup_{x \in A} [\,x\,]_{\mathcal{R}}).$$

(2) *The* $\mathcal{P}(X) - \mathcal{P}(X)$ *mapping* $\mathfrak{n}_{\mathcal{R}}$ *is defined by*

$$(\forall A \in \mathcal{P}(X))(\mathfrak{n}_{\mathcal{R}}(A) \stackrel{\mathrm{def}}{=} \bigcap_{x \in \mathrm{co}A} \mathrm{co}[\,x\,]_{\mathcal{R}}).$$

One readily verifies that these mappings satisfy some very special properties, the most important of which are summed up in the next theorem. The proof is fairly straightforward and is therefore left implicit.

Theorem 1.2. *Let A and B be arbitrary subsets of X, and let $(A_j \mid j \in J)$ be an arbitrary family of elements of $\mathcal{P}(X)$.*

(1) $\mathfrak{p}_{\mathcal{R}}(A) = \mathrm{co}\,\mathfrak{n}_{\mathcal{R}}(\mathrm{co}A)$.
(2) $\mathfrak{p}_{\mathcal{R}}(A) = \bigcap \{\, B \mid B \in \mathcal{R}$ *and* $A \subseteq B \,\}$.
(3) $\mathfrak{n}_{\mathcal{R}}(A) = \bigcup \{\, B \mid B \in \mathcal{R}$ *and* $B \subseteq A \,\}$.

Furthermore,

(4) $A \subseteq \mathfrak{p}_{\mathcal{R}}(A)$;
(5) $A \subseteq B \Rightarrow \mathfrak{p}_{\mathcal{R}}(A) \subseteq \mathfrak{p}_{\mathcal{R}}(B)$;
(6) $\mathfrak{p}_{\mathcal{R}}(\mathfrak{p}_{\mathcal{R}}(A)) = \mathfrak{p}_{\mathcal{R}}(A)$;
(7) $\mathfrak{p}_{\mathcal{R}}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \mathfrak{p}_{\mathcal{R}}(A_j)$;
(8) $A = \mathfrak{p}_{\mathcal{R}}(A) \Leftrightarrow A \in \mathcal{R}$.

On the other hand,

(9) $\mathfrak{n}_{\mathcal{R}}(A) \subseteq A$;
(10) $A \subseteq B \Rightarrow \mathfrak{n}_{\mathcal{R}}(A) \subseteq \mathfrak{n}_{\mathcal{R}}(B)$;
(11) $\mathfrak{n}_{\mathcal{R}}(\mathfrak{n}_{\mathcal{R}}(A)) = \mathfrak{n}_{\mathcal{R}}(A)$;
(12) $\mathfrak{n}_{\mathcal{R}}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \mathfrak{n}_{\mathcal{R}}(A_j)$;
(13) $A = \mathfrak{n}_{\mathcal{R}}(A) \Leftrightarrow A \in \mathcal{R}$.

From this theorem, we may draw the following conclusions: (7) and (8) imply that $\mathfrak{p}_{\mathcal{R}}$ is a $(\mathcal{R}, \subseteq)$-possibility measure on $\mathcal{P}(X)$; and (12) and (13) lead to the conclusion that $\mathfrak{n}_{\mathcal{R}}$ is a $(\mathcal{R}, \subseteq)$-necessity measure on $\mathcal{P}(X)$. Also, $\mathfrak{p}_{\mathcal{R}}$ is the closure operator associated with the closure system (or Moore family) $\mathcal{R}$ (points (4)–(6) and (8)) and $\mathfrak{n}_{\mathcal{R}}$ is the dual closure operator associated with the dual closure system $\mathcal{R}$ (points (9)–(11) and (13)).

Moreover, for arbitrary A in $\mathcal{P}(X)$, (2) tells us that $\mathfrak{p}_{\mathcal{R}}(A)$ is the topological closure of A, and (3) means that $\mathfrak{n}_{\mathcal{R}}(A)$ is the topological interior of A. Not surprisingly therefore, (4)–(7) are closely related to Kuratowski's closure axioms, that can in general be used to define a topology starting from a "closure operator" (see for instance [7]).

This topological interpretation of ample fields brings us to the notion of *coarseness*. Let us consider two ample fields $\mathcal{R}_1$ and $\mathcal{R}_2$ on X. We shall say that $\mathcal{R}_1$ is coarser than $\mathcal{R}_2$ if the topology $\mathcal{R}_1$ is coarser than the topology $\mathcal{R}_2$, i.e., if $\mathcal{R}_1 \subseteq \mathcal{R}_2$ (see for instance [5]). Interestingly, $\mathcal{R}_1$ is coarser than $\mathcal{R}_2$ if and only if the partition $X_{\mathcal{R}_2}$ of X is finer than the partition $X_{\mathcal{R}_1}$ of X, i.e., $(\forall x \in X)([\,x\,]_{\mathcal{R}_2} \subseteq [\,x\,]_{\mathcal{R}_1})$.

To end this section, we want to point out that the mappings $\mathfrak{p}_{\mathcal{R}}$ and $\mathfrak{n}_{\mathcal{R}}$ can be made to play an important role in *rough set theory* [9]. To introduce the notion of a rough set, one normally starts with an equivalence relation E on a universe X. With an arbitrary subset A of X, there is associated a lower approximation $\underline{A} \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x/_E \subseteq A \,\}$ and an upper approximation $\overline{A} \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x/_E \cap A \neq \emptyset \,\}$, where, of course, $x/_E$ is the equivalence class of E containing x. The couple $(\underline{A}, \overline{A})$ is called the rough approximation of A, associated with E. Now, the quotient set $X/_E$ is a partition of X. Let us denote by $\mathcal{R}_E$ the ample field for which $X_{\mathcal{R}_E} = X/_E$ (i.e., the ample field generated by $X/_E$, see definition 3.1 furtheron). Then, of course $\underline{A} = \mathfrak{n}_{\mathcal{R}_E}(A)$ and $\overline{A} = \mathfrak{p}_{\mathcal{R}_E}(A)$.

2. Ample Fields and Atomic Complete Boolean Lattices

There exists a very close relationship between the notion of an ample field and that of an atomic complete Boolean lattice. In this section, we shall explore this relationship and prove a representation theorem that is a generalization of the well-known representation theorem for finite Boolean lattices (see for instance [3] theorem 8.3). We start this section with a definition, reminding the reader of a few basic notions. These are necessary for the proper understanding of the subsequent discussion. For the origins of these notions, we refer to [1] section VIII.9, [3] p. 42.

Definition 2.1.

(1) *An atom λ of a lattice $(L, \leq)$ with a smallest element 0, is an element of L that covers 0, i.e., $\lambda > 0$ and $(\forall \mu \in L)(0 < \mu \leq \lambda \Rightarrow \mu = \lambda)$.*
(2) *The set of the atoms of a lattice $(L, \leq)$ is denoted by $\mathcal{A}(L, \leq)$.*
(3) *A lattice $(L, \leq)$ is atomic if and only if $\mathcal{A}(L, \leq)$ is join-dense in $(L, \leq)$, i.e., if and only if every element μ of L is a supremum of atoms of $(L, \leq)$, and therefore is the supremum of all the atoms λ of $(L, \leq)$ satisfying $\lambda \leq \mu$.*

In order to be able to prove the above-mentioned representation theorem, we first prove a few rather technical lemmas, and a proposition that tells us that ample fields, ordered by set inclusion, are atomic complete Boolean lattices.

Lemma 2.1. *Let $(L, \leq)$ be a lattice with smallest element 0, λ an atom of $(L, \leq)$ and μ an element of L. Then $\lambda \leq \mu \Leftrightarrow \lambda \frown \mu > 0$.*

Proof. Assume that $\lambda \leq \mu$. It follows that $\lambda \frown \mu = \lambda > 0$, since λ is an atom of $(L, \leq)$. Conversely, assume that $\lambda \frown \mu > 0$. Since in this case $0 < \lambda \frown \mu \leq \lambda$, we have $\lambda \frown \mu = \lambda$, taking into account definition 2.1. The consistence property in lattices then yields $\lambda \leq \mu$. □

Lemma 2.2. *Let $(L, \leq)$ be a complete Boolean lattice. Let λ be an element of $\mathcal{A}(L, \leq)$ and let A be a subset of L. Then $\lambda \leq \sup A \Leftrightarrow (\exists \mu \in A)(\lambda \leq \mu)$.*

Proof. Assume on the one hand that there exists a μ in A for which $\lambda \leq \mu$. Since $\sup A$ is in particular an upper bound of A, we have that $\mu \leq \sup A$, whence $\lambda \leq \sup A$, using the transitivity of $\leq$.

Assume on the other hand that $\lambda \leq \sup A$. Let us prove *ex absurdo* that $(\exists \mu \in A)(\lambda \leq \mu)$. If $(\forall \mu \in A)(\lambda \not\leq \mu)$, the previous lemma implies that $(\forall \mu \in A)(\lambda \frown \mu = 0)$, whence

$$\sup_{\mu \in A}(\lambda \frown \mu) = 0.$$

Since in any complete Boolean lattice $\frown$ is completely distributive w.r.t. sup (see for instance [1] theorem V.16) it follows that $\lambda \frown \sup A = 0$, whence, taking into account the previous lemma, $\lambda \not\leq \sup A$, a contradiction. □

Proposition 2.1. *$(\mathcal{R}, \subseteq)$ is an atomic complete Boolean lattice, and furthermore $X_{\mathcal{R}} = \mathcal{A}(\mathcal{R}, \subseteq)$, i.e., the atoms of the ample field $\mathcal{R}$ are also the atoms of the lattice $(\mathcal{R}, \subseteq)$.*

Proof. It is immediately verified that $(\mathcal{R}, \subseteq)$ is a complete Boolean sublattice of $(\mathcal{P}(X), \subseteq)$. Let us therefore show that $\mathcal{A}(\mathcal{R}, \subseteq) = X_{\mathcal{R}}$. Let A be an element of $X_{\mathcal{R}}$, then theorem 1.1(3) implies that $A \in \mathcal{R}$ and $\emptyset \subset A$. Consider an arbitrary element B of $\mathcal{R}$ and assume that $\emptyset \subset B \subseteq A$. For an arbitrary element x of B we then have on the one hand that $[\,x\,]_{\mathcal{R}} \subseteq B$, taking into account definition 1.2. On the other hand we have that $[\,x\,]_{\mathcal{R}} = A$, taking into account $x \in A$ and theorem 1.1(2). Therefore $A \subseteq B$, whence $A = B$. We may conclude that the smallest element $\emptyset$ of $(\mathcal{R}, \subseteq)$ is covered by A, whence $A \in \mathcal{A}(\mathcal{R}, \subseteq)$. This implies that $X_{\mathcal{R}} \subseteq \mathcal{A}(\mathcal{R}, \subseteq)$.

Conversely, let A be an arbitrary element of $\mathcal{A}(\mathcal{R}, \subseteq)$. In order that A be an element of $X_{\mathcal{R}}$, it is, taking into account theorem 1.1(3), sufficient that A is undivisible in $\mathcal{R}$. We give a proof by contradiction. Assume that there exists a B in $\mathcal{R}$ such that $B \cap A \neq \emptyset$ and $\mathrm{co}B \cap A \neq \emptyset$. Lemma 2.1 then implies that $A \subseteq B$ and $A \subseteq \mathrm{co}B$. This is a contradiction, since $A \neq \emptyset$. We may therefore conclude that $A \in X_{\mathcal{R}}$, whence $\mathcal{A}(\mathcal{R}, \subseteq) \subseteq X_{\mathcal{R}}$ and therefore also $X_{\mathcal{R}} = \mathcal{A}(\mathcal{R}, \subseteq)$. This, together with theorem 1.1(4), tells us that the complete Boolean lattice $(\mathcal{R}, \subseteq)$ is atomic. $\square$

This finally brings us to the most important result of this section.

Theorem 2.1 (Representation Theorem). *A partially ordered set $(L, \leq)$ is an atomic complete Boolean lattice if and only if it is order-isomorphic to an ample field of sets, ordered by set inclusion.*

Proof. Taking into account the previous proposition, it suffices to show that if $(L, \leq)$ is an atomic complete Boolean lattice, this structure is order-isomorphic to an ample field of sets, ordered by set inclusion. We give a proof by construction. Let $(L, \leq)$ be an atomic complete Boolean lattice. The complement operator of this Boolean lattice will be denoted by $'$. We define the $L - \mathcal{P}(\mathcal{A}(L, \leq))$ mapping $\mathfrak{P}$ as

$$(\forall \mu \in L)(\mathfrak{P}(\mu) \stackrel{\text{def}}{=} \{\, \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \leq \mu \,\}).$$

Since $(L, \leq)$ is assumed atomic, by definition, $(\forall \mu \in L)(\mu = \sup \mathfrak{P}(\mu))$. Let us first show that $\mathfrak{P}$ is a bijection between L and $\mathfrak{P}(L)$. It clearly suffices to show that $\mathfrak{P}$ is an injection. Consider arbitrary μ_1 and μ_2 in L and suppose that $\mu_1 \neq \mu_2$. Should $\mathfrak{P}(\mu_1) = \mathfrak{P}(\mu_2)$, this would imply that $\mu_1 = \sup \mathfrak{P}(\mu_1) = \sup \mathfrak{P}(\mu_2) = \mu_2$, a contradiction.

We proceed to show that $\mathfrak{P}(L)$ is an ample field on $\mathcal{A}(L, \leq)$. First of all, $\mathfrak{P}(0) = \emptyset$ whence $\emptyset \in \mathfrak{P}(L)$. Let furthermore μ be an arbitrary element of $(L, \leq)$, then

$$\begin{aligned} \mathrm{co}\mathfrak{P}(\mu) &= \mathrm{co}\{\, \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \leq \mu \,\} \\ &= \{\, \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \not\leq \mu \,\}, \end{aligned}$$

and, taking into account the properties of complement operators in Boolean lattices (see for instance [1] theorem I.16, formula I.(7)),

$$= \{\, \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \frown \mu' > 0 \,\},$$

whence, taking into account lemma 2.1,

$$= \{ \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \leq \mu' \} = \mathfrak{P}(\mu').$$

This implies that $\mathrm{co}\mathfrak{P}(\mu) \in \mathfrak{P}(L)$.

In a fairly analogous way, we have for an arbitrary non-empty subset A of L that

$$\begin{aligned} \bigcup_{\mu \in A} \mathfrak{P}(\mu) &= \bigcup_{\mu \in A} \{ \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \leq \mu \} \\ &= \{ \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } (\exists \mu \in A)(\lambda \leq \mu) \} \end{aligned}$$

and, taking into account lemma 2.2,

$$= \{ \lambda \mid \lambda \in \mathcal{A}(L, \leq) \text{ and } \lambda \leq \sup A \} = \mathfrak{P}(\sup A). \tag{2.1}$$

This implies that $\bigcup_{\mu \in A} \mathfrak{P}(\mu) \in \mathfrak{P}(L)$. We may therefore conclude that $\mathfrak{P}(L)$ is an ample field on $\mathcal{A}(L, \leq)$. Formula (2.1) also implies that $\mathfrak{P}$ is an order-isomorphism between the atomic complete Boolean lattices $(L, \leq)$ and $(\mathfrak{P}(L), \subseteq)$. □

3. Measurability of Fuzzy Sets

In this section, we shall denote by $(L, \leq)$ an arbitrary complete lattice, with greatest element 1 and smallest element 0. It is also assumed that $0 \neq 1$. Furthermore, by a $(L, \leq)$-fuzzy set in X we mean a $X - L$ mapping. This implies that our definition of a fuzzy set is more general than the one given by Zadeh [11], that is now generally used.

The intersection of an arbitrary family of ample fields on X is of course again an ample field on X. This implies that the set of the ample fields on X is a *closure system.* With this closure system, we can associate a closure operator τ on $\mathcal{P}(X)$, i.e., a $\mathcal{P}(\mathcal{P}(X)) - \mathcal{P}(\mathcal{P}(X))$ mapping, defined as follows.

Definition 3.1 [10]. *Let $\mathcal{C}$ be a subset of $\mathcal{P}(X)$, then*

$$\tau(\mathcal{C}) \stackrel{\text{def}}{=} \bigcap \{ \mathcal{R} \mid \mathcal{R} \textit{ is an ample field on } X \textit{ and } \mathcal{C} \subseteq \mathcal{R} \},$$

and $\tau(\mathcal{C})$ is called the ample field on X, generated by $\mathcal{C}$, or also the coarsest ample field containing $\mathcal{C}$.

Theorem 3.1 [10]. *$\tau(X_{\mathcal{R}}) = \mathcal{R}$, i.e., every ample field is generated by the set of its atoms.*

With an arbitrary $(L, \leq)$-fuzzy set h on X, we can associate a partition of the universe X: X can be divided into those disjoint subsets of X that h can distinguish between, i.e., $\{ h^{-1}(\{\lambda\}) \mid \lambda \in h(X) \}$. This partition generates an ample field on X, whence the following definition.

Definition 3.2. *Let h be an arbitrary $(L, \leq)$-fuzzy set on X. The ample field generated by h is denoted by $\tau(h)$ and defined as*

$$\tau(h) \stackrel{\text{def}}{=} \tau(\{\, h^{-1}(\{\lambda\}) \mid \lambda \in h(X) \,\}).$$

The ample field generated by a $(L, \leq)$-fuzzy set satisfies a few interesting properties, that are gathered in the next theorem. The proof is rather straightforward and is therefore omitted.

Theorem 3.2. *Let h be an arbitrary $(L, \leq)$-fuzzy set on X. Then the following propositions hold.*

(1) $\tau(h) = h^{-1}(\mathcal{P}(L)) = \{\, h^{-1}(B) \mid B \in \mathcal{P}(L) \,\}$.
(2) $(\forall x \in X)([\, x\,]_{\tau(h)} = h^{-1}(h(\{x\})))$.
(3) *For an arbitrary subset A of X: $A \in \tau(h) \Leftrightarrow h^{-1}(h(A)) = A$.*

There is a very natural notion of measurability of fuzzy sets (or mappings in general, for that matter) with respect to ample fields. Indeed, theorem 3.1 tells us that an ample field $\mathcal{R}$ can be constructed using its atoms. $X_{\mathcal{R}}$ is the finest 'texture' that $\mathcal{R}$ can reveal on X. We will therefore demand that a measurable $(L, \leq)$-fuzzy set h not reveal a finer texture on X than $\mathcal{R}$ does, i.e., that h be constant on the atoms of $\mathcal{R}$.

Definition 3.3. *We call a $(L, \leq)$-fuzzy set h on X measurable w.r.t. $\mathcal{R}$ if and only if h is constant on the atoms of $\mathcal{R}$, i.e.,*

$$(\forall x \in X)(\forall y \in [\, x\,]_{\mathcal{R}})(h(y) = h(x)).$$

To conclude this section, we study a few aspects of this measurability. In proposition 3.1, it is shown that the measurability of fuzzy sets is a natural extension of the measurability of ordinary sets. Theorem 3.3 gives equivalent formulations of the measurability condition. For instance, the equivalence of points (1) and (4) tells us that $\tau(h)$ is the coarsest ample field on X with respect to which h is still measurable.

Proposition 3.1. *Let A be an arbitrary subset of X. Then*

$$A \in \mathcal{R} \Leftrightarrow \chi_A \text{ is measurable w.r.t. } \mathcal{R},$$

where χ_A is the characteristic $X - L$ mapping of the set A.

Theorem 3.3. *Let h be an arbitrary $(L, \leq)$-fuzzy set on X. Then the following propositions are equivalent:*

(1) *h is measurable w.r.t. $\mathcal{R}$;*
(2) $(\forall x \in X)(h(x) = \inf\limits_{y \in [\, x\,]_{\mathcal{R}}} h(y))$;
(3) $(\forall x \in X)(h(x) = \sup\limits_{y \in [\, x\,]_{\mathcal{R}}} h(y))$;
(4) $\tau(h) \subseteq \mathcal{R}$;
(5) $(\forall \lambda \in L)(h^{-1}(\{\lambda\}) \in \mathcal{R})$;
(6) $(\forall \lambda \in L)(h^{-1}([0, \lambda]) \in \mathcal{R})$;
(7) $(\forall \lambda \in L)(h^{-1}([\lambda, 1]) \in \mathcal{R})$.

Proof. The proof of the equivalence of (1), (2) and (3) is immediate. Let us therefore prove the equivalence of (1) and (4). Since, by theorem 3.2(2), $(\forall x \in X)([\,x\,]_{\tau(h)} = h^{-1}(h(\{x\})))$, we also have that

$$(\forall x \in X)(\forall y \in X)(y \in [\,x\,]_{\tau(h)} \Leftrightarrow h(x) = h(y)).$$

Assume that h is measurable w.r.t. $\mathcal{R}$. Then for arbitrary x in X and y in $[\,x\,]_{\mathcal{R}}$ we have by definition that $h(x) = h(y)$, whence $y \in [\,x\,]_{\tau(h)}$. This implies that $[\,x\,]_{\mathcal{R}} \subseteq [\,x\,]_{\tau(h)}$, whence $\tau(h) \subseteq \mathcal{R}$. Conversely, if $\tau(h) \subseteq \mathcal{R}$, or equivalently, if $(\forall x \in X)([\,x\,]_{\mathcal{R}} \subseteq [\,x\,]_{\tau(h)})$, it is obvious that h is constant on $[\,x\,]_{\mathcal{R}}$, for all x in X.

We shall now prove the equivalence of (1), (5), (6) and (7).
(1)⇒(5): Let h be measurable w.r.t. $\mathcal{R}$. Consider an arbitrary λ in L. There are two possibilities. Either $\lambda \notin h(X)$, whence $h^{-1}(\{\lambda\}) = \emptyset \in \mathcal{R}$. Or $\lambda \in h(X)$, and, since h is assumed measurable w.r.t. $\mathcal{R}$ and taking into account theorem 3.2(2) and the equivalence of (1) and (4), $(\forall x \in X)([\,x\,]_{\mathcal{R}} \subseteq h^{-1}(\{h(x)\}))$. On the one hand, this implies that

$$\bigcup_{x \in h^{-1}(\{\lambda\})} [\,x\,]_{\mathcal{R}} \subseteq h^{-1}(\{\lambda\}).$$

On the other hand

$$h^{-1}(\{\lambda\}) = \bigcup_{x \in h^{-1}(\{\lambda\})} \{x\} \subseteq \bigcup_{x \in h^{-1}(\{\lambda\})} [\,x\,]_{\mathcal{R}},$$

whence finally

$$h^{-1}(\{\lambda\}) = \bigcup_{x \in h^{-1}(\{\lambda\})} [\,x\,]_{\mathcal{R}},$$

which is equivalent to $h^{-1}(\{\lambda\}) \in \mathcal{R}$, taking into account theorem 1.1(4).
(5)⇒(6): Assume that $(\forall \lambda \in L)(h^{-1}(\{\lambda\}) \in \mathcal{R})$. Then for arbitrary μ in L

$$h^{-1}([0,\mu]) = h^{-1}(\bigcup_{\lambda \in [0,\mu]} \{\lambda\}) = \bigcup_{\lambda \in [0,\mu]} h^{-1}(\{\lambda\})) \in \mathcal{R},$$

since $\mathcal{R}$ is closed under arbitrary unions.
(5)⇒(7): Analogous to the previous point.
(7)⇒(1): Assume that $(\forall \lambda \in L)(h^{-1}([\lambda,1]) \in \mathcal{R})$. Consider an arbitrary x in X and arbitrary y in $[\,x\,]_{\mathcal{R}}$. We must show that $h(x) = h(y)$. It is easily shown that $(\forall A \in \mathcal{R})(x \in A \Rightarrow y \in A)$, whence in particular

$$(\forall \lambda \in L)(x \in h^{-1}([\lambda,1]) \Rightarrow y \in h^{-1}([\lambda,1])).$$

If we put $\mu \stackrel{\text{def}}{=} h(x)$ and $\nu \stackrel{\text{def}}{=} h(y)$, then certainly $h(x) \in [\mu,1]$, and this implies that $x \in h^{-1}([\mu,1])$, whence $y \in h^{-1}([\mu,1])$. This implies that $\nu \geq \mu$. But, $y \in [\,x\,]_{\mathcal{R}}$ is equivalent to $x \in [\,y\,]_{\mathcal{R}}$, and an analogous line of reasoning now leads to $\nu \leq \mu$. Therefore, $h(x) = h(y)$, and h is measurable w.r.t. $\mathcal{R}$.
(6)⇒(1): Analogous to the previous point. □

Reference

1. G. Birkhoff, *Lattice Theory*, A.M.S. Colloquium Publications, Volume XXV, Providence, Rhode Island, 1967.
2. G. de Cooman, *Evaluation Sets and Mappings—An Order-Theoretic Approach to Vagueness and Uncertainty*, Doctoral dissertation, Private publication, Universiteit Gent, 1993. (In Dutch)
3. B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*, Cambridge University Press, Cambridge, 1990.
4. D. Dubois and H. Prade, *Théorie des possibilités*, Masson, Paris, 1985.
5. M. C. Gemignani, *Elementary Topology*, 1972, Herdruk 1990, Dover Publications, New York.
6. G. Grätzer, *General Lattice Theory*, Mathematische Reihe 52, Birkhäuser Verlag, Basel and Stuttgart, 1978.
7. S. Lipschutz, *Theory and Problems of General Topology*, McGraw-Hill, New York, 1965.
8. E. Mendelson, *Theory and Problems of Boolean Algebra and Switching Circuits*, McGraw-Hill, New York, 1970.
9. Z. Pawlak, Rough Sets, *Internat. J. Inform. Comput. Sci.* 5 (1982), 341–356.
10. Wang Pei-Zhang, Fuzzy Contactability and Fuzzy Variables, *Fuzzy Sets and Systems* 8 (1982), 81–92.
11. L. A. Zadeh, Fuzzy Sets, *Information and Control* 8 (1965), 338–353.
12. L. A. Zadeh, Fuzzy Sets as a Basis for a Theory of Possibility, *Fuzzy Sets and Systems* 1 (1978), 3–28.

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