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Practical implementation of possibilistic probability mass functions

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Abstract Probability assessments of events are often linguistic in nature. We model them by means of possibilistic probabilities (a version of Zadeh's fuzzy probabilities with a behavioural interpretation) with a suitable shape for practical implementation (on a computer). Employing the tools of interval analysis and the theory of imprecise probabilities we argue that the verification of coherence for these possibilistic probabilities, the corrections of non-coherent to coherent possibilistic probabilities and their extension to other events and gambles can be performed by finite and exact algorithms. The model can furthermore be transformed into an imprecise first-order model, useful for decision making and statistical inference.

1 Introduction

Consider a football match in which the three possible outcomes are win (w), draw (d) and loss (l) for the home team. Suppose we have the following probability judgements for a specific match: win is likely to occur, draw and loss both have a chance of about $\frac{1}{4}$.

This is an example of a situation which occurs very often: an event (random variable) is assumed to have a 'true' probability (distribution) but the modeller is uncertain about what it is because the only information about the true probability available to him is linguistic and imprecise in nature. The modeller's uncertainty is called *second-order uncertainty*, to distinguish it from the first-order uncertainty, which is represented by the ideal model. For the kind of situations where the second-order uncertainty is linguistic, Zadeh proposed using so-called fuzzy probabilities [19]. Fuzzy probabilities can be interpreted as a special type of hierarchical uncertainty model. Hierarchical models are quite common in uncertainty theory. They arise whenever there is some 'ideal' uncertainty model that represents the uncertainty about a phenomenon of interest, but the modeller is uncertain about what it is. Probably the best-known hierarchical model is the Bayesian one [1, 2, 10], where second-order uncertainty is represented by a probability measure. This is often unrealistic in problems where the modeller does not have very specific information about the correct first-order model. Such is for instance the case in the example mentioned above, where we propose to use possibilistic probabilities to model the linguistic probability assessments. Possibilistic probabilities can be interpreted as a model where the second-order uncertainty is represented by a possibility measure, which is a special type of coherent imprecise probability model [7]. A justification for using a possibility measure for modelling linguistic assessments can be found in [4, 16, 18].

Two types of possibilistic second-order models can be distinguished: *global* models, as in [17], where the second-order uncertainty concerns an unknown probability *measure*; and *local* models, as in [4], where the uncertainty concerns the probability of particular events or the prevision of certain gambles. A local model is typically the result of linguistic probability assessments. This observation led De Cooman, in [4], to start with a general possibilistic local model and investigate under what conditions it can be represented by a possibilistic global second-order uncertainty model. The reason for considering such a global representation is that it can be transformed, using the methods described in [17], into a behaviourally equivalent first-order imprecise probability model, which in its turn can be used as prior information for decision making and statistical reasoning. Let us give a brief exposition of De Cooman's theory, of which the present work is a logical continuation in the direction of its application to practical problems. More details can be found in [4–6, 8].

2 Possibilistic previsions

Assume that a subject wants to model his beliefs about a certain phenomenon of interest. His *possibility space* Ω is the non-empty set of the mutually exclusive, relevant states ω of the world. A bounded real-valued function X on Ω is called a

Send offprint requests to: Gert de Cooman *gamble*. It can be interpreted as a (possibly negative) reward that is uncertain, because it depends on ω, the (unknown) state of the world. Special gambles are (indicator functions of) events, or subsets of Ω, which only assume the value 0 and 1; and the constant gambles, which will be denoted by the unique value they assume in $\mathbb{R}$. The set of all gambles is denoted by $\mathcal{L}(\Omega)$. A positive linear functional P on $\mathcal{L}(\Omega)$ with $P(1) = 1$ is called a *linear prevision*. It is a prevision in the sense of de Finetti [9], and $P(X)$ can be interpreted as a 'fair price' for the gamble X. The set of all linear previsions with domain $\mathcal{L}(\Omega)$ is denoted by $\mathbb{P}$.

Also assume that with the phenomenon of interest, there is associated a true linear prevision P_T describing the uncertainty associated with it, but that the subject does not have enough information, resources, time, ..., in order to identify it precisely. P_T may therefore be interpreted as a random variable in $\mathbb{P}$.

In practice it will reasonably often be the case that for a number of gambles X, belonging to a set $\mathcal{K}$, there is linguistic information about the value which the actual prevision, or fair price, $P_T(X)$ of X assumes in $\mathbb{R}$ and that, for a given X, this information can be represented by a normal possibility measure on $\mathbb{R}$ whose distribution we shall denote by $\mathfrak{p}(X)$. Note that $\mathfrak{p}(X)$ is a possibility distribution on $\mathbb{R}$, i.e. a map taking $\mathbb{R}$ into $[0,1]$. It can be formally interpreted as a fuzzy quantity, i.e., a fuzzy set on the reals. The assessments $\mathfrak{p}(X)$ for all gambles $X \in \mathcal{K}$ determine a mapping $\mathfrak{p}$, called a *possibilistic prevision*, from the set $\mathcal{K}$ to the set of possibility distributions on $\mathbb{R}$ (or fuzzy quantities). In order to make clear what its possibility space and domain are, the possibilistic prevision will also be denoted by $(\Omega, \mathcal{K}, \mathfrak{p})$. In case all gambles in $\mathcal{K}$ are indicator functions of events, $\mathfrak{p}$ will also be called a *possibilistic probability*. If the local possibility distributions $\mathfrak{p}(X)$, $X \in \mathcal{K}$, can be represented by a normal global possibility distribution π on $\mathbb{P}$ such that the value $\mathfrak{p}(X) \cdot x$ of the possibility distribution $\mathfrak{p}(X)$ in the real number x is given by $\sup\{\pi(P)\colon P(X) = x\}$, — that is, the possibility that the true prevision or fair price $P_T(X)$ for X equals x — then we call the possibilistic prevision $\mathfrak{p}$ *representable*, and π a *representation* of $\mathfrak{p}$. If $(\Omega, \mathcal{K}, \mathfrak{p})$ is not representable but satisfies some minimal conditions (called *normality*) there is a technique called *natural extension* that corrects $\mathfrak{p}$ on its domain $\mathcal{K}$ to make it representable, with minimal behavioural implications. The same technique allows us to extend $(\Omega, \mathcal{K}, \mathfrak{p})$ to the greatest representable possibilistic prevision on $\mathcal{L}(\Omega)$, denoted by $\mathfrak{e}$, that is dominated by $\mathfrak{p}$ on $\mathcal{K}$. $\mathfrak{e}$ is called the natural extension of $\mathfrak{p}$. If $\mathfrak{p}$ is representable then $\mathfrak{p}$ coincides with $\mathfrak{e}$ on $\mathcal{K}$.

There is a special and very important class of possibilistic previsions for which the local assessments $\mathfrak{p}(X)$, $X \in \mathcal{K}$, are normal bounded fuzzy closed intervals. A normal bounded fuzzy closed interval is a fuzzy quantity $f\colon \mathbb{R} \to [0,1]$ whose α-cut sets $f_\alpha = \{x \in \mathbb{R}\colon f(x) \geq \alpha\}$, $(0 < \alpha \leq 1)$ are non-empty bounded closed intervals. $\mathfrak{p}(X)$ is therefore completely characterised by the family of pairs $(\underline{\mathfrak{p}}_\alpha(X), \overline{\mathfrak{p}}_\alpha(X))$ of smallest and greatest elements of its α-cut sets

$$\mathfrak{p}(X)_\alpha = [\underline{\mathfrak{p}}_\alpha(X), \overline{\mathfrak{p}}_\alpha(X)].$$

For a fixed α and variable X these pairs define a lower and upper prevision on $\mathcal{K}$ as defined in Walley's theory of imprecise probabilities [15]. They are called the *lower and upper cut previsions* of $\mathfrak{p}$. It turns out that for the families of lower and upper cut previsions, the respective notions of avoiding sure loss, coherence and natural extension from the theory of imprecise probabilities completely determine the normality, representability and natural extension of the possibilistic prevision $(\Omega, \mathcal{K}, \mathfrak{p})$. Indeed, $\mathfrak{p}$ is normal if and only if the cut previsions $(\underline{\mathfrak{p}}_\alpha, \overline{\mathfrak{p}}_\alpha)$, $\alpha \in]0,1]$, of $\mathfrak{p}$ avoid sure loss and $\mathfrak{p}$ is representable if and only if the cut previsions $(\underline{\mathfrak{p}}_\alpha, \overline{\mathfrak{p}}_\alpha)$, $\alpha \in]0,1]$, of $\mathfrak{p}$ form coherent pairs of lower and upper previsions. Furthermore, assume that $\mathfrak{p}$ is normal so that its natural extension $(\Omega, \mathcal{L}, \mathfrak{e})$ is defined. Then the cut previsions of its natural extension are the natural extension of its pair of cut previsions, or in other words, for any $\alpha \in]0,1]$ and any $X \in \mathcal{L}(\Omega)$: $\underline{\mathfrak{e}}_\alpha(X) = \underline{E}_\alpha(X)$ and $\overline{\mathfrak{e}}_\alpha(X) = \overline{E}_\alpha(X)$, where $(\underline{E}_\alpha, \overline{E}_\alpha)$ is the natural extension of the pair $(\underline{\mathfrak{p}}_\alpha, \overline{\mathfrak{p}}_\alpha)$.

In conclusion, using normal bounded fuzzy closed intervals allows us to reduce possibilistic previsions, through their α-cut sets, to nested families of upper and lower previsions.

3 Normality, representability and natural extension

It would be interesting, for example for computer implementation in practical applications, to have at our disposal efficient algorithms for checking normality and representability of a possibilistic prevision $(\Omega, \mathcal{K}, \mathfrak{p})$ and for constructing its natural extension $(\Omega, \mathcal{L}(\Omega), \mathfrak{e})$. This leads us to the subject of the present work. Let us assume that the possibility space $\Omega = \{\omega_1, \ldots, \omega_n\}$ is finite (or alternatively, that it is partitioned into a finite number of subsets) and that we have at our disposal probability assessments in the form of local possibility distributions that are defined on the atoms of Ω, i.e., $\mathcal{K} = \{\{\omega_1\}, \ldots, \{\omega_n\}\}$. $\mathfrak{p}$ could be called a possibilistic probability mass function, as it models information in terms of possibility distributions about the probability mass function associated with the phenomenon of interest. We assume in addition that the local assessments $\mathfrak{p}(\{\omega_i\})$, $i \in \{1, \ldots, n\}$, are normal bounded fuzzy closed intervals.

The results proposed in the following theorems are based on the fact that normality, representability and natural extension of a possibilistic prevision whose values are normal bounded fuzzy closed intervals are completely determined by avoiding sure loss, coherence and natural extension of its corresponding family of lower and upper cut previsions. Part of Walley's theory of imprecise probabilities [15] concerns the investigation of avoiding sure loss, coherence and natural extension of lower and upper previsions. In the specific situation of lower and upper probabilities defined on the atoms of a finite possibility space Ω, these results can be combined with the theory of probability intervals of de Campos, Huete and Moral [3] to give a very practical characterisation of avoiding sure loss, coherence and natural extension of the lower and upper probabilities. It is useful to give a brief exposition of these characterisations, as we shall see that our later results are generalisations of the results obtained here.

We recall that $\Omega = \{\omega_1, \dots, \omega_n\}$. Let $\mathbb{P}$ be the set of linear previsions on $\mathcal{L}(\Omega)$, or equivalently [15], let $\mathbb{P}$ be the set of finitely additive probabilities on $\mathcal{P}(\Omega)$. A family of intervals $L = ([l_i, u_i], i \in \{1, \dots, n\})$, verifying $0 \leq l_i \leq u_i \leq 1, \forall i$, can be interpreted as a set of bounds on the 'true' probabilities $P_T(\{\omega_i\})$. In other words it defines a lower and upper probability $\underline{P}$ and $\overline{P}$, defined on the atoms of Ω, such that $\underline{P}(\{\omega_i\}) = l_i$ and $\overline{P}(\{\omega_i\}) = u_i$. This family L generates a set $\mathcal{M}$ of possible probabilities associated with L:

$$\mathcal{M} = \{P \in \mathbb{P}\colon l_i \leq P(\{\omega_i\}) \leq u_i, \forall i\}.$$

Walley has proven that $\underline{P}$ and $\overline{P}$ avoid sure loss if and only if the set $\mathcal{M}$ is non-empty. De Campos et al. have shown that this is the case if and only if (since $\sum_{i=1}^{n} P(\{\omega_i\}) = 1$)

$$\sum_{i=1}^{n} l_i \leq 1 \text{ and } \sum_{i=1}^{n} u_i \geq 1. \tag{1}$$

A family of probability intervals which satisfies condition (1) is called *proper*. In summary, lower and upper probabilities, defined by a family of probability intervals, avoid sure loss if and only if the family is proper. With the set $\mathcal{M}$ (and thus with the family of probability intervals L) we can also associate a lower and upper probability $\underline{P}'$ and $\overline{P}'$, given for any $A \subseteq \Omega$ by

$$\underline{P}'(A) = \inf_{P \in \mathcal{M}} P(A) \text{ and } \overline{P}'(A) = \sup_{P \in \mathcal{M}} P(A).$$

In order to maintain the consistency between probability intervals and upper and lower probabilities it is important for the restriction of $\underline{P}'$ and $\overline{P}'$ to the atoms of Ω to be equal to the original bounds, i.e.,

$$\underline{P}'(\{\omega_i\}) = \inf_{P \in \mathcal{M}} P(\{\omega_i\}) = l_i$$
$$\overline{P}'(\{\omega_i\}) = \sup_{P \in \mathcal{M}} P(\{\omega_i\}) = u_i.$$

These equalities do not hold in general: they are true if and only if

$$\sum_{j \neq i} l_j + u_i \leq 1 \text{ and } \sum_{j \neq i} u_j + l_i \geq 1, \quad \forall i. \tag{2}$$

If for a family of probability intervals L, these equalities hold we call L *reachable*. Walley proved that in this case the corresponding upper and lower probabilities $\underline{P}'$ and $\overline{P}'$ are coherent. In summary, lower and upper probabilities, defined by a family of probability intervals, are coherent if and only if the family is reachable.

A proper family of probability intervals $L = ([l_i, u_i], i \in \{1 \dots n\})$ can always be transformed into a reachable family of probability intervals $L' = ([l_i', u_i'], i \in \{1 \dots n\})$, without altering the corresponding set of probability measures $\mathcal{M}$, by modifying the lower and upper bounds in the following way:

$$l_i' = \max(l_i, 1 - \sum_{j \neq i} u_j) \text{ and } u_i' = \min(u_i, 1 - \sum_{j \neq i} l_j), \quad \forall i. \tag{3}$$

For a reachable family of probability intervals L, the values $\underline{P}'(A)$ and $\overline{P}'(A)$ for all $A \in \mathcal{P}(\Omega)$ can be computed easily from the values l_i and u_i:

$$\begin{cases} \underline{P}'(A) = \max(\sum_{\omega_i \in A} l_i, 1 - \sum_{\omega_i \notin A} u_i) \\ \overline{P}'(A) = \min(\sum_{\omega_i \in A} u_i, 1 - \sum_{\omega_i \notin A} l_j) \end{cases} \tag{4}$$

Finally, de Campos et al. have shown that the lower and upper probability measures $\underline{P}'$ and $\overline{P}'$ associated with a reachable family of probability intervals are 2-monotone respectively 2-alternating.

Notice that for a fixed α and variable i, the family of intervals $[\underline{\mathfrak{p}}_\alpha(\{\omega_i\}), \overline{\mathfrak{p}}_\alpha(\{\omega_i\})]$ determine a family of probability intervals as defined above. So it is easy to check whether the cut previsions $\underline{\mathfrak{p}}_\alpha$ and $\overline{\mathfrak{p}}_\alpha$ of $\mathfrak{p}$ avoid sure loss and/or are coherent. Since the results must hold at each α-level, we shall in the present section try and get rid of the α's, and formulate conditions for normality, representability and natural extension of the possibilistic prevision in terms of the normal bounded fuzzy closed intervals $\mathfrak{p}(\{\omega_i\})$. For a better understanding, we give a short overview of the orderings and algebraic operations on normal bounded fuzzy closed intervals that will be used in the following theorems.

It turns out that performing the usual algebraic operations on normal bounded fuzzy closed intervals amounts to doing interval arithmetic on their cut sets. Let $\mathfrak{a}$ and $\mathfrak{b}$ be normal bounded fuzzy closed intervals with cut sets $\mathfrak{a}_\alpha = \{x \in \mathbb{R}\colon \mathfrak{a}(x) \geq \alpha\}$ and $\mathfrak{b}_\alpha = \{x \in \mathbb{R}\colon \mathfrak{b}(x) \geq \alpha\}$, for $\alpha \in]0, 1]$. These cut sets are bounded closed intervals and we use the notation $\mathfrak{a}_\alpha = [\mathfrak{a}_\alpha^l, \mathfrak{a}_\alpha^r]$.

The *sum* $\mathfrak{a} + \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is the normal bounded fuzzy closed interval whose cut sets are the interval sums of the cut sets of $\mathfrak{a}$ and $\mathfrak{b}$: $(\mathfrak{a}+\mathfrak{b})_\alpha = \mathfrak{a}_\alpha + \mathfrak{b}_\alpha$, or in terms of the interval boundaries:

$$(\mathfrak{a} + \mathfrak{b})_\alpha^l = \mathfrak{a}_\alpha^l + \mathfrak{b}_\alpha^l \text{ and } (\mathfrak{a} + \mathfrak{b})_\alpha^r = \mathfrak{a}_\alpha^r + \mathfrak{b}_\alpha^r.$$

The *difference* $\mathfrak{a} - \mathfrak{b}$ of $\mathfrak{a}$ and $\mathfrak{b}$ is the normal bounded fuzzy closed interval whose cut sets are: $(\mathfrak{a} - \mathfrak{b})_\alpha = \mathfrak{a}_\alpha - \mathfrak{b}_\alpha$, or in terms of the interval boundaries

$$(\mathfrak{a} - \mathfrak{b})_\alpha^l = \mathfrak{a}_\alpha^l - \mathfrak{b}_\alpha^r \text{ and } (\mathfrak{a} - \mathfrak{b})_\alpha^r = \mathfrak{a}_\alpha^r - \mathfrak{b}_\alpha^l.$$

The *pointwise ordering* of normal bounded fuzzy closed intervals is denoted by $\leq$ and defined by $\mathfrak{a} \leq \mathfrak{b}$ if and only if $\mathfrak{a}(x) \leq \mathfrak{b}(x)$ for all $x \in \mathbb{R}$, or equivalently if and only if for all $\alpha \in]0, 1]$, $\mathfrak{a}_\alpha \subseteq \mathfrak{b}_\alpha$.

Suppose there is a real number c such that $\mathfrak{a} \geq c$ and $\mathfrak{b} \geq c$, where we identify c with the normal bounded fuzzy closed interval that maps c to 1 and any other real number to 0. Then we define the *pointwise minimum* of $\mathfrak{a}$ and $\mathfrak{b}$, denoted by $\min(\mathfrak{a},\mathfrak{b})$ as the normal bounded fuzzy closed interval given by $(\min(\mathfrak{a},\mathfrak{b}))(x)=\min(\mathfrak{a}(x),\mathfrak{b}(x))$, $\forall x\in\mathbb{R}$, or in terms of the α-cut sets

$$(\min(\mathfrak{a},\mathfrak{b}))_\alpha=[\max(\mathfrak{a}^l_\alpha,\mathfrak{b}^l_\alpha),\min(\mathfrak{a}^r_\alpha,\mathfrak{b}^r_\alpha)],\quad \alpha\in\,]0,1].$$

The reason we require that $\mathfrak{a}\geq c$ and $\mathfrak{b}\geq c$ is that otherwise the pointwise minimum as defined above is not a normal bounded fuzzy closed interval.

The expressions for normality, representability and the natural extension of a possibilistic prevision can now be easily understood and motivated.

Theorem 1 *Let $\mathcal{K}=\{\{\omega_1\},\dots,\{\omega_n\}\}$ and let $(\Omega,\mathcal{K},\mathfrak{p})$ be a possibilistic prevision whose values $\mathfrak{p}(\{\omega_i\})$, $\{\omega_i\}\in\mathcal{K}$, are normal bounded fuzzy closed intervals.*

1. *$(\Omega,\mathcal{K},\mathfrak{p})$ is normal if and only if*
$$\sum_{i=1}^n \mathfrak{p}(\{\omega_i\})\geq 1, \tag{5}$$
2. *$(\Omega,\mathcal{K},\mathfrak{p})$ is representable if and only if*
$$\Big(\forall i\in\{1,\dots,n\}\Big)\Big(\sum_{j\neq i}\mathfrak{p}(\{\omega_j\})\geq 1-\mathfrak{p}(\{\omega_i\})\Big). \tag{6}$$

Proof (Short justification of the theorem) $\sum$ denotes the fuzzy sum of normal bounded fuzzy closed intervals and $\geq$ denotes the pointwise ordering of normal bounded fuzzy closed intervals, where the real number 1 is interpreted as the normal bounded fuzzy closed interval that maps 1 into 1 and any other real number into 0.

Remark that (5) is a generalisation of (1). Indeed the fuzzy sum and the pointwise ordering of normal bounded fuzzy closed intervals in (5) can be transformed into the following condition that must hold at each α-level

$$\sum_{i=1}^n \underline{\mathfrak{p}}_\alpha(\{\omega_i\})\leq 1 \text{ and } \sum_{i=1}^n \overline{\mathfrak{p}}_\alpha(\{\omega_i\})\geq 1.$$

This is the specific form of condition (1) in this situation. (6) is the corresponding generalisation of (2).

A normal possibilistic prevision $(\Omega,\mathcal{K},\mathfrak{p})$ that is not representable can be corrected by means of the technique of natural extension.

Theorem 2 *Let $\mathcal{K}=\{\{\omega_1\},\dots,\{\omega_n\}\}$ and let $(\Omega,\mathcal{K},\mathfrak{p})$ be a normal possibilistic prevision whose values $\mathfrak{p}(\{\omega_i\})$, $\{\omega_i\}\in\mathcal{K}$, are normal bounded fuzzy closed intervals. Then the restriction of the natural extension $\mathfrak{e}$ to $\mathcal{K}$ is given by, for $i=1,\dots,n$:*

$$\mathfrak{e}(\{\omega_i\})=\min\Big(\mathfrak{p}(\{\omega_i\}),1-\sum_{j\neq i}\mathfrak{p}(\{\omega_j\})\Big). \tag{7}$$

Proof (Short justification of the theorem) (7) is an immediate generalisation of (3).

Natural extension does not only allow to correct a normal possibilistic prevision on its domain but enables us to extend the possibilistic prevision to larger domains.

Theorem 3 *Let $\mathcal{K}=\{\{\omega_1\},\dots,\{\omega_n\}\}$ and let $(\Omega,\mathcal{K},\mathfrak{p})$ be a representable possibilistic prevision whose values $\mathfrak{p}(\{\omega_i\})$, $\{\omega_i\}\in\mathcal{K}$, are normal bounded fuzzy closed intervals. Then the natural extension of $\mathfrak{p}$ to $\mathcal{P}(\Omega)$, also denoted by $\mathfrak{p}$, is given by, for all $A\subseteq\Omega$:*

$$\mathfrak{p}(A)=\min\big(\sum_{\omega_i\in A}\mathfrak{p}(\{\omega_i\}),1-\sum_{\omega_i\notin A}\mathfrak{p}(\{\omega_i\})\big). \tag{8}$$

Proof (Short justification of the theorem) (8) is a straightforward generalisation of (4).

Finally, we can extend $\mathfrak{p}$ to all gambles $X\in\mathcal{L}(\Omega)$.

Theorem 4 *Let $\mathcal{K}=\{\{\omega_1\},\dots,\{\omega_n\}\}$ and let $(\Omega,\mathcal{K},\mathfrak{p})$ be a representable possibilistic prevision whose values $\mathfrak{p}(\{\omega_i\})$, $\{\omega_i\}\in\mathcal{K}$, are normal bounded fuzzy closed intervals Then the natural extension $\mathfrak{e}$ of $\mathfrak{p}$ to $\mathcal{L}(\Omega)$ is given by, for all $X\in\mathcal{L}(\Omega)$:*

$$\begin{aligned}\mathfrak{e}(X)&=\int_{\mathbb{R}}X\,\mathrm{d}\mathfrak{p}\\&=\inf X+\int_{\inf X}^{\sup X}\mathfrak{p}(\{\omega\colon X(\omega)\geq x\})\mathrm{d}x,\end{aligned} \tag{9}$$

where the values $\mathfrak{p}(A)$, $A\in\mathcal{P}(\Omega)$, are defined as in the preceding theorem.

Proof (Short justification of the theorem) For a fixed α-level, we know that the lower and upper probabilities $(\Omega,\mathcal{P}(\Omega),\underline{\mathfrak{p}}_\alpha)$ and $(\Omega,\mathcal{P}(\Omega),\overline{\mathfrak{p}}_\alpha)$ are coherent. Moreover they are respectively 2-monotone and 2-alternating. Using [15, section 3.2], the natural extension of $\underline{\mathfrak{p}}_\alpha$ and $\overline{\mathfrak{p}}_\alpha$ to $\mathcal{L}(\Omega)$ can be written as

$$\begin{aligned}\underline{E}_\alpha(X)&=\int_{\mathbb{R}}X\,\mathrm{d}\underline{\mathfrak{p}}_\alpha\\&=\inf X+\int_{\inf X}^{\sup X}\underline{\mathfrak{p}}_\alpha(\{\omega\colon X(\omega)\geq x\})\mathrm{d}x,\end{aligned} \tag{10}$$

and

$$\begin{aligned}\overline{E}_\alpha(X)&=\int_{\mathbb{R}}X\,\mathrm{d}\overline{\mathfrak{p}}_\alpha\\&=\inf X+\int_{\inf X}^{\sup X}\overline{\mathfrak{p}}_\alpha(\{\omega\colon X(\omega)\geq x\})\mathrm{d}x.\end{aligned} \tag{11}$$

It is easy to show that the bounded closed intervals

$$\Big[\int_{\inf X}^{\sup X}\underline{\mathfrak{p}}_\alpha(\{\omega\colon X(\omega)\geq x\})\mathrm{d}x,\quad \int_{\inf X}^{\sup X}\overline{\mathfrak{p}}_\alpha(\{\omega\colon X(\omega)\geq x\})\mathrm{d}x\Big]$$

are the α-cut sets of a unique normal bounded fuzzy closed interval. We denote this normal bounded fuzzy closed interval by

$$\int_{\inf X}^{\sup X} \mathfrak{p}(\{\omega\colon X(\omega) \geq x\})\mathrm{d}x$$

and call it a fuzzy Riemann integral. Note that the defined integral can also be interpreted as a generalisation of the Aumann integral [13]. In the same manner, we denote by $\int_{\mathbb{R}} X\mathrm{d}\mathfrak{p}$ the unique normal bounded fuzzy closed interval whose α-cut sets are given by

$$\Big[\int_{\mathbb{R}} X\mathrm{d}\underline{\mathfrak{p}}_\alpha, \int_{\mathbb{R}} X\mathrm{d}\overline{\mathfrak{p}}_\alpha\Big].$$

We call this normal bounded fuzzy closed interval the fuzzy Choquet integral.

As we already mentioned in the introduction $\underline{E}_\alpha(X)$ and $\overline{E}_\alpha(X)$ are the cut sets of the natural extension $(\Omega, \mathcal{L}(\Omega), \mathfrak{e})$ of $(\Omega, \mathcal{P}(\Omega), \mathfrak{p})$. So the equalities (10) and (11) that hold on each α-level can be summarised by (9).

We have found that starting from a normal possibilistic prevision $\mathfrak{p}$, defined on the atoms of Ω and for which the values $\mathfrak{p}(\{\omega_i\})$ are normal bounded fuzzy closed intervals, we can correct this prevision on its domain so that is becomes representable and extend it to larger domains. The expressions for these correction and extensions are generalisations of the results that already exist for first-order upper and lower probabilities.

4 Continuous piecewise linear normal bounded fuzzy closed intervals

Although general normal bounded fuzzy closed intervals are interesting from a theoretical point of view, they are still too complicated as far as practical implementation is concerned, since they cannot be represented and manipulated in a finitary way. *Continuous piecewise linear* normal bounded fuzzy closed intervals are on the one hand general enough to cover many practical situations, and they are on the other hand simple enough to allow for finitary and efficient representation and computation, since they may be represented by means of a finite list of numbers

$$((x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n))$$

where for all $i \in \{1, \ldots, n\}$ $\quad x_i \in \mathbb{R}$ and $f_i \in [0, 1]$ and $x_i < x_{i+1}$, for all $i \in \{1, \ldots, n-1\}$. Moreover f_1 and f_n must be zero and there must be at least one $i \in \{2, \ldots, n-1\}$ such that $f_i = 1$. Suppose $f_c = 1$ then the sequence $(f_i)_{i=1}^c$ must be non-decreasing and $(f_i)_{i=c}^n$ must be non-increasing.

An explicit expression for the continuous piecewise linear normal bounded fuzzy closed interval $\mathfrak{a}$ with list representation $((x_1, f_1), (x_2, f_2), \ldots, (x_n, f_n))$ is given by

$$\mathfrak{a}(x) = \begin{cases} 0 & \text{if } x < x_1 \text{ or } x > x_n \\ f_i + (f_{i+1} - f_i)\dfrac{x - x_i}{x_{i+1} - x_i} & \text{if } x_i \leq x \leq x_{i+1}, \end{cases} \tag{12}$$

where $i \in \{1, \ldots, n-1\}$. Another interesting property of continuous piecewise linear normal bounded fuzzy probabilities is that they are closed under most so-called fuzzy-set theoretic operations, in particular minimum, addition and subtraction, the ones that we happen to need in our present approach. Algorithms for working with the corresponding list representations are available in the literature [11,12,14]. The convenient properties of continuous piecewise linear fuzzy quantities enable us to translate the expressions for normality, representability and natural extension in terms of the finite list representations of the $\mathfrak{p}(\{\omega_i\})$.

Theorem 5 *Let $\mathcal{K} = \{\{\omega_1\}, \ldots, \{\omega_n\}\}$. The natural extension of a possibilistic prevision $(\Omega, \mathcal{K}, \mathfrak{p})$ whose values $\mathfrak{p}(\{\omega_i\})$, $\{\omega_i\} \in \mathcal{K}$ are continuous piecewise linear normal bounded fuzzy closed intervals, is a representable possibilistic prevision $(\Omega, \mathcal{L}(\Omega), \mathfrak{e})$ for which the $\mathfrak{e}(X)$, $X \in \mathcal{L}(\Omega)$ are still continuous piecewise linear normal bounded fuzzy closed intervals.*

This property is again very interesting from a practical point of view.

Example: Football match

Let us take a closer look at the example we started with. It is clear that $\Omega = \{w, d, l\}$. Suppose the linguistic information is modelled by continuous piecewise linear normal bounded fuzzy closed intervals whose list representations are

$$\mathfrak{p}(\{w\}) : ((\frac{3}{8}, 0), (\frac{1}{2}, 1), (\frac{7}{8}, 1), (1, 0));$$
$$\mathfrak{p}(\{l\}) = \mathfrak{p}(\{d\}) : ((0, 0), (\frac{1}{8}, 1), (\frac{3}{8}, 1), (\frac{1}{2}, 0)).$$

This means that we have at our disposal a possibilistic prevision $(\Omega, \mathcal{K}, \mathcal{P}(\Omega))$ with $\mathcal{K} = \{\{w\}, \{d\}, \{l\}\}$.

Since $\mathfrak{p}(\{w\}) + \mathfrak{p}(\{d\}) + \mathfrak{p}(\{l\}) \geq 1$, this possibilistic prevision is normal. Indeed the list representation of the sum is

$$((\frac{3}{8}, 0), (\frac{3}{4}, 1), (\frac{13}{8}, 1), (2, 0)).$$

The possibilistic prevision is not representable since $\mathfrak{p}(\{d\}) + \mathfrak{p}(\{l\}) \geq 1 - \mathfrak{p}(\{w\})$ does not hold. Indeed, compare the list representations

$$\mathfrak{p}(\{d\}) + \mathfrak{p}(\{l\}) : ((0, 0), (\frac{1}{4}, 1), (\frac{3}{4}, 1), (1, 0));$$
$$1 - \mathfrak{p}(\{w\}) : ((0, 0), (\frac{1}{8}, 1), (\frac{1}{2}, 1), (\frac{5}{8}, 0)).$$

So, we have to correct $\mathfrak{p}$ on $\{w\}$. The correction is given by

$$\mathfrak{e}(\{w\}) = \min(\mathfrak{p}(\{w\}), 1 - \mathfrak{p}(\{d\}) - \mathfrak{p}(\{l\})).$$

The list representations are given by

$$1 - \mathfrak{p}(\{d\}) - \mathfrak{p}(\{l\}) : ((0, 0), (\frac{1}{4}, 1), (\frac{3}{4}, 1), (1, 0));$$
$$\mathfrak{e}(\{w\}) : ((\frac{3}{8}, 0), (\frac{1}{2}, 1), (\frac{3}{4}, 1), (1, 0)).$$

Now we can extend $\mathfrak{p}$ to $\mathcal{P}(\Omega)$. Let us for example consider the event A that the home team doesn't lose, meaning that $A = \{w, d\}$. Then $\mathfrak{e}(A)$ is given by $\min(\mathfrak{p}(\{w\}) + \mathfrak{p}(\{d\}), 1 - \mathfrak{p}(\{l\}))$. The corresponding list representations are

$$\begin{aligned} 1 - \mathfrak{p}(\{l\}) &: ((\frac{1}{2}, 0), (\frac{5}{8}, 1), (\frac{7}{8}, 1), (1, 0)); \\ \mathfrak{p}(\{w\}) + \mathfrak{p}(\{d\}) &: ((\frac{3}{8}, 0), (\frac{5}{8}, 1), (\frac{5}{4}, 1), (\frac{3}{2}, 0)); \\ \mathfrak{e}(A) &: ((\frac{1}{2}, 0), (\frac{5}{8}, 1), (\frac{7}{8}, 1), (1, 0)). \end{aligned}$$

5 Construction of an imprecise first order model

From the global model $(\Omega, \mathcal{L}(\Omega), \mathfrak{e})$ we can construct, using the techniques proposed and justified in [17], a behaviourally equivalent first-order lower prevision $\underline{E}$, suitable as an imprecise prior for decision making and statistical reasoning. For $X \in \mathcal{L}(\Omega)$, $\underline{E}(X)$ is computed by taking the uniform average over $\alpha \in]0, 1]$ of the cut-previsions $\mathfrak{e}_\alpha(X)$

$$\underline{E}(X) = \int_0^1 \underline{\mathfrak{e}}_\alpha(X) \mathrm{d}x$$

In the special case we are considering here, it is possible to prove that this expression can be transformed into

$$\underline{E}(X) = \inf X + \int_{\inf X}^{\sup X} \underline{P}(\{\omega \colon X(\omega) \geq x\}) \mathrm{d}x,$$

with $\underline{P}$ the restriction of $\underline{E}$ to $\mathcal{P}(\Omega)$, meaning that $\underline{E}$ can be obtained by extending $\underline{P}$ using Choquet integration.

Thus, in order to obtain a first-order lower prevision in our approach, where the assessments $\mathfrak{p}(\{\omega_i\})$, $i = 1, \ldots, n$ are continuous piecewise linear normal bounded fuzzy closed intervals, it turns out to be sufficient to extend the possibilistic prevision $(\Omega, \mathcal{K}, \mathfrak{p})$ to $\mathcal{P}(\Omega)$, which is a finitary operation, and then transform this second-order model into the first-order lower prevision $\underline{E}$, which involves the Riemann integration of specific continuous piecewise linear functions and is a finitary operation as well.

In conclusion, we may state that by applying the general theory of possibilistic previsions to the special case of linguistic assessments of a probability mass function involving possibility distributions that are continuous piecewise linear normal bounded fuzzy closed intervals, exact and finite algorithms can be found for using these assessments to obtain prior information useful in decision making and statistical reasoning.

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