Constructing Possibility Measures

Bernard De Baets* and Gert de Cooman†
Department of Applied Mathematics and Computer Science
Department of Electrical Engineering
University of Gent, Krijgslaan 281 (S9), B-9000 Gent, Belgium
E-mail: {bernard.debaets,gert.decooman}@rug.ac.be

Abstract

In this paper, we address some aspects of the extension problem for possibility measures: given the values that a (fuzzy) set mapping takes on a family of (fuzzy) sets, is it possible to extend this mapping to a possibility measure? This problem is shown to be equivalent to a special system of relational equations. When the family of sets considered is a (semi)partition, two important solutions are identified. It is shown that these solutions, and their fuzzifications, play a central part in the treatment of the more general extension problem. This role is shown to be even more conspicuous when the family of fuzzy sets considered is a $\mathcal{T}$-(semi)partition, a notion introduced and studied for the first time in this paper.

1 Introduction

Let us start by introducing a number of basic notions. We shall consider a nonempty set, or universe of discourse, X. A *fuzzy set* in X is a mapping from X to the real unit interval $[0,1]$. The set of all fuzzy sets in X is denoted by $\mathcal{F}(X)$. A fuzzy set is an obvious generalization of the characteristic mapping χ_C of a crisp subset C of X. We can use the natural order relation $\leq$ on $[0,1]$ to define a partial order relation $\sqsubseteq$ on $\mathcal{F}(X)$. For A and B in $\mathcal{F}(X)$, $A \sqsubseteq B$ iff $(\forall x \in X)(A(x) \leq B(x))$. Of course, $(\mathcal{F}(X), \sqsubseteq)$ is a complete lattice, with top χ_X and bottom $\chi_\emptyset$. Supremum and infimum in this structure can be taken pointwise, i.e., for any family $(A_i \mid i \in I)$ of elements of $\mathcal{F}(X)$, $(\sup_{i\in I} A_i)(x) = \sup_{i\in I} A_i(x)$ and $(\inf_{i\in I} A_i)(x) = \inf_{i\in I} A_i(x)$. A fuzzy set A in X is called sup-*normal*, or simply *normal*, if $\sup_{x\in X} A(x) = 1$. The *support* of A is the subset supp$A$ of X defined as supp$A = \{x \mid A(x) > 0\}$. The *kernel* of A is the subset ker$A$ of X defined as ker$A = \{x \mid A(x) = 1\}$. A is called *modal* if its kernel is nonempty.

A *triangular norm* or *t-norm* $\mathcal{T}$ is a binary operator on $[0,1]$ that has increasing partial mappings, is associative and commutative, and has 1 as a neutral element: $(\forall x \in [0,1])(\mathcal{T}(1,x) = x)$ [10]. A *border implicator* $\mathcal{I}$ is a binary operator on $[0,1]$ that is hybrid monotonous, i.e., has decreasing first and increasing second partial mappings, and satisfies both the neutrality principle $(\forall x \in [0,1])(\mathcal{I}(1,x) = x)$ and $(\forall x \in [0,1])(\mathcal{I}(0,x) = 1)$ [2]. We will often assume that a t-norm $\mathcal{T}$ is left-continuous. The reason for this is that such a $\mathcal{T}$ is completely distributive w.r.t. sup, or in other words, that for any a in $[0,1]$ and for any family $(b_i \mid i \in I)$ of elements of $[0,1]$, $\mathcal{T}(a, \sup_{i\in I} b_i) = \sup_{i\in I} \mathcal{T}(a, b_i)$.

With a t-norm $\mathcal{T}$ we associate two binary operators $\mathcal{I}_\mathcal{T}$ and $\mathcal{L}_\mathcal{T}$ on $[0,1]$ defined by [2, 4]:

$$\mathcal{I}_\mathcal{T}(\alpha,\beta) = \sup\{\gamma \mid \gamma \in [0,1] \wedge \mathcal{T}(\alpha,\gamma) \leq \beta\}$$
$$\mathcal{L}_\mathcal{T}(\alpha,\beta) = \inf\{\gamma \mid \gamma \in [0,1] \wedge \mathcal{T}(\alpha,\gamma) \geq \beta\}.$$

It is well known that the operator $\mathcal{I}_\mathcal{T}$ is a border implicator [2]. Also, we define the binary operator $\mathcal{E}_\mathcal{T}$ on $[0,1]$, also called *biresiduation* of $\mathcal{T}$ [9], as follows

$$\mathcal{E}_\mathcal{T}(\alpha,\beta) = \min(\mathcal{I}_\mathcal{T}(\alpha,\beta), \mathcal{I}_\mathcal{T}(\beta,\alpha)).$$

Notice that $\mathcal{E}_\mathcal{T}(\alpha,\beta) = \mathcal{I}_\mathcal{T}(\max(\alpha,\beta), \min(\alpha,\beta))$.

A *possibility measure* Π on X [14] is a mapping from the power class $\wp(X)$ of X to the real unit interval $[0,1]$, satisfying, for any family $(C_i \mid i \in I)$ of elements of $\wp(X)$: $\Pi(\bigcup_{i\in I} C_i) = \sup_{i\in I} \Pi(C_i)$. It has a *(possibility) distribution* π, i.e., a $X - [0,1]$-mapping π, defined by $\pi(x) = \Pi(\{x\})$. Notice that for any C in $\wp(X)$, $\Pi(C) = \sup_{x\in C} \pi(x)$.

Given a left-continuous t-norm, we may also define the possibility of a fuzzy set A in X by [6, 7]:

$$\Pi_\mathcal{T}(A) = \sup_{x\in X} \mathcal{T}(A(x), \pi(x)).$$

*Research Assistant of the Belgian National Fund for Scientific Research.

†Senior Research Assistant of the Belgian National Fund for Scientific Research.

Of course, $\Pi_{\mathcal{T}}$ can be considered as a $\mathcal{F}(X) - [0,1]$-mapping. For any crisp subset C of X, $\Pi_{\mathcal{T}}(\chi_C) = \Pi(C)$. For any family $(A_i \mid i \in I)$ of elements of $\mathcal{F}(X)$, $\Pi_{\mathcal{T}}(\sup_{i\in I} A_i) = \sup_{i\in I} \Pi_{\mathcal{T}}(A_i)$. In this sense, $\Pi_{\mathcal{T}}$ can also be interpreted as an *extended* possibility measure, and is also called the $\mathcal{T}$*-extension* of Π.

Given a family $(C_i \mid i \in I)$ of elements of $\wp(X)$ and a corresponding family $(\beta_i \mid i \in I)$ of elements of $[0,1]$, we may ask whether there exists a possibility measure Π on X, or equivalently, a possibility distribution π, such that

$$(\forall i \in I)(\Pi(C_i) = \sup_{x\in C_i} \pi(x) = \beta_i). \qquad (1)$$

This problem has been completely solved by Wang [12]. In particular, if the family considered is a *semi-partition* of X, i.e., if its elements are mutually disjoint, it is easily proven that there always exists at least one π that satisfies (1). In fact, the greatest solution π^g of (1) is given by

$$\pi^g(x) = \inf_{i\in I} \mathcal{I}(\chi_{C_i}(x), \beta_i) = \begin{cases} \beta_i & \text{, if } x \in C_i \\ 1 & \text{, elsewhere.} \end{cases}$$

In this expression, $\mathcal{I}$ is an arbitrary border implicator. There is also a class of minimal solutions, the supremum of which is again a solution, and given by

$$\pi^p(x) = \sup_{i\in I} \mathcal{T}(\chi_{C_i}(x), \beta_i) = \begin{cases} \beta_i & \text{, if } x \in C_i \\ 0 & \text{, elsewhere.} \end{cases}$$

In this expression, $\mathcal{T}$ is an arbitrary t-norm. Moreover, if the family considered is a *partition* of X, i.e., if it is a semi-partition that furthermore covers X, then $\pi^p = \pi^g$. Finally, if $(C_i \mid i \in I)$ is not a semi-partition, then there is a solution to (1) if and only if π_g is a (the greatest) solution.

In this paper, we investigate some important aspects of the extension (or fuzzification) of problem (1), and deal with extended possibilities and families of fuzzy sets. In other words, we shall consider a left-continuous t-norm $\mathcal{T}$, a family $(A_i \mid i \in I)$ of fuzzy sets in X, a corresponding family $(\beta_i \mid i \in I)$ of elements of $[0,1]$, and look for solutions π of the system of equations

$$(\forall i \in I)(\Pi_{\mathcal{T}}(A_i) = \sup_{x\in X} \mathcal{T}(A_i(x), \pi(x)) = \beta_i).$$

In particular, we will investigate whether appropriate 'fuzzifications' of π^g and π^p are still solutions of this system. First of all, it should be noted that this system of equations can be interpreted as a system of relational equations.

2 Root systems

Of capital importance in relational equation solving is the order-theoretic concept of a root system [1, 2, 4, 5]. A root system is a particular union of closed intervals with the same ending point, as is formalized in the following definition.

Definition 1 (Root systems) [5] *A subset R of an ordered set $(P, \leq)$ is called a root system if and only if there exists an element σ in P and an antichain O in $\downarrow \sigma = \{\alpha \mid \alpha \leq \sigma\}$ such that $R = \bigcup_{\omega\in O} [\omega, \sigma]$.*

The element σ is called the *stem* of the root system. The elements of the antichain O are called the *offshoots* of the root system. A root system is called *finitely generated* if the set of offshoots is finite.

Theorem 1 [5] *Let $(R_i \mid i \in I)$ be a family of finitely generated root systems of a complete lattice $(L, \leq)$ with stem σ_i and set of offshoots O_i. If the intersection $\bigcap_{i\in I} R_i$ is nonempty, then it is a root system with stem $\sigma = \inf_{i\in I} \sigma_i$ and as offshoots the minimal elements of the set*

$$\{\sup_{i\in I} \omega_i \mid (\forall i \in I)(\omega_i \in O_i \wedge \omega_i \leq \sigma)\}.$$

Note that if the intersection of a finite family of finitely generated root systems of a complete lattice is nonempty, then it is a finitely generated root system.

3 Systems of sup-$\mathcal{T}$ equations

3.1 Sup-$\mathcal{T}$ equations

Consider a left-continuous t-norm $\mathcal{T}$, a fuzzy set A in a universe X and a β in $[0,1]$, then we want to determine the solution set of the equation

$$\sup_{x\in X} \mathcal{T}(A(x), U(x)) = \beta \qquad (2)$$

in the unknown fuzzy set U in X. This problem has been solved in a more general setting in [2, 4].

Proposition 1 [2] *The solution set Γ of equation (2) is nonempty if and only if the fuzzy set G in X, defined by $G(x) = \mathcal{I}_{\mathcal{T}}(A(x), \beta)$, is a solution. If $\Gamma \neq \emptyset$, then G is the greatest solution.*

Proposition 2 [2] *If $\mathcal{T}$ is continuous, then the solution set of equation (2) is nonempty if and only if $\sup_{x\in X} A(x) \geq \beta$.*

Proposition 3 [2] *If $\mathcal{T}$ is continuous and furthermore $(\exists x \in X)(A(x) \geq \beta)$, then the solution set Γ of equation (2) contains the root system of $(\mathcal{F}(X), \sqsubseteq)$*

with as stem the fuzzy set G in X, defined by $G(x) = \mathcal{I}_\mathcal{T}(A(x), \beta)$, and as offshoots the elements of the set

$$W = \{M_u \mid A(u) \geq \beta\}$$

with M_u the fuzzy set in X defined by

$$M_u(x) = \begin{cases} \mathcal{L}_\mathcal{T}(A(u), \beta) & , \text{ if } x = u \\ 0 & , \text{ elsewhere.} \end{cases}$$

If X is finite, then Γ coincides with this root system, which is moreover finitely generated.

3.2 Systems of sup-$\mathcal{T}$ equations

Consider a left-continuous t-norm $\mathcal{T}$, a family $(A_i \mid i \in I)$ of fuzzy sets in a universe X and a family $(\beta_i \mid i \in I)$ in $[0, 1]$, then we want to determine the solution set of the system of equations

$$\sup_{x \in X} \mathcal{T}(A_i(x), U(x)) = \beta_i, \quad i \in I \tag{3}$$

in the unknown fuzzy set U in X.

Proposition 4 [2] *The solution set Γ of system (3) is nonempty if and only if the fuzzy set G in X, defined by*

$$G(x) = \inf_{i \in I} \mathcal{I}_\mathcal{T}(A_i(x), \beta_i),$$

is a solution. If $\Gamma \neq \emptyset$, then G is the greatest solution.

Proposition 5 [2] *If X is finite, $\mathcal{T}$ is continuous and the solution set Γ of system (3) is nonempty, then Γ is a root system of $(\mathcal{F}(X), \sqsubseteq)$ with as stem the fuzzy set G in X, defined by*

$$G(x) = \inf_{i \in I} \mathcal{I}_\mathcal{T}(A_i(x), \beta_i),$$

and as offshoots the minimal elements of the set

$$\{\sup_{i \in I} M_u^i \mid (\forall i \in I)(M_u^i \in W_i')\}$$

with $W_i' = \{M_u^i \mid A_i(u) \geq \beta_i \wedge M_u^i \sqsubseteq G\}$ and M_u^i the fuzzy set in X defined by

$$M_u^i(x) = \begin{cases} \mathcal{L}_\mathcal{T}(A_i(u), \beta_i) & , \text{ if } x = u \\ 0 & , \text{ elsewhere.} \end{cases}$$

If I is finite, then this root system is moreover finitely generated.

4 Constructing possibility measures

Consider a left-continuous t-norm $\mathcal{T}$, a family $(A_i \mid i \in I)$ of fuzzy sets in a universe X and a family $(\beta_i \mid i \in I)$ in $[0, 1]$, then we want to determine the possibility distributions π such that

$$(\forall i \in I)(\Pi_\mathcal{T}(A_i) = \sup_{x \in X} \mathcal{T}(A_i(x), \pi(x)) = \beta_i). \tag{4}$$

Proposition 6 *The solution set Γ of the system of equations (4) is nonempty if and only if the possibility distribution π^g defined by*

$$\pi^g(x) = \inf_{i \in I} \mathcal{I}_\mathcal{T}(A_i(x), \beta_i)$$

is a solution. If $\Gamma \neq \emptyset$, then π^g is the greatest solution.

Example 1 Consider as t-norm the minimum operator M, then the (potential) greatest possibility distribution π^g is given by

$$\pi^g(x) = \inf_{i \in I, \beta_i < A_i(x)} \beta_i.$$

Proposition 7 *If $(\forall i \in I)(A_i$ is modal$)$ and*

$$(\forall (i, j) \in I^2)(i \neq j \Rightarrow \ker A_i \cap \text{supp} A_j = \emptyset),$$

then the solution set Γ of the system of equations (4) is nonempty.

Proposition 8 *If $(\forall i \in I)(A_i$ is normal$)$, $\mathcal{T}$ is continuous and*

$$(\forall (i, j) \in I^2)(i \neq j \Rightarrow \text{supp} A_i \cap \text{supp} A_j = \emptyset),$$

then the solution set Γ of the system of equations (4) is nonempty.

Remark the close analogy with the crisp results for semipartitions described in the introduction. The conditions imposed on the family $(A_i \mid i \in I)$ in the previous propositions indicate two possible ways of introducing the concept of a fuzzy semi-partition, namely as a set of modal fuzzy sets with pairwise disjoint kernels and supports, or as a set of normal fuzzy sets with pairwise disjoint supports.

Notice that the greatest solution π^g in the fuzzy case is similar to (a fuzzification of) the greatest solution in the crisp case, and requires the choice of a specific border implicator, namely the residual implicator $\mathcal{I}_\mathcal{T}$.

For a finite universe X, the complete solution set can be determined, as is expressed in the following theorem.

Theorem 2 *If X is finite, $\mathcal{T}$ is continuous and the solution set Γ of the system of equations (4) is nonempty, then Γ is a root system of $(\mathcal{F}(X), \sqsubseteq)$ with stem π^g defined by*

$$\pi^g(x) = \inf_{i \in I} \mathcal{I}_\mathcal{T}(A_i(x), \beta_i)$$

and as offshoots the minimal elements of the set

$$\{\sup_{i \in I} \delta_u^i \mid (\forall i \in I)(\delta_u^i \in W_i')\}$$

with $W_i' = \{\delta_u^i \mid A_i(u) \geq \beta_i \wedge \delta_u^i \sqsubseteq \pi^g\}$ and δ_u^i the fuzzy set in X defined by

$$\delta_u^i(x) = \begin{cases} \mathcal{L}_\mathcal{T}(A_i(u), \beta_i) & , \text{ if } x = u \\ 0 & , \text{ elsewhere.} \end{cases}$$

Notice that the possibility distributions δ_u^i can be considered as distributions of Dirac measures [7].

5 When does fuzzification work?

In the previous section, we have seen that an obvious fuzzification π^g of the classical candidate for the greatest solution, plays a similar part in the fuzzified problem. Let us now also take a look at the possible solution π^p. Consider a left-continuous t-norm $\mathcal{T}$, a family $(A_i \mid i \in I)$ of fuzzy sets in a universe X and a family $(\beta_i \mid i \in I)$ in $[0,1]$, then we want to establish the necessary and sufficient conditions under which the possibility distribution π^p defined by

$$\pi^p(x) = \sup_{i \in I} \mathcal{T}(A_i(x), \beta_i)$$

is a solution of the system $(\forall i \in I)(\Pi_\mathcal{T}(A_i) = \beta_i)$.

Proposition 9 *If π^p is a solution, then the following property holds*

$$(\forall x \in X)(\forall (i,j) \in I^2)(\mathcal{T}(A_i(x), \beta_i) \leq \mathcal{I}_\mathcal{T}(A_j(x), \beta_j)).$$

Theorem 3 *If I is finite and $\mathcal{T}$ is continuous, then π^p is a solution if and only if*

(A) $(\forall (i,j) \in I^2)(\alpha_{i,j}^\mathcal{T} \leq \mathcal{E}_\mathcal{T}(\beta_i, \beta_j))$;

(B) $(\forall j \in I)(\exists i \in I)(\beta_j \leq \beta_i \wedge \mathcal{L}_\mathcal{T}(\beta_i, \beta_j) \leq \alpha_{i,j}^\mathcal{T})$,

where $\alpha_{i,j}^\mathcal{T}$ is defined as

$$\alpha_{i,j}^\mathcal{T} = \sup_{x \in X} \mathcal{T}(A_i(x), A_j(x)).$$

Condition (A) can be interpreted as follows: the height of the intersection (w.r.t. $\mathcal{T}$) of A_i and A_j should not be larger than the degree of equality of β_i and β_j. From this theorem it immediately follows that a set of *sufficient* conditions for π^p to be a solution is given by

(A) $(\forall (i,j) \in I^2)(\alpha_{i,j}^\mathcal{T} \leq \mathcal{E}_\mathcal{T}(\beta_i, \beta_j))$;

(C) $(\forall i \in I)(\mathcal{L}_\mathcal{T}(\beta_i, \beta_i) \leq \alpha_{i,i}^\mathcal{T})$.

Example 2

1. Consider as t-norm the minimum operator M, then condition (A) can be written as

$$\alpha_{i,j}^M \leq \begin{cases} 1 & , \text{ if } \beta_i = \beta_j \\ \min(\beta_i, \beta_j) & , \text{ elsewhere.} \end{cases}$$

2. Consider as t-norm the algebraic product P, then condition (A) can be written as

$$\alpha_{i,j}^P \leq \begin{cases} 1 & , \text{ if } \beta_i = \beta_j \\ \min(\frac{\beta_i}{\beta_j}, \frac{\beta_j}{\beta_i}) & , \text{ if } \min(\beta_i, \beta_j) > 0 \\ 0 & , \text{ elsewhere.} \end{cases}$$

3. Consider the Luckasiewicz t-norm W (defined by $W(x,y) = \max(x+y-1, 0)$), then condition (A) can be written as

$$\alpha_{i,j}^W \leq 1 - \mid \beta_i - \beta_j \mid .$$

The finiteness condition on the index set I, the right-continuity of the partial mappings of $\mathcal{T}$ and condition (B) in Theorem 3 can be omitted in case of modal fuzzy sets.

Proposition 10 *If $(\forall i \in I)(A_i$ is modal$)$, then the following property holds*

$$(\forall i \in I)(\forall x \in \ker A_i)(\pi^p(x) \geq \beta_i).$$

Theorem 4 *If $(\forall i \in I)(A_i$ is modal$)$, then π^p is a solution if and only if condition (A) holds.*

Proposition 11 *If $(\forall i \in I)(A_i$ is modal$)$ and π^p is a solution, then the following property holds*

$$(\forall i \in I)(\forall x \in \ker A_i)(\pi^p(x) = \beta_i).$$

Corollary 1 *If $(\forall i \in I)(A_i$ is modal$)$, then a necessary condition for π^p to be a solution, is*

$$(\forall (i,j) \in I^2)(\beta_i \neq \beta_j \Rightarrow \ker A_i \cap \ker A_j = \emptyset).$$

6 $\mathcal{T}$-equivalences and $\mathcal{T}$-partitions

6.1 Equivalence relations and partitions

A binary relation E in a universe X is called an equivalence relation on X if and only if it is reflexive, symmetric and transitive. Given an equivalence relation E on X, the equivalence class of $x \in X$ is the set $[x]_E$ defined as

$$[x]_E = \{y \mid (x,y) \in E\}.$$

The quotient set X/E is then defined as

$$X/E = \{[x]_E \mid x \in X\}.$$

It is well known that the elements of X/E form a partition of X.

Conversely, to a partition $\mathcal{A}$ of X corresponds an equivalence relation E on X defined as

$$E = \{(x,y) \mid (\exists A \in \mathcal{A})((x,y) \in A^2)\}.$$

It then holds that $X/E = \mathcal{A}$.

6.2 $\mathcal{T}$-equivalences and $\mathcal{T}$-partitions

Definition 2 *Consider a t-norm $\mathcal{T}$. A binary fuzzy relation E in a universe X is called a $\mathcal{T}$-equivalence on X if and only if it is reflexive, symmetric and $\mathcal{T}$-transitive, i.e., if and only if for any x, y and z in X:*

(i) $E(x,x) = 1$;

(ii) $E(x,y) = E(y,x)$;

(iii) $\mathcal{T}(E(x,y), E(y,z)) \leq E(x,z)$.

$\mathcal{T}$-equivalences are also called fuzzy equalities [8], equality relations [9] or indistinguishability operators [11]. M-equivalences are called similarity relations [13] and W-equivalences are called likeness relations.

Definition 3 [9] *Consider a t-norm $\mathcal{T}$ and a $\mathcal{T}$-equivalence E on a universe X. A fuzzy set A in X is called extensional (w.r.t. E) if and only if*

$$(\forall (x,y) \in X^2)(\mathcal{T}(A(x), E(x,y)) \leq A(y)).$$

Definition 4 [9] *Let us consider a t-norm $\mathcal{T}$, a $\mathcal{T}$-equivalence E on a universe X and a fuzzy set A in X. The extensional hull of A (w.r.t. E) is the fuzzy set $[A]_E$ in X defined by*

$$[A]_E(x) = \sup_{y \in X} \mathcal{T}(A(y), E(x,y)).$$

In particular, we can consider extensional hulls of crisp sets (identified with their characteristic mapping). The extensional hull $[C] = [\chi_C]$ of a crisp set C in X w.r.t. a $\mathcal{T}$-equivalence E in X is given by

$$[C]_E(x) = \sup_{y \in C} E(x,y).$$

For a singleton $\{x_0\}$ we obtain $[\{x_0\}]_E(x) = E(x_0, x)$. We will further write $[\{x_0\}]_E = [x_0]_E$. Extensional hulls of singletons are clearly generalizations of the concept of an equivalence class (see also [13]).

This shows that a fuzzy set can be induced by a crisp value in a vague environment that is characterized by a $\mathcal{T}$-equivalence. Höhle [8], Klawonn and Kruse [9] have addressed the following interesting problem: given a family $(A_i \mid i \in I)$ of fuzzy sets in a universe X and a family $(x_i \mid i \in I)$ in X, under what conditions can the fuzzy sets A_i be interpreted as representations of the crisp elements x_i in a vague environment characterized by a suitable $\mathcal{T}$-equivalence E, i.e., $(\forall i \in I)([x_i]_E = A_i)$.

Before recalling these results, we introduce the notion of $\mathcal{T}$-semi-partition and of a $\mathcal{T}$-partition.

Definition 5 *Consider a t-norm $\mathcal{T}$ and a universe X. A subset $\mathcal{A}$ of $\mathcal{F}(X)$ is called a $\mathcal{T}$-semi-partition of X if and only if there exists a $\mathcal{T}$-equivalence E on X such that*

$$\mathcal{A} \subseteq \{[x]_E \mid x \in X\}.$$

Definition 6 *Consider a t-norm $\mathcal{T}$ and a universe X. A subset $\mathcal{A}$ of $\mathcal{F}(X)$ is called a $\mathcal{T}$-partition of X if and only if there exists a $\mathcal{T}$-equivalence E on X such that*

$$\mathcal{A} = \{[x]_E \mid x \in X\}.$$

The $\mathcal{T}$-equivalence corresponding to a $\mathcal{T}$-partition is clearly unique.

Example 3 *Consider a finite partition $\{C_1, \ldots, C_n\}$ of a universe X and the $X \to \{1, \ldots, n\}$ mapping k defined by: $k(x) = i \Leftrightarrow x \in C_i$. Construct the subset $\mathcal{A} = \{A_1, \ldots, A_n\}$ of $\mathcal{F}(X)$ as follows, with $\epsilon \in]0,1[$ and x in X:*

$$A_i(x) = \epsilon^{|i-k(x)|},$$

then $\mathcal{A}$ is a P-partition of X. The corresponding P-equivalence E is defined by

$$E(x,y) = \epsilon^{|k(x)-k(y)|}.$$

Proposition 12 *Consider a t-norm $\mathcal{T}$ and a $\mathcal{T}$-equivalence E on a universe X, then the following properties hold, for any x, y, z in X:*

(i) $[x]_E = [y]_E \Leftrightarrow E(x,y) = 1$;

(ii) $\sup_{z \in X} \mathcal{T}([x]_E(z), [y]_E(z)) \leq E(x,y)$;

(iii) $(\forall u \in \ker[x]_E)([u]_E = [x]_E)$.

This proposition implies that when $E(x,y) \neq 1$, or equivalently $[x]_E \neq [y]_E$, it holds that

$$\ker[x]_E \cap \ker[y]_E = \emptyset.$$

Proposition 13 *Consider a t-norm $\mathcal{T}$ and a universe X. In a $\mathcal{T}$-semi-partition $\mathcal{A}$ of X, the following properties hold:*

(i) $(\forall A \in \mathcal{A})(A \text{ is modal})$;

(ii) $(\forall A \in \mathcal{A})(\forall x \in \ker A)([x]_E = A)$.

Proposition 14 *Consider a t-norm $\mathcal{T}$ and a universe X. The set $\{\ker A \mid A \in \mathcal{A}\}$ corresponding to a $\mathcal{T}$-partition $\mathcal{A}$ forms a partition of X.*

Using the notions above, we can restate the results of Klawonn and Kruse in the following way.

Theorem 5 [9] *Consider a left-continuous t-norm $\mathcal{T}$ and a family $\mathcal{A} = (A_i \mid i \in I)$ of modal fuzzy sets in a universe X, then $\mathcal{A}$ is a $\mathcal{T}$-semi-partition of X if and only if*

$$(\forall (i,j) \in I^2)(\alpha_{i,j}^{\mathcal{T}} \leq \inf_{x \in X} \mathcal{E}_{\mathcal{T}}(A_i(x), A_j(x))). \qquad (5)$$

Condition (5) can be interpreted as follows: the height of the intersection of A_i and A_j (w.r.t. $\mathcal{T}$) should not be larger than their degree of equality .

The pairwise disjointness of the fuzzy sets A_i (w.r.t. $\mathcal{T}$), i.e., $(\forall x \in X)(\mathcal{T}(A_i(x), A_j(x)) = 0)$ is a sufficient requirement for condition (5) to hold, as is of course also the pairwise disjointness of their supports.

Given a $\mathcal{T}$-semi-partition $\mathcal{A}$, the corresponding $\mathcal{T}$-equivalence is not necessarily uniquely determined. The greatest (E^g) and smallest (E^s) corresponding $\mathcal{T}$-equivalence are given by [9]:

$$E^g(x,y) = \inf_{i \in I} \mathcal{E}_{\mathcal{T}}(A_i(x), A_i(y))$$
$$E^s(x,y) = \begin{cases} \sup_{i \in I} \mathcal{T}(A_i(x), A_i(y)) & \text{, if } x \neq y \\ 1 & \text{, if } x = y. \end{cases}$$

Let $X' = \bigcup_{i \in I} \text{supp} A_i$, then in the case of $\mathcal{T} = M$, the following property holds [9]:

$$(\forall (x,y) \in {X'}^2)(E^g(x,y) = E^s(x,y)).$$

When $X' = X$, it follows that the M-equivalence is uniquely determined.

6.3 Further results on fuzzification

Consider a left-continuous t-norm $\mathcal{T}$, a $\mathcal{T}$-semi-partition $\mathcal{A} = \{A_i \mid i \in I\}$ and a family $(\beta_i \mid i \in I)$ in $[0,1]$, then we want to establish necessary and sufficient conditions under which the possibility distribution π^p defined by

$$\pi^p(x) = \sup_{i \in I} \mathcal{T}(A_i(x), \beta_i)$$

is a solution of the system $(\forall i \in I)(\Pi_{\mathcal{T}}(A_i) = \beta_i)$.

Proposition 15 *If the following condition holds*

$$(\forall (i,j) \in I^2)(\inf_{x \in X} \mathcal{E}_{\mathcal{T}}(A_i(x), A_j(x)) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j)),$$

then π^p is a solution.

The sufficient condition in this proposition can be interpreted as follows : the degree of equality of A_i and A_j should not be larger than the degree of equality of β_i and β_j.

Proposition 16 *Let E be a $\mathcal{T}$-equivalence corresponding to $\mathcal{A}$, then the following property holds for any i and j in I:*

$$(\forall x \in \ker A_i)(\forall y \in \ker A_j)(E(x,y) = \alpha_{i,j}^{\mathcal{T}}).$$

This proposition also indicates that for a $\mathcal{T}$-semi-partition the corresponding $\mathcal{T}$-equivalence is uniquely determined on the set $\left(\bigcup_{i \in I} \ker A_i\right)^2$. It allows us to rewrite Theorem 4 in the following way.

Theorem 6 *Let E be a $\mathcal{T}$-equivalence corresponding to $\mathcal{A}$, then π^p is a solution if and only if for any i and j in I*

$$(\forall x \in \ker A_i)(\forall y \in \ker A_j)(E(x,y) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j)).$$

In case of a $\mathcal{T}$-semi-partition, the necessary condition of Proposition 11 also turns into a sufficient condition.

Theorem 7 *π^p is a solution if and only if*

$$(\forall i \in I)(\forall x \in \ker A_i)(\pi^p(x) = \beta_i).$$

The final theorem of this paper shows that for a $\mathcal{T}$-partition, as in the case of a classical partition, the possibility distributions π^g and π^p coincide, provided that they are indeed solutions. That this is not always the case, is demonstrated in an example following the theorem.

Theorem 8 *If $\mathcal{A}$ is a $\mathcal{T}$-partition and there exists a solution, then π^p is the greatest solution, i.e., $\pi^p = \pi^g$.*

Corollary 2 *If $\mathcal{A}$ is a $\mathcal{T}$-partition and π^p is a solution, then the following property holds:*

$$(\forall (i,j) \in I^2)(\forall x \in \ker A_i)(\mathcal{I}_\mathcal{T}(A_j(x), \beta_j) \geq \beta_i).$$

Example 4 Consider the P-partition from Example 3, and let $k \in \{1, \ldots, n-1\}$ and $l = k+1$. When taking $\beta_k = 0$ and $\beta_l \neq 0$, the condition

$$(\forall x \in \ker A_k)(\forall y \in \ker A_l)(E(x,y) \leq \mathcal{E}_P(\beta_k, \beta_l))$$

reduces to

$$(\forall x \in \ker A_k)(\forall y \in \ker A_l)(\epsilon \leq 0),$$

which is clearly not fulfilled. According to Theorem 6, π^p is not a solution, and hence, using Theorem 8, we can conclude that no solutions exist.

References

[1] B. De Baets and E. E. Kerre, "A representation of solution sets of fuzzy relational equations", in: *Cybernetics and Systems '94*, Proceedings of the Twelfth European Meeting on Cybernetics and Systems Research, ed. R. Trappl, pp. 287–294, World Scientific, Singapore, 1994.

[2] B. De Baets, *Oplossen van vaagrelationele vergelijkingen: een ordetheoretische benadering*, PhD. thesis, University of Gent, 1995.

[3] B. De Baets, "Model implicators and their characterization", in: *Proceedings of ISFL '95*, First ICSC International Symposium on Fuzzy Logic (Zürich, Switzerland, May 26–27, 1995), ed. N. Steele, pp. A42–A49, ICSC Academic Press, 1995.

[4] B. De Baets, "An order-theoretic approach to solving sup-$\mathcal{T}$ equations", in: *Fuzzy Set Theory and Advanced Mathematical Applications*, ed. D. Ruan, pp. 67–87, Kluwer Academic, Boston, 1995.

[5] B. De Baets, "Crowns and root systems", *Algebra Universalis*, in preparation.

[6] G. de Cooman and E. E. Kerre, "Possibility and necessity integrals", accepted for publication in *Fuzzy Sets and Systems*, 1995.

[7] G. de Cooman, E. E. Kerre and F. Vanmassenhove, "Possibility theory: An integral theoretic approach", *Fuzzy Sets and Systems*, Vol. 46, pp. 287–300, 1992.

[8] U. Höhle, "Fuzzy equalities and indistinguishability", in: *Proceedings of EUFIT'93*, First European Congress on Fuzzy and Intelligent Technologies (Aachen, Germany, September 7–10, 1993), ed. H.-J. Zimmermann, pp. 358–363, European Laboratory for Intelligent Techniques Engineering (ELITE), 1993.

[9] F. Klawonn and R. Kruse, "From fuzzy sets to indistinguishability and back", in: *Proceedings of ISFL '95*, First ICSC International Symposium on Fuzzy Logic (Zürich, Switzerland, May 26–27, 1995), ed. N. Steele, pp. A57–A59, ICSC Academic Press, 1995.

[10] B. Schweizer and A. Sklar, "Associative functions and abstract semi-groups", *Publ. Math. Debrecen*, Vol. 10, pp. 69–84, 1963.

[11] E. Trillas and L. Valverde, "An inquiry into indistinguishability operators", in: *Aspects of vagueness*, eds. H. Skala, S. Termini and E. Trillas, pp. 231–256, Reidel, Dordrecht, 1984.

[12] Z. Wang and G. J. Klir, *Fuzzy Measure Theory*, Plenum Press, New York, 1992.

[13] L. A. Zadeh, "Similarity relations and fuzzy orderings", *Information Sciences*, Vol. 3, pp. 177–200, 1971.

[14] L. A. Zadeh, "Fuzzy sets as a basis for a theory of possibility", *Fuzzy Sets and Systems*, Vol. 1, pp. 3–28, 1978.