POSSIBILITY THEORY III
POSSIBILISTIC INDEPENDENCE

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Dedicated to Prof. dr. Etienne E. Kerre

The introduction of the notion of independence in possibility theory is a problem of long-standing interest. Many of the measure-theoretic definitions that have up to now been given in the literature face some difficulties as far as interpretation is concerned. Also, there are inconsistencies between the definition of independence of measurable sets and possibilistic variables. After a discussion of these definitions and their shortcomings, a new measure-theoretic definition is suggested, which is consistent in this respect, and which is a formal counterpart of the definition of stochastic independence in probability theory. In discussing the properties of possibilistic independence, I draw from the measure- and integral-theoretic treatment of possibility theory, discussed in Part I of this series of three papers. I also investigate the relationship between this definition of possibilistic independence and the definition of conditional possibility, discussed in detail in Part II of this series. Furthermore, I show that in the special case of classical, two-valued possibility the definition given here has a straightforward and natural interpretation.

INDEX TERMS: Possibility theory, possibilistic independence, logical independence, conditional possibility.

1 POSSIBILISTIC INDEPENDENCE: A SURVEY

This is the third in a series of three papers on the measure- and integral-theoretic aspects of possibility theory, in which I deal with the notion of possibilistic independence. In the second part of this series I have briefly discussed the notions 'noninteractivity' and 'possibilistic independence', as introduced by Zadeh [1978] and Hisdal [1978]. Both notions constitute a first attempt at introducing a counterpart for the notion of stochastic independence in possibility theory. Remark that Zadeh and Hisdal concentrate on the independence of variables, and leave the independence of events and fuzzy events undealt with. Furthermore, I have indicated in Part II that Hisdal's definition of possibilistic independence suffers from a number of shortcomings. When these are eliminated, her notion of independence essentially coincides with Zadeh's noninteractivity.

Possibilistic independence was also studied by Nahmias [1978], and in his footsteps by Rao and Rashed [1981]. It was briefly discussed by Wang [1982]; and more recently, Dubois, Prade and co-workers [Benferhat *et al.*, 1994] [Dubois *et al.*, 1994] studied possibilistic independence in a logical setting. Let me give a concise summary of their ideas and results.

Nahmias [1978] considers a universe X and what we have called in parts I and II a normal $([0,1],\leq)$-possibility measure Π on $(X,\wp(X))$ – he himself calls it a *scale*. He calls the elements $A_1, \ldots, A_n$, $n\in\mathbb{N}\setminus\{0\}$, of $\wp(X)$ *mutually unrelated* iff for any k in $\mathbb{N}\setminus\{0\}$ with $k\leq n$, and for arbitrary and different $j_1, \ldots, j_k$ in $\{1,\ldots,n\}$:

$$\Pi(A_{j_1}\cap\cdots\cap A_{j_k}) = \min_{\ell=1}^{k}\Pi(A_{j_\ell}). \tag{1}$$

His source of inspiration is the formally analogous formula for the stochastic independence of events in probability theory [Burrill, 1972]. Also, he calls two $X-\mathbb{R}$-mappings f_1 and f_2 – *fuzzy variables* in his terminology – *unrelated* iff

$$(\forall(a,b)\in\mathbb{R}^2)(f_1^{-1}(\{a\}) \text{ and } f_2^{-1}(\{b\}) \text{ are mutually unrelated}), \tag{2}$$

or equivalently,

$$(\forall(a,b)\in\mathbb{R}^2)(\Pi(f_1^{-1}(\{a\})\cap f_2^{-1}(\{b\})) = \min(\Pi(f_1^{-1}(\{a\})),\Pi(f_2^{-1}(\{b\})))). \tag{3}$$

Rao and Rashed [1981] point out that according to Eq. (1) any element A of $\wp(X)$ is mutually unrelated to itself, since $\Pi(A\cap A)=\min(\Pi(A),\Pi(A))=\Pi(A)$. With reason, they stress that in probability theory this is generally not the case, and that this property creates semantical problems. They therefore propose to use the phrase 'min-*related*' instead of 'unrelated'.

Wang [1982] generalizes Nahmias' approach by considering an ample field $\mathcal{R}$ on X and calling an $X-\mathbb{R}$-mapping a fuzzy variable only if it is $\mathcal{R}-\wp(\mathbb{R})$-measurable (see also Part I, Definition 4.1). His definition of *independence* for such fuzzy variables is essentially the same as Nahmias' definition of unrelatedness.

In my terminology, the fuzzy variables of Nahmias (and Wang) are possibilistic variables in $(\mathbb{R},\wp(\mathbb{R}))$, where X is considered as a basic space, provided with an ample field $\wp(X)$. Using the notations of Part I, subsection 4.2, Eq. (3) may be rewritten as

$$(\forall(a,b)\in\mathbb{R}^2)(\pi_{(f_1,f_2)}(a,b)=\min(\pi_{f_1}(a),\pi_{f_2}(b))), \tag{4}$$

This formula has an obvious probabilistic counterpart (see, for instance, [Burrill, 1972] Theorem 11-4A with corollary).

In my opinion, the approach of Nahmias [1978] for the independence of events faces a number of difficulties. On the one hand, there is the interpretational difficulty already laid bare by Rao and Rashed [1981]. That any event can be called mutually unrelated to itself – indeed, to any of its subsets – is, to say the least, a little strange. It seems to me that giving the notion another name in order to evade this interpretational difficulty misses the point, because it fails to explain how such a radical difference in interpretation can emerge between stochastic independence and this new notion.

On the other hand, we know that events can always be associated through their characteristic mappings with special fuzzy (or, in my terminology, possibilistic) variables. None of the above-mentioned authors answers the question whether there exists a relationship between the mutual unrelatedness of events and the unrelatedness (or independence) of their characteristic mappings. The existence of such a relationship is a central idea in probability theory. Moreover, it will be shown further on that such a relationship does not generally exist, starting from Eqs. (1) and (2). Interestingly, this difficulty appears to be linked with the interpretational problem discussed above.

In this paper, I intend to construct a more general theory of possibilistic independence, and at the same time provide a solution for the above-mentioned difficulties. My guiding principle in doing so will be the (formal) analogy with probability theory. My tools will be the measure- and integral-theoretic treatment of possibility theory, developed in parts I and II of this series. At the same time, it will be shown that the definition of possibilistic independence given here has an interesting interpretation when classical possibility is considered.

In two interesting and important papers [Benferhat *et al.*, 1994] [Dubois *et al.*, 1994] Benferhat, Dubois, Fariñas del Cerro, Herzig and Prade discuss the independence of events (or propositions) in possibility theory in a logical setting. In all, they discuss three types of independence. The first is based upon Zadeh's notion of noninteractivity for variables (see also Part II, section 1). In [Dubois *et al.*, 1994] two events A and B are called *unrelated in Zadeh's sense* iff

$$\Pi(A \cap B) = \min(\Pi(A), \Pi(B)). \tag{5}$$

This definition is essentially the same as the one given by Nahmias, see Eq. (1). In [Benferhat *et al.*, 1994] however, the authors correctly note that Zadeh's definition of noninteractivity for variables implies, when events are identified with their characteristic mappings, that A and B are unrelated iff

$$\begin{cases} \Pi(A \cap B) = \min(\Pi(A), \Pi(B)) \\ \Pi(A \cap \mathrm{co}B) = \min(\Pi(A), \Pi(\mathrm{co}B)) \\ \Pi(\mathrm{co}A \cap B) = \min(\Pi(\mathrm{co}A), \Pi(B)) \\ \Pi(\mathrm{co}A \cap \mathrm{co}B) = \min(\Pi(\mathrm{co}A), \Pi(\mathrm{co}B)) \end{cases} \tag{6}$$

a criterion which I derived independently in my doctoral dissertation [De Cooman, 1993], and a generalization of which the reader will also find in section 4. However, this first kind of independence in not studied in much detail by the above-mentioned authors. They devote much more time and effort to two other forms of independence for events, which they call *weak* and *strong independence*. These definitions are based upon the notion of conditioning. As also discussed in Part II, section 1 Dubois and Prade have defined the conditional possibility $\Pi(A \mid B)$

of A given B as the *maximal* solution of the equation

$$\Pi(A \cap B) = \min(\Pi(A \mid B), \Pi(B)) \tag{7}$$

or in other words,

$$\Pi(A \mid B) = \begin{cases} 1 & ; \quad \Pi(A \cap B) = \Pi(B) \\ \Pi(A \cap B) & ; \quad \Pi(A \cap B) < \Pi(B). \end{cases} \tag{8}$$

The *necessity* $\mathrm{N}(A)$ of A is defined as $\mathrm{N}(A) = 1 - \Pi(\mathrm{co}A)$ and the *conditional necessity* $\mathrm{N}(A \mid B)$ of A given B as $\mathrm{N}(A \mid B) = 1 - \Pi(\mathrm{co}A \mid B)$. The event A is called *strongly independent of* B iff [Dubois *et al.*, 1994]

$$\mathrm{N}(A \mid B) = \mathrm{N}(A) > 0.$$

Strong independence defines a binary relation on the set of events, which in contrast to unrelatedness, is not a symmetric relation. Dubois *et al.* [1994] show that this relation completely determines the qualitative possibility relation induced by Π.

The event A is called *weakly independent* of B iff [Dubois *et al.*, 1994]

$$\mathrm{N}(A \mid B) > 0 \text{ and } \mathrm{N}(A) > 0.$$

Strong independence clearly implies weak independence, and Dubois *et al.* [1994] show that the binary relation of weak independence can also be used to completely characterize the qualitative possibility relation induced by Π. They also integrate the notion of weak independence in the framework of belief revision.

I feel that the notion of unrelatedness of events, as defined by Eq. (6) deserves a similar study as the notions of strong and weak independence. Note that Dubois *et al.* [1994] make such a study for the notion of unrelatedness as defined by Eq. (5), which is in my view, not the right definition, as I have already argued before. I also want to remark that the definitions and properties of weak and strong independence heavily rely on the specific solution (8) of Eq. (7) for $\Pi(A \mid B)$, which is based upon the *meta*-theoretical *principle of minimum specificity* (see also the discussion in Part II, sections 1 and 5) and on the specific choice of the operator min in the defining equation (7).

In what follows, I deal with the possibilistic independence of possibilistic variables, fuzzy variables (fuzzy events) and measurable sets (events). The independence of possibilistic variables is treated first in section 2. As a special case, the possibilistic independence of fuzzy events is studied in section 3. Yet a further specialization, the independence of events, is dealt with in section 4. In section 5, I discuss the special case of classical possibility, where possibilistic independence and logical independence turn out to be closely related notions.

Let me conclude this introduction with a number of notational conventions. In what follows, $(\Omega, \mathcal{R}_\Omega, \Pi_\Omega)$ denotes a $(L, \leq)$-possibility space. The possibility distribution of Π_Ω is denoted by π_Ω. It is furthermore assumed that Π_Ω is *normal.* By T we shall mean a triangular norm on $(L, \leq)$, such that $(L, \leq, T)$ is a complete lattice with t-norm. We shall sometimes make use of results and definitions given in the first and second paper of this series, which will be referred to as Parts I and II, respectively.

Finally, for most of the results derived here, the reader will find explicit references to their probabilistic counterparts. This should help her find her way in some of the more abstract lines of reasoning that follow.

2 INDEPENDENCE OF POSSIBILISTIC VARIABLES

We start this treatment of possibilistic independence with a basic definition, of which all the other independence definitions turn out to be special cases.

Definition 2.1. Consider a nonempty family $\{\mathcal{O}_j \mid j \in J\}$ of subsets of $\mathcal{R}_\Omega$. This family is called (Π_Ω, T)-independent iff for any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, for any F_k in $\mathcal{O}_{j_k}$, and for any G_k in $\{F_k, \mathrm{co}F_k\}$, $k = 1, \ldots, n$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} G_k) = T_{k=1}^{n} \Pi_\Omega(G_k). \tag{9}$$

In this case, we also say that the sets of events $\mathcal{O}_j$, $j \in J$, are (Π_Ω, T)-independent. Whenever we do not want to mention the $(L, \leq)$-possibility measure Π_Ω and/or the t-norm T explicitly, we simply speak of (possibilistic) independence instead of (Π_Ω, T)-independence.

Why exactly this definition is proposed becomes clear when we take a closer look at the definition of stochastic independence of event sets in probability theory (see, for instance, [Jacobs, 1978] Definition VI.6.1, [Burrill, 1972] section 11-5).

Definition 2.2. Let $(\Omega, \mathcal{S}_\Omega, \mathrm{Pr}_\Omega)$ be a probability space. Consider a nonempty family $\{\mathcal{O}_j \mid j \in J\}$ of subsets of $\mathcal{S}_\Omega$. This family is called independent (for Pr_Ω) iff for any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, and for any F_k in $\mathcal{O}_{j_k}$, $k = 1, \ldots, n$:

$$\mathrm{Pr}_\Omega(\bigcap_{k=1}^{n} F_k) = \prod_{k=1}^{n} \mathrm{Pr}_\Omega(F_k). \tag{10}$$

Let me point out that Definition 2.1 very closely resembles this definition, with the exception of one important detail: in the probabilistic definition, the phrase 'for any G_k in $\{F_k, \mathrm{co}F_k\}$' does not appear, and (9) is modified accordingly. Let me briefly explain my reasons for including this phrase in Definition 2.1.

Let A and B be elements of $\mathcal{S}_\Omega$ and assume that $\mathrm{Pr}_\Omega(A \cap B) = \mathrm{Pr}_\Omega(A)\mathrm{Pr}_\Omega(B)$. Using the additivity and the complementation laws for probability measures, it is easily shown that $\mathrm{Pr}_\Omega(A \cap \mathrm{co}B) = \mathrm{Pr}_\Omega(A)\mathrm{Pr}_\Omega(\mathrm{co}B)$. From this we may conclude that the special properties of probability measures render the above-mentioned phrase superfluous. The *independence formula* (10) is *formally invariant* under complementation of an arbitrary number of subsets in the family $\{F_k \mid k = 1, \ldots, n\}$.

On the other hand, as is shown in the following example, an analogous course of reasoning is *not necessarily valid* in the possibilistic case. Therefore, in order to render the definition of possibilistic independence formally invariant under complementation, we must explicitly add the phrase 'for any G_k in $\{F_k, \mathrm{co}F_k\}$' to the independence definition, and accordingly turn the independence formula (10) into (9).

Example 2.3. Let $\Omega = \{1, 2, 3\}$, $\mathcal{R}_\Omega = \wp(\{1, 2, 3\})$, $(L, \leq) = ([0, 1], \leq)$ and $T = \min$. The $([0, 1], \leq)$-possibility measure Π_Ω is completely determined by $\pi_\Omega(1) = 1$, $\pi_\Omega(2) = 1$ and $\pi_\Omega(3) = \frac{1}{2}$. Let furthermore $A = \{1\}$ and $B = \{1, 2\}$. Then $A \cap B = \{1\}$ and $A \cap \mathrm{co}B = \{1\} \cap \{3\} = \emptyset$, whence $\Pi_\Omega(A) = 1$, $\Pi_\Omega(B) = 1$, $\Pi_\Omega(\mathrm{co}B) = \frac{1}{2}$, $\Pi_\Omega(A \cap B) = 1$ and $\Pi_\Omega(A \cap \mathrm{co}B) = 0$. Clearly, $\Pi_\Omega(A \cap B) = \min(\Pi_\Omega(A), \Pi_\Omega(B))$, whereas $\Pi_\Omega(A \cap \mathrm{co}B) \neq \min(\Pi_\Omega(A), \Pi_\Omega(\mathrm{co}B))$.

From Definition 2.1 we derive the following definition for the possibilistic independence of possibilistic variables (probabilistic counterpart: [Jacobs, 1978] Definition VI.6.5). In Corollary 2.5 we give a number of criteria for this new form of possibilistic independence. Since we are going to work with possibilistic variables, it deserves to be mentioned that the set Ω will henceforth be considered as a *basic space*.

Definition 2.4. Consider an nonempty family $\{X_j \mid j \in J\}$ of universes. For every j in J we consider an ample field $\mathcal{R}_j$ on X_j and a $\mathcal{R}_\Omega - \mathcal{R}_j$-measurable $\Omega - X_j$-mapping f_j, i.e., f_j is a possibilistic variable in $(X_j, \mathcal{R}_j)$. We call the family $\{f_j \mid j \in J\}$ of possibilistic variables (Π_Ω, T)-independent iff the family $\{f_j^{-1}(\mathcal{R}_j) \mid j \in J\}$ of subsets of $\mathcal{R}_\Omega$ is (Π_Ω, T)-independent. In this case we also say that the possibilistic variables f_j, $j \in J$, are (Π_Ω, T)-independent. Whenever we do not want to mention the $(L, \leq)$-possibility measure Π_Ω and/or the t-norm T explicitly, we simply speak of (possibilistic) independence instead of (Π_Ω, T)-independence.

Corollary 2.5. *The following propositions are equivalent.*

(i) The family $\{f_j \mid j \in J\}$ of possibilistic variables is (Π_Ω, T)-independent.

(ii) For any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, and for any A_k in $\mathcal{R}_{j_k}$, $k = 1, \ldots, n$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}(A_k)) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}(A_k)).$$

(iii) For any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, and for any x_k in X_{j_k}, $k = 1, \ldots, n$:

$$\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})).$$

Proof. Let us first show that (i) and (ii) are equivalent. First, assume that (i) holds. Consider arbitrary n in $\mathbb{N} \setminus \{0\}$, arbitrary and different $j_1, \ldots, j_n$ in J and arbitrary A_k in $\mathcal{R}_{j_k}$, $k = 1, \ldots, n$. By definition, $f_{j_k}^{-1}(A_k) \in f_{j_k}^{-1}(\mathcal{R}_{j_k})$. Since the family $\{f_j^{-1}(\mathcal{R}_j) \mid j \in J\}$ is by assumption (Π_Ω, T)-independent, it in particular follows from Definition 2.1 that (choose $G_k = F_k = f_{j_k}^{-1}(A_k)$, $k = 1, \ldots, n$) $\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}(A_k)) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}(A_k))$. We therefore conclude that (ii) holds.

Conversely, assume that (ii) holds. Consider arbitrary n in $\mathbb{N} \setminus \{0\}$, arbitrary and different $j_1, \ldots, j_n$ in J, arbitrary F_k in $f_{j_k}^{-1}(\mathcal{R}_{j_k})$ and arbitrary G_k in $\{F_k, \mathrm{co}F_k\}$, $k = 1, \ldots, n$. For $k = 1, \ldots, n$ we have that $G_k \in f_{j_k}^{-1}(\mathcal{R}_{j_k})$, since $f_{j_k}^{-1}(\mathcal{R}_{j_k})$ is clearly an ample field on Ω. This implies that there exists a A_k in $\mathcal{R}_{j_k}$ such that $G_k = f_{j_k}^{-1}(A_k)$. By assumption, we have that $\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}(A_k)) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}(A_k))$, or equivalently, $\Pi_\Omega(\bigcap_{k=1}^{n} G_k) = T_{k=1}^{n} \Pi_\Omega(G_k)$. We may therefore conclude that the family $\{f_j^{-1}(\mathcal{R}_j) \mid j \in J\}$ is (Π_Ω, T)-independent. This proves the equivalence of (i) and (ii).

Let us now prove that (ii) and (iii) are equivalent. First, assume that (ii) holds. Consider arbitrary n in $\mathbb{N} \setminus \{0\}$, arbitrary and different $j_1, \ldots, j_n$ in J and arbitrary x_k in X_{j_k}, $k = 1, \ldots, n$. For the choice $A_k = [\,x_k\,]_{\mathcal{R}_{j_k}}$, $k = 1, \ldots, n$, we have by assumption that $\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}(A_k)) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}(A_k))$, or equivalently, $\Pi_\Omega(\bigcap_{k=1}^{n} f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = T_{k=1}^{n} \Pi_\Omega(f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}}))$, whence indeed (iii). Conversely, assume that (iii) holds. Consider arbitrary n in $\mathbb{N} \setminus \{0\}$, arbitrary and different $j_1, \ldots, j_n$ in J and arbitrary A_k in $\mathcal{R}_{j_k}$, $k = 1, \ldots, n$. For any x_k in X_{j_k}, $k = 1, \ldots, n$, we have by assumption that $\Pi_\Omega(\bigcap_{k=1}^n f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = T_{k=1}^n \Pi_\Omega(f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}}))$. If we take the supremum on both sides of this equality, we find that

$$\sup_{(x_1,\dots,x_n)\in A_1\times\cdots\times A_n} \Pi_\Omega(\bigcap_{k=1}^n f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = \sup_{(x_1,\dots,x_n)\in A_1\times\cdots\times A_n} T_{k=1}^n \Pi_\Omega(f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})).$$

For the left hand side of this equation we may write that, taking into account Part I, Eq. (1),

$$\sup_{(x_1,\dots,x_n)\in A_1\times\cdots\times A_n} \Pi_\Omega(\bigcap_{k=1}^n f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = \Pi_\Omega(\bigcap_{k=1}^n f_{j_k}^{-1}(A_k)).$$

For the right hand side we find that, again taking into account Part I, Eq. (1),

$$\sup_{(x_1,\dots,x_n)\in A_1\times\cdots\times A_n} T_{k=1}^n \Pi_\Omega(f_{j_k}^{-1}([\,x_k\,]_{\mathcal{R}_{j_k}})) = T_{k=1}^n \Pi_\Omega(f_{j_k}^{-1}(A_k)).$$

Therefore $\Pi_\Omega(\bigcap_{k=1}^n f_{j_k}^{-1}(A_k)) = T_{k=1}^n \Pi_\Omega(f_{j_k}^{-1}(A_k))$, whence (ii). □

In Theorem 2.6, we consider the special case of two possibilistic variables. The analogy with the probabilistic case is striking. From statement (ii) of this theorem we may conclude that the definition of the possibilistic independence of possibilistic variables given here generalizes the definitions of Nahmias *et al.*, mentioned in the previous section, see Eqs. (2)–(4).

Theorem 2.6 may also be interpreted a formalization of the case considered by Zadeh and Hisdal, discussed in Part II, section 1. In statement (iv) we clearly distinguish a generalization of Part II, Eq. (5): Zadeh's noninteractivity is special case of the possibilistic independence discussed here. When the t-norm T is weakly invertible, we may therefore expect that there exists a relationship between possibilistic independence and conditional possibility. This relation is expressed by statements (v)–(vii). They are a generalization of Part II, Eq. (15) and *not* of the Eqs. (9) and (10) in Part II, that were originally proposed by Hisdal.

Before formulating this theorem, we must define our notation. We assume that X_1 and X_2 are two universes, provided with the respective ample fields $\mathcal{R}_1$ and $\mathcal{R}_2$. We consider a possibilistic variable $f_1\colon \Omega \to X_1$ in $(X_1, \mathcal{R}_1)$ and a possibilistic variable $f_2\colon \Omega \to X_2$ in $(X_2, \mathcal{R}_2)$. We know from Part II, Proposition 4.1 that the $\Omega - X_1 \times X_2$-mapping (f_1, f_2) is a possibilistic variable in $(X_1 \times X_2, \mathcal{R}_1 \times \mathcal{R}_2)$. Let us denote the possibility distributions of f_1, f_2 and (f_1, f_2) by Π_{f_1}, Π_{f_2} and $\Pi_{(f_1,f_2)}$ respectively, and their possibility distribution functions by π_{f_1}, π_{f_2} and $\pi_{(f_1,f_2)}$ respectively.

Theorem 2.6. *We shall assume where necessary, i.e., for the statements (v)–(vii) below, that the t-norm T is weakly invertible, so that we may rightfully speak of conditional $(L, \leq, T)$-possibility. The following propositions are equivalent.*

(i) f_1 *and* f_2 *are* (Π_Ω, T)*-independent.*

(ii) For any A_1 *in* $\mathcal{R}_1$ *and* A_2 *in* $\mathcal{R}_2$, $\Pi_{(f_1,f_2)}(A_1 \times A_2) = T(\Pi_{f_1}(A_1), \Pi_{f_2}(A_2))$.

(iii) For any x_1 *in* X_1 *and* x_2 *in* X_2, $\pi_{(f_1,f_2)}(x_1, x_2) = T(\pi_{f_1}(x_1), \pi_{f_2}(x_2))$.

(iv) $\Pi_{(f_1,f_2)} = \Pi_{f_1} \times_T \Pi_{f_2}$.

(v) For any A_1 *in* $\mathcal{R}_1$, $\Pi_{f_1|f_2}(A_1 \mid \cdot) \overset{(\Pi_{f_2},T)}{=} \underline{\Pi_{f_1}(A_1)}$.

(vi) *For any* A_2 *in* $\mathcal{R}_2$, $\Pi_{f_2|f_1}(A_2 \mid \cdot) \overset{(\Pi_{\underline{f_1}},T)}{=} \underline{\Pi_{f_2}(A_2)}$.

(vii) *For any* x_1 *in* X_1 *and* x_2 *in* X_2,

$$T(\pi_{f_1}(x_1), \pi_{f_2}(x_2)) = T(\pi_{f_1|f_2}(x_1 \mid x_2), \pi_{f_2}(x_2)) = T(\pi_{f_2|f_1}(x_2 \mid x_1), \pi_{f_1}(x_1)).$$

Proof. The equivalence of (i), (ii) and (iii) follows immediately from Corollary 2.5. The equivalence of (ii) and (iv) follows from Part I, Theorem 8.2 and Definition 8.3. Let us now prove that (iv) and (v) are equivalent. First, assume that (iv) holds. Consider an arbitrary A_1 in $\mathcal{R}_1$. Then by definition and by assumption, for any A_2 in $\mathcal{R}_2$,

$$(T)\int_{A_2} \Pi_{f_1|f_2}(A_1 \mid \cdot)\mathrm{d}\Pi_{f_2} = \Pi_{(f_1,f_2)}(A_1 \times A_2) = T(\Pi_{f_1}(A_1), \Pi_{f_2}(A_2)) = (T)\int_{A_2} \underline{\Pi_{f_1}(A_1)}\mathrm{d}\Pi_{f_2},$$

also taking into account Part I, Eq. (6). We conclude from Part I, Proposition 6.4(iii) that (v) holds. Conversely, assume that (v) holds. Consider arbitrary A_1 in $\mathcal{R}_1$ and A_2 in $\mathcal{R}_2$. By definition, and taking into account the assumption and Part I, Proposition 6.4(iii),

$$\Pi_{(f_1,f_2)}(A_1 \times A_2) = (T)\int_{A_2} \Pi_{f_1|f_2}(A_1 \mid \cdot)\mathrm{d}\Pi_{f_2} = (T)\int_{A_2} \underline{\Pi_{f_1}(A_1)}\mathrm{d}\Pi_{f_2} = T(\Pi_{f_1}(A_1), \Pi_{f_2}(A_2)),$$

also taking into account Part I, Eq. (6). We conclude that (ii) holds, and therefore also, from the discussion above, that (iv) holds.

The proof of the equivalence of (iv) and (vi) is completely analogous. Let us therefore complete this proof by showing that (iii) and (vii) are equivalent. For any x_1 in X_1 and x_2 in X_2 we have indeed, taking into account Part II, Eq. (26), that $\pi_{(f_1,f_2)}(x_1, x_2) = T(\pi_{f_1}(x_1), \pi_{f_2}(x_2))$ is equivalent to $T(\pi_{f_1}(x_1), \pi_{f_2}(x_2)) = T(\pi_{f_1|f_2}(x_1 \mid x_2), \pi_{f_2}(x_2)) = T(\pi_{f_2|f_1}(x_2 \mid x_1), \pi_{f_1}(x_1))$. □

3 INDEPENDENCE OF FUZZY EVENTS

Formally, fuzzy variables are special possibilistic variables (see also Part I, section 5). In defining the possibilistic independence of fuzzy variables, we may therefore make use of Definition 2.4. Corollary 3.2 gives us a number of criteria for this new form of independence. In this section, we use the notations established in Part I, section 5.

Definition 3.1. Consider an nonempty family $\{h_j \mid j \in J\}$ of $(L, \leq)$-fuzzy variables in $(\Omega, \mathcal{R}_\Omega)$. This family is called (Π_Ω, T)-independent iff the family $\{h_j \mid j \in J\}$ of possibilistic variables in $(L, \wp(L))$ is (Π_Ω, T)-independent. In this case we also say that the fuzzy variables h_j, $j \in J$, are (Π_Ω, T)-independent. Whenever we do not want to mention the $(L, \leq)$-possibility measure Π_Ω and/or the t-norm T explicitly, we simply speak of (possibilistic) independence instead of (Π_Ω, T)-independence.

Corollary 3.2. *The following propositions are equivalent.*

(i) *The family* $\{h_j \mid j \in J\}$ *of* $(L, \leq)$*-fuzzy variables in* $(\Omega, \mathcal{R}_\Omega)$ *is* (Π_Ω, T)*-independent.*

(ii) *For any* n *in* $\mathbb{N} \setminus \{0\}$, *for arbitrary and different* $j_1, \ldots, j_n$ *in* J *and for any* B_k *in* $\wp(L)$, $k = 1, \ldots, n$:

$$\Gamma_{(h_{j_1},\ldots,h_{j_n})}(B_1 \times \cdots \times B_n) = T_{k=1}^{n} \Gamma_{h_{j_k}}(B_k).$$

(iii) For any n in $\mathbb{N}\setminus\{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J and for any λ_k in L, $k = 1, \ldots, n$:

$$\gamma_{(h_{j_1},\ldots,h_{j_n})}(\lambda_1, \ldots, \lambda_n) = T_{k=1}^{n}\gamma_{h_{j_k}}(\lambda_k).$$

Proof. This corollary is a special case of Corollary 2.5. □

Theorem 3.3 provides a criterion for the possibilistic independence of a finite number of fuzzy variables. The formal analogy between these results and the well-known formulas for real stochastic variables – the counterparts of these fuzzy variables – in probability theory is striking (see, for instance, [Burrill, 1972] Theorem 11-4A and corollary).

Theorem 3.3. *The following statements are equivalent.*

(i) The $(L, \leq)$-fuzzy variables $h_1, \ldots, h_m$, $m \in \mathbb{N}\setminus\{0\}$, in $(\Omega, \mathcal{R}_\Omega)$ are (Π_Ω, T)-independent.

(ii) For any $(B_1, \ldots, B_m)$ in $\wp(L)^m$, $\Gamma_{(h_1,\ldots,h_m)}(B_1 \times \cdots \times B_m) = T_{k=1}^{m}\Gamma_{h_k}(B_k)$.

(iii) For any $(\lambda_1, \ldots, \lambda_m)$ in L^m, $\gamma_{(h_1,\ldots,h_m)}(\lambda_1, \ldots, \lambda_m) = T_{k=1}^{m}\gamma_{h_k}(\lambda_k)$.

Proof. It is easily proven that (ii) and (iii) are equivalent. Let us therefore concentrate on the equivalence of (i) and (ii). Corollary 3.2 already tells us that (ii) follows from (i). Let us therefore show that (i) follows from (ii). To this end, assume that (ii) holds, and consider an arbitrary n with $1 \leq n \leq m$. Also consider arbitrary and different $j_1, \ldots, j_n$ in $\{1, \ldots, m\}$ and arbitrary $(B_{j_1}, \ldots, B_{j_n})$ in $\wp(L)$. Then, taking into account Corollary 3.2, it must be shown that

$$\Gamma_{(h_{j_1},\ldots,h_{j_n})}(B_{j_1} \times \cdots \times B_{j_n}) = T_{k=1}^{n}\Gamma_{h_{j_k}}(B_{j_k}).$$

There are two possibilities. Either $n = m$, in which case the string of numbers $j_1 \ldots j_n$ is a permutation of the string $1 \ldots m$. For such a permutation it is easily verified that on the one hand $\Gamma_{(h_{j_1},\ldots,h_{j_n})}(B_{j_1} \times \cdots \times B_{j_n}) = \Gamma_{(h_1,\ldots,h_m)}(B_1 \times \cdots \times B_m)$. On the other hand, we clearly also have that, taking into account the commutativity of T, $T_{k=1}^{n}\Gamma_{h_{j_k}}(B_{j_k}) = T_{k=1}^{m}\Gamma_{h_k}(B_k)$. It now follows from (ii) that $\Gamma_{(h_1,\ldots,h_m)}(B_1 \times \cdots \times B_m) = T_{k=1}^{m}\Gamma_{h_k}(B_k)$, whence indeed, taking into account the equalities derived above, $\Gamma_{(h_{j_1},\ldots,h_{j_n})}(B_{j_1} \times \cdots \times B_{j_n}) = T_{k=1}^{n}\Gamma_{h_{j_k}}(B_{j_k})$.

Or[1] we have that $n < m$. We shall denote the elements of $\{1, \ldots, m\} \setminus \{j_1, \ldots, j_n\}$ as j_{n+1}, $\ldots$, j_m. Once again, the string of numbers $j_1 \ldots j_m$ is a permutation of the string $1 \ldots m$. For the choice $B_{j_k} = L$, $k = n+1, \ldots, m$, we may write, since for these k also $h_{j_k}^{-1}(B_{j_k}) = \Omega$,

$$\begin{aligned}\Gamma_{(h_1,\ldots,h_m)}(B_1 \times \cdots \times B_m) &= \Pi_\Omega\Big(\bigcap_{k=1}^{m} h_k^{-1}(B_k)\Big)\\ &= \Pi_\Omega\Big(\bigcap_{k=1}^{m} h_{j_k}^{-1}(B_{j_k})\Big)\\ &= \Pi_\Omega\Big(\bigcap_{k=1}^{n} h_{j_k}^{-1}(B_{j_k})\Big)\\ &= \Gamma_{(h_{j_1},\ldots,h_{j_n})}(B_{j_1} \times \cdots \times B_{j_n}).\end{aligned}$$

[1]When $m = 1$ this case can be excluded.

On the other hand, taking into account the commutativity and associativity of T,

$$\begin{aligned}
T_{k=1}^m \Gamma_{h_k}(B_k) &= T_{k=1}^m \Pi_\Omega(h_k^{-1}(B_k))\\
&= T_{k=1}^m \Pi_\Omega(h_{j_k}^{-1}(B_{j_k}))\\
&= T(T_{k=1}^n \Pi_\Omega(h_{j_k}^{-1}(B_{j_k})), T_{k=n+1}^m \Pi_\Omega(h_{j_k}^{-1}(L)))\\
&= T(T_{k=1}^n \Pi_\Omega(h_{j_k}^{-1}(B_{j_k})), T_{k=n+1}^m 1_L)\\
&= T_{k=1}^n \Pi_\Omega(h_{j_k}^{-1}(B_{j_k}))\\
&= T_{k=1}^n \Gamma_{h_{j_k}}(B_{j_k}).
\end{aligned}$$

From the assumption, we furthermore deduce that $\Gamma_{(h_1,\dots,h_m)}(B_1 \times \cdots \times B_m) = T_{k=1}^m \Gamma_{h_k}(B_k)$. Using the equalities derived above, we find that, $\Gamma_{(h_{j_1},\dots,h_{j_n})}(B_{j_1} \times \cdots \times B_{j_n}) = T_{k=1}^n \Gamma_{h_{j_k}}(B_{j_k})$. We conclude that the $(L, \leq)$-fuzzy variables $h_1, \dots, h_m$ in $(\Omega, \mathcal{R}_\Omega)$ are (Π_Ω, T)-independent. □

Theorem 3.4 also has a probabilistic counterpart: the expectation of a product of independent real stochastic variables equals the product of the expectations of those variables (see, for instance, [Burrill, 1972] Theorem 11-4C). For the notations, I refer to Part I, Eq. (10).

Theorem 3.4. *If the $(L, \leq)$-fuzzy variables $h_1, \dots, h_m$, $m \in \mathbb{N} \setminus \{0\}$, in $(\Omega, \mathcal{R}_\Omega)$ are (Π_Ω, T)-independent, we have that*

$$(\Pi_\Omega)_T\,(T_{k=1}^m h_k) = T_{k=1}^m (\Pi_\Omega)_T(h_k).$$

Proof. It is easily verified that $T_{k=1}^m h_k$ is $\mathcal{R}_\Omega$-measurable. Furthermore, if we denote any associative extension of T also by T, we have by definition that

$$\begin{aligned}
(\Pi_\Omega)_T\,(T_{k=1}^m h_k) &= (T)\!\int_\Omega T_{k=1}^m h_k \mathrm{d}\Pi_\Omega\\
&= (T)\!\int_\Omega T \circ (h_1, \dots, h_m)\mathrm{d}\Pi_\Omega\\
&= (T)\!\int_{L^m} T \mathrm{d}\Gamma_{(h_1,\dots,h_m)}\\
&= \sup_{(\lambda_1,\dots,\lambda_m)\in L^m} T(T(\lambda_1, \dots, \lambda_m), \gamma_{(h_1,\dots,h_m)}(\lambda_1, \dots, \lambda_m)),
\end{aligned}$$

also taking into account Part I, Theorem 5.5 and Part I, Eq. (5). From the assumption and Theorem 3.3, we deduce that, also taking into account the associativity and the commutativity of T,

$$\begin{aligned}
&= \sup_{(\lambda_1,\dots,\lambda_m)\in L^m} T\,(T_{k=1}^m \lambda_k, T_{k=1}^m \gamma_{h_k}(\lambda_k))\\
&= \sup_{(\lambda_1,\dots,\lambda_m)\in L^m} T_{k=1}^m T(\lambda_k, \gamma_{h_k}(\lambda_k))\\
&= \sup_{\lambda_1\in L} \dots \sup_{\lambda_m\in L} T_{k=1}^m T(\lambda_k, \gamma_{h_k}(\lambda_k))\\
&= T_{k=1}^m \sup_{\lambda_k\in L} T(\lambda_k, \gamma_{h_k}(\lambda_k)),
\end{aligned}$$

whence, taking into account Part I, Corollary 5.6,

$$= T_{k=1}^m (\Pi_\Omega)_T(h_k). \quad \square$$

We conclude this section with a number of results that express the relationship between the possibilistic independence of possibilistic and fuzzy variables. For the probabilistic counterpart of Theorem 3.6 I refer to [Burrill, 1972] Theorem 15-1J(e), and a counterpart for Theorem 3.7 can be found in [Burrill, 1972] Theorem 15-3C(b).

Lemma 3.5. *Let h and g be $(L, \leq)$-fuzzy variables in $(X, \mathcal{R}_\Omega)$. If the fuzzy variables h and g in $(\Omega, \mathcal{R}_\Omega)$ are (Π_Ω, T)-independent, then the fuzzy variables h and $\chi_{g^{-1}(B)}$ in $(\Omega, \mathcal{R}_\Omega)$ are (Π_Ω, T)-independent, for any B in $\wp(L)$.*

Proof. Let λ_1 and λ_2 be arbitrary elements of L and let B be an arbitrary subset of $\wp(L)$. Then

$$\chi^{-1}_{g^{-1}(B)}(\{\lambda_2\}) = \begin{cases} g^{-1}(B) & ; \quad \lambda_2 = 1_L \\ g^{-1}(\text{co}B) & ; \quad \lambda_2 = 0_L \\ \emptyset & ; \quad \text{elsewhere,} \end{cases}$$

and by definition also

$$\gamma_{\chi_{g^{-1}(B)}}(\lambda_2) = \Pi_\Omega(\chi^{-1}_{g^{-1}(B)}(\{\lambda_2\})) = \begin{cases} \Pi_\Omega(g^{-1}(B)) = \Gamma_g(B) & ; \quad \lambda_2 = 1_L \\ \Pi_\Omega(g^{-1}(\text{co}B)) = \Gamma_g(\text{co}B) & ; \quad \lambda_2 = 0_L \\ 0_L & ; \quad \text{elsewhere.} \end{cases}$$

By definition, we also have that

$$\gamma_{(h,\chi_{g^{-1}(B)})}(\lambda_1, \lambda_2) = \Pi_\Omega(h^{-1}(\{\lambda_1\}) \cap \chi^{-1}_{g^{-1}(B)}(\{\lambda_2\})),$$

and, taking into account the equalities derived above,

$$= \begin{cases} \Gamma_{(h,g)}(\{\lambda_1\} \times B) & ; \quad \lambda_2 = 1_L \\ \Gamma_{(h,g)}(\{\lambda_1\} \times \text{co}B) & ; \quad \lambda_2 = 0_L \\ 0_L & ; \quad \text{elsewhere.} \end{cases}$$

Assume that h and g are (Π_Ω, T)-independent. It now follows from Theorem 3.3 that

$$= \begin{cases} T(\gamma_h(\lambda_1), \Gamma_g(B)) & ; \quad \lambda_2 = 1_L \\ T(\gamma_h(\lambda_1), \Gamma_g(\text{co}B)) & ; \quad \lambda_2 = 0_L \\ 0_L & ; \quad \text{elsewhere,} \end{cases}$$

and once again taking into account the equalities derived above

$$= T(\gamma_h(\lambda_1), \gamma_{\chi_{g^{-1}(B)}}(\lambda_2)).$$

From Theorem 3.3 we deduce that h and $\chi_{g^{-1}(B)}$ are (Π, T)-independent. □

Theorem 3.6. *Let the t-norm T be weakly invertible, so that we may rightfully speak of conditional $(L, \leq, T)$-possibility. Let h and g be $(L, \leq)$-fuzzy variables in $(\Omega, \mathcal{R}_\Omega)$. If h and g are (Π_Ω, T)-independent,*

$$\Pi_\Omega(h \mid g = \cdot) \overset{(\Gamma_g, T)}{=} \underline{(\Pi_\Omega)_T(h)}.$$

Proof. Consider an arbitrary B in $\wp(L)$. By definition, and taking into account Part I, Eq. (8) and $T(\chi_{g^{-1}(B)}, h) = \chi_{g^{-1}(B)} \frown h$,

$$(T)\!\!\int_B \Pi_\Omega(h \mid g = \cdot)\mathrm{d}\Gamma_g = (T)\!\!\int_{g^{-1}(B)} h\mathrm{d}\Pi_\Omega,$$
$$= (T)\!\!\int_\Omega T(\chi_{g^{-1}(B)}, h)\mathrm{d}\Pi_\Omega.$$

Taking into account the previous lemma, Theorem 3.4 and Part I, Eqs.(6) and (9), we find that

$$\begin{aligned}
&= T\left((T)\!\!\int_\Omega \chi_{g^{-1}(B)}\mathrm{d}\Pi_\Omega, (T)\!\!\int_\Omega h\mathrm{d}\Pi_\Omega\right)\\
&= T(\Pi_\Omega(g^{-1}(B)), (\Pi_\Omega)_T(h))\\
&= T(\Gamma_g(B), (\Pi_\Omega)_T(h))\\
&= (T)\!\!\int_B \underline{(\Pi_\Omega)_T(h)}\mathrm{d}\Gamma_g.
\end{aligned}$$

Proposition 6.4(iii) in Part I now tells us that the proof is complete. □

Theorem 3.7. *Let the t-norm T be weakly invertible, so that we may rightfully speak of conditional $(L, \leq, T)$-possibility. Let h and g be $(L, \leq)$-fuzzy variables in $(\Omega, \mathcal{R}_\Omega)$. Then h and g are (Π_Ω, T)-independent if and only if*

$$(\forall \lambda \in L)(\gamma_{h|g}(\lambda \mid \cdot) \overset{(\Gamma_g, T)}{=} \underline{\gamma_h(\lambda)}), \tag{11}$$

or equivalently,

$$(\forall(\lambda, \mu) \in L^2)(T(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = T(\gamma_h(\lambda), \gamma_g(\mu)).$$

Proof. Assume on the one hand that h and g are (Π_Ω, T)-independent. It follows from Lemma 3.5 that, for any λ in L, $\chi_{h^{-1}(\{\lambda\})}$ and g are (Π_Ω, T)-independent. Therefore, (11) follows from the previous theorem, taking into account (see also Part II, Definition 3.6)

$$\Pi_\Omega(\chi_{h^{-1}(\{\lambda\})} \mid g = \cdot) = \Pi_\Omega(h^{-1}(\{\lambda\}) \mid g = \cdot) = \gamma_{h|g}(\lambda \mid \cdot),$$

and (see also Part I, Eq. (9))

$$(\Pi_\Omega)_T(\chi_{h^{-1}(\{\lambda\})}) = (T)\!\!\int_\Omega \chi_{h^{-1}(\{\lambda\})}\mathrm{d}\Pi_\Omega = \Pi_\Omega(h^{-1}(\{\lambda\})) = \gamma_h(\lambda).$$

On the other hand, assume that (11) holds, whence for any λ and μ in L, $T(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = T(\gamma_h(\lambda), \gamma_g(\mu))$. By Part II, Theorem 3.8(ii), we find $T(\gamma_{h|g}(\lambda \mid \mu), \gamma_g(\mu)) = \gamma_{(h,g)}(\lambda, \mu)$, whence finally $\gamma_{(h,g)}(\lambda, \mu) = T(\gamma_h(\lambda), \gamma_g(\mu))$. Theorem 3.3 tells us that h and g are (Π_Ω, T)-independent. □

4 INDEPENDENCE OF EVENTS

A fuzzy variable in $(\Omega, \mathcal{R}_\Omega)$ is a generalization of a $\mathcal{R}_\Omega$-measurable subset of Ω: an arbitrary element A of $\mathcal{R}_\Omega$ can be identified with the $(L, \leq)$-fuzzy variable χ_A in $(\Omega, \mathcal{R}_\Omega)$. In Definition 4.1 we use this identification to introduce the possibilistic independence of events. In Theorem 4.2 we deduce a criterion for this independence (probabilistic counterpart: [Burrill, 1972] section 11-4).

Definition 4.1. Consider a family $\{A_j \mid j \in J\}$ of elements of $\mathcal{R}_\Omega$. This family is called (Π_Ω, T)-independent iff the family $\{\chi_{A_j} \mid j \in J\}$ of $(L, \leq)$-fuzzy variables in $(\Omega, \mathcal{R}_\Omega)$ is (Π_Ω, T)-independent. In this case we also say that the elements A_j, $j \in J$, of $\mathcal{R}_\Omega$ are (Π_Ω, T)-independent. Whenever we do not want to mention the $(L, \leq)$-possibility measure Π_Ω and/or the t-norm T explicitly, we simply speak of (possibilistic) independence instead of (Π_Ω, T)-independence.

Theorem 4.2. *A family $\{A_j \mid j \in J\}$ of elements of $\mathcal{R}_\Omega$ is (Π_Ω, T)-independent if and only if*

(A) *for any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, for any F_k in $\{A_{j_k}, \mathrm{co}A_{j_k}\}$, $k = 1, \ldots, n$, $\Pi_\Omega(\bigcap_{k=1}^n F_k) = T_{k=1}^n \Pi_\Omega(F_k)$.*

Proof. The family $\{A_j \mid j \in J\}$ of elements of $\mathcal{R}_\Omega$ is, taking into account Corollary 3.2, (Π_Ω, T)-independent if and only if for any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J and for any B_k in $\wp(L)$, $k = 1, \ldots, n$, $\Pi_\Omega(\bigcap_{k=1}^n \chi^{-1}_{A_{j_k}}(B_k)) = T_{k=1}^n \Pi_\Omega(\chi^{-1}_{A_{j_k}}(B_k))$. Now we have for $k = 1, \ldots, n$ that

$$\chi^{-1}_{A_{j_k}}(B_k) = \begin{cases} A_{j_k} & ; \quad 1_L \in B_{j_k} \text{ and } 0_L \notin B_{j_k} \\ \mathrm{co}A_{j_k} & ; \quad 0_L \in B_{j_k} \text{ and } 1_L \notin B_{j_k} \\ \Omega & ; \quad 1_L \in B_{j_k} \text{ and } 0_L \in B_{j_k} \\ \emptyset & ; \quad 1_L \notin B_{j_k} \text{ and } 0_L \notin B_{j_k}. \end{cases}$$

This implies that the family $\{A_j \mid j \in J\}$ is (Π_Ω, T)-independent if and only if

(B) for any n in $\mathbb{N} \setminus \{0\}$, for arbitrary and different $j_1, \ldots, j_n$ in J, for any G_k in $\{A_{j_k}, \mathrm{co}A_{j_k}, \emptyset, \Omega\}$, $k = 1, \ldots, n$, $\Pi_\Omega(\bigcap_{k=1}^n G_k) = T_{k=1}^n \Pi_\Omega(G_k)$.

It is clear that (A) follows from (B). The proof is therefore complete if we can show that (B) follows from (A). Assume that (A) holds. Consider an arbitrary n in $\mathbb{N} \setminus \{0\}$, arbitrary and different $j_1, \ldots, j_n$ in J and arbitrary G_k in $\{A_{j_k}, \mathrm{co}A_{j_k}, \emptyset, \Omega\}$. It must be shown that

$$\Pi_\Omega(\bigcap_{k=1}^n G_k) = T_{k=1}^n \Pi_\Omega(G_k). \tag{12}$$

For the choice of G_k, $k = 1, \ldots, n$, made above, we now define

$$\begin{aligned} M_1 &= \{\, k \mid k \in \{1, \ldots, n\} \text{ and } G_k = \emptyset \,\} \\ M_2 &= \{\, k \mid k \in \{1, \ldots, n\} \text{ and } G_k = \Omega \,\} \\ M_3 &= \{1, \ldots, n\} \setminus (M_1 \cup M_2). \end{aligned}$$

When $M_1 \neq \emptyset$, (12) trivially holds. Let us therefore assume that $M_1 = \emptyset$. In this case there are two possibilities. The first is that $M_2 \neq \emptyset$. If furthermore $M_3 = \emptyset$, this implies that $M = M_2$, and (12) can therefore be rewritten as $\Pi_\Omega(\bigcap_{k=1}^n \Omega) = T_{k=1}^n \Pi_\Omega(\Omega)$. Since Π_Ω is normal, this is equivalent to the trivially true proposition '$1_L = 1_L$'. Let us therefore assume that $M_3 \neq \emptyset$. Eq. (12) can in this case, with obvious notations, be rewritten as

$$\Pi_\Omega((\bigcap_{k \in M_3} G_k) \cap (\bigcap_{k \in M_2} G_k)) = T(T_{k \in M_3} \Pi_\Omega(G_k), T_{k \in M_2} \Pi_\Omega(G_k)),$$

which is clearly equivalent to

$$\Pi_\Omega(\bigcap_{k \in M_3} G_k) = T_{k \in M_3} \Pi_\Omega(G_k). \tag{13}$$

Since $(\forall k \in M_3)(G_k \in \{\text{co}A_{j_k}, A_{j_k}\})$, it follows from (A) that in this case (13) and therefore also (12) holds.

The second possibility is that $M_2 = \emptyset$. We then have that $G_k \in \{\text{co}A_{j_k}, A_{j_k}\}$, $k \in \{1, \ldots, n\}$, and therefore (12) immediately follows from (A). □

In proposition 4.3 we formulate an independence criterion for two events. That this criterion has four conditions instead of one in its well-known probabilistic counterpart, is necessary in order to make sure that it would be invariant under the complementation of the events involved. This is also apparent from Proposition 4.4. Note that these results generalize Eq. (6).

Proposition 4.3. *Two elements O_1 and O_2 of $\mathcal{R}_\Omega$ are (Π_Ω, T)-independent if and only if*

$$\begin{cases} \Pi_\Omega(O_1 \cap O_2) &= T(\Pi_\Omega(O_1), \Pi_\Omega(O_2)) \\ \Pi_\Omega(O_1 \cap \text{co}O_2) &= T(\Pi_\Omega(O_1), \Pi_\Omega(\text{co}O_2)) \\ \Pi_\Omega(\text{co}O_1 \cap O_2) &= T(\Pi_\Omega(\text{co}O_1), \Pi_\Omega(O_2)) \\ \Pi_\Omega(\text{co}O_1 \cap \text{co}O_2) &= T(\Pi_\Omega(\text{co}O_1), \Pi_\Omega(\text{co}O_2)). \end{cases}$$

Furthermore, for any O in $\mathcal{R}_\Omega$, $\emptyset$, O and Ω are (Π_Ω, T)-independent.

Proposition 4.4. *Let $\{A_j \mid j \in J\}$ be a family of elements of $\mathcal{R}_\Omega$ and let $\{A'_j \mid j \in J\}$ be a family of elements of $\mathcal{R}_\Omega$, obtained by the substitution for their complements of an arbitrary number of elements of the first family. Then the family $\{A_j \mid j \in J\}$ is (Π_Ω, T)-independent if and only if the family $\{A'_j \mid j \in J\}$ is (Π_Ω, T)-independent.*

Proof. Immediately from Theorem 4.2 and the invariance of the independence criterion (A) for complementation. □

It should be noted that this definition of the possibilistic independence of events is different from Nahmias' definition of mutual unrelatedness, because it is invariant under complementation, as the previous proposition indicates. In Example 2.3 we have shown that this is not the case for Nahmias' definition.

Also remark that this invariance follows from the definition of the possibilistic independence of fuzzy variables, and therefore indirectly from that of possibilistic variables. Nahmias' unrelatedness for his type of fuzzy variables is a special case of the definition of the possibilistic independence of possibilistic variables given here. Furthermore, the 'trick' we use to convert the independence of fuzzy events into the independence of events is precisely the identification of an event with its characteristic mapping – a fuzzy event. In probability theory, an analogous approach is followed [Burrill, 1972].

Finally, consider an arbitrary element A of $\mathcal{R}_\Omega$. Taking into account Proposition 4.3, we find that the events A and A are (Π_Ω, T)-independent if and only if

$$\begin{cases} T(\Pi_\Omega(A), \Pi_\Omega(A)) &= \Pi_\Omega(A) \\ T(\Pi_\Omega(\text{co}A), \Pi_\Omega(\text{co}A)) &= \Pi_\Omega(\text{co}A) \\ T(\Pi_\Omega(A), \Pi_\Omega(\text{co}A)) &= 0_L. \end{cases}$$

If we consider, for instance, $(L, \leq) = ([0,1], \leq)$ and $T = \min$, this may be rewritten as $\min(\Pi_\Omega(A), \Pi_\Omega(\text{co}A)) = 0$, or equivalently, $\Pi_\Omega(A) = 0$ or $\Pi_\Omega(\text{co}A) = 0$, which is comparable to the probabilistic case. These remarks indicate that the approach described here solves the problems associated with Nahmias' definition of unrelatedness, discussed in the introduction.

In the following theorem we show that there exists an even closer relationship between the independence of possibilistic variables, fuzzy events and events, than their definitions would make us suspect. Let us borrow the notations of Theorem 2.6.

Theorem 4.5. *The following statements are equivalent.*

(i) f_1 *and* f_2 *are* (Π_Ω, T)*-independent.*

(ii) For every h_1 *in* $\mathcal{G}^{\mathcal{R}_1}_{(L,\leq)}(X_1)$ *and* h_2 *in* $\mathcal{G}^{\mathcal{R}_2}_{(L,\leq)}(X_2)$*, the cylindric extensions* $\overline{h_1}$ *and* $\overline{h_2}$ *to* $X_1 \times X_2$ *are* $(\Pi_{(f_1,f_2)}, T)$*-independent.*

(iii) For every A_1 *in* $\mathcal{R}_1$ *and* A_2 *in* $\mathcal{R}_2$*,* $A_1 \times X_2$ *and* $X_1 \times A_2$ *are* $(\Pi_{(f_1,f_2)}, T)$*-independent.*

Proof. We give a circular proof. For a start, let us show that (i) implies (ii). Assume that f_1 and f_2 are (Π_Ω, T)-independent. Consider an arbitrary h_1 in $\mathcal{G}^{\mathcal{R}_1}_{(L,\leq)}(X_1)$ and an arbitrary h_2 in $\mathcal{G}^{\mathcal{R}_2}_{(L,\leq)}(X_2)$. Clearly, the cylindric extensions $\overline{h_1}$ and $\overline{h_2}$ to $X_1 \times X_2$ are $\mathcal{R}_1 \times \mathcal{R}_2$-measurable (see also Part I, Proposition 8.8(i)). By definition, for any λ_1 and λ_2 in L,

$$\begin{aligned}\gamma_{(\overline{h_1},\overline{h_2})}(\lambda_1,\lambda_2) &= \Pi_{(f_1,f_2)}((\overline{h_1},\overline{h_2})^{-1}(\{(\lambda_1,\lambda_2)\}))\\ &= \Pi_{(f_1,f_2)}(\overline{h_1}^{-1}(\{\lambda_1\}) \cap \overline{h_2}^{-1}(\{\lambda_2\}))\\ &= \Pi_{(f_1,f_2)}(h_1^{-1}(\{\lambda_1\}) \times X_2 \cap X_1 \times h_2^{-1}(\{\lambda_2\}))\\ &= \Pi_{(f_1,f_2)}(h_1^{-1}(\{\lambda_1\}) \times h_2^{-1}(\{\lambda_2\}))\end{aligned}$$

and, taking into account the assumption and Theorem 2.6,

$$= T(\Pi_{f_1}(h_1^{-1}(\{\lambda_1\})), \Pi_{f_2}(h_2^{-1}(\{\lambda_2\})))$$

and, taking into account Part II, Proposition 4.3(i) and (ii),

$$\begin{aligned}&= T(\Pi_{(f_1,f_2)}(h_1^{-1}(\{\lambda_1\}) \times X_2), \Pi_{(f_1,f_2)}(X_1 \times h_2^{-1}(\{\lambda_2\})))\\ &= T(\Pi_{(f_1,f_2)}(\overline{h_1}^{-1}(\{\lambda_1\})), \Pi_{(f_1,f_2)}(\overline{h_2}^{-1}(\{\lambda_2\})))\\ &= T(\gamma_{\overline{h_1}}(\lambda_1), \gamma_{\overline{h_2}}(\lambda_2)).\end{aligned}$$

From Theorem 3.3 we deduce that $\overline{h_1}$ and $\overline{h_2}$ are $(\Pi_{(f_1,f_2)}, T)$-independent.

Next, we prove that (iii) follows from (ii). Assume that (ii) holds. Consider arbitrary A_1 in $\mathcal{R}_1$ and A_2 in $\mathcal{R}_2$, and the corresponding $(L,\leq)$-fuzzy variables χ_{A_1} in $(X_1,\mathcal{R}_1)$ and χ_{A_2} in $(X_2,\mathcal{R}_2)$. Taking into account (ii), their cylindric extensions $\overline{\chi_{A_1}}$ and $\overline{\chi_{A_2}}$ are $(\Pi_{(f_1,f_2)}, T)$-independent. Since furthermore $\overline{\chi_{A_1}} = \chi_{A_1 \times X_2}$ and $\overline{\chi_{A_2}} = \chi_{X_1 \times A_2}$, it follows by definition that $A_1 \times X_2$ and $X_1 \times A_2$ are $(\Pi_{(f_1,f_2)}, T)$-independent.

To conclude the proof, it must be shown that (i) follows from (iii). Assume that (iii) holds. Consider an arbitrary (x_1,x_2) in $X_1 \times X_2$. By assumption, $[\,x_1\,]_{\mathcal{R}_1} \times X_2$ and $X_1 \times [\,x_2\,]_{\mathcal{R}_2}$ are $(\Pi_{(f_1,f_2)}, T)$-independent, and Theorem 4.2 in particular implies that

$$\Pi_{(f_1,f_2)}([\,x_1\,]_{\mathcal{R}_1} \times X_2 \cap X_1 \times [\,x_2\,]_{\mathcal{R}_2}) = T(\Pi_{(f_1,f_2)}([\,x_1\,]_{\mathcal{R}_1} \times X_2), \Pi_{(f_1,f_2)}(X_1 \times [\,x_2\,]_{\mathcal{R}_2}))$$

whence, by Part II, Proposition 4.3(i) and (ii), $\pi_{(f_1,f_2)}(x_1,x_2) = T(\pi_{f_1}(x_1), \pi_{f_2}(x_2))$. From Theorem 2.6 it follows that (i) holds. □

5 AN EXAMPLE: CLASSICAL POSSIBILITY

In the previous sections, I have shown that the definition of possibilistic independence studied here is an improvement of the existing definitions from the formal mathematical point of view: not only does there exist a close relation between the independence of possibilistic variables and the independence of events, but also in the well-known special case studied by Nahmias a measurable set is not as a rule independent of itself.

In this section, I argue that, at least in one special case, this definition is also meaningful from the interpretational point of view. To do so, let us study its meaning in *classical, two-valued possibility.* We therefore choose, as in Part II, subsection 3.4, $(L, \leq) = (\{0,1\}, \leq)$, $T = \wedge$ and $\Pi_\Omega = \Pi_A$, with A an arbitrary but fixed element of $\mathcal{R}_\Omega \setminus \{\emptyset\}$. More explicitly, for any B in $\mathcal{R}_\Omega$:

$$\Pi_A(B) = \begin{cases} 1 & ; \quad A \cap B \neq \emptyset \\ 0 & ; \quad A \cap B = \emptyset, \end{cases}$$

For a discussion of the meaning of the possibility measure Π_A, I refer to Part II, subsection 3.4. For any B in $\mathcal{R}_\Omega$, we call B *necessary* iff $\Pi_A(\text{co}B) = 0$, *impossible* iff $\Pi_A(B) = 0$ and *uncertain* otherwise, i.e., iff $\Pi_A(B) = \Pi_A(\text{co}B) = 1$.

Let us first look for an interpretation of the $(\Pi_A, \wedge)$-independence of two arbitrary events B and C in $\mathcal{R}_\Omega$. According to Proposition 4.3, a necessary and sufficient condition for the independence of B and C is

$$\begin{cases} \Pi_A(B \cap C) &= \Pi_A(B) \wedge \Pi_A(C) \\ \Pi_A(\text{co}B \cap C) &= \Pi_A(\text{co}B) \wedge \Pi_A(C) \\ \Pi_A(B \cap \text{co}C) &= \Pi_A(B) \wedge \Pi_A(\text{co}C) \\ \Pi_A(\text{co}B \cap \text{co}C) &= \Pi_A(\text{co}B) \wedge \Pi_A(\text{co}C). \end{cases} \tag{14}$$

Let us now turn (14) into an equivalent form that is more easily interpretable. For any D and E in $\mathcal{R}_\Omega$, it is easily proven that the statement '$\Pi_A(D \cap E) = \Pi_A(D) \wedge \Pi_A(E)$' is equivalent to '$A \cap D \cap E \neq \emptyset$ or $A \cap D = \emptyset$ or $A \cap \text{co}D = \emptyset$ or $A \cap E = \emptyset$ or $A \cap \text{co}E = \emptyset$'. This implies, after some elementary set theoretic manipulations, that B and C are $(\Pi_A, \wedge)$-independent if and only if

$$\left. \begin{aligned} \Pi_A(B) = 1 \\ \Pi_A(C) = 1 \\ \Pi_A(\text{co}B) = 1 \\ \Pi_A(\text{co}C) = 1 \end{aligned} \right\} \Rightarrow \begin{cases} A \cap B \cap C \neq \emptyset \\ A \cap \text{co}B \cap C \neq \emptyset \\ A \cap B \cap \text{co}C \neq \emptyset \\ A \cap \text{co}B \cap \text{co}C \neq \emptyset. \end{cases}$$

This means that on the one hand the events B and C are possibilistically independent as soon as one of them is *impossible* or *necessary.* On the other hand, whenever both events are *uncertain*, they are possibilistically independent if and only if

$$A \cap C_1 \neq \emptyset, A \cap C_2 \neq \emptyset, A \cap C_3 \neq \emptyset, A \cap C_4 \neq \emptyset, \tag{15}$$

where the events $C_1 = B \cap C$, $C_2 = \text{co}B \cap C$, $C_3 = B \cap \text{co}C$ and $C_4 = \text{co}B \cap \text{co}C$ are called the *constituents* of B and C (see, for instance, [de Finetti, 1974]). The constituents different from $\emptyset$ make up a partition of Ω.

Let us now assume that the occurrence of B and C is possible but not necessary – and therefore uncertain –, and look for an interpretation of (15). From this assumption, we easily deduce that

$$A \not\subseteq C_1, A \not\subseteq C_2, A \not\subseteq C_3, A \not\subseteq C_4. \tag{16}$$

Indeed, assume that for instance $A \subseteq C_3$, or equivalently, $\emptyset = A \cap \mathrm{co}(B \cap \mathrm{co}C) = (A \cap \mathrm{co}B) \cup (A \cap C)$. This implies that $A \cap \mathrm{co}B = \emptyset$ and $A \cap C = \emptyset$, which contradicts our assumption that B and C are uncertain. We may therefore conclude that in this case (15) holds if and only if the constituents C_1, C_2, C_3 and C_4 are *uncertain* events. In this case, the uncertain events B and C are in the literature called *logically independent* (see, for instance, [de Finetti, 1974] section 2.7). *This logical independence means that additional knowledge about the occurrence of either event B or C can on no account change the existing uncertainty about the occurrence of the other event.*

To illustrate this, let us assume that (15) holds and that we know that the event B occurs. We now ask ourselves if this additional information can remove the uncertainty about the occurrence of C. This question must be answered in the negative: taking into account (15), we have that $A \cap B \cap C \neq \emptyset$ and $A \cap B \cap \mathrm{co}C \neq \emptyset$, which means that *a priori* both C and co$C$ are uncertain. The uncertainty about the occurrence of C is preserved due to (15).

We may therefore conclude that two arbitrary events B and C are $(\Pi_A, \wedge)$-independent if and only if at least one of them is either necessary or impossible, or, whenever they are both uncertain, if and only if they are logically independent.

Let us now turn to the (Π_A)-independence of two *possibilistic variables*, and find out whether this also has a simple interpretation. We again borrow the notations from Theorem 2.6. Let us first turn the criterion for the $(\Pi_A, \wedge)$-independence of the possibilistic variables f_1 and f_2, given in Theorem 2.6(iii),

$$(\forall (x_1, x_2) \in X_1 \times X_2)(\pi_{(f_1,f_2)}(x_1, x_2) = \pi_{f_1}(x_1) \wedge \pi_{f_2}(x_2)) \tag{17}$$

into a form that is more readily interpreted. It is easily shown that, using the notations introduced in Part I, Eq. (4), $\Pi_{f_1} = \Pi_{p_{\mathcal{R}_1}(f_1(A))}$ and $\pi_{f_1} = \chi_{p_{\mathcal{R}_1}(f_1(A))}$; $\Pi_{f_2} = \Pi_{p_{\mathcal{R}_2}(f_2(A))}$ and $\pi_{f_2} = \chi_{p_{\mathcal{R}_2}(f_2(A))}$; $\Pi_{(f_1,f_2)} = \Pi_{p_{\mathcal{R}_1 \times \mathcal{R}_2}((f_1,f_2)(A))}$ and $\pi_{(f_1,f_2)} = \chi_{p_{\mathcal{R}_1 \times \mathcal{R}_2}((f_1,f_2)(A))}$. These formulas have a natural interpretation: since we know that in the universe $\mathcal{R}_\Omega$ the event A occurs, we also know that in the universe X_1 the event $p_{\mathcal{R}_1}(f_1(A))$ occurs. It should not surprise us that we find $p_{\mathcal{R}_1}(f_1(A))$ instead of $f_1(A)$: we have no guarantee that $f_1(A) \in \mathcal{R}_1$, whereas $p_{\mathcal{R}_1}(f_1(A))$ is the smallest $\mathcal{R}_1$-measurable set including $f_1(A)$. For f_2 and (f_1, f_2) analogous interpretations can be given. It turns out that there is a clear correspondence between what we would intuitively expect, and what the theory, using projections and transformations of possibility measures, tells us.

The criterion (17) can now be rewritten as

$$p_{\mathcal{R}_1 \times \mathcal{R}_2}((f_1, f_2)(A)) = p_{\mathcal{R}_1}(f_1(A)) \times p_{\mathcal{R}_2}(f_2(A)). \tag{18}$$

Let us now try and interpret this. We first observe that a number of straightforward manipulations allow us to derive the following statement from this criterion:

$$(\forall A_1 \in \mathcal{R}_1)(A_1 \cap p_{\mathcal{R}_1}(f_1(A)) \neq \emptyset \Rightarrow p_{\mathcal{R}_2}(f_2(A)) \subseteq p_{\mathcal{R}_2}(f_2(f_1^{-1}(p_{\mathcal{R}_1}(f_1(A)) \cap A_1)))). \tag{19}$$

Now, assume that f_1 and f_2 are $(\Pi_A, \wedge)$-independent, so that we know that (19) holds. Assume furthermore that we have additional information about the value that f_1 assumes in X_1, say, that f_1 with certainty assumes a value in the set A_1 in $\mathcal{R}_1$, or, in other words, that the event A_1 occurs. Is it possible to deduce from this fact information about the value which f_2 assumes in X_2, that is more specific than the information we already have, namely, that f_2 with certainty takes a value in $p_{\mathcal{R}_2}(f_2(A))$? Let us now show that this is *impossible*, due to (19). We know on the one hand that f_1 with certainty assumes a value in $p_{\mathcal{R}_1}(f_1(A)) \cap A_1$. In order that the extra information would make sense, it must be that $p_{\mathcal{R}_1}(f_1(A)) \cap A_1 \neq \emptyset$. Let us assume that this is indeed the case. Then, by combining both chunks of information, we find that f_1 must take a value in $p_{\mathcal{R}_1}(f_1(A)) \cap A_1$, which implies that the event $f_1^{-1}(p_{\mathcal{R}_1}(f_1(A)) \cap A_1)$ in $\mathcal{R}_\Omega$ occurs. We may therefore deduce that f_2 assumes a value in $f_2(f_1^{-1}(p_{\mathcal{R}_1}(f_1(A)) \cap A_1))$. Since, however, only the elements of $\mathcal{R}_2$ are 'measurable' (or visible to us), we may actually only deduce from this that f_2 must assume a value in $p_{\mathcal{R}_2}(f_2(f_1^{-1}(p_{\mathcal{R}_1}(f_1(A)) \cap A_1)))$. But (19) tells us that

$$p_{\mathcal{R}_2}(f_2(A)) \subseteq p_{\mathcal{R}_2}(f_2(f_1^{-1}(p_{\mathcal{R}_1}(f_1(A)) \cap A_1))),$$

which means that the additional information about the value assumed by f_1 gives us additional information about the value assumed by f_2 that can never be more restrictive than the information already contained in Π_{f_2}. To put it more succinctly, additional knowledge about the values assumed by either one of both possibilistic variables cannot reduce the uncertainty about the values assume by the other. If this is the case, the variables f_1 and f_2 are called *logically independent* (see, for instance, [de Finetti, 1974] subsection 2.7.5 concerning the logical independence of stochastic variables).

We conclude that *possibilistic independence* for classical possibility measures is intricately linked with the notion of *logical independence*.

6 CONCLUSION

The results in paper tell us that it is possible to develop a measure- and integral-theoretic account of possibilistic independence using possibility integrals and the material given in Parts I and II of this series of three papers. It turns out that the formal analogy with the treatment of stochastic independence can be preserved, and that this approach generalizes existing results in the literature, while at the same time removing some inconsistencies.

This paper concludes a series of three on the measure- and integral-theoretic aspects of possibility theory. I have tried to show that it is possible to use the possibility integral to formally develop a possibility theory in very much the same way as a measure- and integral account of probability theory can be given using Lebesgue integrals. An overview of the analogy between probability theory and this account of possibility theory is given in Table 1.

The results discussed in this series seem to indicate that at least formally, the fuzzy integral is a good candidate for the title of *the* possibility integral, that is, an integral ideally suited for possibility measures. Whether its claims also receive corroboration from semantical considerations, is the subject of current research.

PROBABILITY THEORY	POSSIBILITY THEORY
σ-field	ample field
unit interval $[0, 1]$	complete lattice $(L, \leq)$
addition	supremum
multiplication	t-seminorm, t-norm
probability measure (σ-additivity)	possibility measure (supitivity)
Lebesgue integral	seminormed fuzzy integral (= possibility integral)
stochastic variable	possibilistic variable
real stochastic variable	fuzzy variable (= measurable fuzzy set)
probability distribution function	possibility distribution function
expectation of a real stochastic variable	possibility of a fuzzy variable
almost everywhere equality	almost everywhere equality (modified, more general form)
product probability measure	product possibility measure
conditional probability	conditional possibility
stochastic independence	possibilistic independence

Table 1: Overview of the formal analogy between probability and possibility theory

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