CONFIDENCE RELATIONS AND QUALITATIVE POSSIBILITY

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1 Introduction

In this paper, we deal with the order-theoretic characterization of possibility measures. We define the notion of a qualitative possibility ordering, which is a generalization of Dubois' qualitative possibility relations [10, 11] in the following sense: they are defined on arbitrary, not necessarily finite universes, and they allow for incomparability. At the same time, we show that any qualitative possibility ordering is not necessarily determined by its distribution relation, and that in general, special extra conditions must be imposed in order to make sure that it would be.

2 Preliminary results and definitions

Let us begin by giving a brief overview of the material needed for properly understanding what follows. We shall work with a *universe of discourse* X, assumed to be non-empty. We also consider an *ample field* $\mathcal{R}$ [9, 13] on X, which is a class of subsets of X that is closed under arbitrary unions and under complementation. The *atom* $[x]_{\mathcal{R}}$ of $\mathcal{R}$ containing x is the element of $\mathcal{R}$ defined by $[x]_{\mathcal{R}} = \bigcap\{A \mid A \in \mathcal{R} \text{ and } x \in A\}$. The set of the atoms of $\mathcal{R}$ will be denoted by $X_{\mathcal{R}}$, and constitutes a partition of X. Note that for any $A \subseteq X$, $A \in \mathcal{R} \Leftrightarrow A = \bigcup_{x\in A}[x]_{\mathcal{R}}$.

A generic complete lattice will be denoted by $(L, \leq)$. Its top is denoted by 1_L, its bottom by 0_L. It will be assumed that $0_L \neq 1_L$. For more information about the specifically order-theoretic concepts used rather liberally in this paper, we refer to [1, 2].

If $(\mathbb{B}, \leq)$ is a Boolean lattice, then a binary relation $\preceq$ on $\mathbb{B}$ is called a *confidence relation* on $\mathbb{B}$ iff

$$a \preceq b \text{ and } b \preceq c \Rightarrow a \preceq c \qquad \text{(transitivity)} \qquad \text{(C1)}$$

$$a \leq b \Rightarrow a \preceq b \qquad \text{(superorder)} \qquad \text{(C2)}$$

$$1_{\mathbb{B}} \not\preceq 0_{\mathbb{B}} \qquad \text{(non-triviality)} \qquad \text{(C3)}$$

where a, b and c are arbitrary elements of $\mathbb{B}$, $1_{\mathbb{B}}$ is the top of $(\mathbb{B}, \leq)$ and $0_{\mathbb{B}}$ its bottom. If $\preceq$ satisfies (C2), we say that it is a *superorder* on $\mathbb{B}$. By (C1) and (C2), any confidence relation is in particular a partial preorder relation. Let us also define the following derived binary relations on $\mathbb{B}$, where $\preceq$ is any binary relation on $\mathbb{B}$, and a and b are elements of $\mathbb{B}$: $a \approx b \Leftrightarrow (a \preceq b \text{ and } b \preceq a)$, $a \prec b \Leftrightarrow (a \preceq b \text{ and } b \not\preceq a)$ and $a \parallel b \Leftrightarrow (a \not\preceq b \text{ and } b \not\preceq a)$.

Now, let $\preceq$ be a confidence relation on $\mathbb{B}$. It is clear that $\approx$ is an equivalence relation on $\mathbb{B}$. Also, for any a in $\mathbb{B}$, $0_{\mathbb{B}} \preceq a \preceq 1_{\mathbb{B}}$. Note that we do *not* require that $\preceq$ should be *complete*, or equivalently, that $\parallel$ should be empty. It also follows rather easily that there always exists at least one bounded poset $(P, \leq)$ with bottom 0_P and top 1_P, $1_P \neq 0_P$, and a $\mathbb{B} - P$-mapping v such that $v(0_{\mathbb{B}}) = 0_P$, $v(1_{\mathbb{B}}) = 1_P$ and for any a and b in $\mathbb{B}$, $a \preceq b \Leftrightarrow v(a) \leq v(b)$. By (C2), v is increasing, i.e., $v(a) \leq v(B)$ if $a \leq b$. $(P, \leq)$ will be called an *evaluation set* of $\preceq$, and v the corresponding *evaluation mapping*. $((P, \leq), v)$ is called an *evaluation structure* of $\preceq$. Interestingly, there always exists a canonical evaluation structure $((M, \leq), m)$ of $\preceq$ such that m is onto (surjective) and such that for any other evaluation structure $((P, \leq), v)$ of $\preceq$, there exists a top and bottom preserving order-embedding ξ of $(M, \leq)$ into $(P, \leq)$, with $v = \xi \circ m$. For obvious reasons, such an evaluation structure $((M, \leq), m)$ will be called *minimal*. Minimal evaluation structures are of course uniquely determined up to an order-isomorphism.

A detailed mathematical study of confidence relations can be found in [3, 7, 8]. In what follows, we shall mainly be working with confidence relations on the (complete) Boolean lattice $(\mathcal{R}, \subseteq)$, where $\subseteq$ denotes the inclusion of subsets of X.

A mapping Π from an ample field $\mathcal{R}$ to a complete lattice $(L, \leq)$ is called a $(L, \leq)$-*possibility measure* (or simply possibility measure) on $(X, \mathcal{R})$ iff it is a complete join-morphism between the complete lattices $(\mathcal{R}, \subseteq)$ and $(L, \leq)$, i.e., iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$, $\Pi(\bigcup_{j\in J})A_j = \sup_{j\in J} \Pi(A_j)$. Note that this definition implies that $\Pi(\emptyset) = 0_L$. Π is called *normal* iff $\Pi(X) = 1_L$. In this paper, we only work with normal possibility measures. A $X - L$-mapping π is called a *distribution* of Π iff it is constant on the atoms of $\mathcal{R}$ ($\mathcal{R}$-measurable) and if for any A in $\mathcal{R}$, $\Pi(A) = \sup_{x\in A} \pi(x)$. Such a distribution is unique and given by $\pi(x) = \Pi([x]_{\mathcal{R}})$, $x \in X$. A possibility measure is therefore completely determined by its distribution and *vice versa*. For more information about possibility measures, we refer to [4, 5, 6, 12, 14].

Finally, if Π is a normal $(L, \leq)$-possibility measure on $(X, \mathcal{R})$, then the binary relation $\preceq$ on $\mathcal{R}$, defined by $A \preceq B \Leftrightarrow \Pi(A) \leq \Pi(B)$, for any A and B in $\mathcal{R}$, is obviously a confidence relation on $\mathcal{R}$. We shall say that $\preceq$ is the confidence relation which *corresponds* to Π. Of course, $((L, \leq), \Pi)$ is an evaluation structure of $\preceq$, and $((\Pi(\mathcal{R}), \leq), \Pi)$ a minimal evaluation structure.

3 Union-closedness and qualitative possibility orderings

Let $\preceq$ be a confidence relation on $\mathcal{R}$ and consider a family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$ and any B in $\mathcal{R}$. If $(\forall j \in J)(A_j \preceq B)$, it follows that $\bigcap_{j\in J} A_j \preceq B$ since $\preceq$ is a superorder on $\mathcal{R}$. Nevertheless, it does not necessarily hold that $\bigcup_{j\in J} A_j \preceq B$. This observation leads to the following definition, which plays a central role in our order-theoretic characterization of the notion of possibility.

Definition 1. Let $\preceq$ be a confidence relation on $\mathcal{R}$. We call $\preceq$ *union-closed* iff for any family $(A_j \mid j \in J)$ of elements of $\mathcal{R}$ and any B in $\mathcal{R}$: $(\forall j \in J)(A_j \preceq B) \Rightarrow \bigcup_{j\in J} A_j \preceq B$.

An alternative characterization of union-closedness, using the atoms of the ample field $\mathcal{R}$ is given in the following proposition.

Proposition 2. *Let $\preceq$ be a confidence relation on $\mathcal{R}$. Then $\preceq$ is union-closed iff for any A and B in $\mathcal{R}$, $(\forall x \in A)([x]_{\mathcal{R}} \preceq B) \Rightarrow A \preceq B$.*

The following result gives us a first indication of why union-closed confidence relations are important for the characterization of qualitative (ordinal) possibility.

Proposition 3. *A confidence relation $\preceq$ on $\mathcal{R}$ is union-closed iff it has a complete lattice $(L, \leq)$ as a minimal evaluation set and a normal $(L, \leq)$-possibility measure on $(X, \mathcal{R})$ as the associated minimal evaluation mapping.*

Moreover, for any confidence relation $\preceq$ on $\mathcal{R}$, the following statements are easily shown to be equivalent:

$$(\forall (A, B) \in \mathcal{R}^2)(A \preceq B \Rightarrow A \cup B \approx B) \quad (\Pi1)$$

$$(\forall (A, B, C) \in \mathcal{R}^3)(B \preceq C \Rightarrow A \cup B \preceq A \cup C) \quad (\Pi2)$$

$$(\forall (A, B, C) \in \mathcal{R}^3)(B \preceq A \text{ and } C \preceq A \Rightarrow (B \cup C) \preceq A). \quad (\Pi3)$$

Note that for a *finite* universe of discourse X, (Π3) is equivalent with the defining condition for union-closedness. On the other hand, the conditions (Π1) and (Π2) were used by Dubois [10, 11] to characterize his qualitative possibility relations on finite universes of discourse. Indeed, Dubois calls a binary relation $\preceq$ on the power set $\wp(X)$ of a finite universe of discourse X a *qualitative possibility relation* (QPR) iff it satisfies (Π1) or (Π2) and

$\emptyset \prec X$	(non-triviality)	(A0)
$A \preceq B$ or $B \preceq A$	(comparability, completeness)	(A1)
$A \preceq B$ and $B \preceq C \Rightarrow A \preceq C$	(transitivity)	(A2)
$\emptyset \preceq A$		(A3)

where A, B and C are arbitrary subsets of X. Note that Dubois has shown that (A0)–(A3) also imply the equivalence of (Π1) and (Π2). Since it is easily verified that a QPR is a superorder on $\wp(X)$, and therefore, by (A0) and (A2), a confidence relation on $\wp(X)$, we are led to the following conclusion.

Proposition 4. *Let X be a finite universe of discourse. A binary relation $\preceq$ on $\wp(X)$ is a qualitative possibility relation iff it is a complete and union-closed confidence relation on $\wp(X)$.*

The discussion above and in particular propositions 3 and 4 tell us that if we want to define qualitative possibility on *arbitrary* universes that are *not necessarily complete*, union-closed confidence relations are very good candidates.

Definition 5. A binary relation $\preceq$ on $\mathcal{R}$ is called a *qualitative possibility ordering* (QPO) on $\mathcal{R}$ iff it is a union-closed confidence relation on $\mathcal{R}$.

Indeed, a confidence relation is qualitative possibility ordering iff its minimal evaluation sets are complete lattices and the corresponding evaluation mappings are normal possibility measures.

It is now an obvious but nevertheless interesting question whether for qualitative possibility orderings there exists an ordinal equivalent to the notion of a distribution. In other words, consider a qualitative possibility ordering $\preceq$ on $\mathcal{R}$, and define the binary relation $\preceq_X$ on X as follows: for x and y in X,

$$x \preceq_X y \Leftrightarrow [x]_{\mathcal{R}} \preceq [y]_{\mathcal{R}}. \tag{1}$$

We call $\preceq_X$ the *distribution relation* of $\preceq$. Then clearly $\preceq_X$ will be the ordinal counterpart of the notion of a distribution iff we can recover $\preceq$ *uniquely* from $\preceq_X$, or in other words, if $\preceq$ is the only QPO on $\mathcal{R}$ which has $\preceq_X$ as its distribution relation.

Dubois [10, 11] has dealt with this question in the case of his qualitative possibility relations, i.e., complete qualitative possibility orderings on finite universes of discourse. Let $\preceq$ be such a QPR on the finite set X. Since in this case $\mathcal{R} = \wp(X)$, $\preceq_X$ is defined by $x \preceq_X y \Leftrightarrow \{x\} \preceq \{y\}$, for any x and y in X. For any subset A of X, define $\mathcal{M}(A) = \{x \mid x \in A \text{ and } \{x\} \approx A\}$. Then Dubois has shown that for arbitrary *non-empty* subsets A and B of X: $A \preceq B \Leftrightarrow (\forall x \in \mathcal{M}(A))(\forall y \in \mathcal{M}(B))(x \preceq_X y)$, and that for any *non-empty* subset C of X: $\mathcal{M}(C) = \{x \mid x \in C \text{ and } (\forall y \in C)(x \preceq_X y \Rightarrow y \preceq_X x)\}$ is the set of the maximal (non-dominated) elements of C. Thus, since for non-empty C, $\mathcal{M}(C)$ is completely determined by the distribution relation $\preceq_X$, Dubois' results tell us that for any A and B in $\wp(X)$ the relation $\preceq_X$ determines whether $A \preceq B$, *provided that* $B \neq \emptyset$. In general, there seems to be no way to determine whether $A \preceq \emptyset$ from the point information $\preceq_X$ alone, as we shall corroborate further on. *This leads to the conclusion that in the case of Dubois' qualitative possibility relations,* $\preceq$ *is not in general completely determined by* $\preceq_X$.

In what follows, we discuss the problem in the more general setting of arbitrary universes X that are not necessarily finite, and qualitative possibility orderings $\preceq$ that are not necessarily complete. Let us first of all mention that Dubois' analysis using the sets $\mathcal{M}(C)$ does not carry over to this more general problem, because it hinges on the completeness of his qualitative possibility relations. In order to make the problem more tractable, let us reformulate it in such a way that we preserve only what is essential. We again consider a qualitative possibility ordering $\preceq$ on $\mathcal{R}$, and its distribution relation $\preceq_X$. Since $\preceq_X$ is a partial preorder relation on X which is not necessarily antisymmetric, we rephrase everything in terms of its canonical partial order. Define the binary relation $\sim$ on X as follows: $x \sim y \Leftrightarrow (x \preceq_X y \text{ and } y \preceq_X x)$. Clearly, $\sim$ is an equivalence relation, and its equivalence classes define a partition $X/\sim$ of X. Note that $X_{\mathcal{R}}$ is a finer partition than $X/\sim$. Consider the smallest ample field $\mathcal{R}_\sim$ on X which includes $X/\sim$, i.e., $\mathcal{R}_\sim = \bigcap\{\mathcal{S} \mid \mathcal{S} \text{ is an ample field on } X \text{ and } X/\sim \subseteq \mathcal{S}\}$. Then the atoms of $\mathcal{R}_\sim$ are precisely the equivalence classes of $\sim$. It should be obvious now that $\mathcal{R}_\sim \subseteq \mathcal{R}$. Since $\preceq_X$ is a partial preorder, it is always possible to define a binary relation $\trianglelefteq$ on $X/\sim$ as follows:

$$(\forall (x,y) \in X^2)(x \preceq_X y \Leftrightarrow [x]_{\mathcal{R}_\sim} \trianglelefteq [y]_{\mathcal{R}_\sim}). \tag{2}$$

$(X/\sim, \trianglelefteq)$ is a partially ordered set, which by (2) completely determines $\preceq_X$. We will assume in the sequel that $X/\sim$ is non-trivial, in that it contains more than one element. The discussion above tells us that *specifying a distribution relation is completely equivalent to specifying a partial order relation on a partition of* X. For this reason, we call $(X/\sim, \trianglelefteq)$ the *distribution structure* of $\preceq$. By combining (1) and (2), we find, since $\preceq$ is union-closed confidence relation on $\mathcal{R}$ and since $\mathcal{R}_\sim \subseteq \mathcal{R}$, that for any x in X, $[x]_{\mathcal{R}_\sim} \approx [x]_{\mathcal{R}}$, or equivalently,

$$(\forall x \in X)([x]_{\mathcal{R}_\sim} \preceq [x]_{\mathcal{R}}). \tag{3}$$

Our problem can now be formulated as follows: *Given a non-trivial partition* $\mathcal{A} \subset \mathcal{R}$ *of* X *and a partial order relation* $\leq_{\mathcal{A}}$ *on* $\mathcal{A}$*, does there always exist a unique qualitative possibility ordering on* $\mathcal{R}$ *which has* $(\mathcal{A}, \leq_{\mathcal{A}})$ *as its distribution structure?*

Note that it is readily verified using the results given above that the QPO $\preceq$ on $\mathcal{R}$ has $(\mathcal{A}, \leq_{\mathcal{A}})$ as its distribution structure iff $(\forall (A,B) \in \mathcal{A}^2)(A \preceq B \Leftrightarrow A \leq_{\mathcal{A}} B)$ and $(\forall C \in \mathcal{A})(\forall x \in C)(C \preceq [x]_{\mathcal{R}})$.

4 The classical case

Let us first consider the ordinal representation of classical possibility measures. Consider the Boolean chain $(\{0,1\},\leq)$, a non-empty element E of $\mathcal{R}$ and the $(\{0,1\},\leq)$-possibility measure Π_E on $(X,\mathcal{R})$, defined by, for any A in $\mathcal{R}$:

$$\Pi_E(A) = \begin{cases} 1 & ; \quad A \cap E \neq \emptyset \\ 0 & ; \quad A \cap E = \emptyset. \end{cases}$$

Let us define $\mathcal{B}_E = \{A \mid A \in \mathcal{R} \text{ and } \Pi_E(A) = 0\}$ and the corresponding binary relation $\preceq_E$ on $\mathcal{R}$ by $A \preceq_E B \Leftrightarrow \Pi_E(A) \leq \Pi_E(B)$, for any A and B in $\mathcal{R}$. Since $E \neq \emptyset$, $\mathcal{B}_E$ is the non-trivial principal ideal of $(\mathcal{R},\subseteq)$ generated by $\operatorname{co} E$, i.e., $\mathcal{B}_E = \{A \mid A \in \mathcal{R} \text{ and } A \subseteq \operatorname{co} E\}$, and $\preceq_E$ is a qualitative possibility ordering on $\mathcal{R}$, also given by $\preceq_E = (\mathcal{R} \setminus \mathcal{B}_E \times \mathcal{R} \setminus \mathcal{B}_E) \cup (\mathcal{B}_E \times \mathcal{R})$. The distribution of Π_E is the characteristic $X - \{0,1\}$-mapping χ_E of E. The ordinal counterpart of this distribution is the distribution structure $(\mathcal{A}_E, \leq_E)$ of $\preceq_E$, where $\mathcal{A}_E = \{\operatorname{co} E, E\}$ and $\operatorname{co} E <_E E$. Note that $(\mathcal{A}_E, \leq_E)$ is a Boolean chain. Since χ_E completely determines Π, it is now very natural to ask whether $(\mathcal{A}_E, \leq_E)$ determines $\preceq_E$. Note that $\leq_E$ constitutes the nucleus of a confidence relation, and we can therefore try to extend $\leq_E$ towards a full-fledged confidence relation $\preceq$ on $\mathcal{R}$, where, of course, $\preceq \cap (\mathcal{A}_E)^2 = \leq_E$. There are clearly many such extensions, and our problem therefore consists in trying to find a set of extra conditions to be imposed on $\preceq$ such that $\preceq = \preceq_E$. This is done in the following theorem.

Theorem 6. *Let E be an element of $\mathcal{R} \setminus \{\emptyset\}$ and let $\preceq$ be an arbitrary confidence relation on $\mathcal{R}$. Then $\preceq = \preceq_E$ iff the following four conditions hold:*

1. *$\preceq$ extends $\leq_E$ to $\mathcal{R}$, i.e., $A \leq_E B \Leftrightarrow A \preceq B$, for any A and B in $\mathcal{A}_E$;*
2. *$\preceq$ is union-closed;*
3. *$(\forall C \in \mathcal{A}_E)(\forall x \in C)(C \preceq [x]_{\mathcal{R}})$;*
4. *$m(\mathcal{A}_E)$ is meet-dense in $(M,\leq)$, where $((M,\leq),m)$ is a minimal evaluation structure of $\preceq$.*

The interpretation of conditions 1 and 2 is obvious: they ensure that $\preceq$ is a qualitative possibility ordering that agrees with $\leq_E$ on $\mathcal{A}_E$. Condition 3 makes sure that the confidence expressed in $\preceq$ cannot be finer-grained than $\mathcal{A}_E$. This condition obviously relates to (3). Taken together, conditions 1–3 state that $\preceq$ has $(\mathcal{A}_E, \leq_E)$ as its distribution structure. Condition 4 essentially makes the minimal evaluation sets of $\preceq$ as simple as possible.

It should be noted that condition 4 cannot be omitted. Indeed, as is easily verified, conditions 1–3 leave undetermined whether $\emptyset \prec \operatorname{co} E$ or $\emptyset \approx \operatorname{co} E$, and we need condition 4 to make sure that $\emptyset \approx \operatorname{co} E$. Therefore, even in this simplest of all cases, the QPO $\preceq_E$ is not completely determined by its distribution structure $(\mathcal{A}_E, \leq_E)$!

5 The general case

In the previous section we started from the order relation $\leq_E$ on $\mathcal{A}_E$ and extended it *uniquely* to the confidence relation $\preceq_E$ on $\mathcal{R}$ by imposing a number of additional conditions. Using immediate generalizations of these extra conditions, this course of reasoning can be significantly generalized.

To this end, consider the ample field $\mathcal{R}$ and a partition $\mathcal{A}$ of the universe of discourse X, such that $\mathcal{A} \subset \mathcal{R}$. We also assume that $\mathcal{A}$ is non-trivial, in that it has more than one element. Define

$$\mathcal{R}_{\mathcal{A}} = \bigcap \{\mathcal{S} \mid \mathcal{S} \text{ is an ample field on } X \text{ and } \mathcal{A} \subseteq \mathcal{S}\}$$

as the smallest ample field on X containing $\mathcal{A}$. Then $\mathcal{R}_{\mathcal{A}} \subseteq \mathcal{R}$, or equivalently $(\forall x \in X)([x]_{\mathcal{R}} \subseteq [x]_{\mathcal{R}_{\mathcal{A}}})$. Moreover, the atoms of $\mathcal{R}_{\mathcal{A}}$ are precisely the elements of $\mathcal{A}$, i.e., $(\forall A \in \mathcal{A})(\forall x \in A)(A = [x]_{\mathcal{R}_{\mathcal{A}}})$.

We also consider a partial order relation $\leq_{\mathcal{A}}$ on $\mathcal{A}$. We intend the poset $(\mathcal{A}, \leq_{\mathcal{A}})$ to serve as an ordinal counterpart of a possibility distribution. It is an immediate generalization of the chain $(\mathcal{A}_E, \leq_E)$ discussed in the previous section. As in that section, we shall now try to uniquely extend the nucleus of confidence $\leq_{\mathcal{A}}$ from $\mathcal{A}$ towards a full-fledged confidence relation on $\mathcal{R}$, by imposing a number of additional conditions which this relation must satisfy.

An immediate generalization of the conditions expressed in Theorem 6 leads to the following definition.

Definition 7. Let $\mathcal{R}$ be an ample field on X, for which $\mathcal{R}_\mathcal{A} \subseteq \mathcal{R}$. Let $\preceq$ be a confidence relation on $\mathcal{R}$. We call $\preceq$ a *possibilistic extension* of $\leq_\mathcal{A}$ in $\mathcal{R}$ iff $\preceq$ satisfies the following four conditions:

1. $\preceq$ extends $\leq_\mathcal{A}$ to $\mathcal{R}$, i.e., $A \leq_\mathcal{A} B \Leftrightarrow A \preceq B$, for any A and B in $\mathcal{A}$;
2. $\preceq$ is union-closed;
3. $(\forall x \in X)([x]_{\mathcal{R}_\mathcal{A}} \preceq [x]_\mathcal{R})$;
4. $m(\mathcal{A})$ is meet-dense in $(M, \leq)$, where $((M, \leq), m)$ is a minimal evaluation structure of $\preceq$.

Conditions 1–3 state that $\preceq$ is a QPO on $\mathcal{R}$ which has $(\mathcal{A}, \leq_\mathcal{A})$ as its distribution structure. The following proposition gives an alternative formulation for condition 4, and shows that it is indeed independent of the actual choice of the minimal evaluation structure.

Proposition 8. *Let $\mathcal{R}$ be an ample field on X for which $\mathcal{R}_\mathcal{A} \subseteq \mathcal{R}$, and let $\preceq$ be a confidence relation on $\mathcal{R}$. Let $((M, \leq), m)$ be a minimal evaluation structure of $\preceq$. Then $m(\mathcal{A})$ is meet-dense in $(M, \leq)$ iff for any A and B in $\mathcal{R}$, $(\forall C \in \mathcal{A})(A \preceq C \Rightarrow B \preceq C) \Rightarrow B \preceq A$.*

We now state the main theorem, which tells us that the conditions imposed in definition 7 are indeed sensible.

Theorem 9. *Let $\mathcal{R}$ be an ample field on X such that $\mathcal{R}_\mathcal{A} \subseteq \mathcal{R}$. Then there exists a unique possibilistic extension of $\leq_\mathcal{A}$ in $\mathcal{R}$.*

The proof of this theorem is, in our view, non-trivial and uses the theory of Dedekind-MacNeille extensions. Due to limitations of space, we cannot give it here, but we want to mention one of its facets which is instrumental in actually constructing the possibilistic extension $\preceq$ on $\mathcal{R}$ of $(\mathcal{A}, \leq_\mathcal{A})$. Conditions 1–4 in the definition of a possibilistic extension indeed imply that any minimal evaluation set of $\preceq$ is order-isomorphic to the Dedekind-MacNeille extension of $(\mathcal{A}, \leq_\mathcal{A})$.

Since, as in the classical case, condition 4 in the definition of a possibilistic extension is necessary in order to ensure the uniqueness, the theorem above also tells us that *in general a qualitative possibility ordering is not determined by its distribution structure* (or by its distribution relation). If $(\mathcal{A}, \leq_\mathcal{A})$ is a chain, the indeterminacy will typically be concentrated in couples $(A, \emptyset)$, $A \in \mathcal{R}$. If $(\mathcal{A}, \leq_\mathcal{A})$ is not a chain, it may be more severe and located in other couples as well.

To complete the results in this section, we briefly discuss how, given the possibilistic extension of $\leq_\mathcal{A}$ in one ample field, we can find the possibilistic extension in another ample field. Since $\mathcal{R}_\mathcal{A}$ is the smallest ample field on which a possibilistic extension $\preceq_\mathcal{A}$ of $\leq_\mathcal{A}$ can be defined, these results will in particular show us how to find the possibilistic extension $\preceq$ of $\leq_\mathcal{A}$ in $\mathcal{R}$, where $\mathcal{R}_\mathcal{A} \subseteq \mathcal{R}$, if $\preceq_\mathcal{A}$ is given.

We first discuss a way to extend a confidence relation $\preceq$ from an ample field $\mathcal{R}$ to finer ample fields $\mathcal{R}_f$, i.e., $\mathcal{R} \subseteq \mathcal{R}_f$. Consider the $\wp(X) - \mathcal{R}$-mapping $\mathfrak{p}_\mathcal{R}$, defined by, for any $A \subseteq X$, $\mathfrak{p}_\mathcal{R}(A) = \bigcup_{x \in A} [x]_\mathcal{R}$. Then $\mathfrak{p}_\mathcal{R}$ is a (topological) closure operator associated with the system $\mathcal{R}$ of closed (and open) subsets of X. It should be obvious from its definition that $\mathfrak{p}_\mathcal{R}$ is a $(\mathcal{R}, \subseteq)$-possibility measure on $(X, \wp(X))$.

Given the confidence relation $\preceq$ on $\mathcal{R}$, we now use $\mathfrak{p}_\mathcal{R}$ to define a binary relation $\preceq_{\mathfrak{p}_\mathcal{R}}$ on $\mathcal{R}_f$ as follows: for any A and B in $\mathcal{R}_f$, $A \preceq_{\mathfrak{p}_\mathcal{R}} B \Leftrightarrow \mathfrak{p}_\mathcal{R}(A) \preceq \mathfrak{p}_\mathcal{R}(B)$. Note that $\preceq_{\mathfrak{p}_\mathcal{R}}$ is an extension of $\preceq$ to $\mathcal{R}_f$.

Proposition 10. *Let $\mathcal{R}$ and $\mathcal{R}_f$ be ample fields on X, where $\mathcal{R} \subseteq \mathcal{R}_f$. Let $\preceq$ be a confidence relation on $\mathcal{R}$. Then $\preceq_{\mathfrak{p}_\mathcal{R}}$ is a confidence relation on $\mathcal{R}_f$ which coincides with $\preceq$ on $\mathcal{R}$. Moreover, $(\forall x \in X)([x]_\mathcal{R} \preceq_{\mathfrak{p}_\mathcal{R}} [x]_{\mathcal{R}_f})$.*

This proposition in particular tells us that the confidence relation $\preceq_\mathfrak{p}$ extends the confidence relation $\preceq$ in such a way that it does not allow a refining of confidence: elements of $\mathcal{R}_f$ which are included in an atom of $\mathcal{R}$, are given the same confidence as this atom, as far as $\preceq_{\mathfrak{p}_\mathcal{R}}$ is concerned.

It turns out that there is a close connection between the extension method discussed above and the notion of union-closedness. This is made clear in the following proposition, which in a sense gives a characterization of this extension method.

Proposition 11. *Let $\mathcal{R}$ and $\mathcal{R}_f$ be ample fields on X, where $\mathcal{R} \subseteq \mathcal{R}_f$. Let $\preceq$ be a qualitative possibility ordering on $\mathcal{R}$, and let R be an arbitrary binary relation on $\mathcal{R}_f$. Then $R = \preceq_{\mathfrak{p}_\mathcal{R}}$ iff R is a qualitative possibility ordering on $\mathcal{R}_f$ that coincides with $\preceq$ on $\mathcal{R}$, and further satisfies $(\forall x \in X)([x]_\mathcal{R} R [x]_{\mathcal{R}_f})$.*

This result immediately implies the following proposition.

Proposition 12. *Let $\mathcal{R}$ and $\mathcal{R}_f$ be ample fields on X, where $\mathcal{R}_\mathcal{A} \subseteq \mathcal{R} \subseteq \mathcal{R}_f$. Let $\preceq$ be the possibilistic extension of $\leq_\mathcal{A}$ in $\mathcal{R}$. Then $\preceq_{\mathfrak{p}_\mathcal{R}}$ is the possibilistic extension of $\leq_\mathcal{A}$ in $\mathcal{R}_f$.*

To end this section, we show how, given the possibilistic extension of $\leq_{\mathcal{A}}$ in an ample field $\mathcal{R}$, we can obtain the possibilistic extension of $\leq_{\mathcal{A}}$ in the coarser ample field $\mathcal{R}_c$, i.e., $\mathcal{R}_c \subseteq \mathcal{R}$.

Proposition 13. *Let $\mathcal{R}_c$ and $\mathcal{R}$ be ample fields on X, where $\mathcal{R}_c \subseteq \mathcal{R}$. Let $\preceq$ be a qualitative possibility ordering on $\mathcal{R}$. Then the restriction $\preceq \cap\, \mathcal{R}_c^2$ of $\preceq$ to $\mathcal{R}_c$ is a qualitative possibility ordering on $\mathcal{R}_c$.*

Proposition 14. *Let $\mathcal{R}_c$ and $\mathcal{R}$ be ample fields on X, where $\mathcal{R}_{\mathcal{A}} \subseteq \mathcal{R}_c \subseteq \mathcal{R}$. Let $\preceq$ be the possibilistic extension of $\leq_{\mathcal{A}}$ in $\mathcal{R}$. Then the restriction $\preceq \cap\, \mathcal{R}_c^2$ of $\preceq$ to $\mathcal{R}_c$ is the possibilistic extension of $\leq_{\mathcal{A}}$ in $\mathcal{R}_c$.*

6 Conclusion

The results, discussed in this paper, are both positive and negative. On the positive side, we have shown that it is indeed possible to give a very general order-theoretic characterization of the notion of possibility, which extends Dubois' findings. On the negative side, our work indicates that for our qualitative possibility orderings as well as for Dubois' qualitative possibility relations, there is in general *no* order-theoretic equivalent of a distribution. At the same time, we have found an extra condition such that if it holds for a QPO $\preceq$ on $\mathcal{R}$, this $\preceq$ is uniquely determined by its distribution relation: for any minimal evaluation structure $((M,\leq),m)$, it must be that $m(X_{\mathcal{R}})$ is meet-dense in $(M,\leq)$. Clearly, it may be possible that other extra conditions lead to the same result, and further research along these lines is necessary.

Of course, the discussion given here is by needs very restricted. We have had to limit ourselves to succinctly stating the most important results of our research in this area. For the proofs and a more detailed discussion, we refer to a forthcoming paper, where the extension from ample fields to fields, and the dual notion of qualitative necessity orderings are discussed as well.

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