The construction of possibility measures from samples on $\mathcal{T}$-semi-partitions [1]

Bernard De Baets [a], Gert de Cooman [b,2] and Etienne Kerre [a]

[a] *Department of Applied Mathematics and Computer Science, University of Ghent, Krijgslaan 281 (S9), B-9000 Gent, Belgium*

[b] *Vakgroep Elektrische Energietechniek, Universiteit Gent, Technologiepark - Zwijnaarde 9, B-9052 Zwijnaarde, Belgium*

We address the (generalized) extension problem for possibility measures: given a map defined on a family of (fuzzy) sets, is it possible to extend it to a (generalized) possibility measure? The extension problem for possibility measures is known to be equivalent to a system of sup-$\mathcal{T}$ equations, with $\mathcal{T}$ a t-norm. A key role is played by the greatest solution (of type inf-$\mathcal{I}$, with $\mathcal{I}$ a border implicator). When the family of sets considered is a semi-partition, another important solution (of type sup-$\mathcal{T}$, with $\mathcal{T}$ a t-norm) can be identified. In the treatment of the generalized possibilistic extension problem, we show that a fuzzification of the greatest solution also plays a central role. On the other hand, an immediate fuzzification of the sup-$\mathcal{T}$ type solution is investigated. General necessary and sufficient conditions for this fuzzification to be a solution are established. This fuzzification is then further discussed in the case of a $\mathcal{T}$-semi-partition or a $\mathcal{T}$-partition. Finally, we investigate possible criteria for extendability, inspired by Wang's classical criterion of P-consistency.

Key words: P-consistency, possibilistic extension problem, possibility measure, sup-$\mathcal{T}$ equation, $\mathcal{T}$-semi-partition.

1 Problem description

In measure theory, an important issue is whether a given set function, defined on a certain domain, can be extended to a larger domain in such a way that

[1] A preliminary version [8] of this paper was presented at ISUMA–NAFIPS '95 (University of Maryland, USA, September 1995).

[2] Postdoctoral Fellow of the Belgian National Fund for Scientific Research (NFWO).

Preprint submitted to Elsevier Preprint *10 July 1996*

its extension satisfies a number of properties. To give only one example, a positive set function μ that is (finitely) additive and continuous from below on a field of sets $\mathcal{S}$ can be uniquely extended to a monotone set function γ on the closure from below $\overline{\mathcal{S}}$ of $\mathcal{S}$, and can be uniquely extended to a measure γ^* defined on the Caratheodory algebra of the outer set function γ^* of γ [18].

Similarly, in the context of possibility theory [13–15,20,26], it is very natural to ask whether a given set function defined on a particular class of subsets of a universe of discourse X, can be extended to a possibility measure defined on the power class $\wp(X)$. As we shall briefly explain further on, this so-called *possibilistic extension problem* has been solved by Wang [25] for possibility measures taking values in the unit interval, and more generally by Boyen et al. [1] for possibility measures taking values in complete lattices.

In this paper, we look at a more general problem. Indeed, it is possible to define the possibility of fuzzy sets, as a generalization of the possibility of sets [13,16,17,20,26]. This at once leads to a *generalized possibilistic extension problem*: given a function defined on a class of fuzzy sets in X, can we extend it to a possibility measure on $\mathcal{F}(X)$, the set of all fuzzy sets in X?

Let us start our discussion of this problem by giving a brief overview of the most important basic notions and notations that will be used throughout this paper. We shall consider a nonempty set, or *universe of discourse*, X. A *fuzzy set* in X is a map from X to the real unit interval $[0, 1]$. The set of all fuzzy sets in X is denoted by $\mathcal{F}(X)$. A fuzzy set is an obvious generalization of the characteristic map χ_C of a subset C of X. We can use the natural order relation $\leq$ on $[0, 1]$ to define a partial order relation $\sqsubseteq$ on $\mathcal{F}(X)$ as follows: for A and B in $\mathcal{F}(X)$, $A \sqsubseteq B$ iff $(\forall x \in X)(A(x) \leq B(x))$. Of course, $(\mathcal{F}(X), \sqsubseteq)$ is a complete lattice, with top χ_X and bottom $\chi_\emptyset$. Supremum and infimum in this structure can be taken pointwise, i.e. for any family $(A_i \mid i \in I)$ of elements of $\mathcal{F}(X)$ and $x \in X$, $(\sup_{i\in I} A_i)(x) = \sup_{i\in I} A_i(x)$ and $(\inf_{i\in I} A_i)(x) = \inf_{i\in I} A_i(x)$. A fuzzy set A in X is called sup-*normal*, or simply *normal*, iff $\sup_{x\in X} A(x) = 1$. The *support* of A is the subset $\mathrm{supp}A$ of X defined as $\mathrm{supp}A = \{x \mid A(x) > 0\}$. The *kernel* of A is the subset $\ker A$ of X defined as $\ker A = \{x \mid A(x) = 1\}$. A is called *modal* iff its kernel is nonempty, and A is called *nonempty* iff it differs from $\chi_\emptyset$, or in other words, iff its support is nonempty. Any modal fuzzy set is normal and any normal fuzzy set is nonempty.

A *triangular norm* or *t-norm* $\mathcal{T}$ is a binary operator on $[0, 1]$ that has increasing partial maps, is associative and commutative, and has 1 as a neutral element: $(\forall x \in [0, 1])(\mathcal{T}(1, x) = x)$ [23]. A *border implicator* $\mathcal{I}$ is a binary operator on $[0, 1]$ that is hybrid monotonous, i.e. has decreasing first and increasing second partial maps, and satisfies both the neutrality principle $(\forall x \in [0, 1])(\mathcal{I}(1, x) = x)$ and $(\forall x \in [0, 1])(\mathcal{I}(0, x) = 1)$ [4]. We shall often assume that a t-norm $\mathcal{T}$ has left-continuous partial maps. The reason for this is the following. Consider a $[0,1]-[0,1]$-map f. If f is increasing (as are the partial maps of a t-norm and the second partial maps of a border implicator), left-continuity respectively right-continuity of f is equivalent to $f(\sup_{i\in I} a_i) = \sup_{i\in I} f(a_i)$ respectively $f(\inf_{i\in I} a_i) = \inf_{i\in I} f(a_i)$ for any nonempty family $(a_i \mid i \in I)$ in $[0,1]$. On the other hand, if f is decreasing (as are the first partial maps of a border implicator), left-continuity respectively right-continuity of f is equivalent to $f(\sup_{i\in I} a_i) = \inf_{i\in I} f(a_i)$ respectively $f(\inf_{i\in I} a_i) = \sup_{i\in I} f(a_i)$ for any nonempty family $(a_i \mid i \in I)$ in $[0,1]$.

A *possibility measure* Π on X [26] is a map from the power class $\wp(X)$ of X to the real unit interval $[0,1]$, satisfying, for any family $(C_i \mid i \in I)$ of elements of $\wp(X)$: $\Pi(\bigcup_{i\in I} C_i) = \sup_{i\in I} \Pi(C_i)$. It has a *distribution* π, i.e. a $X-[0,1]$-map π, defined by $\pi(x) = \Pi(\{x\})$. Notice that $\Pi(\emptyset) = 0$ and that for any C in $\wp(X)$, $\Pi(C) = \sup_{x\in C} \pi(x)$. Therefore, Π is completely determined by π and *vice versa*.

The *possibilistic extension problem* can now be formulated as follows. Given a subset $\mathcal{S}$ of $\wp(X)$ and a $\mathcal{S}-[0,1]$-map P, does there exist a possibility measure Π on $\wp(X)$ such that its restriction $\Pi|\mathcal{S}$ to $\mathcal{S}$ is equal to P? It will be useful in the context of this paper to reformulate this question in the following completely equivalent manner. Given a family $(C_i \mid i \in I)$ of mutually different [3] elements of $\wp(X)$ and a corresponding family $(\beta_i \mid i \in I)$ of elements of $[0,1]$, we may ask whether there exists a possibility measure Π on X, or equivalently, a distribution π, such that

$$\Pi(C_i) = \sup_{x\in C_i} \pi(x) = \beta_i, \quad i \in I, \tag{1}$$

which may also be written as a system of sup-$\mathcal{T}$ equations

$$\sup_{x\in X} \mathcal{T}(\chi_{C_i}(x), \pi(x)) = \beta_i, \quad i \in I,$$

where $\mathcal{T}$ is any t-norm.

This problem has been solved by Wang [25]. Let us call the family $((C_i, \beta_i) \mid i \in I)$ *P-consistent* iff for any $i \in I$ and any $J \subseteq I$,

$$C_i \subseteq \bigcup_{j\in J} C_j \Rightarrow \beta_i \leq \sup_{j\in J} \beta_j.$$

Note that if $C_i = \emptyset$, P-consistency implies that $\beta_i = 0$. Wang has shown that the system of equations (1) has a solution π iff the family $((C_i, \beta_i) \mid i \in I)$ is P-consistent. He has also proven that if there exists a solution, then the

[3] If $C_i = C_j$ it must obviously hold that $\beta_i = \beta_j$, and either C_i or C_j may be eliminated from the family without affecting the problem.

greatest solution π^g is given by

$$\pi^g(x) = \inf_{i\in I, x\in C_i} \beta_i = \inf_{i\in I} \mathcal{I}(\chi_{C_i}(x), \beta_i),$$

where $\mathcal{I}$ is an arbitrary border implicator. Obviously then, we may conclude that there exists a solution to (1) iff π_g is a (the greatest) solution.

If we look at the special case in which the family $(C_i \mid i \in I)$ constitutes a *semi-partition* of X, i.e. a family of mutually disjoint nonempty subsets of X, a number of interesting observations can be made.

Primo, in this special case it is easily proven that there always exists at least one π that satisfies (1). In fact, the greatest solution π^g of (1) is given by

$$\pi^g(x) = \inf_{i\in I} \mathcal{I}(\chi_{C_i}(x), \beta_i) = \begin{cases} \beta_i & ; \quad x \in C_i,\ i \in I \\ 1 & ; \quad \text{elsewhere,} \end{cases}$$

where $\mathcal{I}$ is an arbitrary border implicator.

Secundo, there also exists a class of minimal solutions, the supremum of which is again a solution, and is given by

$$\pi^p(x) = \sup_{i\in I} \mathcal{T}(\chi_{C_i}(x), \beta_i) = \begin{cases} \beta_i & ; \quad x \in C_i,\ i \in I \\ 0 & ; \quad \text{elsewhere.} \end{cases}$$

In this expression, $\mathcal{T}$ is an arbitrary t-norm.

Tertio, if the family considered is a *partition* of X, i.e. if it is a semi-partition that furthermore covers X, then $\pi^p = \pi^g$.

This discussion suggests that, besides π^g, there also exists a second important *potential* solution π^p. In the most general case, i.e. when $(C_i \mid i \in I)$ is not necessarily a semi-partition of X, it is given by

$$\pi^p(x) = \sup_{i\in I} \mathcal{T}(\chi_{C_i}(x), \beta_i) = \sup_{i\in I, x\in C_i} \beta_i,$$

where again, $\mathcal{T}$ is an arbitrary t-norm. As a matter of fact, it is easily verified that in general, for a family $(C_i \mid i \in I)$ of nonempty subsets of X, π^p is a solution of (1) iff

$$(\forall (i,j) \in I^2)(C_i \cap C_j \neq \emptyset \Rightarrow \beta_i = \beta_j), \tag{2}$$

which is trivially satisfied when $(C_i \mid i \in I)$ is a semi-partition of X.

Given a t-norm $\mathcal{T}$ with left-continuous partial maps, we may also define the *possibility of a fuzzy set* A in X by [13,16,17,20,26]:

$$\Pi_{\mathcal{T}}(A) = \sup_{x\in X} \mathcal{T}(A(x), \pi(x)).$$

Of course, $\Pi_{\mathcal{T}}$ can be considered as a $\mathcal{F}(X) - [0,1]$-map. For any subset C of X, $\Pi_{\mathcal{T}}(\chi_C) = \Pi(C)$. For any family $(A_i \mid i \in I)$ of elements of $\mathcal{F}(X)$, $\Pi_{\mathcal{T}}(\sup_{i\in I} A_i) = \sup_{i\in I} \Pi_{\mathcal{T}}(A_i)$. In this sense, $\Pi_{\mathcal{T}}$ can also be interpreted as a *generalized* or *extended* possibility measure, and is also called the $\mathcal{T}$*-extension* of Π.

In this paper, we investigate some important aspects of the extension (or fuzzification) of problem (1), and deal with extended possibilities and families of fuzzy sets. In other words, we shall consider a t-norm $\mathcal{T}$ with left-continuous partial maps, a family $(A_i \mid i \in I)$ of mutually different [4] fuzzy sets in X, a corresponding family $(\beta_i \mid i \in I)$ of elements of $[0,1]$, and look for solutions π of the system of (relational) sup-$\mathcal{T}$ equations

$$\Pi_{\mathcal{T}}(A_i) = \sup_{x\in X} \mathcal{T}(A_i(x), \pi(x)) = \beta_i, \quad i \in I.$$

In particular, we shall investigate whether appropriate 'fuzzifications' of π^g and π^p are still solutions of this system. Since we shall mainly be dealing with systems of sup-$\mathcal{T}$ equations, we give a brief overview of the relevant theory in Section 2. In Section 3, we deal with the generalized possibilistic extension problem, give necessary and sufficient conditions for extendability, characterize all possible extensions in the case of a finite X, and investigate when a suitable fuzzification of π^p is a solution for the generalized possibilistic extension problem. Since we want to investigate to what extent the analogy with the classical possibilistic extension problem can be preserved in the generalized case, we need a suitable equivalent for the notion of a (semi-)partition in the fuzzy case. This is dealt with by the introduction of $\mathcal{T}$-semi-partitions and $\mathcal{T}$-partitions in Section 4. We then proceed to prove a number of interesting results concerning the generalized possibilistic extension problem in the special case that the family of fuzzy sets considered makes up a $\mathcal{T}$-semi-partition. In Section 5, we define and study weak and strong P-consistency as possible criteria for extendability in the generalized possibilistic extension problem.

[4] As in the nonfuzzy case, if $A_i = A_j$ it must obviously hold that $\beta_i = \beta_j$, and either A_i or A_j may be eliminated from the family without affecting the problem.

2 Systems of sup-$\mathcal{T}$ equations

2.1 Root systems

Of major importance in relational equation solving is the order-theoretic concept of a root system [2,6,10]. A root system is a special union of closed intervals with the same ending point, as is formalized in the following definition.

Definition 1 *[2] A subset R of an ordered set $(P, \leq)$ is called a root system iff there exists an element σ in P and an antichain O in $\downarrow \sigma = \{\alpha \mid \alpha \leq \sigma\}$ such that $R = \bigcup_{\omega \in O} [\omega, \sigma]$.*

The element σ is called the *stem* of the root system. The elements of the antichain O are called the *offshoots* of the root system. A root system is called *finitely generated* iff the set of offshoots is finite.

Theorem 2 *[2] Let $(R_i \mid i \in I)$ be a family of finitely generated root systems of a complete lattice $(L, \leq)$ with stem σ_i and set of offshoots O_i. If the intersection $\bigcap_{i \in I} R_i$ is nonempty, then it is a root system with stem $\sigma = \inf_{i \in I} \sigma_i$ and as offshoots the minimal elements of the set*

$$\{\sup_{i \in I} \omega_i \mid (\forall i \in I)(\omega_i \in O_i \wedge \omega_i \leq \sigma)\}.$$

Note that if the intersection of a finite family of finitely generated root systems of a complete lattice is nonempty, then it is a finitely generated root system.

2.2 Residual operators of a t-norm

With a t-norm $\mathcal{T}$ we may associate two binary operators $\mathcal{I}_\mathcal{T}$ and $\mathcal{L}_\mathcal{T}$ on $[0, 1]$, called the *residual operators* of $\mathcal{T}$, defined by [6,7,12]:

$$\mathcal{I}_\mathcal{T}(\alpha, \beta) = \sup\{\gamma \mid \mathcal{T}(\alpha, \gamma) \leq \beta\}$$
$$\mathcal{L}_\mathcal{T}(\alpha, \beta) = \inf\{\gamma \mid \mathcal{T}(\alpha, \gamma) \geq \beta\}.$$

It is well known that the operator $\mathcal{I}_\mathcal{T}$ is a border implicator [5]; it is therefore called the *residual implicator* of $\mathcal{T}$. Notice that $\mathcal{I}_\mathcal{T}(\alpha, \beta) = 1$ when $\alpha \leq \beta$.

In the following propositions, we have gathered a number of important results concerning the residual implicator of a t-norm with left-continuous partial maps.

Proposition 3 *[5,12] Let $\mathcal{T}$ be a t-norm with left-continuous partial maps. Then the following properties hold:*

(i) $(\forall(\alpha,\beta)\in[0,1]^2)(\mathcal{I}_\mathcal{T}(\alpha,\beta)=1\Leftrightarrow\alpha\le\beta)$.
(ii) $(\forall(\alpha,\beta,\gamma)\in[0,1]^3)(\mathcal{T}(\alpha,\mathcal{I}_\mathcal{T}(\alpha,\beta))\le\beta)$.

Proposition 4 *[12] A t-norm $\mathcal{T}$ has left-continuous partial maps iff*

$$(\forall(\alpha,\beta,\gamma)\in[0,1]^3)(\mathcal{T}(\alpha,\beta)\le\gamma\Leftrightarrow\beta\le\mathcal{I}_\mathcal{T}(\alpha,\gamma)).$$

Proposition 5 *[5]*

(i) The second partial maps of the residual implicator $\mathcal{I}_\mathcal{T}$ of a t-norm $\mathcal{T}$ are right-continuous.
(ii) If $\mathcal{T}$ is a t-norm with left-continuous partial maps, then the first partial maps of the residual implicator $\mathcal{I}_\mathcal{T}$ are left-continuous.

Proposition 6 *Let $\mathcal{T}$ be a t-norm with left-continuous partial maps. Then the following property holds:*

$$(\forall(\alpha,\beta,\gamma)\in[0,1]^3)(\mathcal{I}_\mathcal{T}(\mathcal{T}(\alpha,\gamma),\mathcal{T}(\beta,\gamma))\ge\mathcal{I}_\mathcal{T}(\alpha,\beta)).$$

PROOF. Using Proposition 4 we find that, by definition,

$$\begin{aligned}\mathcal{I}_\mathcal{T}(\mathcal{T}(\alpha,\gamma),&\mathcal{T}(\beta,\gamma))\ge\mathcal{I}_\mathcal{T}(\alpha,\beta)\\&\Leftrightarrow\mathcal{T}(\mathcal{T}(\alpha,\gamma),\mathcal{I}_\mathcal{T}(\alpha,\beta))\le\mathcal{T}(\beta,\gamma)\\&\Leftrightarrow\mathcal{T}(\mathcal{T}(\alpha,\gamma),\sup\{z\mid\mathcal{T}(\alpha,z)\le\beta\})\le\mathcal{T}(\beta,\gamma),\end{aligned}$$

Table 1
Residual operators for the most important continuous t-norms

$\mathcal{T}$	$\mathcal{T}(x,y)$	$\mathcal{I}_\mathcal{T}(x,y)$	$\mathcal{L}_\mathcal{T}(x,y)$
M	$\min(x,y)$	$\begin{cases}1 & ;\ x\le y\\ y & ;\ \text{elsewhere}\end{cases}$	$\begin{cases}1 & ;\ x<y\\ y & ;\ \text{elsewhere}\end{cases}$
P	xy	$\begin{cases}1 & ;\ x\le y\\ y/x & ;\ \text{elsewhere}\end{cases}$	$\begin{cases}1 & ;\ x<y\\ y/x & ;\ 0<y\le x\\ 0 & ;\ \text{elsewhere}\end{cases}$
W	$\max(x+y-1,0)$	$\min(1-x+y,1)$	$\begin{cases}1 & ;\ x<y\\ 1-x+y & ;\ 0<y\le x\\ 0 & ;\ \text{elsewhere}\end{cases}$

and taking into account the left-continuity of the partial maps of $\mathcal{T}$, and the commutativity and associativity of $\mathcal{T}$,

$$\Leftrightarrow \sup\{\mathcal{T}(\mathcal{T}(\alpha,\gamma),z) \mid \mathcal{T}(\alpha,z) \leq \beta\}) \leq \mathcal{T}(\beta,\gamma)$$
$$\Leftrightarrow \sup\{\mathcal{T}(\mathcal{T}(\alpha,z),\gamma) \mid \mathcal{T}(\alpha,z) \leq \beta\}) \leq \mathcal{T}(\beta,\gamma),$$

and this last inequality holds trivially. □

Furthermore, we define the binary operator $\mathcal{E}_\mathcal{T}$ on $[0,1]$, also called *biresidual operator* of $\mathcal{T}$ [22], as follows

$$\mathcal{E}_\mathcal{T}(\alpha,\beta) = \min(\mathcal{I}_\mathcal{T}(\alpha,\beta), \mathcal{I}_\mathcal{T}(\beta,\alpha)).$$

Notice that in the foregoing definition, the minimum operator can be replaced without problem by the t-norm $\mathcal{T}$.

Definition 7 *[11] Let A and B be two fuzzy sets in X.*

(i) The degree of compatibility $C_\mathcal{T}(A,B)$ of A and B is defined as

$$C_\mathcal{T}(A,B) = \sup_{x\in X} \mathcal{T}(A(x),B(x)).$$

(ii) The degree of equality $E_\mathcal{T}(A,B)$ of A and B is defined as

$$E_\mathcal{T}(A,B) = \inf_{x\in X} \mathcal{E}_\mathcal{T}(A(x),B(x)).$$

2.3 sup-$\mathcal{T}$ *equations*

Consider a t-norm $\mathcal{T}$, a fuzzy set A in X and $\beta \in [0,1]$, then we want to determine the solution set of the equation

$$\sup_{x\in X} \mathcal{T}(A(x),U(x)) = \beta \tag{3}$$

in the unknown fuzzy set U in X. This problem has been solved in a very general setting in [5,6]. A comprehensive treatment of relational equations can also be found in [19].

Proposition 8 *[5,6] If $\mathcal{T}$ has left-continuous partial maps, then the solution set of the inequality*

$$\sup_{x\in X} \mathcal{T}(A(x),U(x)) \leq \beta$$

is the root system R of $(\mathcal{F}(X), \sqsubseteq)$ given by $R = [\chi_\emptyset, G]$, with G defined by

$$G(x) = \mathcal{I}_\mathcal{T}(A(x), \beta).$$

Proposition 9 *[5,6] If $\mathcal{T}$ has left-continuous partial maps, then the solution set of equation (3) is nonempty iff the fuzzy set G in X, defined by*

$$G(x) = \mathcal{I}_\mathcal{T}(A(x), \beta),$$

is a solution. If the solution set is nonempty, then G is the greatest solution.

Proposition 10 *[5,9] If $\mathcal{T}$ has left-continuous partial maps, then the solution set of equation (3) is nonempty iff $\sup_{x\in X} A(x) \geq \beta$.*

Theorem 11 *[5,6] If $\mathcal{T}$ is continuous and $(\exists x \in X)(A(x) \geq \beta)$, then the solution set of equation (3) contains the root system R of $(\mathcal{F}(X), \sqsubseteq)$ with as stem the fuzzy set G in X, defined by $G(x) = \mathcal{I}_\mathcal{T}(A(x), \beta)$, and as offshoots the elements of the set $W = \{M_u \mid A(u) \geq \beta\}$, with M_u the fuzzy set in X defined by*

$$M_u(x) = \begin{cases} \mathcal{L}_\mathcal{T}(A(u), \beta) & ; \quad x = u \\ 0 & ; \quad \textit{elsewhere.} \end{cases}$$

If X is finite, then the solution set coincides with this root system, which is then moreover finitely generated.

2.4 Systems of sup-$\mathcal{T}$ *equations*

Consider a t-norm $\mathcal{T}$, a family $(A_i \mid i \in I)$ of fuzzy sets in X and a family $(\beta_i \mid i \in I)$ in $[0, 1]$, then we want to determine the solution set of the system of equations

$$\sup_{x\in X} \mathcal{T}(A_i(x), U(x)) = \beta_i, \quad i \in I \tag{4}$$

in the unknown fuzzy set U in X.

Proposition 12 *[5] If $\mathcal{T}$ has left-continuous partial maps, then the solution set of the system of inequalities*

$$\sup_{x\in X} \mathcal{T}(A_i(x), U(x)) \leq \beta_i, \quad i \in I$$

is the root system R of $(\mathcal{F}(X), \sqsubseteq)$ given by $R = [\chi_\emptyset, G]$, with G defined by

$$G(x) = \inf_{i\in I} \mathcal{I}_\mathcal{T}(A_i(x), \beta_i).$$

Proposition 13 *[5] If $\mathcal{T}$ has left-continuous partial maps, then the solution set of system* (4) *is nonempty iff the fuzzy set G in X, defined by*

$$G(x) = \inf_{i\in I} \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i),$$

is a solution. If the solution set is nonempty, then G is the greatest solution.

Theorem 14 *[5] If X is finite, $\mathcal{T}$ is continuous and the solution set of system* (4) *is nonempty, then it is the root system R of $(\mathcal{F}(X), \sqsubseteq)$ with as stem the fuzzy set G in X, defined by*

$$G(x) = \inf_{i\in I} \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i),$$

and as offshoots the minimal elements of the set

$$\{\sup_{i\in I} M_u^i \mid (\forall i \in I)(M_u^i \in W_i')\}$$

with $W_i' = \{M_u^i \mid A_i(u) \geq \beta_i \wedge M_u^i \sqsubseteq G\}$ and M_u^i the fuzzy set in X defined by

$$M_u^i(x) = \begin{cases} \mathcal{L}_{\mathcal{T}}(A_i(u), \beta_i) & ; \quad x = u \\ 0 & , \quad \textit{elsewhere.} \end{cases}$$

If I is finite, then the root system R is moreover finitely generated.

3 The generalized possibilistic extension problem

In this section, we consider a t-norm $\mathcal{T}$ with left-continuous partial maps, a family $(A_i \mid i \in I)$ of mutually different fuzzy sets in X and a family $(\beta_i \mid i \in I)$ in $[0, 1]$. We want to determine the distributions π such that

$$\Pi_{\mathcal{T}}(A_i) = \sup_{x\in X} \mathcal{T}(A_i(x), \pi(x)) = \beta_i, \quad i \in I. \tag{5}$$

This problem can be considered as the generalized possibilistic extension problem, and is obviously equivalent to a system of sup-$\mathcal{T}$ equations. Solvability of the system of equations (5) will equivalently be called *extendability*.

3.1 Necessary and sufficient conditions for extendability

Proposition 13 can be rephrased in this context as follows.

Proposition 15 *There exists a solution of the system* (5) *iff the distribution* π^g *defined by*

$$\pi^g(x) = \inf_{i \in I} \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i)$$

is a solution. If there exists a solution, then π^g *is the greatest solution.*

Notice that the greatest solution π^g in the fuzzy case is similar to (in fact, a fuzzification of) the greatest solution in the nonfuzzy case, and requires the choice of a specific border implicator, namely the residual implicator $\mathcal{I}_{\mathcal{T}}$.

Two simple sufficient conditions for extendability are given in the following propositions.

Proposition 16 *If* $(\forall i \in I)(A_i$ *is modal*$)$ *and*

$$(\forall (i,j) \in I^2)(i \neq j \Rightarrow \ker A_i \cap \mathrm{supp} A_j = \emptyset), \tag{6}$$

then there exists a solution of system (5).

PROOF. Let $i \in I$ and $x \in \ker A_i$, then (6) implies that $\pi^g(x) = \beta_i$. Therefore

$$\sup_{x \in X} \mathcal{T}(A_i(x), \pi^g(x)) \geq \sup_{x \in \ker A_i} \mathcal{T}(1, \beta_i) = \beta_i.$$

Since on the other hand, by Proposition 12, $\sup_{x \in X} \mathcal{T}(A_i(x), \pi^g(x)) \leq \beta_i$, the proof is complete. □

Proposition 17 *If* $\mathcal{T}$ *is continuous,* $(\forall i \in I)(A_i$ *is normal*$)$ *and*

$$(\forall (i,j) \in I^2)(i \neq j \Rightarrow \mathrm{supp} A_i \cap \mathrm{supp} A_j = \emptyset), \tag{7}$$

then there exists a solution of system (5).

PROOF. Let $i \in I$ and $x \in \mathrm{supp} A_i$, then (7) leads to $\pi^g(x) = \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i)$. Therefore

$$\begin{aligned}\sup_{x \in X} \mathcal{T}(A_i(x), \pi^g(x)) &= \sup_{x \in \mathrm{supp} A_i} \mathcal{T}(A_i(x), \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i)) \\ &= \sup_{x \in X} \mathcal{T}(A_i(x), \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i)).\end{aligned}$$

Given the normality of A_i and Propositions 9 and 10, the proof is complete. □

Remark the close analogy with the results for semi-partitions described in Section 1. The conditions imposed on the family $(A_i \mid i \in I)$ in the previous propositions indicate two possible ways of introducing the concept of a fuzzy semi-partition, namely as a set of modal fuzzy sets with pairwise disjoint kernels and supports, or as a set of normal fuzzy sets with pairwise disjoint supports. It should also be noted that the above-mentioned conditions do not necessarily imply that the family $(A_i \mid i \in I)$ is a $\mathcal{T}$-semi-partition of X, a concept that we shall introduce in Section 4.

An important necessary condition for extendability is given in the following theorem. It can be interpreted as follows: the degree of equality of A_i and A_j should not be larger than the degree of equality of β_i and β_j.

Theorem 18 *If there exists a solution of the system* (5) *then*

$$(\forall(i,j) \in I^2)(E_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j)). \tag{8}$$

PROOF. Assume that there exists a solution π, and choose arbitrary i and j in I. Then

$$\mathcal{I}_{\mathcal{T}}(\beta_i, \beta_j) = \mathcal{I}_{\mathcal{T}}(\sup_{x\in X} \mathcal{T}(A_i(x), \pi(x)), \sup_{y\in X} \mathcal{T}(A_j(y), \pi(y)))$$

and using the left-continuity of the first partial maps of $\mathcal{I}_{\mathcal{T}}$ (by Proposition 5 (ii)) and Proposition 6,

$$\begin{aligned} &= \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(\mathcal{T}(A_i(x), \pi(x)), \sup_{y\in X} \mathcal{T}(A_j(y), \pi(y))) \\ &\geq \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(\mathcal{T}(A_i(x), \pi(x)), \mathcal{T}(A_j(x), \pi(x))) \\ &\geq \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A_i(x), A_j(x)). \end{aligned}$$

Since moreover i and j are arbitrary, we may reverse the roles of i and j, which leads to

$$\begin{aligned} \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j) &\geq \min(\inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A_i(x), A_j(x)), \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A_j(x), A_i(x))) \\ &= \inf_{x\in X} \mathcal{E}_{\mathcal{T}}(A_i(x), A_j(x)) = E_{\mathcal{T}}(A_i, A_j). \qquad \square \end{aligned}$$

3.2 Complete solution set for a finite X

For a finite X and a continuous t-norm $\mathcal{T}$, the complete solution set of the system (5) can be determined. This is expressed in the following theorem, which at once follows from Theorem 14 (see also [3]).

Theorem 19 *If X is finite, $\mathcal{T}$ is continuous and there exists a solution of the system* (5)*, then the solution set is the root system R of $(\mathcal{F}(X), \sqsubseteq)$ with stem*

π^g defined by

$$\pi^g(x) = \inf_{i \in I} \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i)$$

and as offshoots the minimal elements of the set

$$\{\sup_{i \in I} \delta_u^i \mid (\forall i \in I)(\delta_u^i \in W_i')\}$$

with $W_i' = \{\delta_u^i \mid A_i(u) \geq \beta_i \wedge \delta_u^i \sqsubseteq \pi^g\}$ and δ_u^i the distribution defined by

$$\delta_u^i(x) = \begin{cases} \mathcal{L}_{\mathcal{T}}(A_i(u), \beta_i) & ; \quad x = u \\ 0 & ; \quad \textit{elsewhere.} \end{cases}$$

Notice that the distributions δ_u^i can be considered as distributions of Dirac measures [17].

3.3 The distribution π^p

We have seen that an obvious fuzzification π^g of the classical candidate for the greatest solution of the possibilistic extension problem plays a similar part in the generalized possibilistic extension problem. Let us now also take a look at the possible solution π^p. We want to establish necessary and sufficient conditions under which the distribution π^p defined by

$$\pi^p(x) = \sup_{i \in I} \mathcal{T}(A_i(x), \beta_i) \tag{9}$$

is a solution of the system (5).

Notice that the following inequality always holds: $\mathcal{T}(\alpha, \beta) \leq \mathcal{I}_{\mathcal{T}}(\alpha, \beta)$. A necessary condition for π^p to be a solution of the system (5) is that $\pi^p \sqsubseteq \pi^g$, which can be restated in the following equivalent ways.

Theorem 20 *The following statements are equivalent:*

(i) $\pi^p \sqsubseteq \pi^g$;
(ii) $(\forall x \in X)(\forall (i,j) \in I^2)(\mathcal{T}(A_i(x), \beta_i) \leq \mathcal{I}_{\mathcal{T}}(A_j(x), \beta_j))$;
(iii) $(\forall (i,j) \in I^2)(C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$.

PROOF. It is obvious that (i) and (ii) are equivalent. Let us prove that (ii) and (iii) are equivalent. Proposition 4 and the commutativity and associativity of $\mathcal{T}$ lead to the following chain of equivalences

$$\begin{aligned}\mathcal{T}(A_i(x), A_j(x)) \le \mathcal{I}_\mathcal{T}(\beta_i, \beta_j) &\Leftrightarrow \mathcal{T}(\mathcal{T}(A_i(x), A_j(x)), \beta_i) \le \beta_j \\ &\Leftrightarrow \mathcal{T}(\mathcal{T}(A_i(x), \beta_i), A_j(x)) \le \beta_j \\ &\Leftrightarrow \mathcal{T}(A_i(x), \beta_i) \le \mathcal{I}_\mathcal{T}(A_j(x), \beta_j).\end{aligned}$$

Now assume that (ii) holds, then the equivalences above imply that for any $x \in X$ and $(i,j) \in I^2$, $\mathcal{T}(A_i(x), A_j(x)) \le \mathcal{I}_\mathcal{T}(\beta_i, \beta_j)$ and $\mathcal{T}(A_j(x), A_i(x)) \le \mathcal{I}_\mathcal{T}(\beta_j, \beta_i)$, whence $\mathcal{T}(A_i(x), A_j(x)) \le \mathcal{E}_\mathcal{T}(\beta_i, \beta_j)$. Therefore, (iii) holds.

Conversely, assume that (iii) holds, then it follows that

$$(\forall x \in X)(\forall (i,j) \in I^2)(\mathcal{T}(A_i(x), A_j(x)) \le \mathcal{E}_\mathcal{T}(\beta_i, \beta_j) \le \mathcal{I}_\mathcal{T}(\beta_i, \beta_j)).$$

By the equivalences above, we may deduce (ii). □

Theorem 21 *If I is finite and $\mathcal{T}$ is continuous, then π^p is a solution of the system* (5) *iff*

(i) $(\forall (i,j) \in I^2)(C_\mathcal{T}(A_i, A_j) \le \mathcal{E}_\mathcal{T}(\beta_i, \beta_j))$;
(ii) $(\forall j \in I)(\exists i \in I)(\beta_j \le \beta_i \wedge \mathcal{L}_\mathcal{T}(\beta_i, \beta_j) \le C_\mathcal{T}(A_i, A_j))$.

PROOF. The distribution π^p is a solution iff

$$\sup_{x \in X} \mathcal{T}(A_j(x), \sup_{i \in I} \mathcal{T}(A_i(x), \beta_i)) = \beta_j, \quad j \in I.$$

Using the associativity of $\mathcal{T}$ and the left-continuity of its partial maps, we find that this is equivalent to

$$\sup_{i \in I} \mathcal{T}(\sup_{x \in X} \mathcal{T}(A_j(x), A_i(x)), \beta_i)) = \beta_j, \quad j \in I,$$

which may be rewritten as

$$\sup_{i \in I} \mathcal{T}(\beta_i, C_\mathcal{T}(A_i, A_j)) = \beta_j, \quad j \in I.$$

In other words, π^p is a solution iff $C(i,j) = C_\mathcal{T}(A_i, A_j)$, $(i,j) \in I^2$ is a solution of the family of independent sup-$\mathcal{T}$ equations

$$\sup_{i \in I} \mathcal{T}(\beta_i, U(i,j)) = \beta_j, \quad j \in I \tag{10}$$

in the unknown fuzzy relation U. Since the conditions of Theorem 11 are satisfied, we know that this is equivalent to

$$(\forall (i,j) \in I^2)(C_\mathcal{T}(A_i, A_j) \le \mathcal{I}_\mathcal{T}(\beta_i, \beta_j))$$
$$(\forall j \in I)(\exists i \in I)(\beta_j \le \beta_i \wedge \mathcal{L}_\mathcal{T}(\beta_i, \beta_j) \le C_\mathcal{T}(A_i, A_j)).$$

Due to the symmetry of the degree of compatibility, the first formula is equivalent to $(\forall(i,j) \in I^2)(C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$. $\square$

The first condition in Theorem 21 can be interpreted as follows: the height of the intersection (w.r.t. $\mathcal{T}$) of A_i and A_j – a measure for their overlap – should not be larger than the degree of equality of β_i and β_j. From this theorem it immediately follows that a set of *sufficient* conditions for π^p to be a solution is given by (for finite I and continuous $\mathcal{T}$):

(i) $(\forall(i,j) \in I^2)(C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$;
(ii) $(\forall i \in I)(\mathcal{L}_{\mathcal{T}}(\beta_i, \beta_i) \leq C_{\mathcal{T}}(A_i, A_i))$.

Example 22 *In the interesting special case that the family considered contains only one element A with corresponding β, the necessary and sufficient conditions of Theorem 21 reduce to*

$$\mathcal{L}_{\mathcal{T}}(\beta, \beta) \leq C_{\mathcal{T}}(A, A). \tag{11}$$

Consider $\mathcal{T} = P$ or $\mathcal{T} = W$. If $\beta > 0$ this is, taking into account Table 1, equivalent to $C_{\mathcal{T}}(A, A) = 1$. It is readily verified that this is equivalent to the normality of A. On the other hand, for $\mathcal{T} = M$, (11) reduces to $\sup_{x \in X} A(x) \geq \beta$, which, by Proposition 10, is nothing else but the necessary and sufficient condition for extendability; hence, π^p is a solution of (5) iff there exists a solution.

Additional interesting results can be obtained assuming the modality of the A_i, $i \in I$.

Proposition 23 *If $(\forall i \in I)(A_i$ is modal$)$, then the following properties hold for any $i \in I$ and $x \in \ker A_i$.*

(i) $\pi^g(x) \leq \beta_i \leq \pi^p(x)$.
(ii) If π^p is a solution of the system (5), then $\pi^p(x) = \pi^g(x) = \beta_i$.

PROOF. Let $i \in I$ and $x \in \ker A_i$. Clearly,

$$\pi^p(x) = \sup_{j \in I} \mathcal{T}(A_j(x), \beta_j) \geq \mathcal{T}(A_i(x), \beta_i) = \mathcal{T}(1, \beta_i) = \beta_i$$
$$\pi^g(x) = \inf_{j \in I} \mathcal{I}_{\mathcal{T}}(A_j(x), \beta_j) \leq \mathcal{I}_{\mathcal{T}}(A_i(x), \beta_i) = \mathcal{I}_{\mathcal{T}}(1, \beta_i) = \beta_i.$$

which proves (i). To prove (ii), assume that π^p is a solution. Since in that case $\pi^p \sqsubseteq \pi^g$, statement (ii) immediately follows from (i). $\square$

Corollary 24 *If* $(\forall i \in I)(A_i$ *is modal), then a necessary condition for* π^p *to be a solution of the system* (5)*, is*

$$(\forall (i,j) \in I^2)(\ker A_i \cap \ker A_j \neq \emptyset \Rightarrow \beta_i = \beta_j). \tag{12}$$

Note that (12) can be interpreted as a possible generalization of (2). Contrary to the classical case, however, (12) is only a necessary condition, as is shown in the following counter-example. This example also tells us that modality of the A_i, $i \in I$ together with $(\forall i \in I)(\forall x \in \ker A_i)(\pi^p(x) = \beta_i)$ is not sufficient for π^p to be a solution of (5).

Example 25 *Let the cardinality of* X *be greater than 3, and choose an arbitrary* c *in* X*. Then it is always possible to consider a family* $\{A_1, A_2\}$ *of fuzzy sets in* X *with nonempty and disjoint kernels* $C_1 = \ker A_1$ *and* $C_2 = \ker A_2$*, such that* $X \setminus (C_1 \cup C_2) = \{c\}$, $\operatorname{supp} A_1 = C_1 \cup \{c\}$*, and* $\operatorname{supp} A_2 = C_2 \cup \{c\}$*. The left-continuity of* $\mathcal{T}$ *implies that there exists a* $\delta \in]0,1[$ *such that* $\mathcal{T}(\delta, \delta) > 0$*. Let* $A_1(c) = A_2(c) = \delta$*, and let* $\beta_1 = 1$ *and* $\beta_2 = 0$*. Clearly*

$$\pi^p(x) = \begin{cases} 1 & ; \quad x \in C_1 \\ 0 & ; \quad x \in C_2 \\ \delta & ; \quad x = c \end{cases} \quad \textit{and} \quad \pi^g(x) = \begin{cases} 1 & ; \quad x \in C_1 \\ 0 & ; \quad x \in C_2 \\ \mathcal{I}_{\mathcal{T}}(\delta, 0) & ; \quad x = c. \end{cases}$$

Then $\sup_{x \in X} \mathcal{T}(A_2(x), \pi^p(x)) \geq \mathcal{T}(A_2(c), \pi^p(c)) = \mathcal{T}(\delta, \delta) > 0 = \beta_2$*, which implies that* π^p *is not a solution of* (5)*. On the other hand, Proposition 16 guarantees that* π^g *is a solution.*

In Theorem 21, the finiteness condition on the index set I, the right-continuity of the partial maps of the t-norm $\mathcal{T}$ and condition (ii) can be omitted in the case of modal fuzzy sets, as is shown in the following theorem.

Theorem 26 *If* $(\forall i \in I)(A_i$ *is modal), then the following statements are equivalent:*

(i) π^p *is a solution of the system* (5)*;*
(ii) $\pi^p \sqsubseteq \pi^g$*;*
(iii) $(\forall x \in X)(\forall (i,j) \in I^2)(\mathcal{T}(A_i(x), \beta_i) \leq \mathcal{I}_{\mathcal{T}}(A_j(x), \beta_j))$*;*
(iv) $(\forall (i,j) \in I^2)(C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$*.*

PROOF. By Theorem 20, it need for instance only be proven that (i) and (iv) are equivalent. Assume that π^p is a solution, then it follows, similarly as in Theorem 21, that in particular $C(i,j) = C_{\mathcal{T}}(A_i, A_j)$, $(i,j) \in I^2$, is a solution of the family of independent sup-$\mathcal{T}$ inequalities

$$\sup_{i \in I} \mathcal{T}(\beta_i, U(i,j)) \leq \beta_j, \quad j \in I. \tag{13}$$

Proposition 8 then implies that for any i and j in I, $C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{I}_{\mathcal{T}}(\beta_i, \beta_j)$. Statement (iv) then follows immediately.

Conversely, assume that (iv) holds. According to Proposition 8, $C(i,j) = C_{\mathcal{T}}(A_i, A_j)$, $(i,j) \in I^2$, is a solution of the family of independent sup-$\mathcal{T}$ inequalities (13), and equivalently, π^p is a solution of the system of sup-$\mathcal{T}$ inequalities

$$\sup_{x \in X} \mathcal{T}(A_j(x), \pi(x)) \leq \beta_j, \quad j \in I.$$

On the other hand, let j in I and consider $x' \in \ker A_j$, then Proposition 23 (i) implies that

$$\sup_{x \in X} \mathcal{T}(A_j(x), \pi^p(x)) \geq \mathcal{T}(A_j(x'), \pi^p(x')) = \mathcal{T}(1, \pi^p(x')) = \pi^p(x') \geq \beta_j.$$

This completes the proof. $\square$

Contrasting with Corollary 24 and Eq. (12), this theorem is the appropriate translation of the classical result mentioned in Section 1, namely that π^p is a solution of (5) iff for overlapping A_i and A_j the corresponding β_i and β_j are equal (see (2)).

Example 27 *By looking at Theorems 20, 21 and 26, we see that the condition*

$$(\forall (i,j) \in I^2)(C_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$$

has an important part in the discussion of π^p.

(i) For $\mathcal{T} = M$, *this condition can be written as*

$$C_M(A_i, A_j) \leq \begin{cases} 1 & : \quad \beta_i = \beta_j \\ \min(\beta_i, \beta_j) & ; \quad \textit{elsewhere.} \end{cases}$$

(ii) For $\mathcal{T} = P$, *it can be written as*

$$C_P(A_i, A_j) \leq \begin{cases} 1 & ; \quad \beta_i = \beta_j \\ \min(\frac{\beta_i}{\beta_j}, \frac{\beta_j}{\beta_i}) & ; \quad \min(\beta_i, \beta_j) > 0 \\ 0 & ; \quad \textit{elsewhere.} \end{cases}$$

(iii) For $\mathcal{T} = W$, it can be written as

$$C_W(A_i, A_j) \leq 1- \mid \beta_i - \beta_j \mid .$$

4 Samples on $\mathcal{T}$-semi-partitions

4.1 $\mathcal{T}$-semi-partitions

Consider a t-norm $\mathcal{T}$. The concepts of a $\mathcal{T}$-semi-partition and a $\mathcal{T}$-partition were introduced in [11] in a successful attempt to establish a one-to-one correspondence between fuzzy partitions and fuzzy equivalence relations.

Definition 28 *[11] A set $\mathcal{A}$ of modal fuzzy sets in X is called a $\mathcal{T}$-semi-partition of X iff*

$$(\forall (A, B) \in \mathcal{A}^2)(C_{\mathcal{T}}(A, B) \leq E_{\mathcal{T}}(A, B)).$$

This definition can be interpreted as follows: *the degree of compatibility of any two members of a $\mathcal{T}$-semi-partition should not be greater than their degree of equality*. It is a generalization of the observation that for any two members of a semi-partition, the nonemptiness of their intersection implies their equality.

The pairwise disjointness (w.r.t. the t-norm $\mathcal{T}$) of the members of the set $\mathcal{A}$, i.e. for any A and B in $\mathcal{A}$ and x in X it holds that $\mathcal{T}(A(x), B(x)) = 0$, is a sufficient condition for $\mathcal{A}$ to be a $\mathcal{T}$-semi-partition, as is of course also the pairwise disjointness of their supports.

It was also shown that the inequality in the definition of a $\mathcal{T}$-semi-partition can be replaced by an equality, i.e. for any A and B in a $\mathcal{T}$-semi-partition $\mathcal{A}$ it holds that $C_{\mathcal{T}}(A, B) = E_{\mathcal{T}}(A, B)$.

Note that for a $\mathcal{T}$-semi-partition $\mathcal{A}$ of X the set $\{\ker A \mid A \in \mathcal{A}\}$ forms a semi-partition of X. This naturally leads to the definition of a $\mathcal{T}$-partition.

Definition 29 *[11] A set $\mathcal{A}$ of fuzzy sets in X is called a $\mathcal{T}$-partition of X iff it is a $\mathcal{T}$-semi-partition of X and the set $\{\ker A \mid A \in \mathcal{A}\}$ forms a partition of X.*

A $\mathcal{T}$-partition $\mathcal{A}$ is called *finite* iff $\mathcal{A}$ is a finite set.

Example 30 *[11] An example of a finite $\mathcal{T}$-partition is the following. Consider $X = [0, 1[$. Let $n \in \mathbb{N}_0$ and $a \in [0, 1[$. Consider the fuzzy sets A_i,*

$i = 1, \dots, n$, defined by:

$$A_i(x) = \begin{cases} 1 & ; \quad x \in [\frac{i-1}{n}, \frac{i}{n}[\\ a & ; \quad \textit{elsewhere.} \end{cases}$$

One easily verifies that for $i \neq j$ we have that $C_{\mathcal{T}}(A_i, A_j) = E_{\mathcal{T}}(A_i, A_j) = a$, independent of the t-norm $\mathcal{T}$. This implies that $\mathcal{A} = \{A_i \mid i = 1, \dots, n\}$ forms a $\mathcal{T}$-partition of $[0, 1[$ with respect to any t-norm $\mathcal{T}$.

Example 31 *[11] Consider a finite partition $\{C_1, \dots, C_n\}$, $n \in \mathbb{N}_0$, of X and the $X - \{1, \dots, n\}$-map f defined by: $f(x) = i \Leftrightarrow x \in C_i$. Let $\epsilon \in]0, 1[$. Consider the fuzzy sets A_i, $i = 1, \dots, n$, defined by:*

$$A_i(x) = \epsilon^{|i - f(x)|}.$$

One easily verifies that $C_P(A_i, A_j) = E_P(A_i, A_j) = \epsilon^{|i-j|}$, with P the algebraic product. Hence, $\mathcal{A} = \{A_i \mid i = 1, \dots, n\}$ forms a P-partition of X. If $n \leq 2$, then $\mathcal{A}$ also forms an M-partition of X. However, for $n > 2$ this is no longer the case. Indeed, one easily verifies that for instance

$$C_M(A_1, A_3) = \epsilon > E_M(A_1, A_3) = \epsilon^2,$$

which violates the condition $C_M(A_1, A_3) \leq E_M(A_1, A_3)$.

Definition 32 *[11] A binary fuzzy relation E in X is called a $\mathcal{T}$-equivalence on X iff it is reflexive, symmetric and $\mathcal{T}$-transitive, i.e. iff for any x, y and z in X*

(i) $E(x, x) = 1$;
(ii) $E(x, y) = E(y, x)$;
(iii) $\mathcal{T}(E(x, y), E(y, z)) \leq E(x, z)$.

$\mathcal{T}$-equivalences are also called fuzzy equalities [21], equality relations [22] or indistinguishability operators [24].

Let E be a $\mathcal{T}$-equivalence on X. The $\mathcal{T}$-equivalence class of $x \in X$ is the fuzzy set $[x]_E$ in X defined by $[x]_E(y) = E(x, y)$. The corresponding $\mathcal{T}$-quotient set X/E is then defined as $X/E = \{[x]_E \mid x \in X\}$.

Theorem 33 *[11] A set $\mathcal{A}$ of fuzzy sets in X is a $\mathcal{T}$-partition of X iff there exists a $\mathcal{T}$-equivalence E on X such that $\mathcal{A} = X/E$.*

Consider a t-norm $\mathcal{T}$ with left-continuous partial maps, a $\mathcal{T}$-semi-partition $\mathcal{A} = (A_i \mid i \in I)$ of X and a family $(\beta_i \mid i \in I)$ in $[0, 1]$, then we want to establish necessary and sufficient conditions under which the distribution π^p defined by (9) is a solution of the system (5).

Theorem 34 *The following statements are equivalent:*

(i) *there exists a solution of* (5)*;*
(ii) π^g *is a solution of* (5)*;*
(iii) π^p *is a solution of* (5)*;*
(iv) $(\forall (i, j) \in I^2)(E_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j))$.

PROOF. The equivalence of (i) and (ii) follows at once from Proposition 15. By Theorem 18, (i) implies (iv). Theorem 26 and the definition of a $\mathcal{T}$-semi-partition tells us that (iv) implies (iii). Since (iii) trivially implies (i), the proof is complete. □

Since for a semi-partition, Theorem 34 (iv) trivially holds, we rediscover the classical result that for a semi-partition π^g and π^p are always solutions of (5).

The following theorem shows that for a $\mathcal{T}$-partition, as in the case of a classical partition, the distributions π^g and π^p coincide, provided that they are indeed solutions of (5). That the latter is not always the case, is demonstrated in a counter-example following the theorem. The proof of the theorem is immediate given Proposition 23 (ii) and Theorem 34.

Theorem 35 *If* $\mathcal{A}$ *is a* $\mathcal{T}$*-partition of* X *and there exists a solution of the system* (5)*, then* $\pi^g = \pi^p$.

Example 36 *Consider the* P*-partition from Example 31, and let* k *be an arbitrary element of* $\{1, \ldots, n-1\}$*. When taking* $\beta_k = 0$ *and* $\beta_{k+1} \neq 0$*, the condition*

$$C_P(A_k, A_{k+1}) \leq \mathcal{E}_P(\beta_k, \beta_{k+1})$$

is equivalent to $\epsilon \leq 0$*, which is clearly not fulfilled. According to Theorem 26,* π^p *is not a solution. Using Theorem 34, we can conclude that no solutions of* (5) *exist.*

5 Towards generalized P-consistency?

In the possibilistic extension problem, we have seen that P-consistency is a necessary and sufficient condition for extendability. In this section, we want to verify whether appropriately fuzzified versions of this P-consistency are necessary and sufficient conditions for extendability in the generalized possibilistic extension problem.

As before, $\mathcal{T}$ is a t-norm with left-continuous partial maps, $(A_i \mid i \in I)$ is a family of mutually different fuzzy sets in X, and $(\beta_i \mid i \in I)$ a family in $[0, 1]$, and we are interested in criteria for extendability, i.e. solvability of the system (5).

Definition 37 *We call the family* $((A_i, \beta_i) \mid i \in I)$

(i) weakly P-consistent iff for any $i \in I$ *and any* $J \subseteq I$

$$A_i \sqsubseteq \sup_{j \in J} A_j \Rightarrow \beta_i \leq \sup_{j \in J} \beta_j;$$

(ii) strongly P-consistent iff for any $i \in I$ *and any* $J \subseteq I$

$$\inf_{x \in X} \mathcal{I}_{\mathcal{T}}(A_i(x), \sup_{j \in J} A_j(x)) \leq \mathcal{I}_{\mathcal{T}}(\beta_i, \sup_{j \in J} \beta_j).$$

Obviously, strong P-consistency implies weak P-consistency, and for a family of subsets of X, both notions coincide with P-consistency. The proof of the following proposition is immediate.

Proposition 38 *If the family* $((A_i, \beta_i) \mid i \in I)$ *is strongly P-consistent then*

$$(\forall (i, j) \in I^2)(E_{\mathcal{T}}(A_i, A_j) \leq \mathcal{E}_{\mathcal{T}}(\beta_i, \beta_j)).$$

In the following theorem, we show that strong and weak P-consistency are *necessary* conditions for extendability.

Theorem 39 *If there exists a solution of the system* (5)*, then the family* $((A_i, \beta_i) \mid i \in I)$ *is strongly P-consistent, and therefore weakly P-consistent.*

PROOF. Assume that there exists a solution π, and choose arbitrary $i \in I$ and $J \subseteq I$. Then

$$\mathcal{I}_{\mathcal{T}}(\beta_i, \sup_{j \in J} \beta_j) = \mathcal{I}_{\mathcal{T}}(\sup_{x \in X} \mathcal{T}(A_i(x), \pi(x)), \sup_{j \in J} \sup_{y \in X} \mathcal{T}(A_j(y), \pi(y)))$$

and using the left-continuity of the first partial maps of $\mathcal{I}_{\mathcal{T}}$ (by Proposition 5 (ii)) and Proposition 6,

$$\begin{aligned}
&= \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(\mathcal{T}(A_i(x),\pi(x)), \sup_{y\in X}\sup_{j\in J} \mathcal{T}(A_j(y),\pi(y))) \\
&\geq \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(\mathcal{T}(A_i(x),\pi(x)), \mathcal{T}(\sup_{j\in J} A_j(x),\pi(x))) \\
&\geq \inf_{x\in X} \mathcal{I}_{\mathcal{T}}(A_i(x), \sup_{j\in J} A_j(x)). \qquad \square
\end{aligned}$$

It therefore only remains to be investigated whether strong or weak P-consistency are *sufficient* conditions for extendability. In the following theorem we show that this is indeed the case for strong P-consistency when working with a $\mathcal{T}$-semi-partition $(A_i \mid i \in I)$ of X. It is an immediate consequence of Proposition 38 and Theorems 34 and 39.

Theorem 40 *If $(A_i \mid i \in I)$ is a $\mathcal{T}$-semi-partition of X, then there exists a solution of the system* (5) *iff the family $((A_i, \beta_i) \mid i \in I)$ is strongly P-consistent.*

In the following counter-example, we show that for nonmodal A_i, strong (and *a fortiori* weak) P-consistency is *not* a sufficient condition for extendability.

Example 41 *Let $X = \{a\}$, $I = \{1,2\}$, $A_1(a) = \alpha_1$ and $A_2(a) = \alpha_2$. Then strong P-consistency of the family $((A_i, \beta_i) \mid i \in I)$ is equivalent to $\mathcal{I}_{\mathcal{T}}(\alpha_1, 0) \leq \mathcal{I}_{\mathcal{T}}(\beta_1, 0)$, $\mathcal{I}_{\mathcal{T}}(\alpha_2, 0) \leq \mathcal{I}_{\mathcal{T}}(\beta_2, 0)$, $\mathcal{I}_{\mathcal{T}}(\alpha_1, \alpha_2) \leq \mathcal{I}_{\mathcal{T}}(\beta_1, \beta_2)$ and $\mathcal{I}_{\mathcal{T}}(\alpha_2, \alpha_1) \leq \mathcal{I}_{\mathcal{T}}(\beta_2, \beta_1)$. Choose $\mathcal{T} = M$ and $0 < \alpha_1 < \alpha_2 < \beta_1 < \beta_2$, then obviously the above conditions are satisfied, but $\min(\alpha_1, x) = \beta_1$ has no solution in x.*

One might suspect however, in view of Theorem 40, that for modal A_i strong P-consistency could be a sufficient condition. The following counter-example denies this.

Example 42 *Let $X = \{a,b,c\}$, $I = \{1,2,3\}$, $A_1 = \{(a,1),(b,\alpha_1),(c,\alpha_2)\}$, $A_2 = \{(a,\alpha_3),(b,1),(c,\alpha_4)\}$ and $A_3 = \{(a,\alpha_5),(b,\alpha_6),(c,1)\}$. Choose $\beta_1 > \beta_2 > \beta_3$. Then strong P-consistency of the family $((A_i, \beta_i) \mid i \in I)$ is equivalent to*

$$\begin{cases}
\max(\alpha_3, \alpha_5) \leq \mathcal{I}_{\mathcal{T}}(\beta_1, \beta_2) \\
\min(\alpha_5, \mathcal{I}_{\mathcal{T}}(\alpha_1, \alpha_6)) \leq \mathcal{I}_{\mathcal{T}}(\beta_1, \beta_3) \\
\min(\alpha_6, \mathcal{I}_{\mathcal{T}}(\alpha_3, \alpha_5)) \leq \mathcal{I}_{\mathcal{T}}(\beta_2, \beta_3).
\end{cases}$$

Moreover the potential greatest solution π^g of (5) *is given by*

$$\begin{aligned}
\pi^g(a) &= \min(\beta_1, \mathcal{I}_{\mathcal{T}}(\alpha_3, \beta_2), \mathcal{I}_{\mathcal{T}}(\alpha_5, \beta_3)) \\
\pi^g(b) &= \min(\beta_2, \mathcal{I}_{\mathcal{T}}(\alpha_6, \beta_3)) \\
\pi^g(c) &= \beta_3.
\end{aligned}$$

Choose $\mathcal{T} = M$ and let $\beta_1 > \beta_2 > \alpha_6 > \alpha_3 > \beta_3 > \alpha_5$. Then the above conditions for strong P-consistency are satisfied. It is readily verified that $\pi^g(a) = \beta_1$, $\pi^g(b) = \pi^g(c) = \beta_3$, and consequently

$$\sup_{x\in X} \min(A_2(x), \pi^g(x)) = \max(\alpha_3, \beta_3, \min(\alpha_4, \beta_3)) = \alpha_3 < \beta_2,$$

which implies that π^g is not a solution.

In general therefore, strong and weak P-consistency are necessary but not sufficient conditions for extendability.

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