On modeling possibilistic uncertainty in two-state reliability theory

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Abstract: In this paper, we show how a possibilistic uncertainty model can be used to represent and manipulate uncertainty about the states of a system and of its components. At the same time, we present a thorough study of the incorporation of this possibilistic uncertainty model in classical, two-state reliability theory. The possibilistic reliability of a component or system is introduced and studied. Furthermore, we introduce the important notion of a possibilistic structure function, based upon the concept of a classical, two-valued structure function. Under certain conditions of possibilistic independence, it allows us to calculate the possibilistic reliability of a system in terms of the possibilistic reliabilities of its components. Finally, we give straightforward methods for determining a possibilistic structure function from its classical, two-valued counterpart. In this way, we intend to show that a possibilistic uncertainty model in two-state reliability theory is formally analogous to, and certainly not more complicated than, a probabilistic uncertainty model.

Keywords: Measure theory, possibility theory, reliability theory.

1 Introduction

The classical approach to reliability theory has two important characteristics, which may be briefly described as follows. On the one hand, *two-valued structure functions* are used to model the logical structure of systems. Let us consider a system made up of n components. The *classical state space* of the system and each of its components is the set $\mathcal{S} \overset{\text{def}}{=} \{\mathit{fail}, \mathit{work}\}$. This should be interpreted in the following manner: if a system or component works, its state is *work*, and if it fails, its state is *fail*. An important assumption is then that the state x_s of the system can be expressed as a function ϕ of the states x_k of its components C_k. $\phi\colon \mathcal{S}^n \to \mathcal{S}$ is the *two-valued structure function* of the system under consideration. Of course, $x_s = \phi(x_1, \dots, x_n)$.

On the other hand, *probability theory* is used to represent the available information about the states of the components. This means that for each component the probability r_k is given that the component does not fail during a given time interval. This probability is called the *reliability* of the component. Using standard probability-theoretic methods, the reliability of the system r_s can be calculated from the reliabilities of the components under certain conditions of stochastic independence. Under these conditions, there exists a function $\varphi\colon [0,1]^n \to [0,1]$, such that $r_s = \varphi(r_1, \dots, r_n)$. Furthermore, there is a consistent way in which φ can be determined from the two-valued structure function ϕ of the system under consideration. For more detailed information about classical reliability theory, and more specifically about how to calculate φ is ϕ is known, we refer to the important work of Barlow and Proschan [1, 2].

Obviously, there are two ways in which this classical reliability model can be modified. For a start, it may be recognized that, in some cases, the state space $\{\mathit{fail}, \mathit{work}\}$ is not rich or detailed enough to provide a logical, structural description of a system and its components. On the one hand, there is the phenomenon of *degradation*, or *gradual failure*: a component or system may be working, but only imperfectly. This can be modeled by introducing linearly ordered state sets with more than two elements, such as $(\{1, 2, \dots, m\}, \leq)$ (see, for instance, the work of El-Neweihi, Proschan and Sethuraman [31]) or the real unit interval $([0,1], \leq)$ (see, for instance, the work of Baxter, Kim, Montero, Tejada and Yáñez [4, 5, 6, 7, 36, 42, 43]). On the other hand, it is possible that a component or system is subject to different, *mutually incomparable types of failure*, i.e., where one type of failure is not a degradation of the other. In this case, the state space will be only partially ordered, and can in general be represented by a bounded poset, or perhaps more conveniently, by a complete lattice. For a thorough account of reliability theory with complete lattices as state spaces, we refer to the doctoral dissertation of Cappelle [16].

There is a second way in which the classical theory of reliability may be modified, namely, by changing the uncertainty model. If the information we have about the functioning of components and systems is based upon a *statistical analysis*, then it is fairly obvious that a probabilistic uncertainty model should be used in order to mathematically represent and manipulate that information. However, in some cases, the information we have about the functioning of components and systems is not based on statistics, but is of a *linguistic nature*. This means that it is derived from a number of propositions of the type '(subject) is (predicate)', expressed in natural language, where the predicate involved may be vague or imprecise. Examples of such propositions could be, for instance, 'the water pressure is high', or 'component k is more than two years old'. It is obvious that such propositions convey information about the value that the subject (water pressure, age of component k) assumes in the universe of its possible values. However, this information is not sufficient to unequivocally determine that value, and leaves us therefore in some uncertainty. Zadeh has argued that probability theory cannot be used to represent this type of non-statistical, *linguistic information* [46]. He has instead suggested the use of *possibility measures* as a suitable representation. In our doctoral dissertation [18], we have added further weight to Zadeh's claim that possibility measures can indeed be used to represent linguistic information (see also [19]), using a purely order-theoretic approach. We have also succeeded in providing a general, unifying, measure- and integral-theoretic framework to possibility theory [18, 21, 22, 23, 24, 26, 27, 28]. This framework is to some extent formally analogous to the classical measure- and integral-theoretic description of probability theory, due to Kolmogorov (see, for instance [41]).

In this paper, we concentrate mainly on the second modification to the classical reliability model, and describe how a *possibilistic uncertainty model* can be used in combination with a *classical, two-element state space*. This means that for each component we start from a *possibilistic reliability* π_{x_k}, which is a (possibility) distribution on $\{fail, work\}$, i.e., $\pi_{x_k}\colon \{fail, work\} \to L$, where $(L, \leq)$ is a complete lattice. The set of the (possibility) distributions on $\{fail, work\}$ is denoted by $\tilde{\mathcal{S}}$. It is shown using our measure- and integral-theoretic account of possibility theory that, under a few assumptions, there exists a function $\psi\colon \tilde{\mathcal{S}}^n \to \tilde{\mathcal{S}}$, called possibilistic structure function, such that the possibilistic reliability π_{x_s} of the system is given by $\pi_{x_s} = \psi(\pi_{x_1}, \ldots, \pi_{x_n})$. Furthermore, a consistent method is given for determining ψ from the two-valued structure function ϕ.

The results outlined above give us a practical way of treating the possibilistic aspects of the reliability of a system, stress the formal analogy with the probabilistic approach and show that a possibilistic treatment of reliability need not be more complicated than a probabilistic one. In a way, our general treatment extends and adds new insights to the important work of Cai et al. in this field [13]. It turns out that in some cases, we arrived at similar results independently. This work is a natural continuation of previous work we have done in the domain of possibilistic logic [20, 29]. As a matter of fact, there is a very close link between this account of possibilistic, two-state reliability theory and our approach to possibilistic logic.

Let us conclude this introductory section with a sketch of the contents of this paper. In section 2, we give a brief account of those preliminary notions from possibility theory which are necessary for the understanding of what follows. Section 3 contains a short introduction to the theory of two-valued structure functions, with the results that are most relevant to this discussion. In section 4, we describe how possibility theory can be used to represent possibilistic information in the framework of classical reliability theory. This is done by introducing possibilistic reliabilities, and by showing how the possibilistic reliability of a system can in principle be calculated from the possibilistic reliabilities of its components. Since this calculation may in practice turn out to be a cumbersome task, we show in section 5 how it can be simplified. Section 6 concludes this paper.

2 A possibilistic uncertainty model

In order to be able to discuss the possibilistic approach to reliability theory, we shall in this section give a few preliminary definitions and notational conventions. At the same time, we present a brief outline of a few important notions and ideas in possibility theory, relevant to the subject of this paper.

2.1 Triangular norms and conorms

Throughout this paper, we shall denote by $(L, \leq)$ at least a *bounded partially ordered set*. Since $(L, \leq)$ is bounded, it has a greatest element or *top* $\inf \emptyset$, denoted by 1_L, and a smallest element or *bottom* $\sup \emptyset$, denoted by 0_L.

Sometimes, we shall assume that $(L, \leq)$ is a bounded lattice, which implies that for any two elements λ and μ of L their infimum $\inf(\lambda, \mu)$ and their supremum $\sup(\lambda, \mu)$ exist. We shall denote the binary infimum operator or 'meet' of $(L, \leq)$ by $\frown$, and the binary supremum operator or 'join' by $\smile$, i.e., for any two elements λ and μ of L: $\lambda \frown \mu \overset{\text{def}}{=} \inf(\lambda, \mu)$ and $\lambda \smile \mu \overset{\text{def}}{=} \sup(\lambda, \mu)$.

It will also sometimes be assumed that $(L, \leq)$ is a *complete* (and therefore bounded) *lattice*, that is, for any subset A of L, its least upper bound $\sup A$ and greatest lower bound $\inf A$ exist in L. For a more detailed treatment of the order-theoretic notions used in this paper, we refer to [8].

A binary operator T on a bounded poset $(L, \leq)$ that is commutative, associative, isotonic (or non-decreasing) and that satisfies the boundary condition $(\forall \lambda \in L)(T(\lambda, 1_L) = \lambda)$ is called a *triangular norm*, or *t*-norm, on $(L, \leq)$. Any *t*-norm T on $(L, \leq)$ also satisfies the following additional boundary property: $(\forall \lambda \in L)(T(0_L, \lambda) = 0_L)$.

Dually, a binary operator S on $(L, \leq)$ that is commutative, associative, isotonic and that satisfies the boundary condition $(\forall \lambda \in L)(S(\lambda, 0_L) = \lambda)$ is called a *triangular conorm*, or *t*-conorm, on $(L, \leq)$. Any *t*-conorm S on $(L, \leq)$ also satisfies the following additional boundary property: $(\forall \lambda \in L)(S(1_L, \lambda) = 1_L)$.

Of course, if $(L, \leq)$ is in particular a bounded lattice, $\frown$ is a special triangular norm on $(L, \leq)$: it is the only one that is idempotent. Dually, $\smile$ is the only idempotent triangular conorm on $(L, \leq)$.

If $(L, \leq)$ is a complete lattice and if the triangular norm T on $(L, \leq)$ is *completely distributive* with respect to supremum, i.e., if for any λ in L and for any family $(\mu_j \mid j \in J)$ of elements of L:

$$\sup_{j \in J} T(\lambda, \mu_j) = T(\lambda, \sup_{j \in J} \mu_j),$$

we shall call the structure $(L, \leq, T)$ a *complete lattice with t-norm*. Of course, a complete lattice with *t*-norm $(L, \leq, \frown)$ is a *complete Brouwerian lattice* [8]. For a more detailed treatment of triangular norms and conorms defined on bounded partially ordered sets, we refer to [18, 25].

A mapping h from a universe X to the set L will be called a $(L, \leq)$-*fuzzy set* in X, or, if we want to omit explicit reference to the codomain, a fuzzy set in X. We shall call such a fuzzy set h sup-*normal* iff $\sup_{x \in X} h(x)$ exists in $(L, \leq)$ and furthermore equals 1_L.

The reader will have noticed that in the matter of fuzzy sets, we use Goguen's nomenclature [34]. The membership functions of Zadeh's fuzzy sets [45] are special cases: they are mappings with the real unit interval as their codomain, and can therefore be considered as $([0, 1], \leq)$-fuzzy sets.

2.2 Ample fields and measurability

Probability and possibility *measures* are set mappings. This means that they are defined on specific collections of subsets of a universe. For probability measures these collections are generally σ-fields

[11, 33], although many 'subjectivists' will defend the definition of probability measures on fields [30]. Possibility measures were originally defined on the power set of a universe, but their domains can very easily be extended[1] towards the more general ample fields [18, 24, 44].

An *ample field* $\mathcal{R}$ on a universe X is a collection of subsets of X that is closed under complementation and under arbitrary unions. As a consequence, it contains $\emptyset$ and X, and is also closed under arbitrary intersections. The *atom* of $\mathcal{R}$ containing the element x of X is denoted by $[\,x\,]_{\mathcal{R}}$ and defined as

$$[\,x\,]_{\mathcal{R}} \stackrel{\text{def}}{=} \bigcap\{\,A \mid A \in \mathcal{R} \text{ and } x \in A\,\}.$$

It is easily verified that $x \in [\,x\,]_{\mathcal{R}}$ and $[\,x\,]_{\mathcal{R}} \in \mathcal{R}$. The *power set* $\wp(X)$ of X, i.e., the set of all subsets of X, is a special ample field on X, the atoms of which are singletons.

A subset A of X is called $\mathcal{R}$*-measurable* iff $A \in \mathcal{R}$. A $X - L$ mapping is called $\mathcal{R}$*-measurable* iff $(\forall\lambda \in L)(h^{-1}(\{\lambda\}) \in \mathcal{R})$, i.e., if it is constant on the atoms of $\mathcal{R}$.

Finally, let us consider two universes X and Y, provided with the respective ample fields $\mathcal{R}_X$ and $\mathcal{R}_Y$. Then a $X-Y$-mapping f is called $\mathcal{R}_X-\mathcal{R}_Y$*-measurable* iff $(\forall B \in \mathcal{R}_Y)(f^{-1}(B) \in \mathcal{R}_X)$. In this expression $f^{-1}(B)$ is the *inverse image* of B under f: $f^{-1}(B) = \{\,x \mid x \in X \text{ and } f(x) \in B\,\}$.

2.3 Possibility measures and possibilistic variables

In this subsection, $(L, \leq)$ denotes a complete lattice. Let us consider a universe X and an ample field $\mathcal{R}$ of subsets of X. A $(L, \leq)$*-possibility measure* Π on $(X, \mathcal{R})$ is a complete join-morphism between the complete lattices $(\mathcal{R}, \subseteq)$ and $(L, \leq)$, i.e., Π is a $\mathcal{R} - L$-mapping and for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$: $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$. If we want to omit specific reference to the codomain $(L, \leq)$ of Π, we shall simply call Π a possibility measure. For every $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$, there exists a unique $\mathcal{R}$-measurable $X - L$-mapping π such that for any A in $\mathcal{R}$: $\Pi(A) = \sup_{x\in A} \pi(x)$. This mapping satisfies $\pi(x) = \Pi([\,x\,]_{\mathcal{R}})$ for any x in X, and is called the *distribution* of Π. A $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$ by definition always satisfies $\Pi(\emptyset) = 0_L$. It is called *normal* iff $\Pi(X) = 1_L$. In that case, its distribution is a sup-normal $(L, \leq)$-fuzzy set in X.

Let us now turn to the definition of *possibilistic variables*, which are possibilistic equivalents of the stochastic variables in probability theory (see, for instance, [11]). In order to do this, we shall also consider a universe Ω and an ample field $\mathcal{R}_\Omega$ on Ω. This universe will be called *basic space*. The universe X will be called *sample space*. A $\Omega - X$-mapping that is $\mathcal{R}_\Omega - \mathcal{R}$-measurable, will be called a *possibilistic variable* in $(X, \mathcal{R})$. If we also consider a $(L, \leq)$-possibility measure Π_Ω on $(\Omega, \mathcal{R}_\Omega)$, we can use the possibilistic variable f to transform Π_Ω to a $(L, \leq)$-possibility measure Π_f on $(X, \mathcal{R})$, defined by $(\forall B \in \mathcal{R})(\Pi_f(B) = \Pi_\Omega(f^{-1}(B)))$. Π_f is called the *possibility distribution*[2] *(measure)* of the possibilistic variable f. The distribution π_f of Π_f will be called the *possibility distribution function* of f, and satisfies

$$\pi_f(x) = \sup_{f(\omega)\in[\,x\,]_{\mathcal{R}}} \pi_\Omega(\omega),$$

where x is an arbitrary element of X and π_Ω is the distribution of Π_Ω. For a more detailed account of possibility measures and possibilistic variables, we refer to [18, 21, 22, 23].

2.4 The meaning of possibility

Possibility measures were first introduced by Zadeh in 1978, in an attempt to give a mathematical representation of *linguistic uncertainty*, i.e., the uncertainty associated with imprecise and vague

[1]Very recently, we have shown that consistent definitions of possibility measures can also be given on fields [19].

[2]Note that there is a clear distinction between the *distribution* of a possibility measure on the one hand, and the *possibility distribution (measure)* of a possibilistic variable on the other hand.

propositions. Let us briefly describe the ideas behind his concept of possibility. For a more detailed account, we refer to Zadeh's seminal paper [46]. Consider the proposition 'Antje is tall' about the length of a girl called Antje. Clearly, given this information, we know something more about Antje's length, but not enough to determine it precisely. The only problem is how to mathematically represent this kind of information. Zadeh has suggested that his possibility measures are able to do this. Let us first consider the property 'tall'. It induces an ordering on the universe $\mathcal{L}$ of the body lengths – a suitable subset of the reals. This ordering can be represented by a $([0,1],\leq)$-fuzzy set h_{tall} in $\mathcal{L}$, satisfying, for any two elements x and y of $\mathcal{L}$:

$$h_{\text{tall}}(x) \leq h_{\text{tall}}(y) \Leftrightarrow \text{the length } x \text{ is smaller than or equal to the length } y.$$

The linguistic information 'Antje is tall' is then according to Zadeh represented by the $([0,1],\leq)$-possibility measure Π_{tall} on $(\mathcal{L},\wp(\mathcal{L}))$, that has h_{tall} as its distribution. For an arbitrary subset A of $\mathcal{L}$,

$$\Pi_{\text{tall}}(A) = \sup_{x\in A} h_{\text{tall}}(x)$$

is called the possibility that Antje's length belongs to A.

We want to stress that Zadeh only considers possibility measures that have a power set as their domain and $[0,1]$ as their codomain. The definition of possibility measures given in the previous subsection is a generalization of the one given by Zadeh. Although we shall not go into this too deeply, we want to point out that our extended definition is not merely of academic interest, and is, for instance, very useful and even necessary whenever we want to consider *incomparability* of objects with respect to a certain property, and want to associate possibility measures with the $(L,\leq)$-fuzzy sets introduced by Goguen [34], in order to represent more general forms of linguistic uncertainty [19, 46]. A detailed justification for our definition can be found in [18, 19], where we also show that these generalized possibility measures are indeed suitable mathematical representations of linguistic uncertainty.

There exist other generalizations of the possibility measures introduced by Zadeh. Klement and Weber [37] define possibility measures as set mappings that preserve countable suprema. Cai [12] has very recently introduced Q-scale measures, i.e., set mappings that preserve suprema indexed on a set Q that may be finite, countably or uncountably infinite, and defined on Q-domains, which may in general be interpreted as collections of fuzzy sets that are in particular closed under unions (suprema) indexed on Q.

Let us give a brief justification of why, in the context of this paper, we prefer our own generalization of Zadeh's possibility measures to the above-mentioned definitions. In what follows, the entire line of reasoning is centered on the identification of possibilistic reliabilities with possibility distribution functions of particular possibilistic variables. As explained above, these possibility distribution functions are the *distributions* of possibility measures. We have shown elsewhere [10] that an arbitrary set mapping has such a distribution if and only if it can be extended to a possibility measure defined on an ample field. Since we need the concept of a distribution, it is therefore natural to use possibility measures defined on ample fields in the framework of this paper.

It should also be noted that possibility measures can be defined and interpreted in the framework of the Dempster-Shafer theory of evidence (see, for instance, [39]). It goes without saying that the mathematical results obtained in this paper remain valid in either interpretation, or in any other interpretation that might be given to the possibility measures defined above. In order to emphasize this, we will in the sequel use the term 'possibilistic uncertainty' in stead of 'linguistic uncertainty' to refer to the type of uncertainty that is mathematically represented by possibility measures.

3 The classical state space

In this section, we give a brief overview of those notions and results from the theory of structure functions that will be needed further on.

3.1 The state space and state mappings

Let us consider[3] a system S with n components C_k, $k \in \{1, \dots, n\}$, $n \in \mathbb{N} \setminus \{0\}$. We shall often identify a component C_k with its index k, and therefore simply speak of component k. We shall assume that this system and each of its components can only be in either one of the two possible *states* '*work*' and '*fail*'. The *state set* of the system and of each of its components is therefore $\mathcal{S} \stackrel{\text{def}}{=} \{\mathit{fail}, \mathit{work}\}$. The interpretation of these states is obvious: of a component or system works, then its state is *work*, and if it fails, then its state is *fail*. A total order relation $\preceq$ on $\mathcal{S}$ may be defined as follows:

$$\preceq \stackrel{\text{def}}{=} \{(\mathit{fail}, \mathit{fail}), (\mathit{fail}, \mathit{work}), (\mathit{work}, \mathit{work})\},$$

or equivalently, $\mathit{fail} \prec \mathit{work}$. Of course, $(\mathcal{S}, \preceq)$ is a Boolean chain with top *work* and bottom *fail*. The meet (or Boolean multiplication) of this structure will be denoted by $\wedge$, the join (or Boolean addition) by $\vee$. The complement operator will be denoted by $\neg$. For more details about Boolean lattices, we refer to [8].

In what follows, we shall work with the state space model proposed by Gnedenko, Beliaev and Soloviev [32], and further developed and refined by Cappelle [16, 17]. We shall assume that we can associate with the system, and with each of its components, a finite number of parameters, say $\alpha_1, \dots, \alpha_M$, that characterize those aspects of system and components that are relevant with respect to a given context. These parameters could for instance be considered as a condensation of the physical model of the system and its components. With every different value of the *parameter combination* $p_s = (\alpha_1, \dots, \alpha_M)$, we may assume that there is associated a different physical model, and therefore a different system *cum* components. It follows that there exists a set Par_S of parameter combinations p_s that, relative to the given context, completely determine (or describe) the workings and therefore also the states of the system and of its components. In particular, this implies that we assume the existence of a *system state mapping* $x_s \colon \mathrm{Par}_S \to \mathcal{S}$, which maps any parameter combination p_s to the corresponding state $x_s(p_s)$ of the system. Similarly, we assume that there exist n *component state mappings* $x_k \colon \mathrm{Par}_S \to \mathcal{S}$, which map any parameter combination p_s to the corresponding state $x_k(p_s)$ of component C_k, $k \in \{1, \dots, n\}$. Equivalently, we could assume that there exists an *assembly state mapping* $(x_1, \dots, x_n) \colon \mathrm{Par}_S \to \mathcal{S}^n$, which maps any parameter combination p_s to the corresponding state $(x_1(p_s), \dots, x_n(p_s))$ of the *assembly*[4] $(C_1, \dots, C_n)$.

3.2 Structure functions

We shall also make an assumption which is central in classical, two-state reliability theory, namely, that there exists a mapping $\phi \colon \mathcal{S}^n \to \mathcal{S}$ which maps the states of the components to the corresponding state of the system:

$$x_s = \phi \circ (x_1, \dots, x_n). \tag{1}$$

ϕ is called the *two-valued structure function* of the system S. We want to stress at this point that (1) is an equality of mappings. It is furthermore assumed that such a two-valued structure function ϕ is isotonic – if the components work better, the system as a whole cannot do

[3] In this paper, we shall not give a formal definition of systems, components and states. For a thorough discussion of the subject, we refer for instance to the work of Cappelle [16, 17].

[4] We use the new term 'assembly' here, because the components of the system are considered one by one, without reference to their final place in the system, as it were before the system is assembled.

worse –, that $\phi(\mathit{fail}, \dots, \mathit{fail}) = \mathit{fail}$ – if all the components fail then the system fails – and that $\phi(\mathit{work}, \dots, \mathit{work}) = \mathit{work}$ – if all the components work then the system works. Indeed, we have the following very general definition of a structure function, which is an immediate generalization of Cappelle's definition [16] from complete lattices to bounded posets.

Definition 1 (Structure functions) *Let $(L_i, \leq_i)$, $1 \leq i \leq m$, $m \in \mathbb{N} \setminus \{0\}$ be m bounded posets with respective tops 1_{L_i} and bottoms 0_{L_i}. The product of these bounded posets is a bounded poset with top $(1_{L_1}, \dots, 1_{L_n})$ and bottom $(0_{L_1}, \dots, 0_{L_n})$, and is denoted by $(L_1 \times \cdots \times L_n, \leq_1 \times \cdots \times \leq_n)$ (see, for instance, [8]). Let $(L, \leq)$ be a bounded poset with top 1_L and bottom 0_L. Let σ be a $L_1 \times \cdots \times L_n - L$-mapping satisfying*

(i) *isotonicity:*
$(\forall((\lambda_1, \dots, \lambda_n), (\mu_1, \dots, \mu_n)) \in (L_1 \times \cdots \times L_n)^2)$
$((\lambda_1, \dots, \lambda_n) \leq_1 \times \cdots \times \leq_n (\mu_1, \dots, \mu_n) \Rightarrow \sigma(\lambda_1, \dots, \lambda_n) \leq \sigma(\mu_1, \dots, \mu_n))$;

(ii) *boundary conditions:*
$\sigma(0_{L_1}, \dots, 0_{L_n}) = 0_L$ *and* $\sigma(1_{L_1}, \dots, 1_{L_n}) = 1_L$.

Then σ is called a structure function from $(L_1 \times \cdots \times L_n, \leq_1 \times \cdots \times \leq_n)$ to $(L, \leq)$.

Using this terminology, the two-valued structure function ϕ is a structure function from $(\mathcal{S}^n, \preceq^n)$ to $(\mathcal{S}, \preceq)$, where $\preceq^n$ stands for the product ordering $\preceq \times \cdots \times \preceq$ on $\mathcal{S}^n$.

It should be noted that the two-valued structure function ϕ provides us with a logical-structural rather than a physical model of the system S. This structural model can at least in principle be derived from the physical model, of which the parameter set Par_S is the condensation for the given context.

A component C_k is said to be *irrelevant* iff ϕ is constant in x_k, that is, in its k-th place, and *relevant* otherwise. The structure function ϕ is called *coherent* if all the components C_k are relevant. We shall in this paper not require that ϕ should be coherent. However, taking into account the boundary conditions imposed on ϕ, it is easily verified that ϕ has at least one relevant component, i.e., that ϕ is *non-degenerate*.

By $\overline{\phi}$ we denote the *dual two-valued structure function* of ϕ, i.e., the $\mathcal{S}^n - \mathcal{S}$-mapping defined by $\overline{\phi}(\nu_1, \dots, \nu_n) \stackrel{\text{def}}{=} \neg\phi(\neg\nu_1, \dots, \neg\nu_n)$, for arbitrary $(\nu_1, \dots, \nu_n)$ in $\mathcal{S}^n$. It is easily verified that $\overline{\phi}$ is indeed a two-valued structure function, or in other words a structure function from $(\mathcal{S}^n, \preceq^n)$ to $(\mathcal{S}, \preceq)$. Note that $\overline{\overline{\phi}} = \phi$.

We conclude this subsection with the following definition, that will play an important role in the sequel.

Definition 2 *Let $(L, \leq)$ be a bounded poset with bottom 0_L and top 1_L. Let Φ be a $L^n - L$-mapping, and let ξ be the $\mathcal{S} - L$ mapping, defined by $\xi(\mathit{fail}) = 0_L$ and $\xi(\mathit{work}) = 1_L$. Then Φ is called an extension of ϕ from $(\mathcal{S}, \preceq)$ into $(L, \leq)$ iff*

$$(\forall(\nu_1, \dots, \nu_n) \in \mathcal{S}^n)(\xi(\phi(\nu_1, \dots, \nu_n)) = \Phi(\xi(\nu_1), \dots, \xi(\nu_n))).$$

If we restrict the domain of Φ to the set $\{0_L, 1_L\}^n$, the corresponding range of Φ will be a subset of $\{0_L, 1_L\}$. Indeed, the restricted Φ will be a structure function from $(\{0_L, 1_L\}^n, \leq^n)$ to $(\{0_L, 1_L\}, \leq)$. Finally, if we identify *work* with 1_L, and *fail* with 0_L, then the restricted Φ is identical to ϕ.

3.3 Minimal paths and minimal cuts

Let us introduce the following notation.

Definition 3 *Let $(L, \leq)$ be a bounded poset with top 1_L and bottom 0_L. Let $A \subseteq \{1, \ldots, n\}$ be a set of components of the structure function ϕ. Then we define the element $p_{(L,\leq)}(A) = (\beta_1, \ldots, \beta_n)$ of L^n as follows:*

$$(\forall k \in \{1, \ldots, n\})(\beta_k = 1_L \Leftrightarrow k \in A \text{ and } \beta_k = 0_L \Leftrightarrow k \notin A).$$

A set of components $A \subseteq \{1, \ldots, n\}$ is called a *path (set)* of the two-valued structure function ϕ iff[5] $\phi \cdot p_{(\mathcal{S},\preceq)}(A) = \textit{work}$. In other words, a path is a set of components such that if these components work, the system works. Dually, a set of components $B \subseteq \{1, \ldots, n\}$ is called a *cut (set)* of ϕ iff $\phi \cdot p_{(\mathcal{S},\preceq)}(\text{co}B) = \textit{fail}$. In other words, a cut is a set of components such that if these components fail, the system fails.

A minimal set of components, such that if these components fail, the system S fails, is called a *minimal cut (set)* of the system S. Dually, a minimal set of components, such that if these components work, the system S works, is called a *minimal path (set)* of the system S. The following propositions are fairly obvious taking into account the properties of ϕ. A proof may be found in [35].

Proposition 1 *Let A and B be sets of components, i.e., subsets of $\{1, \ldots, n\}$.*

(i) *Assume that $A \subseteq B$. If A is a path of ϕ then so is B. If A is a cut of ϕ then so is B.*

(ii) *Every path of ϕ contains at least one minimal path of ϕ. Every cut of ϕ contains at least one minimal cut of ϕ.*

(iii) *If A is a path and B is a cut of ϕ then $A \cap B \neq \emptyset$.*

(iv) *A component is relevant if and only if it belongs to a minimal path of ϕ. A component is relevant if it belongs to a minimal cut of ϕ.*

(v) *A is a path of ϕ iff A is a cut of $\overline{\phi}$, and B is a cut of ϕ iff B is a path of $\overline{\phi}$.*

Proposition 2 *Consider the two-valued structure functions ϕ_1 and ϕ_2. If they have the same minimal paths, they are equivalent, i.e., identical after suppression of irrelevant components. Dually, if they have the same minimal cuts, they are equivalent.*

With obvious notations, a minimal cut of ϕ will be denoted by C_s, $s \in \{1, \ldots, n_c\}$, where n_c is the number of minimal cuts of ϕ. Similarly, a minimal path of ϕ will be denoted by P_r, $r \in \{1, \ldots, n_p\}$, where n_p is the number of minimal paths of ϕ. Since we know that ϕ has at least one relevant component, proposition 1(iv) tells us that $n_c \geq 1$ and $n_p \geq 1$.

Since the two-valued structure function ϕ is non-degenerate, we also have the following proposition [35].

Proposition 3 *Let A and B be sets of components, i.e., subsets of $\{1, \ldots, n\}$.*

(i) *A is a cut of ϕ iff $(\forall r \in \{1, \ldots, n_p\})(A \cap P_r \neq \emptyset)$.*

(ii) *B is a path of ϕ iff $(\forall s \in \{1, \ldots, n_c\})(B \cap C_s \neq \emptyset)$.*

[5] In this paper, we shall often use the alternative notation $f \cdot x$ for $f(x)$, in order to make complicated formulas with nested parentheses more readable.

Birnbaum, Esary and Saunders [9] have proven the following fundamental theorem in the theory of two-valued structure functions[6].

Theorem 1 *For every two-valued structure function* ϕ*, there exist the following decompositions of* ϕ *in minimal paths respectively minimal cuts. For any* $(\nu_1, \ldots, \nu_n)$ *in* $\mathcal{S}^n$*:*

$$\phi(\nu_1, \ldots, \nu_n) = \bigvee_{1 \leq r \leq n_p} \bigwedge_{i \in P_r} \nu_i = \bigwedge_{1 \leq s \leq n_c} \bigvee_{i \in C_s} \nu_i. \tag{2}$$

[6]We present here a formulation by Cappelle [16], which is slightly more general than the original version of Birnbaum, Esary and Saunders, because there is no coherence condition imposed on ϕ.

4 Possibilistic reliability

After the discussion of the classical state space and structure functions in section 3, and the possibilistic uncertainty model in section 2, we are now ready to merge the contents of these sections into a possibilistic reliability model.

In this section, we shall denote by $(L, \leq)$ a complete lattice, unless explicitly stated otherwise. Let us again consider the system S with two-valued structure function ϕ, introduced in the previous section. We already know that the parameter set Par_S contains a description of the system: every element p_s of Par_S completely determines a behaviour of the system S and its components with respect to the given context. With each different p_s, there is associated a different behaviour for S and its components. Therefore, if we have *possibilistic information* about the behaviour of the system S and its components, it is fairly natural to mathematically represent this as a *normal* $(L, \leq)$-possibility measure Π_{Par_S} on $(\mathrm{Par}_S, \mathcal{R}_{\mathrm{Par}_S})$, where $\mathcal{R}_{\mathrm{Par}_S}$ is an ample field of measurable subsets of Par_S. This possibility measure Π_{Par_S} will be the *starting point of our course of reasoning*. Consequently, in what follows, Par_S will be considered as a *basic space*. Note that as with probability measures, the normality of the possibility measure Π_{Par_S} is an indication that the universe Par_S is complete enough to contain all the possible parameter combinations. It is therefore natural to assume that Π_{Par_S} is normal. The distribution of Π_{Par_S} will be denoted by π_{Par_S}. It is a sup-normal $\mathcal{R}_{\mathrm{Par}_S}$-measurable $\mathrm{Par}_S - L$-mapping.

4.1 Measurability

First of all, we must deal with some aspects of measurability with respect to the sets and mappings that are relevant to this discussion. On the state space $\mathcal{S}$ of the components and the system, we consider the power set $\wp(\mathcal{S}) = \{\emptyset, \{fail\}, \{work\}, \{fail, work\}\}$ as the ample field of measurable subsets, simply because we want to be able to *distinguish between the states 'work' and 'fail'*. This would of course be impossible in the only other ample field $\{\emptyset, \{fail, work\}\}$ on $\mathcal{S}$.

Now, consider an arbitrary component C_k of the system S, $k \in \{1, \ldots, n\}$. We know from the discussion in section 3 that there exists a state mapping x_k that maps any parameter combination p_s in Par_S to the corresponding state $x_k(p_s)$ of component C_k. In what follows, we shall assume that *x_k is $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable*, which is equivalent to

$$\begin{cases} x_k^{-1}(\{work\}) \in \mathcal{R}_{\mathrm{Par}_S} \\ x_k^{-1}(\{fail\}) \in \mathcal{R}_{\mathrm{Par}_S}, \end{cases}$$

where, for instance, $x_k^{-1}(\{work\}) = \{\, p_s \mid p_s \in \mathrm{Par}_S \text{ and } x_k(p_s) = work \,\}$. This means that in the ample field $\mathcal{R}_{\mathrm{Par}_S}$, we must be able to *distinguish between the values of the parameter combination p_s which make the component C_k work, and the ones which make it fail.* It also means that x_k is a possibilistic variable in the sample space $(\mathcal{S}, \wp(\mathcal{S}))$, where, as mentioned before, the role of the basic space is played by $(\mathrm{Par}_S, \mathcal{R}_{\mathrm{Par}_S})$.

Instead of looking at the state mappings of the components one by one, we may consider the $\mathrm{Par}_S - \mathcal{S}^n$-mapping $(x_1, \ldots, x_n)$, also called assembly state mapping. It is easily verified that this mapping is $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S}^n)$-measurable, because each of the state mappings x_k is assumed to be $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable. Indeed, we have the following proposition, the proof of which is trivial and therefore omitted.

Proposition 4 *The component state mappings x_k, $k \in \{1, \ldots, n\}$, are $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable if and only if the assembly state mapping $(x_1, \ldots, x_n)$ is $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S}^n)$-measurable.*

Let us now also consider the state mapping x_s, which maps any value of the parameter combination p_s in Par_S to the corresponding state $x_s(p_s)$ of the system S. Using (1), it can be shown that x_s is $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable, and therefore a possibilistic variable in $(\mathcal{S}, \wp(\mathcal{S}))$. Indeed, we have the following proposition. Its proof is straightforward, and will be omitted.

Proposition 5 *If the component state mappings x_k, $k \in \{1, \ldots, n\}$, are $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable, then the system state mapping $x_s = \phi \circ (x_1, \ldots, x_n)$ is also $\mathcal{R}_{\mathrm{Par}_S} - \wp(\mathcal{S})$-measurable.*

This means that in the ample field $\mathcal{R}_{\mathrm{Par}_S}$, we shall be able to *distinguish between the values of the parameter combination p_s which make the system S work, and the ones which make it fail.*

At this point, we already have possibilistic information about the behaviour of the system S and its components in the form of a $(L, \leq)$-possibility measure Π_{Par_S} on $(\mathrm{Par}_S, \mathcal{R}_{\mathrm{Par}_S})$. Our aim is now twofold. First of all, we want to convert this into information about the states of the system and its components, in the form of so-called possibilistic reliabilities. Second, we want to find a relationship between the possibilistic reliability of the system and the possibilistic reliabilities of the components.

4.2 Possibilistic reliabilities

In order to obtain information about the states of the system S and its components, we shall use the system and component state mappings discussed above to transform Π_{Par_S} into possibility measures on the respective system and component state spaces. How a measurable mapping can be used to transport (or transform) a possibility measure from one universe to another, has been explained in subsection 2.3. If we use the component state mappings to obtain information about the component states, this leads to the following proposition.

Proposition 6 *For any k in $\{1, \ldots, n\}$, the $\wp(\mathcal{S}) - L$-mapping Π_{x_k}, defined by*

$$(\forall B \in \wp(\mathcal{S}))(\Pi_{x_k}(B) \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S}(x_k^{-1}(B))),$$

is a normal $(L, \leq)$-possibility measure on $(\mathcal{S}, \wp(\mathcal{S}))$. The distribution π_{x_k} of Π_{x_k} is the sup*-normal $\mathcal{S} - L$-mapping, satisfying, for any ν_k in $\mathcal{S}$:*

$$\pi_{x_k}(\nu_k) = \sup_{x_k(p_s)=\nu_k} \pi_{\mathrm{Par}_S}(p_s).$$

Of course, we can also use the assembly state mapping $(x_1, \ldots, x_n)$ to obtain information about the assembly state.

Proposition 7 *The $\wp(\mathcal{S}^n) - L$-mapping $\Pi_{(x_1,\ldots,x_n)}$, defined by*

$$(\forall B \in \wp(\mathcal{S}^n))(\Pi_{(x_1,\ldots,x_n)}(B) \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S}((x_1, \ldots, x_n)^{-1}(B))),$$

is a normal $(L, \leq)$-possibility measure on $(\mathcal{S}^n, \wp(\mathcal{S}^n))$. The distribution $\pi_{(x_1,\ldots,x_n)}$ of $\Pi_{(x_1,\ldots,x_n)}$ is the sup*-normal $\mathcal{S}^n - L$-mapping, satisfying, for any $(\nu_1, \ldots, \nu_n)$ in $\mathcal{S}^n$:*

$$\pi_{(x_1,\ldots,x_n)}(\nu_1, \ldots, \nu_n) = \sup_{(x_1,\ldots,x_n)(p_s)=(\nu_1,\ldots,\nu_n)} \pi_{\mathrm{Par}_S}(p_s).$$

Finally, the system state mapping x_s can be used to obtain information about the system state.

Proposition 8 *The $\wp(\mathcal{S}) - L$-mapping Π_{x_s}, defined by*

$$(\forall B \in \wp(\mathcal{S}))(\Pi_{x_s}(B) \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{Par}_S}(x_s^{-1}(B))),$$

is a normal $(L, \leq)$-possibility measure on $(\mathcal{S}, \wp(\mathcal{S}))$. The distribution π_{x_s} of Π_{x_s} is the sup*-normal $\mathcal{S} - L$-mapping, satisfying, for any ν in $\mathcal{S}$:*

$$\pi_{x_s}(\nu) = \sup_{x_s(p_s)=\nu} \pi_{\mathrm{Par}_S}(p_s).$$

The proofs of these propositions are fairly trivial, taking into account the definitions of possibility measures and their distributions, and the properties of the inverse images x_k^{-1}, x_s^{-1} and $(x_1, \ldots, x_n)^{-1}$ of the respective mappings x_k, x_s and $(x_1, \ldots, x_n)$. Indeed, these propositions are special instances of a general result that can be found in [18, 21]. The distributions π_{x_k}, π_{x_s} and $\pi_{(x_1,\ldots,x_n)}$ are of course the *possibility distribution functions* of the respective possibilistic variables x_k, x_s and $(x_1, \ldots, x_n)$. They completely characterize the possibilistic information we have about the states of system and components. For a component C_k, $\pi_{x_k}(work)$ is the $(L, \leq)$-possibility that this component works, and $\pi_{x_k}(fail)$ the $(L, \leq)$-possibility that it fails. For the system S, analogous conclusions can be drawn regarding $\pi_{x_s}(work)$ and $\pi_{x_s}(fail)$. The analogy with the probabilistic approach inspires the following definition[7].

Definition 4 (Possibilistic reliabilities) *We shall call the possibility distribution function π_{x_k} of the component state mapping x_k the possibilistic reliability of component C_k, $k \in \{1, \ldots, n\}$. The possibility distribution function $\pi_{(x_1,\ldots,x_n)}$ of the assembly state mapping $(x_1, \ldots, x_n)$ will be called the possibilistic reliability of the assembly $(C_1, \ldots, C_k)$. Finally, we shall call the possibility distribution function π_{x_s} of the system state mapping x_s the possibilistic reliability of the system S.*

Since we see that sup-normal $\mathcal{S}-L$-mappings play an important part in this theory, we introduce the following notation.

Definition 5 *Let $(L, \leq)$ be a bounded lattice. We shall denote by $\tilde{\mathcal{S}}$ the set of the* sup-*normal $\mathcal{S} - L$-mappings. Furthermore, we introduce the following important elements of $\tilde{\mathcal{S}}$:*

$$\widetilde{work} \overset{\text{def}}{=} \{(fail, 0_L), (work, 1_L)\} \text{ and } \widetilde{fail} \overset{\text{def}}{=} \{(fail, 1_L), (work, 0_L)\}.$$

Finally, if $t \in \tilde{\mathcal{S}}$, then we define the element $\bar{t}$ of $\tilde{\mathcal{S}}$ by $\bar{t}(\nu) = t(\neg\nu)$, for any ν in $\mathcal{S}$.

The fact that in this definition $(L, \leq)$ is only assumed to be a bounded lattice, and not a complete lattice, seem to make this definition more general than is called for at this moment. The reason for this generality lies in the fact that we shall make ample use of it in the next section.

It will be clear that the operator '$\bar{\cdot}$' can be considered as a permutation of $\tilde{\mathcal{S}}$, that exchanges $\widetilde{work}$ and $\widetilde{fail}$. This operator will also play an important role in the next section.

What we want to prove in this paper, is that possibilistic reliabilities play an analogous part in a possibilistic reliability theory as the well-known (normal) reliabilities do in the probabilistic approach. The reader will notice that the probabilistic reliability r_k of, say, a component k – the probability that the component works – is a *single* real number in the unit interval $[0, 1]$. Its possibilistic counterpart π_{x_k}, however, is a $\mathcal{S} - L$-mapping, or equivalently, a *couple* of elements $(\pi_{x_k}(fail), \pi_{x_k}(work))$ of the complete lattice $(L, \leq)$. Of course, the reason for this difference lies in the fact that from r_k, we can immediately infer the probability $1 - r_k$ that the component fails, whereas in general in the possibilistic approach we cannot determine $\pi_{x_k}(fail)$ from $\pi_{x_k}(work)$ or *vice versa*.

Finally, it should be noted that if the possibilistic reliability of, say, a component C_k is $\widetilde{work}$, this means that failure of this component is impossible, according to the information we have. Dually, if it is equal to $\widetilde{fail}$, the component cannot work, according to our information.

[7] Cai et al. [13] basically use the same idea when they define the *reliability* and the *unreliability* of a component or a system.

4.3 Possibilistic structure functions

Let us now investigate the relations which exist between the possibilistic reliabilities we have introduced thus far. For a start, it is fairly easy to determine the possibilistic reliability of the system from that of the assembly. Indeed, from (1) we deduce, for any B in $\wp(\mathcal{S})$, taking into account the properties of inverse images:

$$x_s^{-1}(B) = (\phi \circ (x_1, \dots, x_n))^{-1}(B) = (x_1, \dots, x_n)^{-1}(\phi^{-1}(B)).$$

Since $\phi^{-1}(B) \in \wp(\mathcal{S}^n)$, we find that, taking into account the definitions of Π_{x_s} and $\Pi_{(x_1,\dots,x_n)}$ (see propositions 7 and 8),

$$\begin{aligned}\Pi_{x_s}(B) &= \Pi_{\mathrm{Par}_S}(x_s^{-1}(B))\\ &= \Pi_{\mathrm{Par}_S}((x_1, \dots, x_n)^{-1}(\phi^{-1}(B)))\\ &= \Pi_{(x_1,\dots,x_n)}(\phi^{-1}(B)).\end{aligned}$$

This has an equivalent formulation in the following proposition, which allows us to determine the possibilistic reliability of the system S if that of the assembly $(C_1, \dots, C_n)$ is known.

Proposition 9 *For any ν in $\mathcal{S}$:*

$$\pi_{x_s}(\nu) = \sup_{\phi(\nu_1,\dots,\nu_n)=\nu} \pi_{(x_1,\dots,x_n)}(\nu_1, \dots, \nu_n).$$

Next, we show that we may also determine the possibilistic reliabilities of the components from the possibilistic reliability of the assembly. It turns out that in order to derive such a result, we must consider the *state projection operators.*

Definition 6 *Let k be an element of $\{1, \dots, n\}$. The $\mathcal{S}^n - \mathcal{S}$-mapping* proj_k*, defined by*

$$(\forall(\nu_1, \dots, \nu_n) \in \mathcal{S}^n)(\mathrm{proj}_k(\nu_1, \dots, \nu_n) \stackrel{\mathrm{def}}{=} \nu_k),$$

is called the k-th state projection operator.

These operators derive their name from the fact that they allow us to derive component states from assembly states. Indeed, they satisfy the functional equality $x_k = \mathrm{proj}_k \circ (x_1, \dots, x_n)$. From this equality, we derive for any B in $\wp(\mathcal{S})$, taking into account the properties of inverse images:

$$x_k^{-1}(B) = (x_1, \dots, x_n)^{-1}(\mathrm{proj}_k^{-1}(B)),$$

and since $\mathrm{proj}_k^{-1}(B) \in \wp(\mathcal{S}^n)$, we find that, taking into account the definitions of Π_{x_k} and $\Pi_{(x_1,\dots,x_n)}$ (see propositions 6 and 7),

$$\begin{aligned}\Pi_{x_k}(B) &= \Pi_{\mathrm{Par}_S}(x_k^{-1}(B))\\ &= \Pi_{\mathrm{Par}_S}((x_1, \dots, x_n)^{-1}(\mathrm{proj}_k^{-1}(B)))\\ &= \Pi_{(x_1,\dots,x_n)}(\mathrm{proj}_k^{-1}(B)).\end{aligned}$$

This is reformulated in an equivalent way in proposition 10, a result which allows us to derive the possibilistic reliabilities of the components from that of the assembly.

Proposition 10 *For any ν_k in $\mathcal{S}$:*

$$\pi_{x_k}(\nu_k) = \sup_{\mathrm{proj}_k(\nu)=\nu_k} \pi_{(x_1,\dots,x_n)}(\nu).$$

Now, we really want to be able to determine the possibilistic reliability π_{x_s} of the system S from the possibilistic reliabilities π_{x_k} of the components C_k. In order to achieve our goal, however, we find that we must be able to derive the possibilistic reliability $\pi_{(x_1,\ldots,x_n)}$ of the assembly $(C_1, \ldots, C_n)$ from the possibilistic reliabilities of the components. This will in general not be possible unless the possibilistic variables x_k are in some way *possibilistically independent*. We shall limit ourselves here to giving a straightforward characterization of the possibilistic independence of the component state mappings x_k. For a more detailed, general treatment of the possibilistic independence of possibilistic variables and events, we refer to [18, 22, 23, 26].

Theorem 2 *Let T be a triangular norm on $(L, \leq)$ such that $(L, \leq, T)$ is a complete lattice with t-norm. The possibilistic variables x_k in $(\mathcal{S}, \wp(\mathcal{S}))$ are $(\Pi_{\mathrm{Par}_S}, T)$-independent (or, in general, possibilistically independent) if and only if*

$$(\forall(\nu_1, \ldots, \nu_n) \in \mathcal{S}^n)(\pi_{(x_1,\ldots,x_n)}(\nu_1, \ldots, \nu_n) = T_{k=1}^n \pi_{x_k}(\nu_k)) \tag{3}$$

Equation (3) tells us that in the case of possibilistic independence, we may determine the reliability of the assembly from the reliabilities of the components. If we combine equation (3) and proposition 9, we find the result we were looking for.

Theorem 3 *Let T be a triangular norm on $(L, \leq)$ such that $(L, \leq, T)$ is a complete lattice with t-norm. If the possibilistic variables x_k are $(\Pi_{\mathrm{Par}_S}, T)$-independent, we have, for any ν in $\mathcal{S}$:*

$$\pi_{x_s}(\nu) = \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n \pi_{x_k}(\nu_k).$$

This result inspires the following definition.

Definition 7 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. The $\tilde{\mathcal{S}}^n - \tilde{\mathcal{S}}$-mapping $\widetilde{\phi}_T$, defined by*

$$\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \nu \stackrel{\text{def}}{=} \sup_{\phi(\nu_1,\ldots,\nu_n)=\nu} T_{k=1}^n t_k(\nu_k), \quad \nu \in \mathcal{S}$$

for any $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$, is called the $(L, \leq, T)$-possibilistic structure function, or simply possibilistic structure function, associated with the two-valued structure function ϕ.

We have made the definition of possibilistic structure functions more general than is at this point called for by requiring that $(L, \leq)$ is only a bounded lattice and not a complete lattice, and that T is only distributive with respect to $\smile$. The reason for this generality will become clear in the following section. It should be noted that definition 7 indeed makes sense if $(L, \leq)$ is a bounded lattice, since only suprema of finite (possibly empty) subsets of L are involved. Why we call $\widetilde{\phi}_T$ a possibilistic *structure function* will also be explained in the following section.

It should also be noted that $\widetilde{\phi}_T$ is an *extension* of the $\mathcal{S}^n - \mathcal{S}$-mapping ϕ towards a $\tilde{\mathcal{S}}^n - \tilde{\mathcal{S}}$-mapping in the following sense: if we identify *work* with $\widetilde{work}$ and *fail* with $\widetilde{fail}$, then ϕ and the restriction of $\widetilde{\phi}_T$ to $\{\widetilde{fail}, \widetilde{work}\}^n$ are essentially the same (see also definition 2). We shall therefore also call $\widetilde{\phi}_T$ the $(L, \leq)$-possibilistic T-extension of ϕ. The reader conversant in fuzzy set theory will have recognized that there is a relation between definition 7 and Zadeh's extension principle. For a detailed exploration of this relationship, we refer to [20].

In the next section, we shall also need the notion of the dual of a possibilistic structure function. Definition 8 is a formal analogon of the definition of the dual of a two-valued structure function, where the role of $\neg$ is taken over by the operator '$\overline{\cdot}$' on $\tilde{\mathcal{S}}$.

Definition 8 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. Then we define the dual $\overline{\widetilde{\phi}_T}$ of the $(L, \leq, T)$-possibilistic structure function $\widetilde{\phi}_T$ as follows. For any $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$:*

$$\overline{\widetilde{\phi}_T}(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \overline{\widetilde{\phi}_T(\overline{t_1}, \ldots, \overline{t_n})}.$$

Corollary 1 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. Then for any $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$:*

$$\widetilde{\overline{\phi}}_T(t_1, \ldots, t_n) = \overline{\widetilde{\phi}_T(\overline{t_1}, \ldots, \overline{t_n})}.$$

Proof. By definition, we have that for any ν in $\mathcal{S}$

$$\begin{aligned}
\widetilde{\overline{\phi}}_T(t_1, \ldots, t_n) \cdot \nu &= \sup_{\overline{\phi}(\nu_1, \ldots, \nu_n) = \nu} T_{k=1}^n t_k(\nu_k) \\
&= \sup_{\phi(\neg\nu_1, \ldots, \neg\nu_n) = \neg\nu} T_{k=1}^n \overline{t_k}(\neg\nu_k) \\
&= \sup_{\phi(\nu_1, \ldots, \nu_n) = \neg\nu} T_{k=1}^n \overline{t_k}(\nu_k) \\
&= \widetilde{\phi}_T(\overline{t_1}, \ldots, \overline{t_n}) \cdot \neg\nu,
\end{aligned}$$

whence indeed $\widetilde{\overline{\phi}}_T(t_1, \ldots, t_n) = \overline{\widetilde{\phi}_T(\overline{t_1}, \ldots, \overline{t_n})}$. □

This corollary tells us that $\widetilde{\overline{\phi}}_T$ is the dual $\overline{\widetilde{\phi}_T}$ of $\widetilde{\phi}_T$. In other words, the operations 'extension' and 'taking the dual' can be exchanged.

In summary, we have reached the following important result: *if the component state mappings are $(\Pi_{\mathrm{Par}_S}, T)$-independent, then it is possible to express the possibilistic reliability of the system in terms of the possibilistic reliabilities of its components:* $\pi_{x_s} = \widetilde{\phi}_T(\pi_{x_1}, \ldots, \pi_{x_n})$. This result is analogous to the classical, probabilistic result: if the component state mappings are stochastically independent, then the reliability r_s of the system is a function of the reliabilities r_k of the components: $r_s = \varphi(r_1, \ldots, r_n)$. In the classical, probabilistic approach, it is fairly easy to calculate φ if ϕ is known (see, for instance, [1, 2]): we start with the minimal path (or minimal cut) decomposition of ϕ (see theorem 1), and in this decomposition change ν_k into r_k, $\wedge$ into the product operator on $[0, 1]$, and $\vee$ into the probabilistic sum on $[0, 1]$. Using the distributivity of product with respect to sum, we then convert this expression into a sum of products of component reliabilities. If in such a product a component reliability, say r_k, occurs more than once, say as a power r_k^m, $m \geq 2$, then this power is replaced by r_k. The expression thus obtained is $\varphi(r_1, \ldots, r_n)$. Note that φ is an extension of ϕ from $(\mathcal{S}, \preceq)$ into $([0, 1], \leq)$.

In order to calculate π_{x_s} from the π_{x_k}, we find that we first have to calculate the possibilistic structure function $\widetilde{\phi}_T$. For complicated ϕ, this can be very cumbersome. In the following section, we will take a closer look at possibilistic structure functions, and derive a number of results which will appreciably facilitate their calculation, and which are fairly analogous to the probabilistic method outlined above.

5 Calculating possibilistic structure functions

5.1 Path and cut transforms

We begin this section with what might seem an abstract mathematical digression. It will, however, yield very useful results, and leads to a straightforward method for calculating $\widetilde{\phi}_T$. We begin by observing that in the probabilistic case, in order to calculate φ from ϕ, the decomposition of ϕ in minimal paths or cuts is taken as a starting point, and this decomposition is 'imported' into the structure $([0,1],\leq)$ by replacing the meet $\wedge$ of $(\mathcal{S},\preceq)$ by the product operator, which is a t-norm on $([0,1],\leq)$; and by replacing the join $\vee$ of $(\mathcal{S},\preceq)$ by the probabilistic sum operator, which is a t-conorm on $([0,1],\leq)$. This idea is generalized in the following definition.

Definition 9 (Path and cut transforms) *Let $(L,\leq)$ be a bounded poset. Let T be a t-norm and S a t-conorm on $(L,\leq)$. We define the $L^n - L$ mappings $\mathcal{P}_{TS}(\phi)$, $\mathcal{P}_{ST}(\phi)$, $\mathcal{C}_{TS}(\phi)$ and $\mathcal{C}_{ST}(\phi)$ as follows. For any $(\lambda_1,\ldots,\lambda_n)$ in L^n:*

$$\begin{cases} \mathcal{P}_{TS}(\phi)(\lambda_1,\ldots,\lambda_n) \stackrel{\text{def}}{=} T_{1\leq r\leq n_p} S_{i\in P_r}\lambda_i \\ \mathcal{P}_{ST}(\phi)(\lambda_1,\ldots,\lambda_n) \stackrel{\text{def}}{=} S_{1\leq r\leq n_p} T_{i\in P_r}\lambda_i \\ \mathcal{C}_{TS}(\phi)(\lambda_1,\ldots,\lambda_n) \stackrel{\text{def}}{=} T_{1\leq s\leq n_c} S_{i\in C_s}\lambda_i \\ \mathcal{C}_{ST}(\phi)(\lambda_1,\ldots,\lambda_n) \stackrel{\text{def}}{=} S_{1\leq s\leq n_c} T_{i\in C_s}\lambda_i \end{cases}$$

$\mathcal{P}_{TS}(\phi)$ will be called the TS-path transform, $\mathcal{P}_{ST}(\phi)$ the ST-path transform, $\mathcal{C}_{TS}(\phi)$ the TS-cut transform and $\mathcal{C}_{ST}(\phi)$ the ST-cut transform of the two-valued structure function ϕ.

It is clear that $\mathcal{P}_{TS}(\phi)$, $\mathcal{P}_{ST}(\phi)$, $\mathcal{C}_{TS}(\phi)$ and $\mathcal{C}_{ST}(\phi)$ are structure functions from $(L^n,\leq^n)$ to $(L,\leq)$.

Since $(\mathcal{S},\preceq)$ is a Boolean chain, and therefore certainly a bounded poset, and $\wedge$ is a t-norm and $\vee$ a t-conorm on $(\mathcal{S},\preceq)$, we may apply this definition for $(L,\leq)=(\mathcal{S},\preceq)$, $T=\wedge$ and $S=\vee$. From theorem 1 and proposition 1(v), we deduce that

$$\phi = \mathcal{P}_{\vee\wedge}(\phi) = \mathcal{C}_{\wedge\vee}(\phi) \qquad \text{and} \qquad \overline{\phi} = \mathcal{P}_{\wedge\vee}(\phi) = \mathcal{C}_{\vee\wedge}(\phi). \tag{4}$$

Essentially, what we shall do in this subsection, is try and find out if and how this interesting result can be generalized towards more general choices for $(L,\leq)$. The following propositions are the first step in this investigation.

Proposition 11 *Let $(L,\leq)$ be a bounded poset. Let T be a t-norm and S a t-conorm on $(L,\leq)$. Let A be an arbitrary set of components, i.e., $A\subseteq\{1,\ldots,n\}$. With A we can associate the following elements of L^n: $p_{(L,\leq)}(A)=(\beta_1,\ldots,\beta_n)$ and $p_{(L,\leq)}(\mathrm{co}A)=(\gamma_1,\ldots,\gamma_n)$, with (see, also definition 3)*

$$\begin{cases} (\forall k\in\{1,\ldots,n\})(\beta_k=1_L \Leftrightarrow k\in A \text{ and } \beta_k=0_L \Leftrightarrow k\notin A) \\ (\forall k\in\{1,\ldots,n\})(\gamma_k=0_L \Leftrightarrow k\in A \text{ and } \gamma_k=1_L \Leftrightarrow k\notin A). \end{cases}$$

(i) *A is a path of ϕ if and only if $\mathcal{P}_{ST}(\phi)\cdot p_{(L,\leq)}(A)=1_L$.*

(ii) *A is a cut of ϕ if and only if $\mathcal{P}_{ST}(\phi)\cdot p_{(L,\leq)}(\mathrm{co}A)=0_L$.*

(iii) *A is a path of ϕ if and only if $\mathcal{P}_{TS}(\phi)\cdot p_{(L,\leq)}(\mathrm{co}A)=0_L$.*

(iv) *A is a cut of ϕ if and only if $\mathcal{P}_{TS}(\phi)\cdot p_{(L,\leq)}(A)=1_L$.*

(v) *A is a path of ϕ if and only if $\mathcal{C}_{ST}(\phi) \cdot p_{(L,\leq)}(\text{co}A) = 0_L$.*

(vi) *A is a cut of ϕ if and only if $\mathcal{C}_{ST}(\phi) \cdot p_{(L,\leq)}(A) = 1_L$.*

(vii) *A is a path of ϕ if and only if $\mathcal{C}_{TS}(\phi) \cdot p_{(L,\leq)}(A) = 1_L$.*

(viii) *A is a cut of ϕ if and only if $\mathcal{C}_{TS}(\phi) \cdot p_{(L,\leq)}(\text{co}A) = 0_L$.*

Proof. As an example, we shall prove (i) and (ii). The proofs of the other statements are completely analogous. First, assume that A is a path of ϕ. Proposition 1(ii) tells us that there exists a minimal path P_r of ϕ such that $P_r \subseteq A$. Therefore, by construction, for any i in P_r, $\beta_i = 1_L$, whence $T_{i\in P_r}\beta_i = 1_L$ and therefore also $\mathcal{P}_{ST}(\phi) \cdot p_{(L,\leq)}(A) = S_{1\leq r\leq n_p}T_{i\in P_r}\beta_i = 1_L$, taking into account the boundary behaviour of triangular norms and conorms. Conversely, assume that $\mathcal{P}_{ST}(\phi) \cdot p_{(L,\leq)}(A) = 1_L$. Since the components β_k of $p_{(L,\leq)}(A)$ only take values in the subset $\{0_L, 1_L\}$ of L, this implies that $(\exists r \in \{1,\ldots,n_p\})(\forall i \in P_r)(\beta_i = 1_L)$, taking into account the boundary behaviour of triangular norms and conorms. This implies that $P_r \subseteq A$, which, according to proposition 1(i), implies that A is a path of ϕ. This proves (i).

Let us now prove (ii) in a slightly different way. We have the following chain of equivalences, taking into account the boundary behaviour of triangular norms and conorms:

$$\begin{aligned}
\mathcal{P}_{ST}(\phi) \cdot p_{(L,\leq)}(\text{co}A) = 0_L &\Leftrightarrow S_{1\leq r\leq n_p}T_{i\in P_r}\gamma_i = 0_L \\
&\Leftrightarrow (\forall r \in \{1,\ldots,n_p\})(T_{i\in P_r}\gamma_i = 0_L) \\
&\Leftrightarrow (\forall r \in \{1,\ldots,n_p\})(\exists i \in P_r)(\gamma_i = 0_L) \\
&\Leftrightarrow (\forall r \in \{1,\ldots,n_p\})(\exists i \in P_r)(i \in A) \\
&\Leftrightarrow (\forall r \in \{1,\ldots,n_p\})(A \cap P_r \neq \emptyset) \\
&\Leftrightarrow A \text{ is a cut of } \phi.
\end{aligned}$$

The last equivalence is based on proposition 3(i). □

Proposition 12 *Let $(L,\leq)$ be a bounded poset. Let T be a t-norm and S a t-conorm on $(L,\leq)$.*

(i) *$\mathcal{P}_{ST}(\overline{\phi}) = \mathcal{C}_{ST}(\phi)$ and $\mathcal{C}_{ST}(\overline{\phi}) = \mathcal{P}_{ST}(\phi)$.*

(ii) *$\mathcal{P}_{TS}(\overline{\phi}) = \mathcal{C}_{TS}(\phi)$ and $\mathcal{C}_{TS}(\overline{\phi}) = \mathcal{P}_{TS}(\phi)$.*

Proof. Immediately from proposition 1(v). □

Proposition 11, together with proposition 2 tells us that $\mathcal{P}_{ST}(\phi)$ and $\mathcal{C}_{TS}(\phi)$ are *extensions* of ϕ from $(\mathcal{S},\preceq)$ into $(L,\leq)$. If we also take into account proposition 12, we see that $\mathcal{P}_{TS}(\phi)$ and $\mathcal{C}_{ST}(\phi)$ are extensions of $\overline{\phi}$ from $(\mathcal{S},\preceq)$ into $(L,\leq)$. This observation implicitly contains the generalization of the first equalities in the equations (4). It also implies that the functions $\mathcal{P}_{TS}(\cdot)$, $\mathcal{C}_{ST}(\cdot)$, $\mathcal{P}_{ST}(\cdot)$ and $\mathcal{C}_{TS}(\cdot)$ are injective, and therefore invertible on their range. The transforms $\mathcal{P}_{ST}(\phi)$ and $\mathcal{C}_{TS}(\phi)$ are generalizations towards bounded posets of the *order norm extensions* introduced by Cappelle [16], and the well-known *extensions of Barlow and Wu* [3].

Let us now look for a generalization of the second equalities in the equations (4). It turns out that such a generalization is indeed possible. In proposition 13, we give a necessary condition for the equality of the functions $\mathcal{P}_{TS}(\cdot)$ and $\mathcal{C}_{ST}(\cdot)$, and $\mathcal{P}_{ST}(\cdot)$ and $\mathcal{C}_{TS}(\cdot)$. In theorem 4, we give sufficient conditions. Both results are then combined into corollary 2.

Proposition 13 *Let $(L,\leq)$ be a bounded lattice. Let T be a t-norm and S a t-conorm on $(L,\leq)$. There is a special necessary condition in order that the transforms $\mathcal{P}_{TS}(\cdot)$ and $\mathcal{C}_{ST}(\cdot)$, and $\mathcal{P}_{ST}(\cdot)$ and $\mathcal{C}_{TS}(\cdot)$ would be identical.*

(i) $(\forall\phi, \phi \text{ two-valued structure function})(\mathcal{P}_{TS}(\phi) = \mathcal{C}_{ST}(\phi)) \Rightarrow \begin{cases} (L, \leq) \text{ is distributive} \\ T = \frown \text{ and } S = \smile. \end{cases}$

(ii) $(\forall\phi, \phi \text{ two-valued structure function})(\mathcal{P}_{ST}(\phi) = \mathcal{C}_{TS}(\phi)) \Rightarrow \begin{cases} (L, \leq) \text{ is distributive} \\ T = \frown \text{ and } S = \smile. \end{cases}$

Proof. We shall prove (ii). The proof of (i) is completely analogous. Assume that the transforms $\mathcal{P}_{ST}(\cdot)$ and $\mathcal{C}_{TS}(\cdot)$ are indeed identical. First of all consider the system S with three components C_1, C_2 and C_3, that is a series connection of C_1 and the parallel connection of C_2 and C_3. The corresponding (coherent) two-valued structure function ϕ has two minimal paths: $n_p = 2$, $P_1 = \{1, 2\}$ and $P_2 = \{1, 3\}$; and two minimal cuts: $n_c = 2$, $C_1 = \{1\}$ and $C_2 = \{2, 3\}$. Therefore, for any $(\lambda_1, \lambda_2, \lambda_3)$ in L^3:

$$\mathcal{P}_{ST}(\phi)(\lambda_1, \lambda_2, \lambda_3) = S(T(\lambda_1, \lambda_2), T(\lambda_1, \lambda_3))$$

and

$$\mathcal{C}_{TS}(\phi)(\lambda_1, \lambda_2, \lambda_3) = T(\lambda_1, S(\lambda_2, \lambda_3)).$$

Since, by assumption $\mathcal{P}_{ST}(\phi) = \mathcal{C}_{TS}(\phi)$, we conclude that T is distributive with respect to S. This in turn implies that $S = \smile$ (see, for instance, [18, 25]).

Next, consider the consider the system S with three components C_1, C_2 and C_3, that is a parallel connection of C_1 and the series connection of C_2 and C_3. The corresponding (coherent) two-valued structure function ϕ has two minimal paths: $n_p = 2$, $P_1 = \{1\}$ and $P_2 = \{2, 3\}$; and two minimal cuts: $n_c = 2$, $C_1 = \{1, 2\}$ and $C_2 = \{1, 3\}$. Therefore, for any $(\lambda_1, \lambda_2, \lambda_3)$ in L^3:

$$\mathcal{P}_{ST}(\phi)(\lambda_1, \lambda_2, \lambda_3) = S(\lambda_1, T(\lambda_2, \lambda_3))$$

and

$$\mathcal{C}_{TS}(\phi)(\lambda_1, \lambda_2, \lambda_3) = T(S(\lambda_1, \lambda_2), S(\lambda_1, \lambda_3)).$$

Since by assumption $\mathcal{P}_{ST}(\phi) = \mathcal{C}_{TS}(\phi)$, we conclude that S is distributive with respect to T. This in turn implies that $T = \frown$ (see, for instance, [18, 25]).

In conclusion, since we find that $T = \frown$ and $S = \smile$ are mutually distributive, we may also conclude that the bounded lattice $(L, \leq)$ is distributive. □

Theorem 4 *If the bounded lattice $(L, \leq)$ is distributive, then for any two-valued structure function ϕ: $\mathcal{P}_{\smile\frown}(\phi) = \mathcal{C}_{\frown\smile}(\phi)$ and $\mathcal{P}_{\frown\smile}(\phi) = \mathcal{C}_{\smile\frown}(\phi)$.*

Proof. As an example, we shall prove that $\mathcal{P}_{\smile\frown}(\phi) = \mathcal{C}_{\frown\smile}(\phi)$. The proof of the other equality is completely analogous. Consider an arbitrary $(\lambda_1, \dots, \lambda_n)$ in L^n. Then we must prove that $\mathcal{P}_{\smile\frown}(\phi)(\lambda_1, \dots, \lambda_n) = \mathcal{C}_{\frown\smile}(\phi)(\lambda_1, \dots, \lambda_n)$. Let us first prove that $\mathcal{P}_{\smile\frown}(\phi)(\lambda_1, \dots, \lambda_n) \leq \mathcal{C}_{\frown\smile}(\phi)(\lambda_1, \dots, \lambda_n)$. We start with the following well-known property of minimal paths and cuts, which is a corollary of proposition 1(iii):

$$(\forall r \in \{1, \dots, n_p\})(\forall s \in \{1, \dots, n_c\})(P_r \cap C_s \neq \emptyset).$$

This immediately implies that

$$(\forall r \in \{1, \dots, n_p\})(\forall s \in \{1, \dots, n_c\}) \left(\inf_{i \in P_r} \lambda_i \leq \sup_{i \in C_s} \lambda_i \right),$$

and therefore also, taking into account the definition of infimum and supremum

$$\sup_{1 \leq r \leq n_p} \inf_{i \in P_r} \lambda_i \leq \inf_{1 \leq s \leq n_c} \sup_{i \in C_s} \lambda_i,$$

or equivalently $\mathcal{P}_{\smile\frown}(\phi)(\lambda_1,\ldots,\lambda_n) \leq \mathcal{C}_{\frown\smile}(\phi)(\lambda_1,\ldots,\lambda_n)$. It now remains to be proven that $\mathcal{P}_{\smile\frown}(\phi)(\lambda_1,\ldots,\lambda_n) \geq \mathcal{C}_{\frown\smile}(\phi)(\lambda_1,\ldots,\lambda_n)$. Let us call m_s the number of elements of C_s, $s = 1,\ldots,n_c$. Remark that $1 \leq m_s \leq n$. We shall now construct sets of components in the following way. Each set is formed by selecting one component from every minimal cut C_s of ϕ and bringing the selected components together. Because the minimal cuts are not necessarily disjoint, it is possible that two component sets formed in this way, are equal. Let us assume that there are N different component sets that can be formed in this way. Of course, $1 \leq N \leq \prod_{1\leq s\leq n_c} m_s$. Let us call these component sets A_ℓ, $\ell = 1,\ldots,N$. Taking into account the distributivity of $\frown$ with respect to $\smile$, it is easily verified that

$$\mathcal{C}_{\frown\smile}(\phi)(\lambda_1,\ldots,\lambda_n) = \inf_{1\leq s\leq n_c} \sup_{i\in C_s} \lambda_i = \sup_{1\leq \ell\leq N} \inf_{j\in A_\ell} \lambda_j.$$

Furthermore, A_k is a path of ϕ, $k = 1,\ldots,N$. Indeed, borrowing the notations from proposition 11, we find that, putting $p_{(L,\leq)}(A_k) = (\delta_1,\ldots,\delta_n)$

$$\mathcal{C}_{\frown\smile}(\phi)\cdot p_{(L,\leq)}(A_k) = \sup_{1\leq \ell\leq N} \inf_{j\in A_\ell} \delta_j \geq \inf_{j\in A_k} \delta_j = 1_L.$$

From proposition 11(vii) we therefore deduce that A_k is a path of ϕ. This implies that there exists at least one minimal path $P_{r(k)}$ of ϕ such that $P_{r(k)} \subseteq A_k$. We may therefore write that

$$\mathcal{C}_{\frown\smile}(\phi)(\lambda_1,\ldots,\lambda_n) = \sup_{1\leq k\leq N} \inf_{j\in A_k} \lambda_j \leq \sup_{1\leq k\leq N} \inf_{j\in P_{r(k)}} \lambda_j.$$

It is obvious that $\{\, r(k) \mid 1 \leq k \leq N \,\} \subseteq \{1,\ldots,n_p\}$, whence

$$\sup_{1\leq k\leq N} \inf_{j\in P_{r(k)}} \lambda_j \leq \sup_{1\leq r\leq n_p} \inf_{j\in P_r} \lambda_j = \mathcal{P}_{\smile\frown}(\phi)(\lambda_1,\ldots,\lambda_n).$$

We may therefore conclude that $\mathcal{P}_{\smile\frown}(\phi)(\lambda_1,\ldots,\lambda_n) \geq \mathcal{C}_{\frown\smile}(\phi)(\lambda_1,\ldots,\lambda_n)$. $\square$

Corollary 2 *Let $(L,\leq)$ be a bounded lattice. Let T be a t-norm and S a t-conorm on $(L,\leq)$. We have the following characterization.*

(i) $(\forall\phi,\phi$ *two-valued structure function*$)(\mathcal{P}_{TS}(\phi) = \mathcal{C}_{ST}(\phi)) \Leftrightarrow \begin{cases} T = \frown \text{ and } S = \smile \\ (L,\leq) \text{ is distributive} \end{cases}$

(ii) $(\forall\phi,\phi$ *two-valued structure function*$)(\mathcal{P}_{ST}(\phi) = \mathcal{C}_{TS}(\phi)) \Leftrightarrow \begin{cases} T = \frown \text{ and } S = \smile \\ (L,\leq) \text{ is distributive.} \end{cases}$

This result also gives us an *a posteriori* explanation why the decomposition in minimal paths $\mathcal{P}_{\vee\wedge}(\phi)$ and the decomposition in minimal cuts $\mathcal{C}_{\wedge\vee}(\phi)$ of ϕ are equal, as theorem 1 assures us. Indeed, the Boolean chain $(\mathcal{S},\preceq)$ is in particular a bounded and distributive lattice, its meet $\wedge$ is the only triangular norm on $(\mathcal{S},\preceq)$, and its join $\vee$ is the only triangular conorm on $(\mathcal{S},\preceq)$.

5.2 Possibilistic structure functions

Let us now exploit the ideas developed in the previous subsection, and try to find easy ways to calculate a possibilistic structure function $\widetilde{\phi}_T$.

Theorem 5 may be considered as a possibilistic analogon of the probabilistic method for calculating φ if ϕ is known, referred to at the end of the previous section. Thus, knowing the minimal paths respectively cuts of the two-valued structure function ϕ, it becomes straightforward to calculate $\widetilde{\phi}_T(t_1,\ldots,t_n)\cdot \textit{work}$ respectively $\widetilde{\phi}_T(t_1,\ldots,t_n)\cdot \textit{fail}$.

Theorem 5 *Let $(L, \leq)$ be a bounded lattice, and let T be a triangular norm on $(L, \leq)$ that T is distributive with respect to $\smile$. For any $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$:*

(i) $\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{work} = \mathcal{P}_{\smile T}(\phi)(t_1(\mathit{work}), \ldots, t_n(\mathit{work})) = \sup_{1 \leq r \leq n_p} T_{i \in P_r} t_i(\mathit{work})$;

(ii) $\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{fail} = \mathcal{C}_{\smile T}(\phi)(t_1(\mathit{fail}), \ldots, t_n(\mathit{fail})) = \sup_{1 \leq s \leq n_c} T_{i \in C_s} t_i(\mathit{fail})$.

Proof. Let us first prove (i). Consider the mapping

$$W\colon \mathcal{S}^n \to \wp(\{1, \ldots, n\})\colon (\nu_1, \ldots, \nu_n) \mapsto \{\, k \mid \nu_k = \mathit{work} \,\}$$

Remark that for any $(\nu_1, \ldots, \nu_n)$ in $\mathcal{S}^n$, $p_{(\mathcal{S}, \preceq)} \cdot W(\nu_1, \ldots, \nu_n) = (\nu_1, \ldots, \nu_n)$. We therefore have, taking into account proposition 1(i)–(ii), that for any $(\nu_1, \ldots, \nu_n)$ in $\mathcal{S}^n$,

$$\begin{aligned}
&\phi(\nu_1, \ldots, \nu_n) = \mathit{work} \\
&\quad \Leftrightarrow W(\nu_1, \ldots, \nu_n) \text{ is a path of } \phi \\
&\quad \Leftrightarrow (\exists r \in \{1, \ldots, n_p\})(P_r \subseteq W(\nu_1, \ldots, \nu_n)) \\
&\quad \Leftrightarrow (\exists r \in \{1, \ldots, n_p\})(\forall i \in P_r)(\nu_i = \mathit{work}) \\
&\quad \Leftrightarrow (\exists r \in \{1, \ldots, n_p\})((\forall i \in P_r)(\nu_i = \mathit{work}) \text{ and } (\forall \ell \in \{1, \ldots, n\} \setminus P_r)(\nu_\ell \in \mathcal{S})).
\end{aligned}$$

We may therefore write

$$\begin{aligned}
\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{work} &= \sup_{\phi(\nu_1, \ldots, \nu_n) = \mathit{work}} T_{k=1}^n t_k(\nu_k) \\
&= \sup_{1 \leq r \leq n_p} \sup_{P_r \subseteq W(\nu_1, \ldots, \nu_n)} T_{k=1}^n t_k(\nu_k)
\end{aligned}$$

and taking into account the commutativity and associativity of T, and the distributivity of T with respect to $\smile$,

$$= \sup_{1 \leq r \leq n_p} T(T_{i \in P_r} t_i(\mathit{work}), T_{\ell \in \{1, \ldots, n\} \setminus P_r} \sup_{\nu_\ell \in \mathcal{S}} t_\ell(\nu_\ell))$$

and since t_k is sup-normal, $k = 1, \ldots, n$, also taking into account the boundary behaviour of t-norms

$$= \sup_{1 \leq r \leq n_p} T_{i \in P_r} t_i(\mathit{work}).$$

This completes the proof of (i). The proof of (ii) now immediately follows from (i), proposition 12 and corollary 1. Indeed

$$\begin{aligned}
\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{fail} &= \overline{\widetilde{\phi}_T(t_1, \ldots, t_n)} \cdot \mathit{work} \\
&= \widetilde{\overline{\phi}}_T(\overline{t_1}, \ldots, \overline{t_n}) \cdot \mathit{work} \\
&= \mathcal{P}_{\smile T}(\overline{\phi})(\overline{t_1}(\mathit{work}), \ldots, \overline{t_n}(\mathit{work})) \\
&= \mathcal{C}_{\smile T}(\phi)(t_1(\mathit{fail}), \ldots, t_n(\mathit{fail})). \quad \square
\end{aligned}$$

Using this result, and going back to section 4, we may now write for the possibilistic reliability π_{x_s} of the system S, that if the component state mappings x_k are $(\Pi_{\mathrm{Par}_S}, T)$-independent:

$$\pi_{x_s}(work) = \sup_{1 \le r \le n_p} T_{i \in P_r} \pi_{x_i}(work)$$

and

$$\pi_{x_s}(fail) = \sup_{1 \le s \le n_c} T_{i \in C_s} \pi_{x_i}(fail),$$

where $(L, \le, T)$ is now assumed to be a complete lattice with t-norm. This makes the calculation of π_{x_s} fairly easy[8].

Theorem 5 is the basis for a further investigation of possibilistic structure functions. First of all, it is fairly obvious that $\wedge$ and $\vee$ are two-valued structure functions, representing respectively a series and a parallel connection of components. Theorem 1 tells us that any system can be represented as a parallel connection of series connections of components, or as a series connection of parallel connections of components. Equivalently, it tells us that any two-valued structure function can be written down using only the two-valued structure functions $\wedge$ and $\vee$. In what follows, we intend to prove a similar result for possibilistic structure functions. Let us therefore take a look at the possibilistic structure functions[9] associated with $\wedge$ and $\vee$.

Proposition 14 *Let $(L, \le)$ be a bounded lattice, and let T be a triangular norm on $(L, \le)$ that is distributive with respect to $\smile$. Let t_1 and t_2 be arbitrary elements of $\tilde{\mathcal{S}}$. Then*

$$\begin{cases} (t_1 \,\widetilde{\wedge}_T\, t_2) \cdot work = T(t_1(work), t_2(work)) \\ (t_1 \,\widetilde{\wedge}_T\, t_2) \cdot fail = t_1(fail) \smile t_2(fail) \end{cases}$$

and

$$\begin{cases} (t_1 \,\widetilde{\vee}_T\, t_2) \cdot work = t_1(work) \smile t_2(work) \\ (t_1 \,\widetilde{\vee}_T\, t_2) \cdot fail = T(t_1(fail), t_2(fail)). \end{cases}$$

Proof. We shall give a proof for $\widetilde{\wedge}_T$. The proof for $\widetilde{\vee}_T$ is completely analogous. The structure function $\wedge$ has one minimal path: $n_p = 1$ and $P_1 = \{1, 2\}$. It has two minimal cuts: $n_c = 2$, $C_1 = \{1\}$ and $C_2 = \{2\}$. Taking this into account, the given expressions for $t_1 \,\widetilde{\wedge}_T\, t_2 \cdot work$ and $t_1 \,\widetilde{\wedge}_T\, t_2 \cdot fail$ follow immediately from theorem 5. □

Now, we make a significant conceptual step: we shall, at least formally, consider $\tilde{\mathcal{S}}$ as a set of states. It turns out that, just as the states in $\mathcal{S}$ are ordered by the relation $\preceq$, the 'states' in $\tilde{\mathcal{S}}$ can be ordered by the relation $\tilde{\preceq}$, introduced in the following definition.

Definition 10 *Let $(L, \le)$ be a bounded lattice. We define the following binary relation $\tilde{\preceq}$ on $\tilde{\mathcal{S}}$. For any t_1 and t_2 in $\tilde{\mathcal{S}}$:*

$$t_1 \,\tilde{\preceq}\, t_2 \Leftrightarrow \begin{cases} t_1(work) \le t_2(work) \\ t_1(fail) \ge t_2(fail). \end{cases}$$

The relation $\tilde{\preceq}$ has the immediate interpretation 'is less reliable than'. The structure $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is investigated in the following proposition.

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[8] As a special case of these formulas, we find the results formerly proven by Cai et al. in [13], theorem 5.4.

[9] If we combine proposition 14 with theorem 3 and definition 7, we find a result that is a generalization of theorems 5.2 and 5.3 proven by Cai et al. in [13].

Proposition 15 (i) *If $(L, \leq)$ is a bounded lattice, then $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is a bounded poset with top $\widetilde{work}$ and bottom $\widetilde{fail}$. Moreover, for any t_1 and t_2 in $\tilde{\mathcal{S}}$: $t_1 \tilde{\preceq} t_2 \Leftrightarrow \overline{t_2} \tilde{\preceq} \overline{t_1}$, which implies that $\bar{\cdot}$ is an involutive dual order-automorphism of $(\tilde{\mathcal{S}}, \tilde{\preceq})$.*

(ii) *Moreover, if $(L, \leq)$ is a distributive bounded lattice, then $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is a distributive bounded lattice, with meet $\tilde{\wedge}_\frown$ and join $\tilde{\vee}_\frown$.*

Proof. First of all, assume that $(L, \leq)$ is a bounded lattice. It is obvious from definition 10 that $\tilde{\preceq}$ is a reflexive, transitive and antisymmetric relation, and therefore a partial order relation on $\tilde{\mathcal{S}}$. Since for any t in $\tilde{\mathcal{S}}$, $0_L \leq t(work) \leq 1_L$ and $1_L \geq t(fail) \geq 0_L$, we find that $\widetilde{fail} \tilde{\preceq} t \tilde{\preceq} \widetilde{work}$, which implies that $\widetilde{fail}$ is the bottom and $\widetilde{work}$ the top of the partially ordered set $(\tilde{\mathcal{S}}, \tilde{\preceq})$. The second part of the proof of (i) is trivial.

Next, assume that $(L, \leq)$ is a distributive bounded lattice. Let us now show that $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is also a distributive bounded lattice, with meet $\tilde{\wedge}_\frown$ and join $\tilde{\vee}_\frown$. Consider an arbitrary *finite* nonempty subset A of $\tilde{\mathcal{S}}$. Let us introduce the subsets $w(A) \stackrel{\text{def}}{=} \{\, t(work) \mid t \in A \,\}$ and $f(A) \stackrel{\text{def}}{=} \{\, t(fail) \mid t \in A \,\}$ of L, and define $i(A) \stackrel{\text{def}}{=} \{(fail, \sup f(A)), (work, \inf w(A))\}$. It is easily verified that $i(A) \in \tilde{\mathcal{S}}$. Indeed, since $(L, \leq)$ is distributive and A is finite,

$$\begin{aligned}
\inf w(A) \smile \sup f(A) &= \left(\inf_{t \in A} t(work)\right) \smile \left(\sup_{s \in A} s(fail)\right) \\
&= \inf_{t \in A} \left(t(work) \smile \sup_{s \in A} s(fail)\right) \\
&= \inf_{t \in A} 1_L = 1_L,
\end{aligned}$$

whence $i(A) \in \tilde{\mathcal{S}}$. It is also obvious that $i(A)$ is a lower bound of A. Let s be another lower bound of A. This implies that $(\forall t \in A)(s(work) \leq t(work))$ and $(\forall t \in A)(s(fail) \geq t(fail))$. Therefore, $s(work)$ is a lower bound for $w(A)$, whence, by definition of infimum, $s(work) \leq \inf w(A)$. Similarly, we find that $s(fail) \geq \sup f(A)$. We conclude that $s \tilde{\preceq} i(A)$, which implies that $i(A)$ is the infimum of A. In a completely analogous way, it can be shown that A has a supremum. Therefore, $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is a lattice. Let us now show that $\tilde{\wedge}_\frown$ is the meet of $(\tilde{\mathcal{S}}, \tilde{\preceq})$. Indeed, consider arbitrary t_1 and t_2 in $\tilde{\mathcal{S}}$. From the course of reasoning above, combined with proposition 14 for $t = \frown$, we deduce that

$$\inf(t_1, t_2) = \{(fail, t_1(fail) \smile t_2(fail)), (work, t_1(work) \frown t_2(work))\} = t_1 \tilde{\wedge}_\frown t_2$$

which implies that $\tilde{\wedge}_\frown$ is indeed the meet of $(\tilde{\mathcal{S}}, \tilde{\preceq})$. In a completely analogous way, it may be shown that $\tilde{\vee}_\frown$ is the join of $(\tilde{\mathcal{S}}, \tilde{\preceq})$. It now remains to be shown that $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is distributive. To this end, consider arbitrary t_1, t_2 and t_3 in $\tilde{\mathcal{S}}$. Then, taking into account proposition 14 and the fact that $(L, \leq)$ is by assumption distributive:

$$\begin{aligned}
(t_1 \tilde{\wedge}_\frown (t_2 \tilde{\vee}_\frown t_3)) \cdot work &= t_1(work) \frown (t_2(work) \smile t_3(work)) \\
&= (t_1(work) \frown t_2(work)) \smile (t_1(work) \frown t_3(work)) \\
&= ((t_1 \tilde{\wedge}_\frown t_2) \tilde{\vee}_\frown (t_1 \tilde{\wedge}_\frown t_3)) \cdot work
\end{aligned}$$

and similarly $(t_1 \tilde{\wedge}_\frown (t_2 \tilde{\vee}_\frown t_3)) \cdot fail = ((t_1 \tilde{\wedge}_\frown t_2) \tilde{\vee}_\frown (t_1 \tilde{\wedge}_\frown t_3)) \cdot fail$, whence

$$t_1 \tilde{\wedge}_\frown (t_2 \tilde{\vee}_\frown t_3)) = (t_1 \tilde{\wedge}_\frown t_2) \tilde{\vee}_\frown (t_1 \tilde{\wedge}_\frown t_3).$$

We may therefore conclude that the lattice $(\tilde{\mathcal{S}}, \tilde{\preceq})$ is distributive [8]. □

Theorem 6 indicates that if we formally consider $(\tilde{\mathcal{S}}, \tilde{\preceq})$ as a set of states, then we may also consider $\tilde{\phi}_T$ as a structure function, in the sense of definition 1. This result is an *a posteriori* justification for our calling $\tilde{\phi}_T$ a possibilistic *structure function*.

Theorem 6 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. Then $\widetilde{\phi}_T$ is a structure function from $(\tilde{\mathcal{S}}^n, \widetilde{\preceq}^n)$ to $(\tilde{\mathcal{S}}, \widetilde{\preceq})$, i.e.,*

(i) $\widetilde{\phi}_T(\widetilde{fail}, \ldots, \widetilde{fail}) = \widetilde{fail}$ *and* $\widetilde{\phi}_T(\widetilde{work}, \ldots, \widetilde{work}) = \widetilde{work}$*;*

(ii) $\widetilde{\phi}_T$ *is isotonic.*

Furthermore, the dual $\overline{\widetilde{\phi}_T}$ *of* $\widetilde{\phi}_T$ *is also a structure function from* $(\tilde{\mathcal{S}}^n, \widetilde{\preceq}^n)$ *to* $(\tilde{\mathcal{S}}, \widetilde{\preceq})$.

Proof. It is obvious from the properties of the operator $\overline{\cdot}$ on $\tilde{\mathcal{S}}$ that $\overline{\widetilde{\phi}_T}$ is a structure function if $\widetilde{\phi}_T$ is. It must therefore only be proven that $\widetilde{\phi}_T$ is a structure function. Let us first prove (i). We shall prove $\widetilde{\phi}_T(\widetilde{fail}, \ldots, \widetilde{fail}) = \widetilde{fail}$. The proof the other boundary condition is completely analogous. Taking into account theorem 5 and the definition of $\widetilde{fail}$, we find that

$$\widetilde{\phi}_T(\widetilde{fail}, \ldots, \widetilde{fail}) \cdot work = \sup_{1 \leq r \leq n_p} T_{i \in P_r} \widetilde{fail}(work) = 0_L$$

and

$$\widetilde{\phi}_T(\widetilde{fail}, \ldots, \widetilde{fail}) \cdot fail = \sup_{1 \leq s \leq n_c} T_{i \in C_s} \widetilde{fail}(fail) = 1_L.$$

We shall now prove (ii). Consider arbitrary $(t_1, \ldots, t_n)$ and $(\tau_1, \ldots, \tau_n)$ in $\tilde{\mathcal{S}}^n$, and assume that $(t_1, \ldots, t_n) \widetilde{\preceq}^n (\tau_1, \ldots, \tau_n)$, or equivalently, $(\forall k \in \{1, \ldots, n\})(t_k \widetilde{\preceq} \tau_k)$, which is by definition equivalent with

$$(\forall k \in \{1, \ldots, n\})(t_k(work) \leq \tau_k(work) \text{ and } \tau_k(fail) \leq t_k(fail)). \tag{5}$$

Taking into account theorem 5(i), equation (5) and the isotonicity of T and sup, we find that, with obvious notations,

$$\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot work = \sup_{1 \leq r \leq n_p} T_{i \in P_r} t_i(work) \leq \sup_{1 \leq r \leq n_p} T_{i \in P_r} \tau_i(work) = \widetilde{\phi}_T(\tau_1, \ldots, \tau_n) \cdot work.$$

Taking into account theorem 5(ii), equation (5) and the isotonicity of T and of sup, we also find that, with obvious notations,

$$\widetilde{\phi}_T(\tau_1, \ldots, \tau_n) \cdot fail = \sup_{1 \leq s \leq n_c} T_{i \in C_s} \tau_i(fail) \leq \sup_{1 \leq s \leq n_c} T_{i \in C_s} t_i(fail) = \widetilde{\phi}_T(t_1, \ldots, t_n) \cdot fail,$$

whence, finally, $\widetilde{\phi}_T(t_1, \ldots, t_n) \widetilde{\preceq} \widetilde{\phi}_T(\tau_1, \ldots, \tau_n)$. □

Proposition 16 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. Then $\widetilde{\wedge}_T$ is a triangular norm and $\widetilde{\vee}_T$ a triangular conorm on $(\tilde{\mathcal{S}}, \widetilde{\preceq})$.*

Proof. As an example, let us prove that $\widetilde{\wedge}_T$ is a triangular norm on $(\tilde{\mathcal{S}}, \widetilde{\preceq})$. For a start, we deduce from theorem 6 that $\widetilde{\wedge}_T$ is isotonic. From proposition 14 and the associativity of both T and $\smile$, we deduce that $\widetilde{\wedge}_T$ is associative. Proposition 14 combined with the commutativity of T and $\smile$ tells us that $\widetilde{\wedge}_T$ is commutative. Combined with the boundary behaviour of the triangular norm T and the boundary behaviour of the triangular conorm $\smile$, this proposition also tells us that $\widetilde{\wedge}_T$ satisfies the boundary condition for triangular norms. □

Let us now assume that $(L, \leq)$ is a bounded lattice and T a t-norm on $(L, \leq)$ that is distributive with respect to $\smile$. Then the previous results tell us that $(\tilde{\mathcal{S}}, \widetilde{\preceq})$ is certainly a bounded poset, and that $\widetilde{\wedge}_T$ is a triangular norm and $\widetilde{\vee}_T$ a triangular conorm on $(\tilde{\mathcal{S}}, \widetilde{\preceq})$. We are therefore able to apply definition 9, and import the minimal path and cut decompositions of ϕ into $(\tilde{\mathcal{S}}, \widetilde{\preceq})$.

Definition 11 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. We define the $\tilde{\mathcal{S}}^n - \tilde{\mathcal{S}}$-mappings $\widetilde{\phi}_T^p$ and $\widetilde{\phi}_T^c$ as follows. Let $(t_1, \ldots, t_n)$ be an arbitrary element of $\tilde{\mathcal{S}}^n$.*

(i) $\widetilde{\phi}_T^p(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \mathcal{P}_{\widetilde{\vee}_T \widetilde{\wedge}_T}(\phi)(t_1, \ldots, t_n) = \underset{1 \leq r \leq n_p}{\widetilde{\vee}_T} \; \underset{i \in P_r}{\widetilde{\wedge}_T} \; t_i$.

(ii) $\widetilde{\phi}_T^c(t_1, \ldots, t_n) \stackrel{\text{def}}{=} \mathcal{C}_{\widetilde{\wedge}_T \widetilde{\vee}_T}(\phi)(t_1, \ldots, t_n) = \underset{1 \leq s \leq n_c}{\widetilde{\wedge}_T} \; \underset{i \in C_s}{\widetilde{\vee}_T} \; t_i$.

Of course, $\widetilde{\phi}_T^p$ and $\widetilde{\phi}_T^c$ are structure functions from $(\tilde{\mathcal{S}}^n, \widetilde{\preceq}^n)$ to $(\tilde{\mathcal{S}}, \widetilde{\preceq})$.

The following proposition tells us that there exists a close relationship between the path and cut transforms for $(\tilde{\mathcal{S}}, \widetilde{\preceq})$, and the path and cut transforms for $(L, \leq)$ introduced in the previous subsection.

Proposition 17 *Let $(L, \leq)$ be a bounded lattice and let T be a triangular norm on $(L, \leq)$ that is distributive with respect to $\smile$. Let $(t_1, \ldots, t_n)$ be an arbitrary element of $\tilde{\mathcal{S}}^n$. Then*

$$\begin{cases} \widetilde{\phi}_T^p(t_1, \ldots, t_n) \cdot \mathit{work} = \mathcal{P}_{\smile T}(\phi)(t_1(\mathit{work}), \ldots, t_n(\mathit{work})) = \sup\limits_{1 \leq r \leq n_p} T_{i \in P_r} t_i(\mathit{work}) \\ \widetilde{\phi}_T^p(t_1, \ldots, t_n) \cdot \mathit{fail} = \mathcal{P}_{T \smile}(\phi)(t_1(\mathit{fail}), \ldots, t_n(\mathit{fail})) = T_{1 \leq r \leq n_p} \sup\limits_{i \in P_r} t_i(\mathit{fail}) \end{cases}$$

and

$$\begin{cases} \widetilde{\phi}_T^c(t_1, \ldots, t_n) \cdot \mathit{work} = \mathcal{C}_{T \smile}(\phi)(t_1(\mathit{work}), \ldots, t_n(\mathit{work})) = T_{1 \leq s \leq n_c} \sup\limits_{i \in C_s} t_i(\mathit{work}) \\ \widetilde{\phi}_T^c(t_1, \ldots, t_n) \cdot \mathit{fail} = \mathcal{C}_{\smile T}(\phi)(t_1(\mathit{fail}), \ldots, t_n(\mathit{fail})) = \sup\limits_{1 \leq s \leq n_c} T_{i \in C_s} t_i(\mathit{fail}) \end{cases}$$

Proof. As an example, we shall prove $\widetilde{\phi}_T^p(t_1, \ldots, t_n) \cdot \mathit{work} = \mathcal{P}_{\smile T}(\phi)(t_1(\mathit{work}), \ldots, t_n(\mathit{work}))$. The proofs of the other equalities are completely similar. Taking into account the definition of $\widetilde{\phi}_T^p$, we have for any $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$ that

$$\widetilde{\phi}_T^p(t_1, \ldots, t_n) \cdot \mathit{work} = \left(\underset{1 \leq r \leq n_p}{\widetilde{\vee}_T} \; \underset{i \in P_r}{\widetilde{\wedge}_T} \; t_i \right) \cdot \mathit{work}$$

and, taking into account proposition 14,

$$\begin{aligned} &= \sup_{1 \leq r \leq n_p} \left(\underset{i \in P_r}{\widetilde{\wedge}_T} \; t_i \right) \cdot \mathit{work} \\ &= \sup_{1 \leq r \leq n_p} T_{i \in P_r} t_i(\mathit{work}) \\ &= \mathcal{P}_{\smile T}(\phi)(t_1(\mathit{work}), \ldots, t_n(\mathit{work})). \quad \square \end{aligned}$$

This result together with theorem 5 already tells us that for arbitrary $(t_1, \ldots, t_n)$ in $\tilde{\mathcal{S}}^n$: $\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{work} = \widetilde{\phi}_T^p(t_1, \ldots, t_n) \cdot \mathit{work}$ and $\widetilde{\phi}_T(t_1, \ldots, t_n) \cdot \mathit{fail} = \widetilde{\phi}_T^c(t_1, \ldots, t_n) \cdot \mathit{fail}$. The next theorem takes this result one step further.

Theorem 7 (Decomposition theorem for possibilistic structure functions) *Let $(L,\leq)$ be a distributive bounded lattice and let T be a triangular norm on $(L,\leq)$ that is distributive with respect to $\smile$. Then $(\forall\phi,\phi$ is a two-valued structure function$)(\widetilde{\phi}_T=\widetilde{\phi}^p_T=\widetilde{\phi}^c_T)\Leftrightarrow T=\frown$.*

Proof. From the proposition above and theorem 5 we deduce that $\widetilde{\phi}^p_T=\widetilde{\phi}^c_T$ is equivalent to $\widetilde{\phi}_T=\widetilde{\phi}^p_T=\widetilde{\phi}^c_T$. Also, since $(L,\leq)$ is a distributive bounded lattice, we know from proposition 15(ii) that $(\tilde{\mathcal{S}},\tilde{\preceq})$ is a distributive bounded lattice with meet $\widetilde{\wedge}_\frown$ and join $\widetilde{\vee}_\frown$. First, assume that for any two-valued structure function ϕ, $\widetilde{\phi}^p_T=\widetilde{\phi}^c_T$, or equivalently, $\mathcal{P}_{\widetilde{\vee}_T\widetilde{\wedge}_T}(\phi)=\mathcal{C}_{\widetilde{\wedge}_T\widetilde{\vee}_T}(\phi)$. According to corollary 2, this implies that $\widetilde{\wedge}_T=\widetilde{\wedge}_\frown$ and $\widetilde{\vee}_T=\widetilde{\vee}_\frown$. Proposition 14 then tells us that $T=\frown$.

Conversely, assume that $T=\frown$. According to proposition 14, this implies that $\widetilde{\wedge}_T=\widetilde{\wedge}_\frown$ and $\widetilde{\vee}_T=\widetilde{\vee}_\frown$. Corollary 2 then tells us that $\mathcal{P}_{\widetilde{\vee}_T\widetilde{\wedge}_T}(\phi)=\mathcal{C}_{\widetilde{\wedge}_T\widetilde{\vee}_T}(\phi)$. $\square$

From this theorem, we deduce that in a distributive bounded lattice $(L,\leq)$

$$\widetilde{\phi}_\frown=\mathcal{P}_{\widetilde{\vee}_\frown\widetilde{\wedge}_\frown}(\phi)=\mathcal{C}_{\widetilde{\wedge}_\frown\widetilde{\vee}_\frown}(\phi),\tag{6}$$

or equivalently, for any $(t_1,\dots,t_n)$ in $\tilde{\mathcal{S}}^n$

$$\widetilde{\phi}_\frown(t_1,\dots,t_n)=\widetilde{\bigvee_{1\leq r\leq n_p}}_{\frown}\ \widetilde{\bigwedge_{i\in P_r}}_{\frown}\ t_i=\widetilde{\bigwedge_{1\leq s\leq n_c}}_{\frown}\ \widetilde{\bigvee_{i\in C_s}}_{\frown}\ t_i.\tag{7}$$

Furthermore, since clearly $\overline{\wedge}=\vee$ and $\overline{\vee}=\wedge$, we deduce from (7) and corollary 1 that

$$\begin{aligned}
\widetilde{\overline{\phi}}_\frown(t_1,\dots,t_n)&=\overline{\widetilde{\phi}_\frown(\overline{t_1},\dots,\overline{t_n})}\\
&=\overline{\widetilde{\bigvee_{1\leq r\leq n_p}}_{\frown}\ \widetilde{\bigwedge_{i\in P_r}}_{\frown}\ \overline{t_i}}\\
&=\widetilde{\overline{\bigvee_{1\leq r\leq n_p}}}_{\frown}\ \overline{\widetilde{\bigwedge_{i\in P_r}}_{\frown}\ \overline{t_i}}\\
&=\widetilde{\overline{\bigvee_{1\leq r\leq n_p}}}_{\frown}\ \widetilde{\overline{\bigwedge_{i\in P_r}}}_{\frown}\ \overline{\overline{t_i}}=\widetilde{\bigwedge_{1\leq r\leq n_p}}_{\frown}\ \widetilde{\bigvee_{i\in P_r}}_{\frown}\ t_i,
\end{aligned}$$

whence $\widetilde{\overline{\phi}}_\frown=\mathcal{P}_{\widetilde{\wedge}_\frown\widetilde{\vee}_\frown}(\phi)$ and similarly $\widetilde{\overline{\phi}}_\frown=\mathcal{C}_{\widetilde{\vee}_\frown\widetilde{\wedge}_\frown}(\phi)$. This may also be deduced from (6) and proposition 12. In summary, taking into account corollary 1, we find that

$$\overline{\widetilde{\phi}_\frown}=\mathcal{P}_{\widetilde{\wedge}_\frown\widetilde{\vee}_\frown}(\phi)=\mathcal{C}_{\widetilde{\vee}_\frown\widetilde{\wedge}_\frown}(\phi).\tag{8}$$

Equations (6) and (8) are a rather interesting, and in our opinion beautiful, generalization of equation (4). Equation (7) tells us that a $(L,\leq,\frown)$-possibilistic structure function can always be decomposed in terms of the $(L,\leq,\frown)$-possibilistic structure functions $\widetilde{\wedge}_\frown$ and $\widetilde{\vee}_\frown$. Furthermore, this decomposition can be derived from the minimal path of minimal cut decomposition by 'fuzzification'. Equation (2) can simply be fuzzified[10], i.e., we substitute the two-valued structure function ϕ by its associated $(L,\leq,\frown)$-possibilistic structure function $\widetilde{\phi}_\frown$, the variables t_k in $\tilde{\mathcal{S}}$ for the variables ν_k in $\mathcal{S}$, $\widetilde{\wedge}_\frown$ for $\wedge$ and $\widetilde{\vee}_\frown$ for $\vee$, and the equality in $\mathcal{S}$ by the equality in $\tilde{\mathcal{S}}$.

In conclusion, if we go back to the results of the previous section, and assume that $(L,\leq)$ is a complete Brouwerian lattice, we now find that if the components state mappings x_k are $(\Pi_{\mathrm{Par}_S},\frown)$-independent, then

$$\pi_{x_s}=\widetilde{\bigvee_{1\leq r\leq n_p}}_{\frown}\ \widetilde{\bigwedge_{i\in P_r}}_{\frown}\ \pi_{x_i}=\widetilde{\bigwedge_{1\leq s\leq n_c}}_{\frown}\ \widetilde{\bigvee_{i\in C_s}}_{\frown}\ \pi_{x_i},$$

[10] In the language of fuzzy set theory, fuzzification formally means turning a mathematical notion into its fuzzy counterpart. From the notational point of view, this is formally often accomplished by writing a tilde on top the non-fuzzy notation.

a result which is formally very closely related with its probabilistic equivalent, discussed at the end of the previous section.

5.3 An example

Let us conclude the discussion in this section with an example, that will emphasize the practical significance and relevance of the established results.

Consider a system with 5 components, whose graphical representation is depicted in figure 1. It is easily verified that for this system, we have $n = 5$, $n_p = 4$ and $n_c = 4$. Its minimal paths and cuts are given by $P_1 = \{1, 4\}$, $P_2 = \{2, 5\}$, $P_3 = \{1, 3, 5\}$, $P_4 = \{2, 3, 4\}$ and $C_1 = \{1, 2\}$, $C_2 = \{4, 5\}$, $C_3 = \{1, 3, 5\}$ $C_4 = \{2, 3, 4\}$. Theorem 1 tells us that the structure function for this system can be written as

$$\phi(\nu_1, \nu_2, \nu_3, \nu_4, \nu_5) = (\nu_1 \wedge \nu_4) \vee (\nu_2 \wedge \nu_5) \vee (\nu_1 \wedge \nu_3 \wedge \nu_5) \vee (\nu_2 \wedge \nu_3 \wedge \nu_4) \tag{9}$$

$$= (\nu_1 \vee \nu_2) \wedge (\nu_4 \vee \nu_5) \wedge (\nu_1 \vee \nu_3 \vee \nu_5) \wedge (\nu_2 \vee \nu_3 \vee \nu_4). \tag{10}$$

Let us denote by t_k the possibilistic reliability of component C_k, and by t_s the possibilistic reliability of the system. In the case of possibilistic $\frown$-independence, the results above tell us that $t_s = \widetilde{\phi}_\frown(t_1, t_2, t_3, t_4, t_5)$ and that $\widetilde{\phi}_\frown$ can be obtained in a straightforward way by 'fuzzifying' either one of the equations (9) or (10):

$$\begin{aligned} t_s &= (t_1 \widetilde{\wedge}_\frown t_4) \widetilde{\vee}_\frown (t_2 \widetilde{\wedge}_\frown t_5) \widetilde{\vee}_\frown (t_1 \widetilde{\wedge}_\frown t_3 \widetilde{\wedge}_\frown t_5) \widetilde{\vee}_\frown (t_2 \widetilde{\wedge}_\frown t_3 \widetilde{\wedge}_\frown t_4) \\ &= (t_1 \widetilde{\vee}_\frown t_2) \widetilde{\wedge}_\frown (t_4 \widetilde{\vee}_\frown t_5) \widetilde{\wedge}_\frown (t_1 \widetilde{\vee}_\frown t_3 \widetilde{\vee}_\frown t_5) \widetilde{\wedge}_\frown (t_2 \widetilde{\vee}_\frown t_3 \widetilde{\vee}_\frown t_4). \end{aligned}$$

Using these expressions, it is very easy to calculate the actual value of t_s if actual values of t_1, t_2, t_3, t_4 and t_5 are given. On the other hand, had we used theorem 3 to calculate t_s, a time-consuming calculation would have led to very complicated expressions.

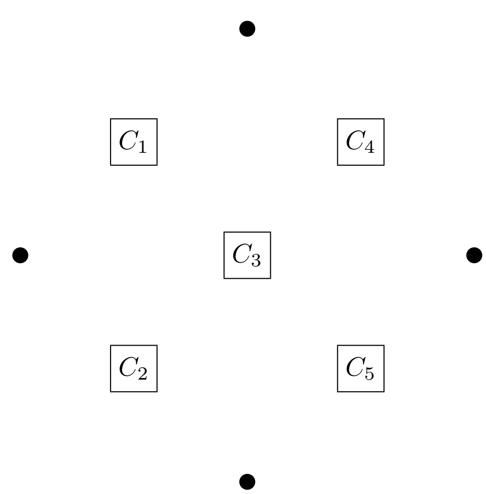

Figure 1: Graphical representation of the system, discussed in subsection 5.3

6 Conclusion

The results in this paper show that it is indeed feasible to represent possibilistic uncertainty in a reliability theoretic context. Of course, they make up only the beginning of what could be called an ambitious program, since we have only concerned ourselves here with *two-state* reliability models. Yet, the approach given here can in principle be extended towards multi-state reliability models. As is the case with probabilistic multi-state reliability theory, we foresee that a possibilistic multi-state reliability model will be much more complicated that its two-state counterpart.

It goes without saying that the results given here can in particular be applied to the complete Brouwerian chain $(L, \leq) = ([0, 1], \leq)$. We have however in this paper chosen for an approach that is as general as possible.

It should be noted that our approach to incorporating possibilistic uncertainty into reliability theory is different from Cappelle's [14, 15, 16]. Indeed, our possibilistic reliabilities are possibility distribution functions that completely characterize the available possibilistic information about the states of system and components. In a multi-state reliability model, this idea would be essentially preserved: possibilistic reliabilities would then be possibility distribution functions defined on state sets with more than two elements.

On the other hand, as Cappelle himself indicates [16], his *reliability mappings* in most cases do not completely characterize the available possibilistic information about system and components, and, however useful, yield only a partial picture. Interestingly, Cappelle's reliability mappings are immediate possibilistic equivalents of *cumulative* distribution functions in probability theory. In that theory, these cumulative distribution functions completely determine the given probabilistic information. The problem then is the following: the formal analogy between probability and possibility theory is not such that the "cumulative distribution functions" in possibility theory completely determine the possibilistic information. Indeed, as we have shown in [22, 23, 21, 18], their role in this respect is taken over [18, 21, 22, 23], their role in this respect is taken over by the possibility distribution functions described in section 2, which lie at the basis of our notion of possibilistic reliability.

We would like to point out explicitly a number of conclusions that may be drawn from the discussion in this paper. First of all, there is the interesting observation that possibilistic two-state reliability theory is to a high extent formally analogous to the probabilistic treatment. Also, the generalization of and justification for the well-known Barlow-Wu extensions in multistate reliability theory, and the natural role they play in possibilistic two-state reliability theory is a new and in our view, rather important result. And finally, the realisation that, formally, possibilistic two-state reliability theory can be considered as a specific multi-state reliability theory, and can be described using multi-valued structure functions, is in our opinion not without importance.

The formal analogy between possibilistic and probabilistic two-state reliability theory naturally leads to the question of how both uncertainty models can be combined. It will indeed often be the case that the available information about a system is a mixture of probabilistic (e.g. statistical) and possibilistic (e.g. linguistic) information. Unfortunately, little research has been done in this domain. We explicitly mention the work by Klir et al. on probability-possibility conversion methods [40], in the framework of Generalized Information Theory [38]. It is, however, at this point still an open question whether and how this and other methods of combining both kinds of uncertainty might fit into a reliability theoretic framework.

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Footnotes

1. Very recently, we have shown that consistent definitions of possibility measures can also be given on fields [19].

2. Note that there is a clear distinction between the *distribution* of a possibility measure on the one hand, and the *possibility distribution (measure)* of a possibilistic variable on the other hand.

3. In this paper, we shall not give a formal definition of systems, components and states. For a thorough discussion of the subject, we refer for instance to the doctoral dissertation of Cappelle [16].

4. We use the new term 'assembly' here, because the components of the system are considered one by one, without reference to their final place in the system, as it were before the system is assembled.

5. In this paper, we shall often use the alternative notation $f \cdot x$ for $f(x)$, in order to make complicated formulas with nested parentheses more readable.

6. We present here a formulation by Cappelle [16], which is slightly more general than the original version of Birnbaum, Esary and Saunders, because there is no coherence condition imposed on ϕ.

7. Cai et al. [13] basically use the same idea when they define the reliability and the unreliability of a component or a system.

8. As a special case of these formulas, we find the results formerly proven by Cai et al. in [13], theorem 5.4.

9. If we combine this result with theorem 3 and definition 7, we find a result that is a generalization of theorems 5.2 and 5.3 proven by Cai et al. in [13].

10. In the language of fuzzy set theory, fuzzification formally means turning a mathematical notion into its fuzzy counterpart. From the notational point of view, this is formally often accomplished by writing a tilde on top the non-fuzzy notation.