# ON THE EXTENSION OF P-CONSISTENT MAPPINGS

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ABSTRACT

In this paper, the notion of P-consistency is extended to complete lattice-valued mappings. It is proven that a P-consistent mapping possesses a distribution if and only if it is extendable to a possibility measure defined on an ample field. A necessary and sufficient condition is given for extendability, and it is shown by counterexamples that this condition is not always satisfied. Finally, sufficient conditions are given under which a P-consistent mapping is always extendable, and it is shown that every complete lattice can be embedded in another complete lattice in such a way that every P-consistent mapping is extendable to a possibility measure taking values in the second complete lattice.

## 1. Introduction

Ever since the introduction of possibility theory by Zadeh[8] in 1978, various attempts have been made to generalize Zadeh's original definition of a possibility measure. Essentially, Zadeh defined a *possibility measure* $\Pi$ as a mapping from the power class $\wp(X)$ of a non-empty set $X$ to the real unit interval $[0, 1]$, satisfying for any family $(A_j)_{j\in J}$ of subsets of $X$:

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j). \tag{1}$$

It will be conceded that the power class $\wp(X)$ is a fairly restrictive structure of subsets of $X$. It is not surprising, therefore, that attempts have been made to define possibility measures on more general collections of subsets, and to generalize the possibility law (1) accordingly.

The problem can be briefly summarized as follows. Given a collection $\mathcal{A}$ of subsets of $X$ and a mapping $\Pi\colon \mathcal{A} \to [0, 1]$, it is reasonable to call $\Pi$ a possibility measure if *it can be extended to a possibility measure in Zadeh's sense*, i.e., if there exists a $\wp(X) - [0, 1]$-mapping $\Pi'$ satisfying (1), such that

$$(\forall A \in \mathcal{A})(\Pi(A) = \Pi'(A)).$$

Then, what are the conditions to be imposed on $\Pi$ in order that it would be thus extendable? Or, in other words, what is the proper generalization of (1)? Wang[6,7] has solved this problem by showing that the $\mathcal{A} - [0, 1]$-mapping $\Pi$ is extendable if and only if it is *P-consistent*, i.e., iff for any $A$ in $\mathcal{A}$ and for any family $(A_j)_{j\in J}$ of elements of $\mathcal{A}$:

$$A \subseteq \bigcup_{j \in J} A_j \Rightarrow \Pi(A) \leq \sup_{j \in J} \Pi(A_j).$$

De Cooman et al.[2,3,4] have further generalized possibility measures by generalizing their codomains from the real unit interval towards complete lattices. In his doctoral dissertation[2], De Cooman has argued that such a generalization is necessary in order to model potential incomparability in linguistic information.

In this paper, we address the extension problem for possibility measures taking values in complete lattices. In section 2, we extend Wang's notion of P-consistency to set mappings taking values in a complete lattice, and study the relation between (possibility) distributions, P-consistency and extendability. It is shown among other things that P-consistency is a necessary condition for extendability. In section 3, we derive necessary and sufficient conditions for a P-consistent mapping to be extendable. In section 4, we construct a number of interesting counterexamples, which tell us that, when incomparability enters the picture, not every P-consistent mapping is extendable to a possibility measure. In section 5, we consider a number of special cases, or special conditions to be imposed on the domain and codomain of a set mapping, in order that its P-consistency would be a sufficient condition for its extendability to a possibility measure. Finally, in section 6, we show that a P-consistent mapping can always be made extendable, by embedding its codomain in a properly chosen new complete lattice.

Let us conclude this introductory section with a number of basic definitions, necessary for understanding the material in the following sections. A *plump field*[7] $\mathcal{L}$ on a non-empty set $X$ is a subset of the power class $\wp(X)$, that is closed under arbitrary unions and intersections. The *atom*[7] of $\mathcal{L}$ containing the element $x$ of $X$ is denoted by $[x]_{\mathcal{L}}$ and is defined as

$$[x]_{\mathcal{L}} = \bigcap \{A \mid A \in \mathcal{L} \text{ and } x \in A\}.$$

It is easily verified[7] that for any $x$ in $X$,

$$[x]_{\mathcal{L}} \in \mathcal{L} \text{ and } x \in [x]_{\mathcal{L}}.$$

Furthermore[7], for an arbitrary subset $A$ of $X$,

$$A \in \mathcal{L} \Leftrightarrow A = \bigcup_{x \in A} [x]_{\mathcal{L}}.$$

Finally, for arbitrary $x$ in $X$ and arbitrary $A$ in $\mathcal{L}$,

$$x \in A \Leftrightarrow [x]_{\mathcal{L}} \subseteq A.$$

An *ample field*[5] $\mathcal{R}$ on a non-empty set $X$ is a plump field on $X$ that is closed under complementation. If $\mathcal{A}$ is a subset of $\wp(X)$, then the smallest ample field which includes $\mathcal{A}$ is denoted by $\tau(\mathcal{A})$.

Throughout this paper, let, unless stated otherwise, $X$ be a non-empty set, $\mathcal{A}$ be an arbitrary non-empty subset of $\wp(X)$, $\mathcal{R}$ be an ample field on $X$ and $(L,\leq)$ be a complete lattice[1].

## 2. Possibility Measures and P-Consistency

Let us start this discussion with a direct and obvious generalization of Zadeh's definition of a possibility measure[8], towards more general domains and codomains.

**Definition 1 (Possibility measure[2,3,4]).** We call a $\mathcal{R}-L$-mapping $\Pi$ a $(L,\leq)$-*possibility measure* on $(X,\mathcal{R})$ iff for any family $(A_j)_{j\in J}$ of elements of $\mathcal{R}$

$$\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j). \tag{2}$$

Of course, a $(L,\leq)$-possibility measure on $(X,\mathcal{R})$ is isotonic w.r.t. the complete lattices $(\mathcal{R},\subseteq)$ and $(L,\leq)$; in fact it is by definition a complete join-morphism between these complete lattices.

It is obvious that in generalizing $\wp(X)$ to $\mathcal{R}$ and $[0,1]$ to $(L,\leq)$, the possibility law can be formally preserved, i.e., (1) and (2) are formally identical. The question we want to answer is, how $(L,\leq)$-possibility measures can be defined on a more general set of subsets $\mathcal{A}$ than an ample field $\mathcal{R}$, and what the corresponding possibility law then turns into. The most natural condition that a mapping $\Pi\colon \mathcal{A}\to L$ must satisfy to be called a $(L,\leq)$-possibility measure, is that it should be extendable.

**Definition 2 (Extendability).** We call a $\mathcal{A}-L$-mapping $\Pi$ *extendable to a* $(L,\leq)$-*possibility measure on* $(X,\mathcal{R})$ iff there exists a $(L,\leq)$-possibility measure $\Pi'$ on $(X,\mathcal{R})$ such that $(\forall A\in\mathcal{A})(\Pi(A)=\Pi'(A))$. $\Pi$ is called *extendable to a* $(L,\leq)$-*possibility measure* iff there exists an ample field $\mathcal{R}$ on $X$ such that $\Pi$ is extendable to a $(L,\leq)$-possibility measure on $(X,\mathcal{R})$.

The following definition extends Wang's definition[6,7] of P-consistency for set mappings with a complete lattice as a codomain.

**Definition 3 (P-consistency).** A $\mathcal{A}-L$-mapping $\Pi$ is called *P-consistent* iff for any family $(A_j)_{j\in J}$ of elements of $\mathcal{A}$ and any element $A$ of $\mathcal{A}$:

$$A\subseteq \bigcup_{j\in J} A_j \Rightarrow \Pi(A)\leq \sup_{j\in J}\Pi(A_j). \tag{3}$$

A classical result by Wang[7] can now be generalized at once. Its proof is trivial.

**Proposition 4.** *Let the* $\mathcal{A}-L$*-mapping* $\Pi$ *be P-consistent. Then* $\Pi$ *is isotonic w.r.t.* $(\mathcal{A},\subseteq)$ *and* $(L,\leq)$*. Furthermore, for any family* $(A_j)_{j\in J}$ *of elements of* $\mathcal{A}$*:*

$$\bigcup_{j\in J} A_j\in\mathcal{A} \Rightarrow \Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J}\Pi(A_j).$$

The next result tells us that P-consistency can be interpreted as a possible generalization of the possibility law (2). Whether it is the generalization we are looking for, i.e., whether P-consistency is a necessary and sufficient condition for extendability, will be answered later in this paper.

**Proposition 5.** *Let* $\Pi$ *be a* $\mathcal{R} - L$*-mapping. Then* $\Pi$ *is a* $(L, \leq)$*-possibility measure on* $(X, \mathcal{R})$ *if and only if* $\Pi$ *is P-consistent.*

It turns out that the notion of a (possibility) distribution has a very important role to play in this discussion of extendability.

**Definition 6 (Distribution).** Let $\Pi$ be a $\mathcal{A} - L$-mapping. A $X - L$-mapping $\pi$ is called a *(possibility) distribution* of $\Pi$ iff for any $A$ in $\mathcal{A}$:

$$\Pi(A) = \sup_{x \in A} \pi(x).$$

Of course, if a $\mathcal{A} - L$-mapping $\Pi$ has a distribution, it is completely determined by it: given the distribution, the image $\Pi(A)$ in any element $A$ of its domain $\mathcal{A}$ can be calculated. It will be clear that not every $\mathcal{A} - L$-mapping has such a distribution, however. The following proposition indeed tells us that only P-consistent mappings can possess a distribution. Its proof is straightforward, and can be omitted.

**Proposition 7.** *Let* $\Pi$ *be a* $\mathcal{A} - L$ *mapping which possesses a distribution. Then* $\Pi$ *is P-consistent.*

De Cooman[2,3,4] has proven that every $(L, \leq)$-possibility measure $\Pi$ on $(X, \mathcal{R})$ possesses a distribution $\pi$ which is given by $(\forall x \in X)(\pi(x) = \Pi([x]_{\mathcal{R}}))$. With this result we can prove the following theorem, which tells us that, using the notion of a distribution, we can formulate a necessary and sufficient condition for the extendability of a $\mathcal{A} - L$-mapping to a $(L, \leq)$-possibility measure.

**Theorem 8.** *Let* $\Pi$ *be a* $\mathcal{A} - L$*-mapping and suppose that* $\mathcal{A} \subseteq \mathcal{R}$*. Then* $\Pi$ *possesses a distribution if and only if* $\Pi$ *is extendable to a* $(L, \leq)$*-possibility measure on* $(X, \mathcal{R})$*.*

*Proof.* For a start, assume that $\Pi$ possesses a distribution $\pi$. Then it is clear that the $\mathcal{R} - L$-mapping $\Pi'$, defined by $(\forall A \in \mathcal{R})(\Pi'(A) = \sup_{x \in A} \pi(x))$, is a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ which extends $\Pi$.

On the other hand, assume that $\Pi$ is extendable to a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$, say $\Pi'$. Since $\Pi'$ possesses a distribution, $\Pi$ also possesses the same distribution. □

From this theorem it easily follows that, if $\mathcal{R}$ includes $\mathcal{A}$, a P-consistent $\mathcal{A} - L$-mapping $\Pi$ is extendable to a $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ if and only if $\Pi$ is extendable to a $(L, \leq)$-possibility measure on $(X, \tau(\mathcal{A}))$.

Furthermore, since only P-consistent mappings can have a distribution, it also tells us that P-consistency is a *necessary* condition for extendability. Whether it is also a sufficient condition is answered in the following two sections.

## 3. Extendability of P-Consistent Mappings

In this section, we derive sufficient and necessary conditions for a P-consistent mapping to be extendable. In order to do this, we first need a number of lemmas and definitions. We shall denote by $\Pi$ a P-consistent $\mathcal{A}-L$-mapping and by $\mathcal{R}$ an ample field that includes $\mathcal{A}$, i.e., $\mathcal{A} \subseteq \mathcal{R}$.

**Lemma 9.** *Let $x$ and $y$ be two elements of $X$ such that $[x]_{\mathcal{R}} = [y]_{\mathcal{R}}$. Then*

$$(\forall A \in \mathcal{A})(x \in A \Leftrightarrow y \in A).$$

*Proof.* For any $A$ in $\mathcal{A}$ we obtain $x \in A \Leftrightarrow [x]_{\mathcal{R}} \subseteq A \Leftrightarrow [y]_{\mathcal{R}} \Leftrightarrow y \in A$. □

**Lemma 10.** *Suppose that $\Pi$ is extendable to a $(L, \leq)$-possibility measure $\Pi'$ on $(X, \mathcal{R})$. Then for any $x$ in $X$:*

$$\Pi'([x]_{\mathcal{R}}) \leq \inf_{\substack{B \in \mathcal{A} \\ x \in B}} \Pi(B).$$

*Proof.* Consider an arbitrary $x$ in $X$. Since $(\forall B \in \mathcal{A})(x \in B \Rightarrow [x]_{\mathcal{R}} \subseteq B)$, it follows from the isotonicity of $\Pi'$ that $(\forall B \in \mathcal{A})(x \in B \Rightarrow \Pi'([x]_{\mathcal{R}}) \leq \Pi'(B) = \Pi(B))$. □

This lemma inspires the following definition. Lemma 9 assures us that this definition is indeed meaningful, provided that $\mathcal{A} \subseteq \mathcal{R}$, or equivalently, $\tau(\mathcal{A}) \subseteq \mathcal{R}$.

**Definition 11.** We shall denote by $\Pi^*_{\mathcal{R}}$ the $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ defined by, for any $x$ in $X$:

$$\Pi^*_{\mathcal{R}}([x]_{\mathcal{R}}) = \inf_{\substack{B \in \mathcal{A} \\ x \in B}} \Pi(B).$$

Since, obviously, $\Pi^*_{\mathcal{R}}([x]_{\mathcal{R}})$ does not depend on the choice of $\mathcal{R}$, provided that $\mathcal{A} \subseteq \mathcal{R}$, i.e., $\tau(\mathcal{A}) \subseteq \mathcal{R}$, we shall also denote this as $\pi^*(x)$. In other words, $\pi^*$ is the distribution of all the $\Pi^*_{\mathcal{R}}$, $\mathcal{A} \subseteq \mathcal{R}$.

**Lemma 12.** $(\forall A \in \mathcal{A})(\Pi^*_{\mathcal{R}}(A) \leq \Pi(A))$.

*Proof.* For any $A$ in $\mathcal{A}$ we obtain that

$$\Pi^*_{\mathcal{R}}(A) = \sup_{x \in A} \Pi^*_{\mathcal{R}}([x]_{\mathcal{R}}) = \sup_{x \in A} \inf_{\substack{B \in \mathcal{A} \\ x \in B}} \Pi(B) \leq \sup_{x \in A} \Pi(A) = \Pi(A). \quad \square$$

We are now able to prove the main result of this paper: a criterion for the extendability of a P-consistent $\mathcal{A}-L$-mapping $\Pi$. Corollary 14 gives an alternative form for this criterion. Its proof is trivial.

**Theorem 13.** $\Pi$ *is extendable to a* $(L,\leq)$*-possibility measure on* $(X,\mathcal{R})$ *if and only if* $\Pi^*_{\mathcal{R}}$ *is an extension of* $\Pi$*:* $(\forall A\in\mathcal{A})(\Pi^*_{\mathcal{R}}(A)=\Pi(A))$.

*Proof.* First of all, assume that $\Pi'$ is a $(L,\leq)$-possibility measure on $(X,\mathcal{R})$ that is an extension of $\Pi$. For any $A$ in $\mathcal{A}$, we obtain, since $\mathcal{A}\subseteq\mathcal{R}$, using lemmas 10 and 12,

$$\Pi(A)=\Pi'(A)=\sup_{x\in A}\Pi'([x]_{\mathcal{R}})\leq\sup_{x\in A}\Pi^*_{\mathcal{R}}([x]_{\mathcal{R}})=\Pi^*_{\mathcal{R}}(A)\leq\Pi(A).$$

Therefore, $\Pi^*_{\mathcal{R}}$ is an extension of $\Pi$. The proof of the reverse implication is trivial. □

**Corollary 14.** $\Pi$ *is extendable to a* $(L,\leq)$*-possibility measure if and only if for any* $A$ *in* $\mathcal{A}$*:*

$$\Pi(A)=\sup_{x\in A}\inf_{\substack{B\in\mathcal{A}\\x\in B}}\Pi(B)=\sup_{x\in A}\pi^*(x).$$

## 4. P-Consistency and Extendability: Counterexamples

In this section it is shown by means of counterexamples that not every P-consistent mapping is extendable to a possibility measure, or in other words, that not every P-consistent mapping satisfies the criteria given in theorem 13 and corollary 14.

*Example 15.* Let $L=\{0,1,a_1,a_2,a_3\}$ be a set consisting of 5 elements and define on $L$ the relation $\leq\,=\{(0,\lambda)\mid\lambda\in L\}\cup\{(\lambda,1)\mid\lambda\in L\}\cup\{(a_i,a_i)\mid i\in\{1,2,3\}\}$. Then $(L,\leq)$ is a complete lattice. Let $X=L$, $\mathcal{A}=\{\{a_1,a_2\},\{a_1,a_3\},\{a_2,a_3\}\}$ and let $\Pi$ be the $\mathcal{A}-L$-mapping defined by

$$\begin{cases}\Pi(\{a_1,a_2\})=a_3,\\\Pi(\{a_1,a_3\})=a_2,\\\Pi(\{a_2,a_3\})=a_1.\end{cases}$$

It is clear that $\Pi$ is a P-consistent $\mathcal{A}-L$-mapping, but $\Pi$ is not extendable to a $(L,\leq)$-possibility measure since

$$\sup(\inf_{\substack{B\in\mathcal{A}\\a_1\in B}}\Pi(B),\inf_{\substack{B\in\mathcal{A}\\a_2\in B}}\Pi(B))=\sup(0,0)=0\neq\Pi(\{a_1,a_2\}).$$

*Example 16.* Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be two sequences, satisfying

$$\begin{cases}(\forall(i,j)\in\mathbb{N}^2)(i\neq j\Rightarrow a_i\neq a_j)\\(\forall(i,j)\in\mathbb{N}^2)(i\neq j\Rightarrow b_i\neq b_j)\\(\forall(i,j)\in\mathbb{N}^2)(a_i\neq b_j).\end{cases}$$

Also consider two different elements 0 and 1 that do not occur either in $(a_i)_{i\in\mathbb{N}}$ nor in $(b_i)_{i\in\mathbb{N}}$. Furthermore, let $L=\{a_i\mid i\in\mathbb{N}\}\cup\{b_i\mid i\in\mathbb{N}\}\cup\{0,1\}$. If we define the relation $\leq$ on $L$ by

$$\begin{aligned}\leq=&\{(a_i,a_j)\mid(i,j)\in\mathbb{N}^2\text{ and }i\geq j\}\cup\{(b_i,b_j)\mid(i,j)\in\mathbb{N}^2\text{ and }i\geq j\}\\&\cup\{(a_i,b_j)\mid(i,j)\in\mathbb{N}^2\text{ and }i>j\}\cup\{(0,x)\mid x\in L\}\cup\{(x,1)\mid x\in L\},\end{aligned}$$

then $(L,\leq)$ is a complete lattice. Let $\mathcal{V}$ be the smallest field on $\overline{\mathbb{N}}=\mathbb{N}\cup\{+\infty\}$ that includes $\{[n,+\infty]\mid n\in\mathbb{N}\}$. Since

$$(\forall A\in\mathcal{V})(\exists n\in\mathbb{N})(A\subseteq[0,n]\text{ or }[n,+\infty]\subseteq A)$$

the following definition is meaningful. Let $\Pi$ be the $\mathcal{V}-L$-mapping defined by

$$\Pi(A)=\begin{cases}0 & \text{if }A=\emptyset,\\ 1 & \text{if }+\infty\in A\text{ and }0\in A\\ b_{\min A-1} & \text{if }+\infty\in A\text{ and }0\notin A\\ a_{\min A} & \text{if }+\infty\notin A\text{ and }A\neq\emptyset.\end{cases}$$

It is trivially verified that $\Pi$ is a P-consistent $\mathcal{V}-L$-mapping. But $\Pi$ is not extendable to a $(L,\leq)$-possibility measure because

$$\sup_{x\in\overline{\mathbb{N}}}\inf_{\substack{B\in\mathcal{V}\\x\in B}}\Pi(B)=\sup(\sup_{x\in\mathbb{N}}\inf_{\substack{B\in\mathcal{V}\\x\in B}}\Pi(B),\inf_{\substack{B\in\mathcal{V}\\+\infty\in B}}\Pi(B))=\sup(\sup_{x\in\mathbb{N}}a_x,0)=a_0\neq1=\Pi(\overline{\mathbb{N}}).$$

*Example 17.* Let $\mathcal{A}$ be the smallest topology on $\mathbb{R}$ that includes $\{]a,b[\mid(a,b)\in\mathbb{R}^2\}$. Since $\mathcal{A}$ is a dual closure system[1] on $\wp(\mathbb{R})$, $(\mathcal{A},\subseteq)$ is a complete lattice in which, for an arbitrary family $(A_j)_{j\in J}$ of elements of $\mathcal{A}$,

$$\sup_{j\in J}A_j=\bigcup_{j\in J}A_j\text{ and }\inf_{j\in J}A_j=\bigcup\{B\mid B\in\mathcal{A}\text{ and }B\subseteq\bigcap_{j\in J}A_j\}.$$

It is easily proven that for any $x$ in $\mathbb{R}$, $\inf\{B\mid B\in\mathcal{A}\text{ and }x\in B\}=\emptyset$. Let $\Pi$ be the identical mapping on $\mathcal{A}$. Then $\Pi$ is a P-consistent $\mathcal{A}-\mathcal{A}$-mapping but $\Pi$ is not extendable to a $(\mathcal{A},\subseteq)$-possibility measure because, for any $A$ in $\mathcal{A}\setminus\{\emptyset\}$,

$$\sup_{x\in A}\inf_{\substack{B\in\mathcal{A}\\x\in B}}\Pi(B)=\sup_{x\in A}\emptyset=\emptyset\neq A=\Pi(A).$$

## 5. Special Cases: Sufficient Conditions for Extendability

It is obvious from corollary 14 that whether or not a P-consistent $\mathcal{A}-L$-mapping $\Pi$ is extendable, depends in an intricate and complicated way on the nature of $(L,\leq)$, $\mathcal{A}$ and $\Pi$. In this section, we shall consider a number of special cases for $(L,\leq)$ and $\mathcal{A}$, for which the notions of extendability and P-consistency coincide.

The following result tells us that if the codomain $(L, \leq)$ of the $\mathcal{A} - L$-mapping $\Pi$ is a complete chain, then $\Pi$ is extendable if and only if it is P-consistent. Since $([0,1], \leq)$ is a complete chain, this result generalizes Wang's extension theorem[6,7]. The proof of this result rather heavily relies on the characterization of supremum and infimum in a chain. Indeed, if $(L, \leq)$ is a chain, then, for any $Q \subseteq L$ and any $\lambda$ in $L$:

$$\lambda < \sup Q \Leftrightarrow (\exists \mu \in Q)(\lambda < \mu) \tag{4}$$

$$\lambda > \inf Q \Leftrightarrow (\exists \mu \in Q)(\lambda > \mu). \tag{5}$$

**Theorem 18.** *Assume that $(L, \leq)$ is a complete chain and that $\Pi$ is a P-consistent $\mathcal{A} - L$-mapping. Then $\Pi$ is extendable to a $(L, \leq)$-possibility measure.*

*Proof.* Assume *ex absurdo* that $\Pi$ is not extendable to a $(L, \leq)$-possibility measure. Then corollary 14 tells us that there exists an element $A$ of $\mathcal{A}$ such that $\sup_{x \in A} \pi^*(x) \neq \Pi(A)$. Since, by definition of $\pi^*$, $\Pi(A)$ is an upper bound of $\{\pi^*(x) \mid x \in A\}$ there exists an element $c$ of $L$ such that $c < \Pi(A)$ and $(\forall x \in A)(\pi^*(x) \leq c)$. Taking into account (5), we conclude that there exists a family $(B_x)_{x \in A}$ of elements of $\mathcal{A}$ such that $(\forall x \in A)(x \in B_x$ and $\Pi(B_x) < \Pi(A))$. Since, by definition of P-consistency and $A \subseteq \bigcup_{x \in A} B_x$, $\Pi(A) = \sup_{x \in A} \Pi(B_x)$, it follows, using (4), that there exists an element $y$ of $A$ such that $c < \Pi(B_y) < \Pi(A)$. So, once again using (5), we find that we can choose a family $(B'_x)_{x \in A}$ of elements of $\mathcal{A}$ such that $(\forall x \in A)(x \in B'_x$ and $\Pi(B'_x) < \Pi(B_y))$. From this we obtain a contradiction since, again by the definition of P-consistency and $A \subseteq \bigcup_{x \in A} B'_x$,

$$\Pi(A) \leq \sup_{x \in A} \Pi(B'_x) \leq \Pi(B_y) < \Pi(A). \quad \square$$

On the other hand, if we concentrate on the domains of P-consistent mappings, we have the following interesting result.

**Theorem 19.** *Let $\mathcal{L}$ be a plump field on $X$ and $\Pi$ a P-consistent $\mathcal{L} - L$-mapping. Then $\Pi$ is extendable to a $(L, \leq)$-possibility measure.*

*Proof.* First, consider an arbitrary element $x$ of $X$. For arbitrary $B$ in $\mathcal{L}$ for which $x \in B$, we have that $[x]_{\mathcal{L}} \in \mathcal{L}$ and $[x]_{\mathcal{L}} \subseteq B$, and the isotonicity of the P-consistent mapping $\Pi$ tells us that $\Pi([x]_{\mathcal{L}}) \leq \Pi(B)$. Therefore,

$$\Pi([x]_{\mathcal{L}}) \leq \inf_{\substack{B \in \mathcal{L} \\ x \in B}} \Pi(B) \leq \Pi([x]_{\mathcal{L}}).$$

Furthermore, let $A$ be an arbitrary element of $\mathcal{L}$. Since in the plump field $\mathcal{L}$ the equality $A = \bigcup_{x \in A} [x]_{\mathcal{L}}$ holds, the P-consistency of $\Pi$ tells us that $\Pi(A) \leq \sup_{x \in A} \Pi([x]_{\mathcal{L}})$. On the other hand, we have for arbitrary $x$ in $A$ that $[x]_{\mathcal{L}} \subseteq A$, and since $\Pi$ is isotonic, $\Pi([x]_{\mathcal{L}}) \leq \Pi(A)$, whence $\sup_{x \in A} \Pi([x]_{\mathcal{L}}) \leq \Pi(A)$. We may therefore conclude that

$$\Pi(A) = \sup_{x \in A} \Pi([x]_{\mathcal{L}}) = \sup_{x \in A} \inf_{\substack{B \in \mathcal{L} \\ x \in B}} \Pi(B).$$

Corollary 14 now tells us that $\Pi$ is extendable to a $(L, \leq)$-possibility measure. $\square$

Finally, we consider another special class of codomains, that will turn out useful in the following section.

**Theorem 20.** *Let $\mathcal{B}$ be a non-empty subset of the power class $\wp(Y)$ of a non-empty set $Y$ such that $(\mathcal{B}, \supseteq)$ is a complete lattice in which, for an arbitrary non-empty family $(A_j)_{j\in J}$, $\sup_{j\in J} A_j = \bigcap_{j\in J} A_j$ and $\inf_{j\in J} A_j = \bigcup_{j\in J} A_j$. In other words, $\mathcal{B}$ must be closed under arbitrary non-empty intersections and unions. Let $\Pi$ be a P-consistent $\mathcal{A} - \mathcal{B}$-mapping. Then $\Pi$ is extendable to a $(\mathcal{B}, \supseteq)$-possibility measure.*

*Proof.* First of all, it should be noted that the bottom 0 of the complete lattice $(\mathcal{B}, \supseteq)$ is not necessarily $Y$, and that its top 1 is not necessarily $\emptyset$, because $\mathcal{B}$ need only be closed under *non-empty* unions and intersections. Consider an arbitrary $A$ in $\mathcal{A}$. Let us, for the sake of notational simplicity, denote $\sup_{x\in A} \inf_{\substack{B\in\mathcal{A}\\ x\in B}} \Pi(B)$ by $C$. Then it must be proven that $\Pi(A) = C$. Two possibilities may occur. Either $A = \emptyset$, whence, from the P-consistency of $\Pi$, $\Pi(A) = 0$. On the other hand, since in the complete lattice $(\mathcal{B}, \supseteq)$, $\sup \emptyset = 0$, we also find that $C = 0$, whence $\Pi(A) = C$. Or $A \neq \emptyset$. Then we may write $C = \bigcap_{x\in A} \bigcup_{\substack{B\in\mathcal{A}\\ x\in B}} \Pi(B)$, and it is obvious that $\Pi(A) \subseteq C$. If $\emptyset \in \mathcal{B}$ then it is possible that $C = \emptyset$, in which case also $\Pi(A) = C = \emptyset$. Let us now assume that $C \neq \emptyset$. Let $a$ be an arbitrary element of $C$. This implies that there exists a family $(B_x)_{x\in A}$ of elements of $\mathcal{A}$ such that for arbitrary $x$ in $A$, $x \in B_x$ and $a \in B_x$. By the definition of P-consistency and $A \subseteq \bigcup_{x\in A} B_x$ we obtain $a \in \bigcap_{x\in A} \Pi(B_x) \subseteq \Pi(A)$. This implies that $C \subseteq \Pi(A)$. □

## 6. Embeddings and Extendability

In this final section, we show that there is always a way in which a P-consistent $\mathcal{A} - L$-mapping can be made extendable, simply by embedding $(L, \leq)$ in a new and appropriately chosen complete lattice $(L', \leq')$ using a mapping $\phi\colon L \to L'$, and by considering $\Pi' = \phi \circ \Pi$ instead of $\Pi$. Of course, there is one important condition which must be imposed on $\phi$. Indeed, assume that for some family $(A_j)_{j\in J}$, $\bigcup_{j\in J} A_j \in \mathcal{A}$. Proposition 4 tells us that $\Pi(\bigcup_{j\in J} A_j) = \sup_{j\in J} \Pi(A_j)$ and we would want to extend this supremum preserving behaviour to $\Pi'$. Of course, this will be the case if $\phi$ is supremum preserving. As a matter of fact, it is readily proven that a supremum preserving mapping even preserves P-consistency.

**Proposition 21.** *Let $(L', \leq')$ be a complete lattice, $\phi$ be a supremum preserving $L - L'$-mapping and $\Pi$ be a P-consistent $\mathcal{A} - L$-mapping. Then $\phi \circ \Pi$ is a P-consistent $\mathcal{A} - L'$-mapping.*

Now, a subset $U$ of the complete lattice $(L, \leq)$ for which

$$(\forall x \in U)(\forall y \in L)(x \leq y \Rightarrow y \in U)$$

is called an up-set of $(L, \leq)$. The set of all up-sets of $(L, \leq)$ is denoted by $\mathcal{U}(L)$. Consider the set $\mathcal{U}^*(L) = \mathcal{U}(L) \setminus \{\emptyset\}$. It is easily proven that $(\mathcal{U}^*(L), \supseteq)$ is a complete lattice that satisfies the conditions of theorem 20. Define, for every $x \in L$,

$$\uparrow x = \{y \mid y \in L \text{ and } x \leq y\}.$$

Since the $L - \mathcal{U}^*(L)$-mapping $\phi$, defined by $(\forall x \in L)(\phi(x) = \uparrow x)$, is a supremum preserving order embedding of $(L, \leq)$ into $(\mathcal{U}^*(L), \supseteq)$, the proposition above tells us that $\phi \circ \Pi$ is P-consistent, and theorem 20 assures us that $\phi \circ \Pi$ is extendable to a $(L', \leq')$-possibility measure. This leads to the following important result.

**Theorem 22.** *The complete lattice $(L, \leq)$ can be embedded using a supremum preserving mapping $\phi$ in a second complete lattice $(L', \leq')$, in such a way that for every P-consistent $\mathcal{A} - L$-mapping $\Pi$, the P-consistent $\mathcal{A} - L'$ mapping $\phi \circ \Pi$ is extendable to a $(L', \leq')$-possibility measure.*

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## 8. References


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