GENERALIZED POSSIBILITY AND NECESSITY MEASURES ON FIELDS OF SETS

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Abstract. We give a generalization of possibility and necessity measures: their domains are extended towards fields of sets, and their codomains towards arbitrary complete lattices. In this way, these measures can be associated with $(Q, \leq)$-fuzzy sets, where $(Q, \leq)$ is at least a poset. An important inconsistency problem, intricately linked with this association, is solved. It is argued that order lies at the basis of a mathematical description of vagueness and linguistic uncertainty. The results obtained here allow us to mathematically represent and manipulate linguistic uncertainty in the presence of incomparability.

Keywords: confidence relation, fuzzy set, linguistic information, necessity measure, possibility measure.

1. Introduction

The notion of a possibility measure was first introduced by Zadeh in 1978 [14], as a mathematical description of the (linguistic) information conveyed by vague propositions such as, for instance, 'John is old'. Zadeh's course of reasoning can be briefly summarized as follows. With the predicate 'old', he associates a fuzzy set on the universe Y of the possible ages, a properly chosen subset of the reals. The membership function μ_{old} of this fuzzy set is a mapping from Y to the real unit interval $[0, 1]$ with the following interpretation: for any age a in Y, $\mu_{\text{old}}(a)$ is the degree to which the age a corresponds with the notion 'old', or, in other words, the degree of membership of a in the corresponding fuzzy set.

The proposition 'John is old' gives us information about John's age, although it still leaves us in some uncertainty about John's actual age. In order to represent this uncertainty, Zadeh suggests that we use a possibility measure Π_{old}, i.e., a $\mathcal{P}(Y) - [0, 1]$-mapping, defined by

$$\Pi_{\text{old}}(A) = \sup_{a \in A} \mu_{\text{old}}(a) \tag{1}$$

for arbitrary $A \in \mathcal{P}(Y)$, where $\mathcal{P}(Y)$ is the power class of Y. It should be noted that Π_{old} is a *complete join-morphism* between the complete lattice $(\mathcal{P}(X), \subseteq)$ and the complete chain $([0, 1], \leq)$, i.e., that for an arbitrary family $(A_j \mid j \in J)$ of elements of $\mathcal{P}(Y)$:

$$\Pi_{\text{old}}(\bigcup_{j \in J} A_j) = \sup_{j \in J} \Pi_{\text{old}}(A_j).$$

The possibility measure Π_{old} is, according to Zadeh, a mathematical representation of the linguistic uncertainty (or, dually, information) contained in the proposition 'John is old'.

Possibility measures have been extensively studied by Dubois and Prade, who also introduced the dual notion of a necessity measure (see, for instance, [10]). Recently, we generalized possibility and necessity measures towards more general domains and codomains (see, [5, 8, 7, 9]). Following a suggestion by Wang [12], the domain of possibility measures was generalized towards ample fields. At the same time, the structure $([0, 1], \leq)$ was replaced by arbitrary complete lattices. Let us now briefly explain what this generalization consists in. For more information about ample fields and their atoms, we refer to [8, 12].

In what follows, we denote by X an arbitrary universe (a non-empty set), and by $\mathcal{R}$ an ample field on X. By $(L, \leq)$ we denote an arbitrary complete lattice. We define a $(L, \leq)$-possibility measure Π on $(X, \mathcal{R})$ as a *complete join-morphism* between the complete lattices $(\mathcal{R}, \subseteq)$ and $(L, \leq)$, and a $(L, \leq)$-necessity measure N as a *complete meet-morphism* between these structures, i.e., for an arbitrary fam-

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ily $(A_j \mid j \in J)$ of elements of $\mathcal{R}$:

$$\begin{cases} \Pi(\bigcup_{j \in J} A_j) = \sup_{j \in J} \Pi(A_j) \\ \mathrm{N}(\bigcap_{j \in J} A_j) = \inf_{j \in J} \mathrm{N}(A_j). \end{cases} \tag{2}$$

Given Π, there exists a *unique* $X - L$-mapping π, such that π is constant on the atoms of $\mathcal{R}$ and, for arbitrary A in $\mathcal{R}$:

$$\Pi(A) = \sup_{x \in A} \pi(x). \tag{3}$$

This π is called the *distribution* of Π, and satisfies $\pi(x) = \Pi([x]_{\mathcal{R}})$, were $x \in X$ and $[x]_{\mathcal{R}}$ is the atom of $\mathcal{R}$ containing x. Analogously, given N, there exists a *unique* $X - L$-mapping ν, such that ν is constant on the atoms of $\mathcal{R}$ and, for arbitrary A in $\mathcal{R}$:

$$\mathrm{N}(A) = \inf_{x \in \mathrm{co}A} \nu(x). \tag{4}$$

This ν is called the *distribution* of N, and satisfies $\nu(x) = \mathrm{N}(\mathrm{co}[x]_{\mathcal{R}})$, were $x \in X$, and co is the complement operator. For more details and the proofs of these statements, we refer to [5]. It therefore turns out that we can introduce $(L, \leq)$-possibility and $(L, \leq)$-necessity measures on ample fields in two equivalent ways: either we start with conditions (2), or we start with the $X - L$-mappings π and ν, and use equations (3) and (4). If we compare (3) and (1), we see that *in principle* the connection Zadeh makes between possibility and fuzzy sets, can be extended towards a connection between our $(L, \leq)$-possibility measures and $(L, \leq)$-fuzzy sets, i.e., $X - L$-mappings.

In this paper, we propose yet a further generalization of the original definition of possibility and necessity measures, *towards* $(L, \leq)$*-possibility and* $(L, \leq)$*-necessity measures defined on fields of sets*. At the same time, we give a motivation and justification for this generalization, and for the association of possibility and necessity measures with linguistic uncertainty. In this way, it becomes possible to associate *in a consistent way* possibility and necessity measures with $(Q, \leq)$-fuzzy sets, where $(Q, \leq)$ is at least a poset, and not just with Zadeh's fuzzy sets. This is not a trivial result, because we shall show that a straightforward generalization of Zadeh's approach from the unit interval to partially ordered sets may lead to serious consistency problems. The results in this paper show the way out of this inconsistency.

The course of reasoning in this paper can be summarized as follows. In section 2 we look at vagueness and fuzzy sets from an order-theoretic point of view, and show that with every vague or clear (non-vague) property on a universe we can associate a partial order relation, defined on a partition of that universe. In section 3, we show how for a clear property, the partially ordered set thus obtained can be used to define classical possibility and necessity measures. How this can be generalized for vague properties, is shown in section 4. The generalization proposed there, however, is not unique, which may lead to inconsistency problems. These problems, together with their solution, are discussed in section 5. Section 6 concludes this paper.

2. Fuzzy Sets and Order

In this section, we want to show that with a clear or vague property on a universe, there can be associated a partial order relation, defined on a partition of that universe. This will lead to a justification for the introduction of $(Q, \leq)$-fuzzy sets [11], where $(Q, \leq)$ is at least a partially ordered set. For a more detailed account of the material treated in this section, we refer to [3, 4].

Consider a property P of elements of the universe X. This property is assumed to be either *clear* or *vague*. Examples could be respectively the property 'less then 10' for the universe of real numbers, and the property 'intelligent' for, say, the universe of Flemish undergraduates. With such a property P, we assume that it is possible to associate a large-preference relation (see, for instance, [13]) R_P, defined as follows: for arbitrary x and y in X

$$xR_Py \Leftrightarrow y \text{ is at least as } P \text{ as } x.$$

We also make the basic assumption that R_P is reflexive and transitive, which is intuitively acceptable. In other words, R_P is a *quasi order relation*, or a partial preorder relation, on X. It thus appears that *a property* P *induces an order*[1] *on the universe* X. With the large-preference relation R_P we can associate an indifference relation $I_P \stackrel{\text{def}}{=} R_P \cap R_P^{-1}$ that is an equivalence relation on X. An equivalence class x/I_P of this relation contains those elements of X that satisfy P equally well as the object x. Together, these equivalence classes make up a

[1] It should be noted that we do not assume that R_P is total, i.e., that any x and y are comparable w.r.t. the property P, or equivalently, $(\forall (x, y) \in X^2)(xR_Py \text{ or } yR_Px)$. We do not wish to *a priori* exclude incomparability of objects with respect to properties.

partition of the universe X:

$$\mathcal{A}_P \stackrel{\text{def}}{=} \{\, x/I_P \mid x \in X \,\}.$$

Using the surjective quotient mapping

$$q_P\colon X \to \mathcal{A}_P\colon x \mapsto x/I_P,$$

it is possible to define a *partial order relation* $\leq_P$ on $\mathcal{A}_P$, on the basis of the following equation:

$$(\forall (x,y) \in X^2)(xR_Py \Leftrightarrow q_P(x) \leq_P q_P(y)).$$

This equation tells us that the poset $(\mathcal{A}_P, \leq_P)$ and the mapping q_P completely characterize the order R_P induced on the universe X by the property P. Any poset $(Q, \leq)$ and $X - Q$ mapping h that characterize R_P, i.e., for which

$$(\forall (x,y) \in X^2)(xR_Py \Leftrightarrow h(x) \leq h(y)),$$

will be called an *evaluation structure* for P in X. $(Q, \leq)$ will be called *evaluation set* and h *evaluation mapping*. It can be proven [5] that for such an evaluation structure, there exists an order-embedding[2] ξ of $(\mathcal{A}_P, \leq_P)$ into $(Q, \leq)$, such that $h = \xi \circ q_P$. In this sense, the evaluation structure $((\mathcal{A}_P, \leq_P), q_P)$ is *minimal* or canonical. Of course, minimal evaluation structures are determined up to an order-isomorphism.

This course of reasoning clearly leads to a justification for the introduction of Goguen's L-fuzzy sets [11]. In what follows, when $((Q, \leq), h)$ is an evaluation structure for the property P in the universe X, then we call h the $(Q, \leq)$*-fuzzy set* associated with P. It should be clear that for a property P in a universe X, there is an infinite class of evaluation structures. All these structures, and the corresponding fuzzy sets, are *equivalent, as far as the characterization of the order R_P induced on X by P is concerned*. If we want to take into account extra-ordinal characteristics of the property P, then only a special subclass of these evaluation structures, able to represent these characteristics, will be appropriate. But in any case, order is always the starting point. In this paper, we take the view that *order is fundamental to the description of imprecision and vagueness*. We therefore want to investigate what picture arises if only this ordering is taken into consideration, and any extra-ordinal aspects of properties are assumed to be either inexistent or irrelevant.

[2]An order-embedding ϕ of a poset $(P, \leq)$ into a poset $(Q, \leq)$ is a $P - Q$-mapping that by definition satisfies $(\forall (\lambda, \mu) \in P^2)(\lambda \leq \mu \Leftrightarrow \phi(\lambda) \leq \phi(\mu))$.

In this section, we have shown how a property on a universe induces a partial order relation on a partition of this universe, and how this partial order can be characterized by Goguen's fuzzy sets[3]. We conclude this section with the remark that, conversely, if we generalize Zadeh's line of reasoning, and *assume* that with a property P we may associate a $(Q, \leq)$-fuzzy set h, then we can also associate in a standard way with this fuzzy set a partial order relation $\leq_h$ on a partition $\mathcal{A}_h$ of the universe X. Indeed,

$$\begin{aligned} &\mathcal{A}_h \stackrel{\text{def}}{=} \{\, h^{-1}(\{\lambda\}) \mid \lambda \in h(X) \,\} \\ &q_h\colon X \to \mathcal{A}_h\colon x \mapsto h^{-1}(\{h(x)\}) \end{aligned} \tag{5}$$

If we introduce the $\mathcal{A}_h - Q$-mapping ξ_h as follows:

$$(\forall C \in \mathcal{A}_h)(\forall x \in X)(\xi_h(C) \stackrel{\text{def}}{=} h(x)), \tag{6}$$

then $h = \xi_h \circ q_h$ and $\leq_h$ is defined by

$$(\forall (C_1, C_2) \in \mathcal{A}_h^2)(C_1 \leq_h C_2 \Leftrightarrow \xi_h(C_1) \leq \xi_h(C_2)). \tag{7}$$

3. Classical Possibility and Necessity

Let us now turn to the study of linguistic uncertainty. Consider a variable ξ in the universe X, and let us assume that the information we have about the values that this variable may take in X, is given by the proposition 'ξ is P'. The question we want to answer in the rest of this paper is how this information can be represented mathematically.

In this section, we assume that the property P is *clear*. For every object x in X we have, in other words, that x either completely satisfies P, or completely does not satisfy P. If we introduce the classical subset

$$A_P \stackrel{\text{def}}{=} \{\, x \mid x \in X \text{ and } x \text{ is } P \,\},$$

of X, then, with the notations of the previous section:

$$\begin{aligned} &\mathcal{A}_P = \{A_P, \mathrm{co}A_P\} \\ &\leq_P = \{(\mathrm{co}A_P, \mathrm{co}A_P), (\mathrm{co}A_P, A_P), (A_P, A_P)\}. \end{aligned}$$

The Boolean chain $(\{0,1\}, \leq)$ is a *minimal* evaluation set for P in X and the characteristic mapping

$$\chi_{A_P}\colon X \to \{0,1\}\colon x \mapsto \begin{cases} 1 & ; \quad x \in A_P \\ 0 & ; \quad x \notin A_P \end{cases}$$

[3]...and by (the membership functions of) Zadeh's fuzzy sets if there is total comparability, i.e., if the ordering induced on this partition is total.

of A_P the corresponding *minimal* evaluation mapping, or $(\{0,1\},\leq)$-fuzzy set in X.

In our doctoral dissertation, we have argued that at the basis of a mathematical description of uncertainty lies the comparison of confidence in the occurrence[4] of events. Let us consider the field $\mathcal{V}$ of subsets of X as a set of measurable sets, or events. The order-theoretic description of uncertainty is given by a large-preference relation R on $\mathcal{V}$, defined as follows. For arbitrary (A,B) in $\mathcal{V}^2$, ARB iff, on the basis of the available information, we have at least as much confidence in the occurrence of B as in that of A. It is intuitively acceptable that R satisfies the following requirements[5]: (i) R is transitive; (ii) $(\forall(A,B)\in\mathcal{V}^2)(A\subseteq B\Rightarrow ARB)$; and (iii) $(X,\emptyset)\notin R$. We have called any relation satisfying these requirements a *confidence relation* on $\mathcal{V}$. Confidence relations model the order-theoretic aspects of the mathematical representation of uncertainty. For a detailed study of this very interesting class of relations, we refer to [5, 6]. Let us mention here that, first of all, a confidence relation R on $\mathcal{V}$ is a partial preorder relation on $\mathcal{V}$, and that in exactly the same way as in the previous section, we can define *evaluation structures* for this R. An evaluation mapping is then a mapping from $\mathcal{V}$ to a suitable poset, and is as such an ordinal precursor to such notions as probability measures, possibility and necessity measures, etc. Next, and generally speaking, the more confidence we have in the occurrence of an event A, the less confidence we can have in the occurrence of its *opposite event* co$A$. This fact allows us to associate with any confidence relation R on $\mathcal{V}$ its *dual confidence relation* R^D, defined by

$$(\forall(A,B)\in\mathcal{V}^2)(AR^DB\Leftrightarrow \mathrm{co}BR\mathrm{co}A).$$

In order to return to the main topic of this section, let us first assume that $\mathcal{A}_P\subseteq\mathcal{V}$, which means that the field $\mathcal{V}$ is detailed enough to allow us to observe the order that P induces on the universe X. Remark that on the basis of the linguistic information 'ξ is P' about the values that the variable ξ may assume in X, we already have an ordering $\leq_P$ on a part $\mathcal{A}_P$ of $\mathcal{V}$. Indeed, the given information tells us on the ordinal level that we should have less confidence in the occurrence of coA_P than in that of A_P. What we now must do, is try and extend this *germ of ordering* towards a full-fledged confidence relation on R. Of course, there will be many confidence relations R that are consistent with this initial information, or in other words, for which $R\cap\mathcal{A}_P^2=\leq_P$. Among these, we shall select precisely those confidence relations having the appropriate possibility respectively necessity measures as minimal evaluation mappings. If this is possible, we shall be able to say that these possibility and necessity measures are indeed, at the ordinal level, mathematical representations of the uncertainty (or information) conveyed by the affirmative sentence 'ξ is P'.

Consider the $\mathcal{A}_P-\{0,1\}$-mapping ϕ_P, defined by $\phi_P(\mathrm{co}A_P)=0$ and $\phi_P(A_P)=1$. ϕ_P is an order-embedding of the germ of order $(\mathcal{A}_P,\leq_P)$ into the evaluation set $(\{0,1\},\leq)$ for P in X. Because we want to extend $\leq_P$ to a confidence relation on $\mathcal{V}$, we shall try and extend this embedding ϕ_P towards a $\mathcal{V}-\{0,1\}$-mapping. This is done by introducing the mappings Π_P and N_P, with, for arbitrary A in $\mathcal{V}$:

$$\Pi_P(A)\stackrel{\text{def}}{=}\sup_{C\in\mathcal{A}_P,C\cap A\neq\emptyset}\phi_P(C) \tag{8}$$

and

$$\mathrm{N}_P(A)\stackrel{\text{def}}{=}\inf_{C\in\mathcal{A}_P,C\not\subseteq A}1-\phi_P(C). \tag{9}$$

Note that

$$(\forall A\in\mathcal{V})(\mathrm{N}_P(A)=1-\Pi_P(\mathrm{co}A)). \tag{10}$$

We define the binary relations R_{pos} and R_{nec} on $\mathcal{V}$ as follows: for arbitrary A and B in $\mathcal{V}$

$$\begin{cases} AR_{\text{pos}}B\Leftrightarrow\Pi_P(A)\leq\Pi_P(B)\\ AR_{\text{nec}}B\Leftrightarrow\mathrm{N}_P(A)\leq\mathrm{N}_P(B).\end{cases}$$

R_{pos} and R_{nec} are dual confidence relations on $\mathcal{V}$. Since $(\forall C\in\mathcal{A}_P)(\Pi_P(C)=\mathrm{N}_P(C)=\phi_P(C))$, we have that $R_{\text{pos}}\cap\mathcal{A}_P^2=R_{\text{nec}}\cap\mathcal{A}_P^2=\leq_P$. For an arbitrary family $(A_j\mid j\in J)$ of elements of $\mathcal{V}$,

$$\begin{aligned}\bigcup_{j\in J}A_j\in\mathcal{V}&\Rightarrow\Pi_P(\bigcup_{j\in J}A_j)=\sup_{j\in J}\Pi_P(A_j)\\ \bigcap_{j\in J}A_j\in\mathcal{V}&\Rightarrow\mathrm{N}_P(\bigcap_{j\in J}A_j)=\inf_{j\in J}\mathrm{N}_P(A_j).\end{aligned}$$

This implies that Π_P is a *conditionally complete join-morphism* and that N_P is a *conditionally complete meet-morphism* between the structures $(\mathcal{V},\subseteq)$ and $(\{0,1\},\leq)$. Furthermore, for arbitrary A in $\mathcal{V}$,

$$\Pi_P(A)=\sup_{x\in A}\chi_{A_P}(x)=\begin{cases}1 & ; & A\cap A_P\neq\emptyset\\ 0 & ; & A\cap A_P=\emptyset\end{cases}$$

is the possibility that A occurs, and

$$\mathrm{N}_P(A)=\inf_{x\in\mathrm{co}A}\chi_{\mathrm{co}A_P}(x)=\begin{cases}1 & ; & A_P\subseteq A\\ 0 & ; & A_P\not\subseteq A.\end{cases}$$

[4] As usual, we say that an event A occurs, if the variable ξ is an element of A.

[5] Remark that we do not *a priori* demand that R should be total: we do not want to preclude incomparability of events on the basis of the confidence we have in their occurrence.

is the necessity that A occurs, given the information that A_P occurs[6]. We therefore call the mapping Π_P a $(\{0,1\},\leq)$-possibility measure or *classical possibility measure* on $(X,\mathcal{V})$, and N_P a $(\{0,1\},\leq)$-necessity measure or *classical necessity measure* on $(X,\mathcal{V})$. Remark that the distribution χ_{A_P} of Π_P is a $(\{0,1\},\leq)$-fuzzy set, associated with P; and that the distribution $\chi_{\mathrm{co}A_P}$ of N_P is a $(\{0,1\},\geq)$-fuzzy set, associated with P.

4. General Possibility and Necessity

In this section, we treat the problem of finding a mathematical representation for the linguistic information conveyed by 'ξ is P', where P is a *vague property.* We know from section 2 that P induces a partial order relation $\leq_P$ on a partition $\mathcal{A}_P$ of the universe X. We again consider a field $\mathcal{V}$ of measurable subsets of X, and assume that the elements of $\mathcal{A}_P$ are $\mathcal{V}$-measurable, i.e., that $\mathcal{A}_P \subseteq \mathcal{V}$. This means that the field $\mathcal{V}$ is detailed enough to allow us to observe the order that P induces on X.

As in the previous section, we have a germ of order $\leq_P$ on a subset $\mathcal{A}_P$ of $\mathcal{V}$. Indeed, if $C_1 \leq_P C_2$, for C_1 and C_2 in $\mathcal{A}_P$, then we should, on the basis of the given information have at least as much confidence in the occurrence of C_2 as in that of C_1. We shall try and extend this germ of confidence towards appropriate confidence relations R_{pos} and R_{nec} on $\mathcal{V}$, in such a way that their evaluation mappings will turn out to be suitable candidates for our generalized possibility respectively necessity measures.

We intend to follow as closely as possible the procedure for classical possibility and necessity. The first step we take, is to consider a complete lattice $(L_1,\leq_1)$ that is an evaluation set for P in X. From the results of section 2, we know that this means that there should exist an order-embedding ϕ_P of the poset $(\mathcal{A}_P,\leq_P)$ into $(L_1,\leq_1)$. Lattice theory tells us that for any poset, there always exists at least one complete lattice into which it can be embedded (see, for instance, [1, 2]). In other words, we can always find at least one (and really infinitely many) complete lattice $(L_1,\leq_1)$ into which the poset $(\mathcal{A}_P,\leq_P)$ can be embedded.

Since $(L_1,\leq_1)$ is a complete lattice, we may now define the following $\mathcal{V}-L_1$-mapping Π_P, with, for arbitrary A in $\mathcal{V}$:

$$\Pi_P(A) \stackrel{\mathrm{def}}{=} \sup_{C\in\mathcal{A}_P, C\cap A\neq\emptyset} \phi_P(C). \tag{11}$$

Of course, if we define the relation R_{pos} on $\mathcal{V}$ as follows:

$$(\forall (A,B)\in\mathcal{V}^2)(AR_{\mathrm{pos}}B \Leftrightarrow \Pi_P(A)\leq_1 \Pi_P(B)),$$

then R_{pos} is a confidence relation on $\mathcal{V}$, and clearly $R_{\mathrm{pos}}\cap\mathcal{A}_P^2 = \leq_P$, since, for C in $\mathcal{A}_P$, $\Pi_P(C) = \phi_P(C)$. Also remark that (11) is an immediate generalization of (8). For an arbitrary family $(A_j \mid j\in J)$ of elements of $\mathcal{V}$, we have, if $\bigcup_{j\in J} A_j\in\mathcal{V}$, that

$$\Pi_P(\bigcup_{j\in J} A_j) = \sup_{j\in J}\Pi_P(A_j).$$

This implies that Π_P is a *conditionally complete join-morphism* between the structures $(\mathcal{V},\subseteq)$ and $(L_1,\leq_1)$. Furthermore, there exists a unique $X-L_1$-mapping π_P, constant on the elements of $\mathcal{A}_P$, such that for arbitrary A in $\mathcal{V}$

$$\Pi_P(A) = \sup_{x\in A}\pi_P(x).$$

π_P is defined by

$$(\forall C\in\mathcal{A}_P)(\forall x\in C)(\pi_P(x) \stackrel{\mathrm{def}}{=} \phi_P(C)).$$

We conclude that the mapping Π_P behaves as we would expect a possibility measure to behave under the given circumstances. Therefore, Π_P will be called a $(L_1,\leq_1)$*-possibility measure on* $(X,\mathcal{V})$, and π_P its *distribution.* It is very important to note that $((L_1,\leq_1),\pi_P)$ is an evaluation structure for P in X, or, in other words, that π_P is a $(L_1,\leq_1)$-fuzzy set in X, associated with P.

Let us now turn our attention to the necessistic side of the problem. In order to introduce possibility measures, we extended the germ of confidence $(\mathcal{A}_P,\leq_P)$ to an appropriate confidence relation R_{pos} on $\mathcal{V}$. In order to introduce necessity measures, we shall try and extend a *dual germ of confidence* towards an appropriate dual confidence relation R_{nec} on $\mathcal{V}$. In order to introduce dual confidence, we saw in section 3 that we must switch from events to their opposite events. Therefore, if we introduce the subset

$$\mathcal{A}'_P \stackrel{\mathrm{def}}{=} \{\,\mathrm{co}C \mid C\in\mathcal{A}_P\,\}$$

of $\mathcal{V}$ and the relation

$$\leq'_P \stackrel{\mathrm{def}}{=} \{\,(\mathrm{co}C_2,\mathrm{co}C_1) \mid (C_1,C_2)\in\leq_P\,\}$$

[6]...or equivalently, that ξ is P.

on $\mathcal{A}'_P$, then the poset $(\mathcal{A}'_P, \leq'_P)$ is the dual germ of confidence we are looking for, and is based on the information 'ξ is P'. Of course, the posets $(\mathcal{A}_P, \leq_P)$ and $(\mathcal{A}'_P, \leq'_P)$ are dually order-isomorphic, which tells us that they indeed contain the same ordinal information. The operator co is the corresponding dual order-isomorphism between these structures. Consider a complete lattice $(L_2, \leq_2)$ into which the poset $(\mathcal{A}'_P, \leq'_P)$ can be embedded. The corresponding order-embedding will be called ϕ'_P. In other words, ϕ'_P is a $\mathcal{A}'_P - L_2$-mapping, and for arbitrary C_1 and C_2 in $\mathcal{A}'_P$:

$$C_1 \leq'_P C_2 \Leftrightarrow \phi'_P(C_1) \leq_2 \phi'_P(C_2).$$

In order to introduce the $\mathcal{V} - L_2$-mapping N_P, we simply take the *dual expression* of the definition of Π_P. Concretely, for arbitrary A in $\mathcal{V}$:

$$\mathrm{N}_P(A) \stackrel{\text{def}}{=} \inf_{C \in \mathcal{A}'_P, C \cup A \neq X} \phi'_P(C). \tag{12}$$

If we define the relation R_{nec} on $\mathcal{V}$ as follows:

$$(\forall (A,B) \in \mathcal{V}^2)(A R_{\text{nec}} B \Leftrightarrow \mathrm{N}_P(A) \leq \mathrm{N}_P(B)),$$

then R_{nec} is a confidence relation on $\mathcal{V}$, and clearly $R_{\text{nec}} \cap (\mathcal{A}'_P)^2 = \leq'_P$, since, for C in $\mathcal{A}'_P$, $\mathrm{N}_P(C) = \phi'_P(C)$. For an arbitrary family $(A_j \mid j \in J)$ of elements of $\mathcal{V}$, we have, if $\bigcap_{j \in J} A_j \in \mathcal{V}$, that

$$\mathrm{N}_P(\bigcap_{j \in J} A_j) = \inf_{j \in J} \mathrm{N}_P(A_j).$$

This implies that N_P is a *conditionally complete meet-morphism* between the structures $(\mathcal{V}, \subseteq)$ and $(L_2, \leq_2)$. Furthermore, there exists a unique $X - L_2$-mapping ν_P, constant on the elements of $\mathcal{A}_P$, such that for arbitrary A in $\mathcal{V}$

$$\mathrm{N}_P(A) = \inf_{x \in \text{co}A} \nu_P(x).$$

ν_P is defined by

$$(\forall C \in \mathcal{A}'_P)(\forall x \in \text{co}C)(\nu_P(x) \stackrel{\text{def}}{=} \phi'_P(C)).$$

We conclude that the mapping N_P behaves as we would expect a necessity measure to behave under the given circumstances. Therefore, N_P will be called a $(L_2, \leq_2)$-necessity measure on $(X, \mathcal{V})$, and ν_P its *distribution*. It is very important to note that $((L_2, \geq_2), \nu_P)$ is an evaluation structure for P in X, or, in other words, that ν_P is a $(L_2, \geq_2)$-fuzzy set associated with P.

In general, there is no immediate counterpart for the relation (10) between classical possibility and necessity. Since, however, $(\mathcal{A}_P, \leq_P)$ and $(\mathcal{A}'_P, \leq'_P)$ are dually order-isomorphic, it is always possible to choose $(L_1, \leq_1)$ and $(L_2, \leq_2)$ in such a way that they are *also dually order-isomorphic*, and that for the dual order-isomorphism n between these structures:

$$(\forall C \in \mathcal{A}_P)(n(\phi_P(C)) = \phi'_P(\text{co}C))$$

This equality tells us that (12) is a generalization of (9). Furthermore, for arbitrary A in $\mathcal{V}$,

$$\begin{aligned} n(\Pi_P(\text{co}A)) &= n(\sup_{C \in \mathcal{A}_P, C \cap \text{co}A \neq \emptyset} \phi_P(C)) \\ &= \inf_{C \in \mathcal{A}_P, C \cap \text{co}A \neq \emptyset} n(\phi_P(C)) \\ &= \inf_{C \in \mathcal{A}_P, C \cap \text{co}A \neq \emptyset} \phi'_P(\text{co}C) \\ &= \inf_{C \in \mathcal{A}'_P, \text{co}C \cap \text{co}A \neq \emptyset} \phi'_P(C) \\ &= \inf_{C \in \mathcal{A}'_P, C \cup A \neq X} \phi'_P(C) = \mathrm{N}_P(A), \end{aligned}$$

which generalizes (10). In this case, R_{pos} and R_{nec} are *dual confidence relations.*

We may conclude that we are to a large extent able to generalize the results of the previous section. There is one important point of difference, however, that calls for our attention. The image set $(\{0,1\}, \leq)$ of classical possibility and necessity measures is a complete lattice that is at the same time a *minimal evaluation set* for the clear property P in X. On the other hand, if P is vague, there is no guarantee that the minimal evaluation set $(\mathcal{A}_P, \leq_P)$ will be a complete lattice. In order to introduce our possibility and necessity measures, we therefore have to embed this minimal evaluation set (or its dual) into a complete lattice. In order to make the analogy between general and classical possibility and necessity more complete, we would have to impose certain *minimality conditions* on the complete lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$. We shall return to this point in the next section.

5. The Consistency Problem

In the previous section, we have shown how the information contained in the proposition 'ξ is P' can be represented by a $(L_1, \leq_1)$-possibility measure Π_P on $(X, \mathcal{V})$, and by a $(L_2, \leq_2)$-necessity measure N_P on $(X, \mathcal{V})$. These measures are evaluation mappings of the respective confidence relations R_{pos} and R_{nec} on $\mathcal{V}$. These dual confidence relations are ordinal representations of the above-mentioned linguistic information.

One important point remains to be clarified, however. It is obvious that there is an infinity of pairs of dually order-isomorphic complete lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$ that can be used to construct the respective mappings Π_P and N_P. This means that the possibility and necessity measures we have constructed to represent linguistic information are *not unique*. However, we have taken order as a starting point, and *take the point of view that possibility and necessity measures are essentially ordinal representations of linguistic information*. The abovementioned non-unicity for these mappings should therefore pose no problem, *provided that the underlying ordinal information, i.e., the confidence relations* R_{pos} *and* R_{nec}*, remain the same for every choice of* $(L_1, \leq_1)$ *and* $(L_2, \leq_2)$.

It is not difficult to prove that this is not necessarily so (for a counter-example, we refer to our doctoral dissertation [5]). This leaves us with a serious problem of consistency, because the ordinal representations R_{pos} and R_{nec} of the linguistic information 'ξ is P' depend on the choice of $(L_1, \leq_1)$ and $(L_2, \leq_2)$. It therefore seems necessary to impose additional conditions on these complete lattices, that eliminate the variation in R_{pos} and R_{nec}. We refer in this respect to the closing remark of the previous section. These additional conditions are given in the following theorem.

Theorem 1 *(i) If the set* $\phi_P(\mathcal{A}_P)$ *is meet-dense*[7] *in the poset* $(\Pi_P(\mathcal{V}), \leq_1)$*, then the ordering* R_{pos} *induced by* Π_P *on* $\mathcal{V}$ *is independent of the choice of* $(L_1, \leq_1)$ *and* ϕ_P*, and only depends on* $(\mathcal{A}_P, \leq_P)$.
(ii) If the set $\phi'_P(\mathcal{A}'_P)$ *is join-dense in the poset* $(\mathrm{N}_P(\mathcal{V}), \leq_2)$*, then the ordering* R_{nec} *induced by* N_P *on* $\mathcal{V}$ *is independent of the choice of* $(L_2, \leq_2)$ *and* ϕ'_P*, and only depends on* $(\mathcal{A}'_P, \leq'_P)$.

Proof. We shall prove (i). The proof of (ii) is completely similar. First of all, if we put for arbitrary D in $\mathcal{V}$, $a(D) \stackrel{\text{def}}{=} \{\, C \mid C \in \mathcal{A}_P \text{ and } C \cap D \neq \emptyset \,\}$, then we see that $a(D)$ is clearly independent of $(L_1, \leq_1)$ and ϕ_P. Furthermore $\Pi_P(D) = \sup \phi_P(a(D))$. Consider arbitrary A and B in $\mathcal{V}$. Since we assume that $\phi_P(\mathcal{A}_P)$ is meet-dense in $(\Pi_P(\mathcal{V}), \leq_1)$,

$$\Pi_P(B) = \inf_{C \in \mathcal{A}_P, \Pi_P(B) \leq \phi_P(C)} \phi_P(C).$$

We therefore find that

$$\begin{aligned}
&A R_{\text{pos}} B \\
&\quad \Leftrightarrow \Pi_P(A) \leq \Pi_P(B) \\
&\quad \Leftrightarrow \sup \phi_P(a(A)) \leq \Pi_P(B) \\
&\quad \Leftrightarrow (\forall C \in a(A))(\phi_P(C) \leq \Pi_P(B)) \\
&\quad \Leftrightarrow (\forall C \in a(A))(\phi_P(C) \leq \inf_{\substack{D \in \mathcal{A}_P \\ \Pi_P(B) \leq \phi_P(D)}} \phi_P(D)) \\
&\quad \Leftrightarrow (\forall C \in a(A))(\forall D \in \mathcal{A}_P) \\
&\qquad (\Pi_P(B) \leq \phi_P(D) \Rightarrow \phi_P(C) \leq \phi_P(D)) \\
&\quad \Leftrightarrow (\forall C \in a(A))(\forall D \in \mathcal{A}_P) \\
&\qquad ((\forall E \in a(B))(\phi_P(E) \leq \phi_P(D)) \\
&\qquad\quad \Rightarrow \phi_P(C) \leq \phi_P(D)) \\
&\quad \Leftrightarrow (\forall C \in a(A))(\forall D \in \mathcal{A}_P) \\
&\qquad ((\forall E \in a(B))(E \leq_P D) \Rightarrow C \leq_P D),
\end{aligned}$$

and this last condition is clearly independent of the choice of $(L_1, \leq_1)$ and ϕ_P, and only depends on $(\mathcal{A}_P, \leq_P)$. □

This theorem clearly removes the inconsistency we were faced with. Indeed, it provides us with the conditions we must impose on the complete lattices $(L_1, \leq_1)$ and $(L_2, \leq_2)$, and on the order-embeddings ϕ_P and ϕ'_P, in order that there would be *unique ordinal representations* R_{pos} and R_{nec} of the linguistic information conveyed by 'ξ is P', that are completely and uniquely determined by the order imposed on the universe X by the property P.

Note that for classical possibility and necessity measures we have $\mathcal{A}_P = \mathcal{A}'_P = \{A_P, \text{co}A_P\}$, $\leq_P = \leq'_P$, $\phi_P(\mathcal{A}_P) = \phi'_P(\mathcal{A}'_P) = \{0, 1\}$ and $\Pi_P(\mathcal{V}) = \mathrm{N}_P(\mathcal{V}) = \{0, 1\}$. We find that in this case $\phi_P(\mathcal{A}_P)$ is meet-dense in $(\Pi_P(\mathcal{V}), \subseteq)$ and $\phi'_P(\mathcal{A}'_P)$ is join-dense in $(\mathrm{N}_P(\mathcal{V}), \subseteq)$. In the classical case, therefore, the *consistency conditions* are satisfied. We may, by the way, also conclude that, in general, the consistency conditions mentioned in the theorem above are also the appropriate *minimality conditions*, mentioned at the end of the previous section.

6. Conclusion

Let us conclude this paper with a discussion of how Zadeh's approach to possibility can be generalized: how do we generalize (1) if the property involved is not described by the membership function of a Zadeh fuzzy set, but by a $(Q, \leq)$-fuzzy set h in X? The answer is now very simple. h induces an order on the universe X: with h, we can associate the partial order relation $\leq_h$ on the partition $\mathcal{A}_h$ of X (see (5)–(7)). It is assumed that $\mathcal{A}_h \subseteq \mathcal{V}$. We look for a complete lattice $(L, \leq)$ such that there exists an order-embedding, say ϕ_h,

[7] A subset P of a poset $(Q, \leq)$ is meet-dense in $(Q, \leq)$ iff $(\forall \lambda \in Q)(\lambda = \inf \{\, \mu \mid \mu \in P \text{ and } \lambda \leq \mu \,\})$. The notion 'join-dense' is dual. For more details, see [2].

of $(\mathcal{A}_h, \leq_h)$ into $(L, \leq)$. Then of course, the $X - L$-mapping $\overline{h} \stackrel{\text{def}}{=} \phi_h \circ q_h$ is a $(L, \leq)$-fuzzy set in X that is equivalent with h from the ordinal point of view: it induces the same order on X as h does. In other words, we have already succeeded in replacing the poset $(Q, \leq)$ by a complete lattice $(L, \leq)$. Let us now define the $\mathcal{V} - L$-mapping Π_h as follows:

$$\Pi_h(A) \stackrel{\text{def}}{=} \sup_{C \in \mathcal{A}_h, C \cap A \neq \emptyset} \phi_h(C),$$

for arbitrary A in $\mathcal{V}$. Clearly,

$$\Pi_h(A) = \sup_{x \in A} \phi_h(q_h(x)) = \sup_{x \in A} \overline{h}(x),$$

which generalizes Zadeh's equation (1). Theorem 1 tells us that our approach is consistent[8] if $\phi_h(\mathcal{A}_h)$ is meet-dense in $(\Pi_h(\mathcal{V}), \leq)$, or equivalently, if $\overline{h}(X)$ is meet-dense in $(\Pi_h(\mathcal{V}), \leq)$. For necessity measures, a similar approach is possible.

It should be noted that the remarks we have made about consistency also apply to Zadeh's possibility measures, under the condition that we assume that the membership function of a Zadeh fuzzy set is only a representation of the ordering induced by a vague property in a universe. If such a Zadeh fuzzy set is taken to represent more than this ordering, then this consistency is no longer an issue.

Let us end this concluding section with a few remarks. First of all, it must be mentioned that in this paper we have mainly approached the problem of representing linguistic uncertainty from the measure-theoretic side, since we have concentrated on possibility and necessity measures, and not on the special confidence relations hiding behind them. It nevertheless is very interesting to treat the same problem using only confidence relations, because then the order-theoretic aspects of (linguistic) uncertainty emerge. This has been done in our doctoral dissertation [5], and will be published elsewere. Secondly, we would also like to point out that possibility and necessity measures defined on ample fields (see section 1) can be defined either using their distributions (see (3) and (4)), or using the fact that they are a complete join-morphism respectively meet-morphism (see (2)). This is no longer so when we want to define these measures on fields. In this case the latter approach turns out to be more general than the former. This will be reported on in more detail elsewere.

[8] Generally, it is not difficult to find a complete lattice $(L, \leq)$ such that all these conditions are satisfied: the Dedekind-MacNeille extension of $(\mathcal{A}_h, \leq_h)$ [2] is always suited.

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