Confidence relations and ordinal information [1]

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We define confidence relations on Boolean lattices, which can be interpreted as ordinal representations of uncertainty or information. The set of the confidence relations on a given Boolean lattice can be ordered by set inclusion and thus is shown to form a complete meet-semilattice. We investigate and identify the maximal elements of this structure. Moreover, we prove that it is in particular an algebraic semilattice (or domain), and that its finite elements are precisely the finitely generated confidence relations. We also investigate the relationship with information systems. We define duality and self-duality for confidence relations and show that similar conclusions can be reached if we restrict ourselves to confidence relations which are in particular self-dual. Finally, we discuss the possible incompleteness of confidence relations, and the relation between the abstract mathematical structures studied here, and other existing uncertainty models.

Key words: Confidence relation, qualitative uncertainty model, algebraic semilattice, CPO, information system, duality, self-duality.

1 Introduction

Consider a universe of discourse Ω, i.e., a non-empty set representing those aspects of the world which are of concern to us in a given problem. The elements ω of Ω could for instance be all the possible outcomes of a particular

[1] Expanded version of a paper presented at the Sixth International Fuzzy Systems Association World Congress (São Paulo, Brazil, July 1995).
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experiment, and we shall indeed use this example as our working paradigm. We require that these outcomes mutually exclude each other. A subset of Ω will be called an *event*. Let us denote by $\mathcal{E}$ the set of those events we want to consider. Typically, $\mathcal{E}$ is assumed to be at least a *field* of subsets of Ω. If the experiment is carried out, its outcome assumes one particular value, say ω_o, in Ω. For an event $A \in \mathcal{E}$, we say that A *occurs* if $\omega_o \in A$ and *does not occur* if $\omega_o \notin A$. Obviously, an event A occurs if and only if its *opposite event* $\Omega \setminus A$ does not occur, and *vice versa*.

If the experiment is not yet carried out, we are in most cases in a state of uncertainty as to which value its outcome will assume in Ω. In general therefore, it will be impossible for an event A to predict *a priori*, i.e., before the experiment is actually carried out, whether it will occur or not. A first and basic step in modelling *a priori* information about the outcome of the experiment consists in trying to order the events on the basis of the confidence we have in their occurrence. We thus try to find (or construct) a binary relation $\preceq$ on the set of events $\mathcal{E}$ such that for any two events A and B, $A \preceq B$ if and only if we have at least as much confidence in the occurrence of B as in the occurrence of A.

The use of binary relations expressing *preference* between events [3] in order to represent information (or uncertainty) has a long tradition. Within the framework of probability theory this approach has led to the notion of *qualitative probability*, for which de Finetti formulated a set of axioms in 1937 [4]. Qualitative probabilities were used by Savage [11] to lay the foundations for a subjective interpretation of probability. Fine has given an overview of results about qualitative probabilities in his book on the foundations of probability theory [8]. Other researchers have tried to relax de Finetti's axioms, and have provided ordinal characterisations of possibility and necessity measures [6,7], and belief and plausibility measures [12,15]. Recently, Walley [13] has introduced a theory of *comparative probability orderings* in a behavioural setting, which is very intricately linked with his theory of imprecise probabilities.

Indeed, there is a connection in the literature between binary preference orderings of events as *ordinal* uncertainty models, and set functions (or 'measures') as *numerical* uncertainty models. A binary preference ordering $\preceq$ between events is said to be *compatible* with a $\mathcal{E} - [0, 1]$-mapping m iff $A \preceq B \Leftrightarrow m(A) \leq m(B)$, for any events A and B in $\mathcal{E}$. Clearly, given any set function $m \colon \mathcal{E} \to [0, 1]$ there exists a unique binary preference ordering compatible with it. On the other hand, given $\preceq$, there will in general be more than one set function m such that $\preceq$ is compatible with m. In this sense, ordinal un-

[3] Some authors consider more in general preference orderings between gambles, rather than events. For a detailed treatment and a good survey of the relevant literature, we refer to [13].

certainty models are said to be more fundamental than numerical ones. Very often in the literature, binary preference orderings of events are used to provide a justification for their numerical counterparts, i.e., set functions such as probability measures, belief and plausibility measures, ... In particular, the main focus has been on the study of the relation between properties of set functions and properties of the binary preference orderings compatible with them (see [15] for a good overview).

It has been argued that probability theory is too restrictive for the representation of uncertainty, and that uncertainty can for instance be modelled by set functions which are not necessarily additive (see [13] for a clear exposition of this point of view). Once this is accepted, one may come to consider a very large class of set functions, or perhaps more fundamentally binary preference orderings between events, as potential mathematical representations of uncertainty. Such preference orderings therefore become important objects of study in themselves.

In this paper, we attempt a formal mathematical study of a very general class of binary relations, which can be interpreted as carriers of ordinal information, and which will be called *confidence relations*. They are defined by a number of basic properties which we feel are fundamental to the mathematical, ordinal representation of uncertainty, and which, by the way, are shared by all preference orderings of events which we encountered in the literature. For one thing, it is fairly natural to require that a binary preference ordering $\preceq$ between the events in $\mathcal{E}$ should be transitive[4]:

$$(\forall(A,B,C) \in \mathcal{E}^3)(A \preceq B \text{ and } B \preceq C \Rightarrow A \preceq C).$$

But this is definitely not the whole story. Let us consider two events A and B for which $A \subseteq B$. If the outcome ω_o of the experiment belongs to A, it must clearly also belong to B. To put it differently, if A occurs, then automatically also B. This implies that we must have at least as much confidence in the occurrence of B as in that of A:

$$(\forall(A,B) \in \mathcal{E}^2)(A \subseteq B \Rightarrow A \preceq B).$$

And, to conclude, the fact that we can have *more* confidence in the occurrence of the certain event Ω than in the occurrence of the impossible event $\emptyset$ also corresponds with our intuition. Since $\emptyset \subseteq \Omega$ and therefore by the previous requirement $\emptyset \preceq \Omega$, this can be expressed as $\Omega \not\preceq \emptyset$. These three basic requirements lie at the origin of our abstract notion of a confidence relation. It should be stressed here that we do not demand that $\preceq$ should be *complete*, or in other words that for any two events A and B in $\mathcal{E}$, $A \preceq B$ or $B \preceq A$. It

[4] In a behavioural context, transitivity follows from rationality criteria, and can be justified by a money pump argument (see, for instance, [13]).

is our conviction that there is no intuitive reason to *a priori* demand for any two events that the confidence in the occurrence of the first should always be comparable to the confidence in the occurrence of the second. One may not be able or willing to state that $A \preceq B$ or $B \preceq A$ on the basis of the given information, particularly if very little information is available. For this reason, we do not want to impose the additional requirement of completeness. On the one hand, this renders our treatment of uncertainty more general, and on the other hand we are thus offered the possibility to consider completeness as an additional property, and if necessary, to investigate its consequences. We shall return to this topic in the last section.

We begin this discussion with the formal definition of *confidence relations* on a Boolean lattice in Section 2. The set of the confidence relations defined on a Boolean lattice is provided with a *natural partial order relation*. A thorough investigation of the structure of the partially ordered set thus formed leads to a number of interesting conclusions. Indeed, we show that this structure is an *algebraic semilattice*, with no top, but containing a set of mutually incomparable *maximal elements*, which can be interpreted as maxima of ordinal information. In practical applications, the notion of a finitely generated confidence relation emerges in a natural way. That these special confidence relations are also important from a theoretical point of view, is made clear in Section 4. Furthermore, in Section 5, the relation between our confidence relations and the well-known notion of an *information system*, originating in the theory of computation [2], is explored in detail. In Section 6 we introduce and study the *duality of confidence relations*. The class of *self-dual* confidence relations is studied in more detail in Section 7. We conclude the paper in Section 8 with a discussion of the potential incompleteness of confidence relations, and point out striking similarities between the fundamental mathematical structures underlying confidence relations and other uncertainty models in the literature.

This is a mathematical paper, but we are convinced that the main results will be of interest to many students of uncertainty. It is possible however that they will be put off by the strong emphasis on rigour and proof, and by the typical mathematical construction using definitions, propositions and theorems. In order to make the paper more accessible to the less mathematically inclined, we have used the following naming system. Any result which we consider important in itself and which provides insight into the representation of uncertainty, has been called a *theorem*. Auxiliary and less important results have been called *propositions*. We therefore encourage readers with only a limited amount of patience and time to restrict themselves to the definitions, the comments in the text and the theorems.

2 Confidence relations and their evaluation structures

This discussion of uncertainty is of an order-theoretic nature, and frequent use is made of a number of notions from order, or lattice, theory. It is not practical to define and discuss all these notions here, and we refer the reader to detailed accounts of order theory in the literature [1,2]. In order to make the paper somewhat self-contained, a number of basic definitions and properties have been gathered in the Appendix.

2.1 Definition of a confidence relation

We start by giving an abstract definition of a confidence relation. Up to now, we have considered a set of events $\mathcal{E}$ as the domain for our binary preference relations. Since we assumed that $\mathcal{E}$ is a field of subsets of Ω, $(\mathcal{E}, \subseteq)$ is a Boolean lattice [1,2]. We want to make the formal mathematical part of this discussion as interpretation-free as possible. We therefore define confidence relations in general on Boolean lattices, and not just on fields of subsets. This will also allow us for instance to directly define confidence relations on sets of propositions (rather than events) which are closed under negation and disjunction (and contain the universal proposition). On the other hand, we shall often want to interpret certain formal definitions and results, and to that effect think of the abstract Boolean lattice as a field of events $\mathcal{E}$, associated with an experiment.

Let us consider a non-empty set $\mathbb{B}$, provided with a partial order relation $\leq$, such that $(\mathbb{B}, \leq)$ is a *Boolean lattice*. The meet of this lattice is denoted by $\frown$, the join by $\smile$ and the complement operator by $\neg$. $(\mathbb{B}, \leq)$ has a top or greatest element $1_{\mathbb{B}}$, and a bottom or smallest element $0_{\mathbb{B}}$. It will be assumed that $0_{\mathbb{B}} \neq 1_{\mathbb{B}}$. Elements of $\mathbb{B}$ are also called *events*. For any event a, $\neg a$ is called the *opposite event* of a.

Definition 1 *We call a binary relation $\preceq$ on $\mathbb{B}$ a confidence relation on $(\mathbb{B}, \leq)$ iff*

(i) $\preceq$ *is transitive:* $(\forall(a, b, c) \in \mathbb{B}^3)((a \preceq b$ *and* $b \preceq c) \Rightarrow a \preceq c)$;
(ii) $\preceq$ *is a superorder:* $(\forall(a, b) \in \mathbb{B}^2)(a \leq b \Rightarrow a \preceq b)$;
(iii) $\preceq$ *is non-trivial:* $1_{\mathbb{B}} \not\preceq 0_{\mathbb{B}}$.

The triple $(\mathbb{B}, \leq, \preceq)$ *is called a confidence structure. The set of the confidence relations on* $(\mathbb{B}, \leq)$ *is denoted by* $\mathcal{V}(\mathbb{B}, \leq)$.

If $\preceq$ satisfies (ii), we say that it is a *superorder* on $(\mathbb{B}, \leq)$, or that it is *monotone*. By (i) and (ii), any confidence relation is in particular a partial preorder, i.e., a reflexive and transitive binary relation. Any binary relation on $\mathbb{B}$ satisfying (i) and (ii) but not (iii), must be equal to $\mathbb{B}^2$.

For any confidence relation $\preceq$ on $(\mathbb{B}, \leq)$, we introduce the following relations derived from it. Let a and b be elements of $\mathbb{B}$, then $a \approx b \Leftrightarrow (a \preceq b$ and $b \preceq a)$; $a \prec b \Leftrightarrow (a \preceq b$ and $b \not\preceq a)$; and $a \parallel b \Leftrightarrow (a \not\preceq b$ and $b \not\preceq a)$. Clearly, $(\prec, \approx, \parallel)$ can be interpreted as a preference structure on $\mathbb{B}$, with *characteristic relation* $\preceq$ [10]. $\prec$ is called the *strict preference relation*, $\approx$ the *indifference relation* and $\parallel$ the *incomparability relation*. It is clear that $\approx$ is an equivalence relation on $\mathbb{B}$. Also, by (ii), for any a in $\mathbb{B}$, $0_{\mathbb{B}} \preceq a \preceq 1_{\mathbb{B}}$. Note again that we do *not* require that $\preceq$ should be *complete*, or equivalently, that $\parallel$ should be empty.

2.2 Evaluation structures for confidence relations

It follows rather easily that $\preceq$ is a confidence relation on $(\mathbb{B}, \leq)$ iff there exists a bounded partially ordered set $(P, \leq)$ with bottom 0_P and top 1_P, $1_P \neq 0_P$, and a $\mathbb{B} - P$-mapping v such that (α) $v(0_{\mathbb{B}}) = 0_P$ and $v(1_{\mathbb{B}}) = 1_P$; (β) $(\forall (a,b) \in \mathbb{B}^2)(a \preceq b \Leftrightarrow v(a) \leq v(b))$; and (γ) v is increasing, i.e., for any a and b in $\mathbb{B}$, $v(a) \leq v(b)$ if $a \leq b$. $(P, \leq)$ is called an *evaluation set* of $\preceq$, and v the corresponding *evaluation mapping*. $((P, \leq), v)$ is called an *evaluation structure* of $\preceq$. From (β) we deduce that an evaluation structure completely characterises a confidence relation.

Evaluation mappings may be interpreted as generalisations of the numerical uncertainty models (or set functions) discussed in the introduction. Indeed, if $\preceq$ is complete and the set of the equivalence classes of $\approx$ is small enough, so that it can be embedded in $[0,1]$, it is possible to choose $([0,1], \leq)$ as an evaluation set, and $\preceq$ is then obviously compatible (in the sense of Section 1) with the corresponding evaluation mapping. On the other hand, if $\preceq$ is not complete, an evaluation set of $\preceq$ will necessarily be only partially ordered. We shall come back to this remark in the last section. It should be emphasised that we introduce and study evaluation structures not because they generalise compatibility and because we believe that compatibility provides a fundamental link between numerical and ordinal uncertainty models, but rather because they are a convenient and often compact characterisation of confidence relations. This is also the reason for including the following considerations.

Let $\preceq$ be a confidence relation on $(\mathbb{B}, \leq)$. Interestingly, there always exists a *canonical* evaluation structure $((M, \leq), m)$ of $\preceq$ such that m is onto (surjective) and such that for any other evaluation structure $((P, \leq), v)$ of $\preceq$, there exists a top and bottom preserving order-embedding of $(M, \leq)$ into $(P, \leq)$, i.e., a $M - P$-mapping ξ satisfying (δ) $\xi(0_M) = 0_P$ and $\xi(1_M) = 1_P$; and (ϵ) $(\forall (\lambda, \mu) \in M^2)(\lambda \leq \mu \Leftrightarrow \xi(\lambda) \leq \xi(\mu))$; such that moreover $v = \xi \circ m$.

Of course, 1_M denotes the top of $(M, \leq)$ and 0_M its bottom. For obvious reasons, such an evaluation structure $((M, \leq), m)$ is called *minimal*. Minimal evaluation structures are uniquely determined up to an order-isomorphism.

A special minimal evaluation structure of $\preceq$ can always be constructed as follows. Let $\mathbb{B}/\!\approx$ denote the set of the equivalence classes of the indifference relation $\approx$, and $q_\preceq \colon \mathbb{B} \to \mathbb{B}/\!\approx$ the corresponding quotient mapping, which maps any a in $\mathbb{B}$ to the corresponding equivalence class $a/\!\approx\, = \{\, b \in \mathbb{B} \mid a \approx b \,\}$. Since $\preceq$ is a partial preorder and $q_\preceq$ is onto, we may define a partial order $\rho_\preceq$ on $\mathbb{B}/\!\approx$ as follows: $q_\preceq(a)\, \rho_\preceq\, q_\preceq(b) \Leftrightarrow a \preceq b$, $(a, b) \in \mathbb{B}^2$. Obviously, $((\mathbb{B}/\!\approx, \rho_\preceq), q_\preceq)$ is a minimal evaluation structure of $\preceq$.

2.3 Ordering confidence relations

Since the elements of $\mathcal{V}(\mathbb{B}, \leq)$ are binary relations on $\mathbb{B}$ and therefore subsets of $\mathbb{B}^2$, we may provide $\mathcal{V}(\mathbb{B}, \leq)$ with a partial order that is the inclusion relation for these subsets. We intend to give a natural interpretation to the partial order $\subseteq$ on $\mathcal{V}(\mathbb{B}, \leq)$, and at the same time investigate the mathematical structure $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. This combination of *interpretation* and *abstract structural analysis* will lead to interesting insights into the ordinal representation of information and uncertainty.

Note that since $\leq$ is transitive and $1_\mathbb{B} \not\leq 0_\mathbb{B}$, $\leq$ is a confidence relation on $(\mathbb{B}, \leq)$, i.e., $\leq\, \in \mathcal{V}(\mathbb{B}, \leq)$. This already implies that $\mathcal{V}(\mathbb{B}, \leq) \neq \emptyset$. The intersection of a *non-empty* family of elements of $\mathcal{V}(\mathbb{B}, \leq)$ satisfies conditions (i)–(iii) of Definition 1 and is therefore a confidence relation on $(\mathbb{B}, \leq)$. This implies that $\mathcal{V}(\mathbb{B}, \leq)$ is an *intersection structure* [2], or equivalently, that the partially ordered set $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is a *complete meet-semilattice* [1], in which intersection plays the role of infimum. Definition 1(ii) tells us that $\leq$ is the bottom of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$.

In order to find an interpretation for the relation $\subseteq$ on $\mathcal{V}(\mathbb{B}, \leq)$, let us go back to the experiment lying at the bottom of this discussion, and take a closer look at the meaning of confidence relations. A confidence relation $\preceq$ represents in a purely ordinal way—that is, on the level of the comparison of confidence in the occurrence of events—potential information about this experiment. An increase of information can at the ordinal level only manifest itself through the addition of couples of events to $\preceq$. This justifies the following interpretation of the relation $\subseteq$ on $\mathcal{V}(\mathbb{B}, \leq)$.

Definition 2 *Let $\preceq_1$ and $\preceq_2$ be confidence relations on $(\mathbb{B}, \leq)$. When $\preceq_1 \subseteq \preceq_2$, we say that $\preceq_2$ contains at least as much ordinal information as $\preceq_1$.*

There exists an *absolute minimum* of ordinal information, namely the partial order $\leq$. This bottom of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is the ordering of events which is always present, even if we have no information at all. Whether $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is also a *complete lattice*, only depends on the existence of a *top* of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$, or in other words, on the existence of an absolute maximum of ordinal information. The question whether or not such a top exists, is given a detailed answer in Section 3, in the form of Theorems 6 and 9. These results tell us that in all non-trivial cases $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ has no top, but does possess a set of mutually incomparable maximal elements, of which they give a complete characterisation.

In Section 4 we take our structural analysis a step further. First of all, we introduce finitely generated confidence relations, i.e., confidence relations based upon a finite number of comparisons of events. It is shown that $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is a complete partially ordered set, or CPO. Using the concept of a finitely generated confidence relation, we then proceed to show that $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is an algebraic semilattice, and that every confidence relation is the union of the finitely generated confidence relations it contains.

Any algebraic semilattice can be interpreted as an information system. In Section 5, we briefly describe the information system associated with $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. This yields some insight into the meaning of a finitely generated confidence relation, and the methods of reasoning that lie behind it.

In Section 7 we show that similar conclusions can be reached if we restrict our attention to confidence relations which are in particular self-dual.

3 Maximal elements

The top of a partially ordered set is always a *maximal element* [1,2]. For this reason, if we want to find out whether $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ has a top, it should help us if we first track down the maximal elements. We already know that $\mathcal{V}(\mathbb{B}, \leq)$ is an *intersection structure* [5] on $\mathbb{B}^2$. Since however $\mathbb{B}^2 \notin \mathcal{V}(\mathbb{B}, \leq)$, $\mathcal{V}(\mathbb{B}, \leq)$ is not a *topped* intersection structure, and is therefore not a *closure system* on $\mathbb{B}^2$. This means that we cannot associate a *closure operator* on $\mathbb{B}^2$ with the intersection structure $\mathcal{V}(\mathbb{B}, \leq)$. In other words, since $\mathbb{B}^2 \notin \mathcal{V}(\mathbb{B}, \leq)$, for an arbitrary binary relation on $\mathbb{B}$ there does not always exist a (smallest) confidence relation that includes it. Proposition 3 should therefore come as no surprise. Note that in what follows, we denote the set-theoretic complement operator in $\mathbb{B}^2$ by co. Also, the transitive closure [6] of a binary relation R is denoted by $\mathrm{tc}(R)$.

[5] For more details about (topped) intersection structures, closure operators and closure systems, we refer to [2] and the Appendix.

[6] The transitive closure (see, for instance, [2]) of a relation R is the smallest transitive relation (w.r.t. inclusion) that includes R. Let us also point out that the set of

Proposition 3 *Let $\preceq$ be a confidence relation on $(\mathbb{B}, \leq)$. Then $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ iff $(\forall (a,b) \in \mathrm{co} \preceq)(\mathrm{tc}(\preceq \cup \{(a,b)\}) = \mathbb{B}^2)$, that is, iff for any a and b such that $a \not\preceq b$, there is no confidence relation on $(\mathbb{B}, \leq)$ which includes $\preceq$ and at the same time relates a to b.*

PROOF. Let $\preceq$ be any element of $\mathcal{V}(\mathbb{B}, \leq)$. Then clearly $\preceq \neq \mathbb{B}^2$, whence $\mathrm{co} \preceq \neq \emptyset$. Let us first assume that $\preceq$ satisfies $\mathrm{tc}(\preceq \cup \{(a,b)\}) = \mathbb{B}^2$, for all $(a,b) \in \mathrm{co} \preceq$. Consider $\preceq'$ in $\mathcal{V}(\mathbb{B}, \leq)$ and assume that $\preceq \subseteq \preceq'$. We must show that $\preceq = \preceq'$. *Ex absurdo*, assume that $\preceq \subset \preceq'$, and consider (a,b) in $\preceq' \setminus \preceq$, i.e., $(a,b) \in \mathrm{co} \preceq$ and $(a,b) \in \preceq'$. Since $\preceq'$ is transitive and includes both $\preceq$ and $\{(a,b)\}$, we must have, by definition of a transitive closure, that $\mathrm{tc}(\preceq \cup \{(a,b)\}) \subseteq \preceq'$. From the assumption, we conclude that $\mathbb{B}^2 \subseteq \preceq'$, whence $\preceq' = \mathbb{B}^2$ and therefore also $\preceq' \notin \mathcal{V}(\mathbb{B}, \leq)$, a contradiction.

Conversely, assume that $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. This already implies that $\preceq \subset \mathbb{B}^2$. Let (a,b) be an element of $\mathrm{co} \preceq$ and consider the relation $\preceq' = \mathrm{tc}(\preceq \cup \{(a,b)\})$ on $\mathbb{B}$. It clearly satisfies $\preceq \subset \preceq'$. Furthermore, $\preceq'$ is transitive and $\leq \subseteq \preceq'$, which means that $\preceq'$ satisfies the first two conditions in the definition of a confidence relation. Let us assume *ex absurdo* that $\preceq$ also satisfies condition (iii). This implies that $\preceq' \in \mathcal{V}(\mathbb{B}, \leq)$, and since $\preceq \subseteq \preceq'$ this leads to $\preceq = \preceq'$, because $\preceq$ is by assumption a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. This contradicts $\preceq \subset \preceq'$, whence $1_{\mathbb{B}} \preceq' 0_{\mathbb{B}}$. We conclude that $\preceq' = \mathbb{B}^2$. $\square$

We now introduce an important notation, which will be used abundantly throughout this paper. Let $\mathcal{B}$ be a subset of $\mathbb{B}$, then we associate with it a binary relation $\preceq_{\mathcal{B}}$ on $\mathbb{B}$, defined as follows:

$$\preceq_{\mathcal{B}} = (\mathcal{B} \times \mathcal{B}) \cup (\mathbb{B} \setminus \mathcal{B} \times \mathbb{B}).$$

Remark that we denote the set-theoretic complement operator in $\mathbb{B}$ by $\mathbb{B}\setminus$. In Proposition 4 we identify an important class of confidence relations $\preceq_{\mathcal{B}}$, where $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$. Theorem 6 tells us that this class is precisely made up of the maximal elements of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. Note that by a *proper* up-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$, we mean an up-set of $(\mathbb{B}, \leq)$ [2] that is also a proper subset of $\mathbb{B}$, i.e., $\mathcal{B} \neq \mathbb{B}$ and $\mathcal{B} \neq \emptyset$.

Proposition 4 *Let $\mathcal{B}$ be a subset of $\mathbb{B}$. Then $\preceq_{\mathcal{B}}$ is a confidence relation on $(\mathbb{B}, \leq)$ iff $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$.*

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the transitive relations on a universe is a closure system, i.e., is closed under arbitrary intersections. The mapping tc that maps an arbitrary relation to its transitive closure is precisely the closure operator associated with this closure system.

PROOF. Assume that $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$. We must show that $\preceq_{\mathcal{B}} \in \mathcal{V}(\mathbb{B}, \leq)$. First of all, since it is easily verified from the definition of $\preceq_{\mathcal{B}}$ that for the composition of $\preceq_{\mathcal{B}}$ with itself, $\preceq_{\mathcal{B}} \circ \preceq_{\mathcal{B}} = \preceq_{\mathcal{B}}$, we conclude that $\preceq_{\mathcal{B}}$ is a transitive[7] relation. Next, it must be shown that $\leq \subseteq \preceq_{\mathcal{B}}$. Consider a and b in $\mathbb{B}$ and assume that $a \leq b$. There are two possibilities. Either $a \in \mathcal{B}$ and since $\mathcal{B}$ is an up-set of $(\mathbb{B}, \leq)$, also $b \in \mathcal{B}$, whence $(a, b) \in \preceq_{\mathcal{B}}$. Or $a \notin \mathcal{B}$, whence immediately $(a, b) \in \preceq_{\mathcal{B}}$. We conclude that $\leq \subseteq \preceq_{\mathcal{B}}$. Finally, it must be proven that $(1_{\mathbb{B}}, 0_{\mathbb{B}}) \notin \preceq_{\mathcal{B}}$. Since $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$, we certainly have that $1_{\mathbb{B}} \in \mathcal{B}$ and $0_{\mathbb{B}} \notin \mathcal{B}$. This implies that $(1_{\mathbb{B}}, 0_{\mathbb{B}}) \in \mathcal{B} \times \mathbb{B} \setminus \mathcal{B}$, or equivalently, $(1_{\mathbb{B}}, 0_{\mathbb{B}}) \in \mathrm{co} \preceq_{\mathcal{B}}$.

Conversely, assume that $\preceq_{\mathcal{B}}$ is a confidence relation on $(\mathbb{B}, \leq)$. Let us first show that in this case $\mathcal{B}$ is an up-set of $(\mathbb{B}, \leq)$. Consider a in $\mathcal{B}$ and b in $\mathbb{B}$, with $a \leq b$. Since by assumption $\leq \subseteq \preceq_{\mathcal{B}}$, we know that $a \preceq_{\mathcal{B}} b$, and it then follows from the definition of $\preceq_{\mathcal{B}}$ that $b \in \mathcal{B}$. Therefore, $\mathcal{B}$ is an up-set of $(\mathbb{B}, \leq)$. Should $\mathcal{B} = \emptyset$ or $\mathcal{B} = \mathbb{B}$, it would follow that $\preceq_{\mathcal{B}} = \mathbb{B}^2$, contradicting the assumption that $\preceq_{\mathcal{B}}$ is a confidence relation on $(\mathbb{B}, \leq)$. We conclude that $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$. $\square$

The next proposition tells us that for every confidence relation there exists a confidence relation of the type described in Proposition 4 that contains at least as much ordinal information.

Proposition 5 *Let $\preceq$ be a confidence relation on $(\mathbb{B}, \leq)$. Then there exists a proper up-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$ for which $\preceq \subseteq \preceq_{\mathcal{B}}$.*

PROOF. We give a proof by construction. Consider a minimal evaluation structure $((M, \leq), m)$ of $\preceq$. By definition, $(M, \leq)$ is a bounded partially ordered set with at least two different elements $m(0_{\mathbb{B}})$ and $m(1_{\mathbb{B}})$. Therefore, this poset has at least one proper up-set, say B. Consider the binary relation $\preceq'$ on $\mathbb{B}$, defined by $a \preceq' b \Leftrightarrow (m(a), m(b)) \in (B \times B) \cup (M \setminus B \times M)$, $(a, b) \in \mathbb{B}^2$. Let $\mathcal{B} = \{ a \in \mathbb{B} \mid m(a) \in B \}$, then clearly $\preceq' = \preceq_{\mathcal{B}}$. Since B is a proper up-set of $(M, \leq)$, we find that $m(1_{\mathbb{B}}) \in B$ and $m(0_{\mathbb{B}}) \notin B$, whence $1_{\mathbb{B}} \in \mathcal{B}$ and $0_{\mathbb{B}} \notin \mathcal{B}$. This implies that $\mathcal{B}$ is a proper subset of $\mathbb{B}$. Let us now prove that $\mathcal{B}$ is an up-set of $(\mathbb{B}, \leq)$. Consider c in $\mathcal{B}$ and d in $\mathbb{B}$. If we assume that $c \leq d$, it follows that $c \preceq d$, since $\preceq$ is by assumption a confidence relation on $(\mathbb{B}, \leq)$. This is equivalent to $m(c) \leq m(d)$. Since furthermore $c \in \mathcal{B}$ is equivalent to $m(c) \in B$, and B is an up-set of $(M, \leq)$, we find that $m(d) \in B$, which is in turn equivalent to $d \in \mathcal{B}$. We are thus led to the conclusion that $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$. We complete the proof by showing that $\preceq \subseteq \preceq_{\mathcal{B}}$. Consider (a, b) in $\mathbb{B}^2$, and let $a \preceq b$, whence $m(a) \leq m(b)$. There are now two possibil-

[7] A binary relation R is transitive iff $R \circ R \subseteq R$ (see, for instance, [14]).

ities. Either $m(a) \in M \setminus B$ whence $a \preceq_{\mathcal{B}} b$. Or $m(a) \in B$, whence $m(b) \in B$, since B is an up-set of $(M, \leq)$. We therefore again find that $a \preceq_{\mathcal{B}} b$. $\square$

Theorem 6 *$\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ iff there exists a proper up-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$ such that $\preceq = \preceq_{\mathcal{B}}$.*

PROOF. Let $\mathcal{B}$ be a proper up-set of $(\mathbb{B}, \leq)$ and assume that $\preceq = \preceq_{\mathcal{B}}$. Taking into account Proposition 4, $\preceq$ is a confidence relation on $(\mathbb{B}, \leq)$. It remains to be shown that $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. To this end, we use Proposition 3. Consider (a, b) in $\mathrm{co}\preceq = \mathcal{B} \times \mathbb{B} \setminus \mathcal{B} \neq \emptyset$, i.e., $a \in \mathcal{B}$ and $b \in \mathbb{B} \setminus \mathcal{B}$. Furthermore, consider $\preceq' = \mathrm{tc}(\preceq \cup \{(a, b)\})$ and any element (c, d) of $\mathrm{co}\preceq$, i.e., $c \in \mathcal{B}$ and $d \in \mathbb{B} \setminus \mathcal{B}$. Then clearly $c \preceq a$ and $b \preceq d$. Since $\preceq \subseteq \preceq'$, we find that $c \preceq' a$ and $b \preceq' d$. Also, since $a \preceq' b$ and $\preceq'$ is transitive, it follows that $c \preceq' d$. This implies that $\mathrm{co}\preceq \subseteq \preceq'$, whence $\preceq' = \mathbb{B}^2$. Taking into account Proposition 3, we conclude that $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$.

Conversely, assume that $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. This means that $\preceq$ is in particular a confidence relation on $(\mathbb{B}, \leq)$, and, taking into account Proposition 5, that there exists a proper up-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$ for which $\preceq \subseteq \preceq_{\mathcal{B}}$. Since $\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$, we conclude that $\preceq = \preceq_{\mathcal{B}}$. $\square$

The theorem above emphasises the relevance of proper up-sets of $(\mathbb{B}, \leq)$ to this theory. There is however no reason why the dual symmetry should be broken here, and why up-sets would have a more important part than down-sets [2]. Indeed, if we note that for any subset $\mathcal{B}$ of $\mathbb{B}$, $\mathcal{B}$ is an up-set of $(\mathbb{B}, \leq)$ iff $\mathbb{B} \setminus \mathcal{B}$ is a down-set of $(\mathbb{B}, \leq)$, and *vice versa*, the proof of Proposition 7 becomes trivial, and the down-sets are rehabilitated.

Proposition 7 *$\preceq$ is a maximal element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ iff there exists a proper down-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$ such that $\preceq = \preceq_{\mathbb{B} \setminus \mathcal{B}}$.*

In what follows, we use the notation $\mathcal{M}(\mathbb{B}, \leq)$ for the set of the maximal elements of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. Also, given any binary relation R on $\mathbb{B}$, let $\mathcal{M}(R) = \{ \preceq_{\mathcal{B}} \in \mathcal{M}(\mathbb{B}, \leq) \mid R \subseteq \preceq_{\mathcal{B}} \}$ denote the set of the maximal elements of $\mathcal{V}(\mathbb{B}, \leq)$ which dominate R. Note that Proposition 5 and Theorem 6 tell us that for any element $\preceq$ of $\mathcal{V}(\mathbb{B}, \leq)$, $\mathcal{M}(\preceq) \neq \emptyset$.

If $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ has a top, it is also its only maximal element. Theorem 9 gives a definite answer to the question about the existence of a top of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$. In order to make the proof of this theorem more readable, a few intermediate results are given in Proposition 8. Its proof is straightforward, and is therefore omitted.

Proposition 8 *Let $\mathcal{B}$, $\mathcal{B}_1$ and $\mathcal{B}_2$ be proper up-sets of $(\mathbb{B}, \leq)$. Then*

(i) $\preceq_{\mathcal{B}_1} \subseteq \preceq_{\mathcal{B}_2} \Leftrightarrow \preceq_{\mathcal{B}_1} = \preceq_{\mathcal{B}_2} \Leftrightarrow \mathcal{B}_1 = \mathcal{B}_2$*;*
(ii) if $\{0_{\mathbb{B}}, 1_{\mathbb{B}}\} \subset \mathbb{B}$*, there exists another proper up-set of* $(\mathbb{B}, \leq)$ *different from* $\mathcal{B}$*.*

Theorem 9 *(i) When* $\mathbb{B} = \{0_{\mathbb{B}}, 1_{\mathbb{B}}\}$*,* $\mathcal{V}(\mathbb{B}, \leq)$ *has only one element* $\leq$*, which is at the same time the top and the bottom of* $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$*.*
(ii) When $\{0_{\mathbb{B}}, 1_{\mathbb{B}}\} \subset \mathbb{B}$*,* $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ *has no top.*

PROOF. The proof of statement (i) is trivial, so we only prove (ii). Assume *ex absurdo* that $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ has a top $\preceq$. Since the top is also a maximal element, it follows from Theorem 6 that there exists a proper up-set $\mathcal{B}$ of $(\mathbb{B}, \leq)$ for which $\preceq = \preceq_{\mathcal{B}}$. Taking into account Proposition 8(ii) there exists at least one other proper up-set $\mathcal{B}_1 \neq \mathcal{B}$ of $(\mathbb{B}, \leq)$. By Proposition 8(i), $\preceq_{\mathcal{B}_1} \neq \preceq$. But Proposition 4 tells us that $\preceq_{\mathcal{B}_1}$ is a confidence relation on $(\mathbb{B}, \leq)$, whence $\preceq_{\mathcal{B}_1} \subseteq \preceq$. Taking into account Theorem 6, $\preceq_{\mathcal{B}_1}$ is a *maximal* element of $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$, which by assumption implies that $\preceq = \preceq_{\mathcal{B}_1}$, a contradiction. □

We are led to the conclusion that in all cases of interest, the complete meet-semilattice $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ has *no top*, and is therefore *not a complete lattice*.

For a number of maximal elements, we have an immediate interpretation [8]. First of all, consider a proper filter $\mathcal{F}$ of $(\mathbb{B}, \leq)$, i.e., a proper up-set of $(\mathbb{B}, \leq)$ that is in particular meet-closed [2]. Such a filter is an order-theoretic model of a *set of certain events*. Indeed, if we use the specific language and notation of Section 1, an event $A \in \mathcal{E}$ is certain if we know, on the basis of the given information, that the outcome ω_o of the experiment belongs to A. Then clearly if A is certain, so is any of its supersets $B \supseteq A$, $B \in \mathcal{E}$. Moreover, if A and B are certain, so is their intersection (or meet). Finally Ω is certain and $\emptyset$ is not, which implies that the set of the certain events should be a proper filter of the Boolean lattice $(\mathcal{E}, \subseteq)$.

Dually, a proper ideal of $(\mathbb{B}, \leq)$, i.e., a proper down-set of $(\mathbb{B}, \leq)$ that is join-closed, is an order-theoretic model of a set of *impossible events*. If we again look at the model of Section 1, it is also obvious that an event is certain iff it opposite event is impossible, which implies that with a set of certain events (proper filter) $\mathcal{F}$, there corresponds a set of impossible events (proper ideal) $\neg\mathcal{F} = \{\, \neg a \mid a \in \mathcal{F} \,\}$. Conversely, with a set of impossible events (proper ideal) $\mathcal{I}$, there corresponds a set of certain events (proper filter) $\neg\mathcal{I} = \{\, \neg a \mid a \in \mathcal{I} \,\}$.

Then $\preceq_{\mathcal{F}}$ expresses in ordinal terms that the events in the proper filter $\mathcal{F}$ are certain, and that the events in $\mathbb{B} \setminus \mathcal{F}$ are less than certain, but not necessarily

[8] Interestingly, similar conclusions can be drawn in Walley's theory of coherent almost-preference relations [13].

impossible! Note indeed that since $\mathcal{F}$ is a *proper* filter, it must be that $\neg\mathcal{F} \subseteq \mathbb{B} \setminus \mathcal{F}$. In the special case that $\neg\mathcal{F} = \mathbb{B} \setminus \mathcal{F}$, or equivalently,

$$\mathcal{F} = \mathbb{B} \setminus \neg\mathcal{F}, \tag{1}$$

there are no uncertain events, i.e., all events are certain or impossible, and *the ordinal model* $\preceq_{\mathcal{F}}$ *describes absolute certainty*. Note that (1) holds iff $\mathcal{F}$ is an ultrafilter of $(\mathbb{B}, \leq)$ [2]. We shall have more to say about this condition in connection with the notion of (self-)duality, see Sections 6 and 7.

Dually, $\preceq_{\mathbb{B}\setminus\mathcal{I}}$ expresses in ordinal terms that the events in the proper ideal $\mathcal{I}$ are impossible, and that the events in $\mathbb{B} \setminus \mathcal{I}$ are possible, but not necessarily certain! Again, since $\mathcal{I}$ is a *proper* ideal, it must be that $\neg\mathcal{I} \subseteq \mathbb{B} \setminus \mathcal{I}$. When $\neg\mathcal{I} = \mathbb{B} \setminus \mathcal{I}$, or equivalently, $\mathcal{I} = \mathbb{B} \setminus \neg\mathcal{I}$, there are no uncertain events, i.e., all events are certain or impossible. The ordinal model $\preceq_{\mathbb{B}\setminus\mathcal{I}}$ then describes absolute certainty. Note that the above condition holds iff $\mathcal{I}$ is a maximal ideal of $(\mathbb{B}, \leq)$ [2].

To give an example, consider an element a of $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$. Then the set $\uparrow a = \{\, b \in \mathbb{B} \mid a \leq b \,\}$ is a proper filter which is called the *principal filter* generated by a. The set $\uparrow a$ can be interpreted as a set of certain events, and the confidence relation $\preceq_{\uparrow a}$ reflects the information that the event a obtains. Similarly, $\downarrow \neg a = \{\, b \in \mathbb{B} \mid b \leq \neg a \,\} = \neg\uparrow a$ is a proper ideal which is called the *principal ideal* generated by $\neg a$. It can be viewed as a set of impossible events, and the confidence relation $\preceq_{\mathbb{B}\setminus\downarrow\neg a}$ models the information that $\neg a$ does not obtain. It is fairly easy to show that $\uparrow a = \mathbb{B} \setminus \neg(\uparrow a)$, or in other words that $\uparrow a$ is an ultrafilter and $\downarrow\neg a$ a maximal ideal, iff a is indivisible in $(\mathbb{B}, \leq)$, i.e. if $(\forall b \in \mathbb{B})(a \leq b \text{ or } a \leq \neg b)$ (so a will be an atom of $(\mathbb{B}, \leq)$). Informally, this states that *we have absolute certainty if all certainty is concentrated in an indivisible (and therefore minimal) element of* $\mathbb{B} \setminus \{0_{\mathbb{B}}\}$.

We conclude this section with a result that states that the maximal elements of $\mathcal{V}(\mathbb{B}, \leq)$ can be used to construct all other confidence relations on $(\mathbb{B}, \leq)$.

Theorem 10 *Any confidence relation* $\preceq$ *on* $(\mathbb{B}, \leq)$ *is the intersection of the maximal confidence relations above it:* $\preceq\; = \bigcap \mathcal{M}(\preceq)$. *In other words, the set* $\mathcal{M}(\mathbb{B}, \leq)$ *of the maximal confidence relations on* $(\mathbb{B}, \leq)$ *is meet-dense in* $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$.

PROOF. Clearly, $\preceq\; \subseteq \bigcap \mathcal{M}(\preceq)$. To prove that $\bigcap \mathcal{M}(\preceq) \subseteq\; \preceq$, consider $(a, b) \notin\; \preceq$. It must be proven that $(a, b) \notin \bigcap \mathcal{M}(\preceq)$. Consider the subset $\mathcal{B} = \{\, c \in \mathbb{B} \mid a \preceq c \,\}$ of $\mathbb{B}$. We first show that $\mathcal{B}$ is a proper up-set of $(\mathbb{B}, \leq)$. Let c be any element of $\mathcal{B}$ and d any element of $\mathbb{B}$. Assume that $c \leq d$, whence $c \preceq d$, since $\preceq$ is monotone. Since also by assumption $a \preceq c$ and $\preceq$ is transitive, we find that $a \preceq d$, which implies that $d \in \mathcal{B}$. Moreover, since $(a, b) \notin\; \preceq$ we find that $b \notin \mathcal{B}$, whence $\mathcal{B} \neq \mathbb{B}$. Also, $a \in \mathcal{B}$, whence $\mathcal{B} \neq \emptyset$. This tells us that $\mathcal{B}$ is indeed a proper up-set of $(\mathbb{B}, \leq)$. Next, we show that $\preceq \subseteq \preceq_{\mathcal{B}}$. Indeed, if (c, d) is any element of $\preceq$, there are two possibilities. Either $c \in \mathbb{B} \setminus \mathcal{B}$, in which case clearly $(c, d) \in \preceq_{\mathcal{B}}$. Or $c \in \mathcal{B}$, whence $a \preceq c$. Since $c \preceq d$ and $\preceq$ is transitive, this implies that $a \preceq d$, whence $d \in \mathcal{B}$, and therefore also $(c, d) \in \preceq_{\mathcal{B}}$. We therefore conclude that $\preceq_{\mathcal{B}} \in \mathcal{M}(\preceq)$. But, since $a \in \mathcal{B}$ and $b \notin \mathcal{B}$, it must also be that $(a, b) \notin \preceq_{\mathcal{B}}$, whence $(a, b) \notin \bigcap \mathcal{M}(\preceq)$. □

4 Finite confidence relations

Confidence relations are a mathematical condensation of relative confidence in events. Starting from the available information, one practical way [9] to find the associated confidence relation would be to actually carry out the pairwise comparison of all the elements of $\mathbb{B}$. But, since $\mathbb{B}$ can be very large or infinite, this is not always feasible. If we want to obtain a confidence relation by this process of carrying out pairwise comparisons, we shall always be forced to limit ourselves to comparing only a *finite* number of elements of $\mathbb{B}$.

For this reason, it seems very natural to consider a finite subset Y of $\mathbb{B}^2$ as a first step towards the construction of a confidence relation on $(\mathbb{B}, \leq)$, and ask ourselves whether there exist confidence relations $\preceq$ on $(\mathbb{B}, \leq)$ that are *compatible* with this *initial information*, i.e., for which $Y \subseteq \preceq$.

Definition 11 *Let Y be a finite subset of $\mathbb{B}^2$. We call Y confidence-consistent iff $\mathcal{D}(Y) = \{ \preceq \in \mathcal{V}(\mathbb{B}, \leq) \mid Y \subseteq \preceq \} \neq \emptyset$. An element of $\mathcal{D}(Y)$ is called compatible with Y. The set of the finite, confidence-consistent subsets of $\mathbb{B}^2$ is denoted by $Con(\mathbb{B}, \leq)$.*

If a finite subset Y of $\mathbb{B}^2$ is confidence-consistent, it is of course perfectly possible that there is more than one confidence relation on $(\mathbb{B}, \leq)$ that is compatible with Y. But, since $\mathcal{V}(\mathbb{B}, \leq)$ is an intersection structure, we can always consider the smallest (w.r.t. inclusion) element $\bigcap \mathcal{D}(Y)$ of $\mathcal{V}(\mathbb{B}, \leq)$ compatible with Y. This is the confidence relation compatible with Y containing the least ordinal information. It enables us to compute the minimal ordinal information compatible with the initial information Y.

Definition 12 *Let Y be a finite, confidence-consistent subset of $\mathbb{B}^2$. We call*

$$\mathfrak{v}(Y) = \bigcap \mathcal{D}(Y) = \bigcap \{ \preceq \in \mathcal{V}(\mathbb{B}, \leq) \mid Y \subseteq \preceq \}$$

the confidence relation generated by Y. We call a confidence relation $\preceq$ on

[9] There are of course other ways to construct or determine a confidence relation, for instance by imposing special conditions that determine it uniquely.

$(\mathbb{B}, \leq)$ finitely generated iff there exists a finite, confidence-consistent subset Y of $\mathbb{B}^2$, such that $\preceq = \mathfrak{v}(Y)$.

In Proposition 13, we study some important properties of finitely generated confidence relations. In particular, statements (i) and (ii) assure us that $\mathfrak{v}(Y)$ is a confidence relation compatible with Y, and (iii) gives a characterisation of $\mathfrak{v}(Y)$ in terms of the transitive closure operator.

Proposition 13 *Let Y, Y_1 and Y_2 be finite, confidence-consistent subsets of $\mathbb{B}^2$. Then*

(i) $\mathfrak{v}(Y) \in \mathcal{V}(\mathbb{B}, \leq)$;
(ii) $Y \subseteq \mathfrak{v}(Y)$;
(iii) $\mathfrak{v}(Y) = \mathrm{tc}(\leq \cup Y)$;
(iv) $Y_1 \subseteq Y_2 \Rightarrow \mathfrak{v}(Y_1) \subseteq \mathfrak{v}(Y_2)$.

PROOF. Since Y is confidence-consistent, $\mathcal{D}(Y) \neq \emptyset$. The intersection of a non-empty set of confidence relations is a confidence relation, and (i) follows. The proof of (ii) is trivial. We now prove (iii). Let us put, for the sake of notational simplicity, $\preceq_o = \mathrm{tc}(\leq \cup Y)$. By construction, $\preceq_o$ is transitive and monotone. Consider an element $\preceq$ of $\mathcal{D}(Y)$. It immediately follows that $Y \cup \leq \subseteq \preceq$, and, taking into account the properties of the transitive closure and the transitivity of $\preceq$, $\preceq_o = \mathrm{tc}(Y \cup \leq) \subseteq \mathrm{tc}(\preceq) = \preceq$. From this inequality, we deduce that $\preceq_o \subseteq \mathfrak{v}(Y)$. This in turn implies that $(1_{\mathbb{B}}, 0_{\mathbb{B}}) \notin \preceq_o$, whence $\preceq_o \in \mathcal{V}(\mathbb{B}, \leq)$. Since also $Y \subseteq \preceq_o$, it follows that $\preceq_o \in \mathcal{D}(Y)$, whence $\mathfrak{v}(Y) \subseteq \preceq_o$. This completes the proof of (iii). The proof of (iv) is again trivial, taking into account (iii) and the properties of transitive closure. □

Finitely generated confidence relations are obviously interesting from a practical point of view. In the rest of this section, we intend to show that they are also important theoretically. In order to do so, we must first deal with *directed sets of confidence relations*, and *finite confidence relations*. These notions are special instances of the directed sets and finite elements defined in the theory of complete partially ordered sets and algebraic semilattices [2].

In the previous section, we have shown that in all cases of interest the complete meet-semilattice $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is not a complete lattice. If we consider a non-empty subset $\mathcal{A}$ of $\mathcal{V}(\mathbb{B}, \leq)$, this means that the supremum $\sup \mathcal{A}$ of $\mathcal{A}$ in $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ does not necessarily exist. But, if it does exist, it is plain to see that

$$\sup \mathcal{A} = \bigcap \{ \preceq \in \mathcal{V}(\mathbb{B}, \leq) \mid \bigcup \mathcal{A} \subseteq \preceq \}, \quad (2)$$

since infimum coincides with intersection in $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ and since supremum is the smallest upper bound. Thus, if $\bigcup \mathcal{A}$ is a confidence relation on $(\mathbb{B}, \leq)$, we find that $\sup \mathcal{A}$ exists and is equal to $\bigcup \mathcal{A}$.

We now introduce special subsets of $\mathcal{V}(\mathbb{B}, \leq)$ with the interesting property that their union is a confidence relation. This is proven in Proposition 16. An intermediate but extremely helpful result is proven in Proposition 15.

Definition 14 *We call a non-empty subset $\mathcal{D}$ of $\mathcal{V}(\mathbb{B}, \leq)$ directed iff for every finite subset $\mathcal{E}$ of $\mathcal{D}$ there exists a $D \in \mathcal{D}$ such that $(\forall S \in \mathcal{E})(S \subseteq D)$.*

Proposition 15 *Let $\mathcal{D}$ be a directed subset of $\mathcal{V}(\mathbb{B}, \leq)$. Let furthermore $Y = \{(a_1, b_1), \ldots, (a_n, b_n)\}$ be a finite subset of $\bigcup \mathcal{D}$. Then there exists a D in $\mathcal{D}$, such that $Y \subseteq D$.*

PROOF. Since $Y \subseteq \bigcup \mathcal{D}$, it is possible for every k in $\{1, \ldots, n\}$ to find a D_k in $\mathcal{D}$, such that $(a_k, b_k) \in D_k$. Consider the finite subset $\{D_1, \ldots, D_n\}$ of $\mathcal{D}$. Since $\mathcal{D}$ is directed, there exists by definition a D in $\mathcal{D}$, such that $D_k \subseteq D$, $k \in \{1, \ldots, n\}$, which of course implies that $\{(a_k, b_k)\} \subseteq D$, $k \in \{1, \ldots, n\}$, and therefore also $Y \subseteq D$. $\square$

Proposition 15 also tells us that any finite subset of the union $\bigcup \mathcal{D}$ of a directed subset $\mathcal{D}$ of $\mathcal{V}(\mathbb{B}, \leq)$ is necessarily confidence-consistent.

Proposition 16 *Let $\mathcal{D}$ be a directed subset of $\mathcal{V}(\mathbb{B}, \leq)$. Then $\bigcup \mathcal{D}$ is a confidence relation on $(\mathbb{B}, \leq)$.*

PROOF. Since conditions (ii) and (iii) of Definition 1 are trivially satisfied for $\bigcup \mathcal{D}$, it remains to be shown that $\bigcup \mathcal{D}$ is transitive. To this end, consider (a, b) and (b, c) in $\bigcup \mathcal{D}$. Since $\{(a, b), (b, c)\} \subseteq \bigcup \mathcal{D}$, Proposition 15 tells us that there exists a D in $\mathcal{D}$, such that $\{(a, b), (b, c)\} \subseteq D$. Using the transitivity of D, we deduce that $(a, c) \in D$, and therefore also that $(a, c) \in \bigcup \mathcal{D}$. $\square$

In general, an intersection structure is called *algebraic* iff it contains the union of any of its directed subsets [2]. The combination of Eq. (2) and Proposition 16 leads to the following important conclusions.

Theorem 17 *$(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is an algebraic intersection structure, and therefore $\sup \mathcal{D} = \bigcup \mathcal{D}$ for any directed subset $\mathcal{D}$ of $\mathcal{V}(\mathbb{B}, \leq)$, where $\sup$ is the supremum in the complete meet-semilattice $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$.*

Proposition 18 *$(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is a CPO (complete partially ordered set), i.e., a partially ordered set satisfying:*

(i) $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ *has a bottom, namely* $\leq$*;*

(ii) *any directed subset* $\mathcal{D}$ *of* $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ *has a supremum in* $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$*.*

Since $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is a CPO, we can apply the definition of a *finite element* [10] in such a complete partially ordered set [2].

Definition 19 *A confidence relation* $\preceq$ *on* $(\mathbb{B}, \leq)$ *is called finite iff for every directed subset* $\mathcal{D}$ *of* $\mathcal{V}(\mathbb{B}, \leq)$*:* $\preceq \subseteq \bigcup \mathcal{D} \Rightarrow (\exists D \in \mathcal{D})(\preceq \subseteq D)$*.*

As Theorem 21 further on indicates, the theoretical notion of a *finite confidence relation*, originating in the theory of complete partially ordered sets, turns out to be equivalent with the more practically inspired notion of a *finitely generated* confidence relation. Before we prove this fact, however, we invoke the help of a few intermediate results, gathered in Proposition 20.

Proposition 20 *Let* $\preceq$ *be a confidence relation on* $(\mathbb{B}, \leq)$*. Consider the subset*

$$FGS(\preceq) = \{\, D \in \mathcal{V}(\mathbb{B}, \leq) \mid D \text{ is finitely generated and } D \subseteq \preceq \,\}$$

of $\mathcal{V}(\mathbb{B}, \leq)$*. Then* $\preceq = \bigcup FGS(\preceq)$ *and* $FGS(\preceq)$ *is a directed subset of* $\mathcal{V}(\mathbb{B}, \leq)$*.*

PROOF. We first prove that $\preceq = \bigcup FGS(\preceq)$. For any (a, b) in $\preceq$, $\{(a, b)\}$ is confidence-consistent. Consider the subset $\mathcal{D} = \{\, \mathfrak{v}(\{(a, b)\}) \mid (a, b) \in \preceq \,\}$ of $\mathcal{V}(\mathbb{B}, \leq)$. Of course, $\mathcal{D} \subseteq FGS(\preceq)$, whence $\bigcup \mathcal{D} \subseteq \bigcup FGS(\preceq)$. Furthermore, by Proposition 13(ii) we find for any (a, b) in $\preceq$ that $\{(a, b)\} \subseteq \mathfrak{v}(\{(a, b)\})$, whence $\preceq = \bigcup_{(a,b) \in \preceq} \{(a, b)\} \subseteq \bigcup_{(a,b) \in \preceq} \mathfrak{v}(\{(a, b)\}) = \bigcup \mathcal{D}$ and therefore also $\preceq \subseteq \bigcup FGS(\preceq)$. On the other hand, since $(\forall D \in FGS(\preceq))(D \subseteq \preceq)$, we find that $\bigcup FGS(\preceq) \subseteq \preceq$.

It remains to be proven that $FGS(\preceq)$ is a directed subset of $\mathcal{V}(\mathbb{B}, \leq)$. Consider a finite subset $\mathcal{E}$ of $FGS(\preceq)$. It must be shown that there exists a D in $FGS(\preceq)$ such that $(\forall S \in \mathcal{E})(S \subseteq D)$. Let us write, for the sake of simplicity, $\mathcal{E} = \{\mathfrak{v}(Y_1), \ldots, \mathfrak{v}(Y_n)\}$, where, of course, n is a natural number and Y_k is a finite, confidence-consistent subset of $\mathbb{B}^2$ with $Y_k \subseteq \preceq$, $k \in \{1, \ldots, n\}$. If we define the finite confidence-consistent subset $Y = \bigcup_{k=1}^{n} Y_k$ of $\mathbb{B}^2$ and take into account that $\preceq$ is transitive and monotone, we find that $\leq \cup Y \subseteq \preceq$ and therefore also $\mathfrak{v}(Y) = \mathrm{tc}(\leq \cup Y) \subseteq \preceq$. We conclude that $\mathfrak{v}(Y) \in FGS(\preceq)$ and $\mathfrak{v}(Y_k) \subseteq \mathfrak{v}(Y)$, $k \in \{1, \ldots, n\}$. $\square$

Theorem 21 *Let* $\preceq$ *be a confidence relation on* $\mathcal{V}(\mathbb{B}, \leq)$*. Then* $\preceq$ *is finite iff* $\preceq$ *is finitely generated.*

[10] This notion of finiteness can be linked to the notion of compactness used in algebra, logic and of course topology [2].

PROOF. Assume that $\preceq$ is finitely generated. This means that there exists a finite, confidence-consistent subset Y of $\mathbb{B}^2$ such that $\preceq = \mathfrak{v}(Y)$. Consider a directed subset $\mathcal{D}$ of $\mathcal{V}(\mathbb{B}, \leq)$, and assume that $\preceq \subseteq \bigcup \mathcal{D}$. This implies that $Y \subseteq \bigcup \mathcal{D}$, and taking into account Proposition 15 we find that there exists a D in $\mathcal{D}$ such that $Y \subseteq D$. By definition, it then follows that $\preceq = \mathfrak{v}(Y) \subseteq D$, which means that $\preceq$ is finite.

Conversely, assume that $\preceq$ is finite. Consider the set $FGS(\preceq)$, introduced in the previous proposition. We already know that $FGS(\preceq)$ is a directed subset of $\mathbb{B}^2$, and that in particular $\preceq \subseteq \bigcup FGS(\preceq)$. On the one hand, since $\preceq$ is finite by assumption, this implies that there exists a D in $FGS(\preceq)$ such that $\preceq \subseteq D$. On the other hand, we have by definition that $D \subseteq \preceq$. We conclude that $\preceq = D$, which implies that $\preceq$ is finitely generated. $\square$

It is a well-known result from the theory of algebraic semilattices that every algebraic intersection structure is an *algebraic semilattice*, that is, a complete semilattice (or consistently complete CPO) for which any element is the supremum of the directed set of finite elements below it [2]. Theorems 17 and 21 therefore also lead to the following immediate conclusion.

Theorem 22 $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ *is an algebraic semilattice, and every confidence relation* $\preceq$ *is the union of the finite (or finitely generated) confidence relations it includes:*

$$\preceq = \bigcup FGS(\preceq) = \bigcup \{ \mathfrak{v}(Y) \mid Y \in Con(\mathbb{B}, \leq) \text{ and } Y \subseteq \preceq \}.$$

This also tells us that $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is join-dense in the set of the finitely generated (finite) confidence relations. This result should be compared with Theorem 10, which states that $(\mathcal{V}(\mathbb{B}, \leq), \subseteq)$ is on the other hand meet-dense in the set of the maximal confidence relations.

We have begun this section with a short description of how confidence relations can be constructed in a practical manner. Assume that we are attempting to construct a confidence relation $\preceq$, i.e., that $\preceq$ is an ordinal representation of the available information. Since it is not feasible or even possible for a very large or infinite $\mathbb{B}$ to explicitly compare all its elements on the basis of the available information, the idea emerged to only compare a finite number of events, and to use this initial information to construct an approximation for the actual confidence relation $\preceq$. Thus, we were led to introduce confidence relations generated by finite, confidence-consistent subsets of $\mathbb{B}^2$. Let Y be such a subset, Then $Y \subseteq \preceq$ and therefore also $\mathfrak{v}(Y) \subseteq \preceq$. $\mathfrak{v}(Y)$ is an *approximation* of $\preceq$ in the sense that it cannot contradict $\preceq$, i.e., contain couples that do not belong to $\preceq$. Furthermore, the approximation $\mathfrak{v}(Y)$ is *conservative* because it is the confidence relation compatible with Y containing the *least* ordinal information. Proposition 13(iv) also tells us that the larger the initial information Y is, the better the approximation $\mathfrak{v}(Y)$. In addition, the theorem above assures us that ultimately, $\preceq$ is the union of all its approximations.

Our results can still be extended somewhat. Recall the notation $\mathcal{M}(R) = \{ \preceq_{\mathcal{B}} \in \mathcal{M}(\mathbb{B}, \leq) \mid R \subseteq \preceq_{\mathcal{B}} \}$, for any binary relation R on $\mathbb{B}$, and call the relation R *confidence-consistent* iff R can be extended to a confidence relation on $(\mathbb{B}, \leq)$. This is in perfect agreement with the definition of confidence consistency for finite subsets of $\mathbb{B}^2$. Obviously, R is confidence-consistent iff $\mathcal{M}(R) \neq \emptyset$. If R is confidence-consistent, then the smallest confidence relation (containing the least ordinal information) which includes R is given by $\mathrm{tc}(\leq \cup R) = \bigcap \mathcal{M}(R)$, which could be called the minimal confidence extension of R. R is therefore a confidence relation iff it coincides with its minimal confidence extension. The analogy between confidence consistency, being a confidence relation, and minimal confidence extension on the one hand, and avoiding sure loss, coherence and natural extension in Walley's theory of imprecise probabilities [13] on the other hand, is more than striking. We shall have more to say about this analogy in Section 8.

5 Information systems

As we have explained before, a confidence relation $\preceq$ on a Boolean lattice $(\mathbb{B}, \leq)$ is a mathematical, order-theoretic representation of a specific kind of information. In this sense, such a relation can be said to *contain ordinal information*. Indeed, the elements (a, b) of such a confidence relation $\preceq$ can be seen as *elementary building blocks*, as indivisible lumps of ordinal information that must be consistently combined to yield the full ordinal picture $\preceq$ of the information that is available. Explicitly, (a, b) stands for the information: 'there is at least as much confidence in b as in a'. In the previous section, we have found out how we can take a finite number of these elementary building blocks and construct a smallest confidence relation compatible with it.

The notions of *finiteness* and *information content* that emerge from this brief discussion make us think at once of the theory of computation, which is pervaded by these ideas. More particularly, the notion of an *information system* [2] has been introduced in that context to provide an abstract representation for many computational (reasoning) systems that have these ideas in common, and which are used all of them in their particular form to find the information contained in a finite collection of elementary building blocks. Such an information system is explicitly defined as follows.

Definition 23 *An information system is a triple $(T, C, \vdash)$ consisting of* *(i)* *a set T of tokens;*

(ii) *a non-empty set C of finite subsets of T, called the finite consistent sets, which satisfy:*

(IS1) $(\forall Y \subseteq T)(\forall Z \subseteq T)(Y \in C$ *and* $Z \subseteq Y \Rightarrow Z \in C)$*,*

(IS2) $(\forall a \in T)(\{a\} \in C)$*;*

(iii) *a binary relation $\vdash$, called entailment, between C and T satisfying*

(IS3) $(\forall a \in T)(\forall Y \in C)(Y \vdash a \Rightarrow Y \cup \{a\} \in C)$*,*

(IS4) $(\forall Y \in C)(\forall a \in Y)(Y \vdash a)$*,*

(IS5) $(\forall (Y, Z) \in C^2)(\forall a \in T)((\forall b \in Z)(Y \vdash b)$ *and* $Z \vdash a \Rightarrow Y \vdash a)$*.*

The tokens in T are intended as abstractions of the elementary building blocks, or indivisible lumps, of information. The finite sets of tokens in C represent consistent finite collections of these lumps, and are abstractions of the finite packets of information that we have at our disposal and must content ourselves with in many real problems. The entailment relation $\vdash$ is of course used to characterise the *implied information*; it allows us to see the full extent of information conveyed by the finite packets. The 'axioms' (IS1)–(IS5) that T, C and $\vdash$ satisfy, have a fairly obvious interpretation. Some rather straightforward characteristics of the notions of consistency and entailment, and how they interrelate, are condensed in axioms (IS1), (IS3) and (IS4). Axiom (IS2) tells us that the tokens in themselves convey information, and axiom (IS5) governs the connection between subsequent entailments, and tells us that entailment is in a sense transitive.

Let us go back to the theory of confidence relations, and see how this fits into the framework created by Definition 23. Can we associate an information system with the notions introduced in the previous sections? We have already interpreted the elements of the confidence relations on a Boolean lattice $(\mathbb{B}, \leq)$ as elementary building blocks of ordinal information. We therefore define a set of tokens as follows:

$$T_{\mathcal{V}(\mathbb{B},\leq)} = \bigcup \mathcal{V}(\mathbb{B}, \leq) = \mathbb{B}^2 \setminus \{(1_{\mathbb{B}}, 0_{\mathbb{B}})\}.$$

If we take a finite subset Y of $T_{\mathcal{V}(\mathbb{B},\leq)}$, we have seen before that this can be a first step towards a full representation of ordinal information, i.e., a confidence relation, if there exists a confidence relation compatible with it. This leads to the following definition of the finite consistent subsets of $T_{\mathcal{V}(\mathbb{B},\leq)}$ as the finite, confidence-consistent subsets [11] of $\mathbb{B}^2$:

$$C_{\mathcal{V}(\mathbb{B},\leq)} = Con(\mathbb{B}, \leq).$$

If we consider an element Y of $C_{\mathcal{V}(\mathbb{B},\leq)}$, then $\mathfrak{v}(Y)$ is the smallest complete representation of ordinal information compatible with it. In this sense, all the tokens in $\mathfrak{v}(Y)$ can be regarded as information conveyed by Y. Let us therefore

[11] Remark that if $(1_{\mathbb{B}}, 0_{\mathbb{B}}) \in Y$, then Y cannot be confidence-consistent.

suggest the following binary relation $\vdash_{\mathcal{V}(\mathbb{B},\leq)}$ between $C_{\mathcal{V}(\mathbb{B},\leq)}$ and $T_{\mathcal{V}(\mathbb{B},\leq)}$ as the appropriate entailment relation:

$$(\forall(a,b) \in T_{\mathcal{V}(\mathbb{B},\leq)})(\forall Y \in C_{\mathcal{V}(\mathbb{B},\leq)})(Y \vdash_{\mathcal{V}(\mathbb{B},\leq)} (a,b) \Leftrightarrow (a,b) \in \mathfrak{v}(Y)).$$

The following theorem tells us that we have in this way indeed formed an information system. It also reveals the reasoning system hidden behind the notion of finitely generated confidence relations.

Theorem 24 *The triple* $(T_{\mathcal{V}(\mathbb{B},\leq)}, C_{\mathcal{V}(\mathbb{B},\leq)}, \vdash_{\mathcal{V}(\mathbb{B},\leq)})$ *is an information system.*

PROOF. We must show that $(T_{\mathcal{V}(\mathbb{B},\leq)}, C_{\mathcal{V}(\mathbb{B},\leq)}, \vdash_{\mathcal{V}(\mathbb{B},\leq)})$ satisfies the defining properties (IS1)–(IS5) of an information system. To prove that (IS1) holds, consider an element Y of $C_{\mathcal{V}(\mathbb{B},\leq)}$ and a subset Z of Y. Then, taking into account Proposition 13(iii) and the properties of a transitive closure, we find that $Z \subseteq \mathrm{tc}(\leq \cup Z) \subseteq \mathrm{tc}(\leq \cup Y) = \mathfrak{v}(Y)$, which implies that Z is confidence-consistent. The proof of (IS2) is trivial, taking into account the definition of $T_{\mathcal{V}(\mathbb{B},\leq)}$. To prove (IS3), consider (a,b) in $T_{\mathcal{V}(\mathbb{B},\leq)}$ and Y in $C_{\mathcal{V}(\mathbb{B},\leq)}$, and assume that $Y \vdash_{\mathcal{V}(\mathbb{B},\leq)} (a,b)$. This means that $(a,b) \in \mathfrak{v}(Y)$, and, taking into account Proposition 13(ii), that $Y \cup \{(a,b)\} \subseteq \mathfrak{v}(Y)$. Therefore, $Y \cup \{(a,b)\}$ is confidence-consistent. We now prove that (IS4) holds. Consider a Y in $C_{\mathcal{V}(\mathbb{B},\leq)}$ and (a,b) in Y. Then clearly, taking into account Proposition 13(ii), $(a,b) \in \mathfrak{v}(Y)$ and therefore $Y \vdash_{\mathcal{V}(\mathbb{B},\leq)} (a,b)$. Finally, we prove (IS5). Consider Y and Z in $C_{\mathcal{V}(\mathbb{B},\leq)}$ and (a,b) in $T_{\mathcal{V}(\mathbb{B},\leq)}$. If we assume that $(\forall(c,d) \in Z)(Y \vdash_{\mathcal{V}(\mathbb{B},\leq)} (c,d))$, this implies that $Z \subseteq \mathfrak{v}(Y)$. Therefore, taking into account Proposition 13(iii) and the properties of a transitive closure, we find that $\mathfrak{v}(Z) \subseteq \mathfrak{v}(Y)$. If we also assume that $Z \vdash_{\mathcal{V}(\mathbb{B},\leq)} (a,b)$, or equivalently, $(a,b) \in \mathfrak{v}(Z)$, we immediately find that $(a,b) \in \mathfrak{v}(Y)$. $\square$

6 Duality of confidence relations

In this section, we intend to show that confidence relations come in dual pairs. Indeed, it appears that the ordinal aspects of information, represented by our confidence relations, have two different, but mutually related sides. The origin of this duality is a special kind of symmetry: if an event occurs, its *opposite event* does not occur, and *vice versa*. This fact may seem tritely obvious, but it has far-reaching consequences. It eventually provides us with an explanation for the curious fact that the well-known measures representing uncertainty always seem to appear in dual pairs: possibility and necessity measures, plausibility and belief measures, ... And if probability measures seem to be an exception to this rule, this is only because the well-known complementation law—really a consequence of the additivity and normalisation—makes a probability measure coincide with its dual counterpart.

In order to see what the order-theoretic consequences of the above-mentioned symmetry are, let us consider a confidence relation $\preceq$ on the Boolean lattice $(\mathbb{B}, \leq) = (\mathcal{E}, \subseteq)$, considered in the introduction. $\preceq$ reflects the ordinal aspects of a certain type of confidence in—or information about—the outcome of an experiment. We also consider two elements a and b of $\mathbb{B} = \mathcal{E}$. If $a \preceq b$, this means that, on the basis of our knowledge about the outcome of the experiment, we have at most as much confidence in the occurrence of the event a as in that of b. Furthermore, as we have indicated above, if the event a occurs, we know that the opposite event $\neg a$ does not occur, and *vice versa*. The duality (or dual order-automorphism) $\neg$ on $(\mathbb{B}, \leq)$ leads in a very natural way to a duality in the occurrence or otherwise of the elements of $\mathbb{B}$. It provides us with the possibility of taking a first type of confidence and using it to derive a second, *dual* type. This is done by concentrating on the opposite events of the events related to each other by the confidence relation $\preceq$: whenever we have at most as much confidence in the occurrence of a as in that of b, we say that we have at most as much *dual confidence* in the occurrence of $\neg b$ as in that of $\neg a$. This leads to the following definition.

Definition 25 *Let $\preceq$ be a confidence relation on $(\mathbb{B}, \leq)$. We call the binary relation $\preceq^D$ on $\mathbb{B}$, defined by $(\forall (a, b) \in \mathbb{B}^2)(a \preceq^D b \Leftrightarrow \neg b \preceq \neg a)$, the dual confidence relation*[12] *of $\preceq$ on $\mathbb{B}$.*

Of course, we are being a bit rash in calling $\preceq^D$ a confidence relation before proving that it is worthy of that name. The following proposition justifies our nomenclature. It also tells us why we can call $\preceq$ and $\preceq^D$ *dual confidence relations*. Its proof is trivial, and is therefore omitted.

Proposition 26 *Let $\preceq$ be a confidence relation on $(\mathbb{B}, \leq)$. Then $\preceq^D$ is a confidence relation on $(\mathbb{B}, \leq)$ as well. Furthermore, $\preceq$ is the dual confidence relation of $\preceq^D$, i.e., $(\preceq^D)^D = \preceq$.*

We thus come to the interesting conclusion that there is an inherent duality in the ordinal representation of uncertainty: the information represented by a confidence relation $\preceq$, also has another, dual representation in the confidence relation $\preceq^D$.

In the following rather obvious definition, we introduce the essential notion of *self-duality* of confidence relations, which will be studied more in detail in the following section.

Definition 27 *A confidence relation $\preceq$ on $(\mathbb{B}, \leq)$ is called self-dual iff $\preceq^D = \preceq$. The set of the self-dual confidence relations is denoted by $\mathcal{V}_z(\mathbb{B}, \leq)$.*

[12] It should be stressed that the *dual confidence relation* $\preceq^D$ of the confidence relation $\preceq$ is not necessarily equal to the *dual relation* $(\mathrm{co} \preceq)^{-1}$ of $\preceq$, as for example defined in the theory of preference relations [10].

Let us investigate which are the dual confidence relations of the *maximal elements* of $(\mathcal{V}(\mathbb{B},\leq),\subseteq)$.

Proposition 28 *Let $\mathcal{B}$ be a proper up-set of $(\mathbb{B},\leq)$ and consider the set of complements $\neg\mathcal{B}=\{\neg a \mid a\in\mathcal{B}\}$. Then $(\preceq_{\mathcal{B}})^D=\preceq_{\mathbb{B}\setminus\neg\mathcal{B}}$, which implies that the dual confidence relation of a maximal confidence relation is again a maximal confidence relation. Furthermore, $\preceq_{\mathcal{B}}$ is self-dual iff $\mathcal{B}=\mathbb{B}\setminus\neg\mathcal{B}$.*

PROOF. It is easily verified that $(\preceq_{\mathcal{B}})^D=(\neg(\mathbb{B}\setminus\mathcal{B})\times\neg(\mathbb{B}\setminus\mathcal{B}))\cup(\neg\mathcal{B}\times\mathbb{B})$, where $\neg(\mathbb{B}\setminus\mathcal{B})=\{\neg a \mid a\in\mathbb{B}\setminus\mathcal{B}\}=\mathbb{B}\setminus\neg\mathcal{B}$. It is furthermore obvious that $\neg\mathcal{B}$ is a proper down-set of $(\mathbb{B},\leq)$, and that consequently $\mathbb{B}\setminus\neg\mathcal{B}$ is a proper up-set of $(\mathbb{B},\leq)$. We conclude from Theorem 6 that $(\preceq_{\mathcal{B}})^D=\preceq_{\mathbb{B}\setminus\neg\mathcal{B}}$ is a maximal confidence relation on $\mathbb{B}$. The rest of the proof follows from Proposition 8(i). $\square$

The reader will have noticed that duality in the sense of confidence relations is certainly not the same thing as duality in the sense of partial order relations. But, as we shall presently show, there exists an interesting relation between these notions. It comes to the fore when we consider the minimal evaluation sets and mappings w.r.t. dual confidence relations. Let us make this relation more explicit. We use the following notations: $((M,\leq),m)$ is a minimal evaluation structure of $\preceq$, and $((M^D,\leq^D),m^D)$ a minimal evaluation structure of $\preceq^D$.

For one thing, the tops 1_M and 1_{M^D}, and bottoms 0_M and 0_{M^D} of the respective minimal evaluation sets $(M,\leq)$ and $(M^D,\leq^D)$ satisfy the following property: for any a in $\mathbb{B}$, $a\in m(1_{\mathbb{B}})\Leftrightarrow\neg a\in m^D(0_{\mathbb{B}})$ and $a\in m(0_{\mathbb{B}})\Leftrightarrow\neg a\in m^D(1_{\mathbb{B}})$. More generally, we have for any a and b in $\mathbb{B}$, with the notation $\approx^D$ for the indifference relation associated with $\preceq^D$,

$$a\approx^D b\Leftrightarrow a\preceq^D b \text{ and } b\preceq^D a\Leftrightarrow\neg b\preceq\neg a \text{ and } \neg a\preceq\neg b\Leftrightarrow\neg b\approx\neg a,$$

or equivalently,

$$m(a)=m(b)\Leftrightarrow m^D(\neg a)=m^D(\neg b) \tag{3}$$

As a consequence, and also since the minimal evaluation mappings m and m^D are by definition surjective, the following definition makes sense.

Definition 29 *Let $\preceq$ be a confidence relation on $(\mathbb{B},\leq)$. We introduce the mapping $\vartheta_{\preceq}\colon M\to M^D\colon\alpha\mapsto\vartheta_{\preceq}(\alpha)$, with $\vartheta_{\preceq}(m(a))=m^D(\neg a)$, $a\in\mathbb{B}$.*

In the following theorem, it is proven that the minimal evaluation sets $(M,\leq)$ and $(M^D,\leq^D)$ of $\preceq$ and $\preceq^D$ are dually order-isomorphic, with $\vartheta_{\preceq}$ serving as the dual order-isomorphism between them.

Theorem 30 *Let $\preceq$ be an confidence relation on $(\mathbb{B}, \leq)$. Then*

(i) $\vartheta_{\preceq}$ *is a dual order-isomorphism between* $(M, \leq)$ *and* $(M^D, \leq^D)$*;*
(ii) $(\forall a \in \mathbb{B})(m^D(a) = \vartheta_{\preceq}(m(\neg a)))$.

PROOF. Statement (ii) follows from the definition of $\vartheta_{\preceq}$. We prove (i). Let us first show that $\vartheta_{\preceq}$ is a bijection between M and M^D. Consider α and β in M and assume that $\vartheta_{\preceq}(\alpha) = \vartheta_{\preceq}(\beta)$. Since m is by definition surjective, there exist a and b in $\mathbb{B}$ such that $m(a) = \alpha$ and $m(b) = \beta$, whence by applying $\vartheta_{\preceq}$, $m^D(\neg a) = m^D(\neg b)$. It follows from (3) that $m(a) = m(b)$, or equivalently, $\alpha = \beta$. $\vartheta_{\preceq}$ is therefore an injection. Consider furthermore γ in M^D. Since m^D is by definition surjective, there exists a c in $\mathbb{B}$ such that $m^D(c) = \gamma$. Then, by definition, $\gamma = m^D(c) = \vartheta_{\preceq}(m(\neg c))$, which implies that $\vartheta_{\preceq}$ is a surjection as well. To complete the proof, let us consider α and β in M. There exist a and b in $\mathbb{B}$ such that $m(a) = \alpha$ and $m(b) = \beta$. Taking into account the definition of $\vartheta_{\preceq}$, we find that

$$\alpha \leq \beta \Leftrightarrow a \preceq b \Leftrightarrow \neg b \preceq^D \neg a \Leftrightarrow m^D(\neg b) \leq^D m^D(\neg a) \Leftrightarrow \vartheta_{\preceq}(\beta) \leq^D \vartheta_{\preceq}(\alpha),$$

which completes the proof of (i). □

The reader will probably have noticed that (ii) in a very general way states that the models m and m^D are each other's dual, in the sense that are for instance upper and lower probabilities [5,13], or plausibility and functions measures [12], or possibility and necessity measures [7].

In the special case that $\preceq$ is self-dual, $(M, \leq)$ and $(M^D, \leq^D)$ are obviously order-isomorphic, and we may identify them. In that case, the common minimal evaluation set $(M, \leq)$ of $\preceq$ and $\preceq^D$ is self-dual, or in other words dually order-isomorphic with itself, with $\vartheta_{\preceq}$ as a dual order-automorphism. For the minimal evaluation mappings we then have that $m = m^D$.

7 Self-dual Confidence Relations

We have seen in the previous section that for every ordinal representation of information $\preceq$, there exists a dual representation $\preceq^D$ that is equivalent to it. To phrase it rather loosely, when information is represented ordinally, it can be looked at from two different angles, which yields two mutually equivalent views. When however $\preceq$ is self-dual, or $\preceq = \preceq^D$, both dual representations of the same ordinal information *coincide* and are represented by one and the same confidence relation. $\preceq$ is then complete enough to accommodate for the fact that a possible comparison of events also allows us compare the opposite events. For this reason *a confidence relation $\preceq$ that is self-dual can be considered a more complete—or more perfect—model than one that is not*. Let us therefore in this section concentrate on the mathematics of self-dual confidence relations [13].

In order to distinguish the self-dual confidence relations from the other ones, we shall call them *z-confidence relations*. As indicated above, we denote the set of the z-confidence relations on $(\mathbb{B}, \leq)$ by $\mathcal{V}_z(\mathbb{B}, \leq)$, and the set of the maximal elements of $(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$ by $\mathcal{M}_z(\mathbb{B}, \leq)$. We introduce the following notation, which extends the one introduced in the previous section: for any subset R of $\mathbb{B}^2$, $R^D = \{\, (\neg b, \neg a) \mid (a, b) \in R \,\}$.

The following unary operator on $\mathcal{V}(\mathbb{B}, \leq)$ has an important role in the ensuing discussion, and is called the *self-dualiser* of $\mathcal{V}(\mathbb{B}, \leq)$, for reasons that will become clear in Proposition 31 below:

$$\mathfrak{z}\colon \mathcal{V}(\mathbb{B}, \leq) \to \mathcal{V}(\mathbb{B}, \leq)\colon \preceq \;\mapsto\; \preceq \cap \preceq^D.$$

Let us summarise the most important properties of this operator in the following proposition. The proof is straightforward and therefore omitted.

Proposition 31 *Let $\preceq$, $\preceq_1$ and $\preceq_2$ be confidence relations on $(\mathbb{B}, \leq)$ and let $(\preceq_j \mid j \in J)$ be a non-empty family of confidence relations on $(\mathbb{B}, \leq)$. Then*

(i) $\mathfrak{z}(\preceq) = \mathfrak{z}(\preceq^D)$;
(ii) $\mathfrak{z}(\bigcap_{j \in J} \preceq_j) = \bigcap_{j \in J} \mathfrak{z}(\preceq_j)$;
(iii) $\mathfrak{z}(\preceq) \in \mathcal{V}_z(\mathbb{B}, \leq)$;
(iv) $\preceq \in \mathcal{V}_z(\mathbb{B}, \leq) \Leftrightarrow \mathfrak{z}(\preceq) = \preceq$;
(v) $\mathfrak{z}(\mathcal{V}(\mathbb{B}, \leq)) = \mathcal{V}_z(\mathbb{B}, \leq)$;
(vi) $\mathfrak{z}(\preceq) \subseteq \preceq$ *and* $\mathfrak{z}(\preceq) \subseteq \preceq^D$;
(vii) $\preceq_1 \subseteq \preceq_2 \Rightarrow \mathfrak{z}(\preceq_1) \subseteq \mathfrak{z}(\preceq_2)$;
(viii) $\mathfrak{z}(\mathfrak{z}(\preceq)) = \mathfrak{z}(\preceq)$.

If we recall that $\mathcal{V}(\mathbb{B}, \leq)$ is an intersection structure and that $\leq$ is a z-confidence relation on $(\mathbb{B}, \leq)$, we find that $(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$ is a complete meet-semilattice with bottom $\leq$. Concerning the maximal elements of this structure, we may easily prove the following theorem.

Theorem 32 $\mathcal{V}_z(\mathbb{B}, \leq) \cap \mathcal{M}(\mathbb{B}, \leq) \subseteq \mathcal{M}_z(\mathbb{B}, \leq) \subseteq \mathfrak{z}(\mathcal{M}(\mathbb{B}, \leq))$. *Moreover, for any $\preceq \in \mathcal{M}(\mathbb{B}, \leq)$, $\mathfrak{z}(\preceq) \in \mathcal{M}(\mathbb{B}, \leq) \Leftrightarrow \preceq \in \mathcal{V}_z(\mathbb{B}, \leq)$. Also, let $\mathcal{B}$ be a proper up-set of $(\mathbb{B}, \leq)$. Then $\preceq_{\mathcal{B}} \in \mathcal{V}_z(\mathbb{B}, \leq)$ and therefore $\preceq_{\mathcal{B}} \in \mathcal{M}_z(\mathbb{B}, \leq)$*

[13] Note that the qualitative probabilities discussed in Section 1 are self-dual; and so are the qualitative probability orderings considered by Walley in his theory of imprecise probabilities [13].

iff $\mathcal{B} = \mathbb{B} \setminus \neg\mathcal{B}$, *or equivalently,*

$$(\forall a \in \mathbb{B})(a \in \mathcal{B} \Leftrightarrow \neg a \notin \mathcal{B}). \tag{4}$$

Finally, if $\mathcal{B}$ *is a filter of* $(\mathbb{B}, \leq)$*, then* $\preceq_{\mathcal{B}} \in \mathcal{V}_z(\mathbb{B}, \leq)$ *and therefore* $\preceq_{\mathcal{B}} \in \mathcal{M}_z(\mathbb{B}, \leq)$ *iff* $\mathcal{B}$ *is an ultrafilter of* $(\mathbb{B}, \leq)$.

PROOF. Let $\preceq_o$ be an element of $\mathcal{V}_z(\mathbb{B}, \leq) \cap \mathcal{M}(\mathbb{B}, \leq)$ and let $\preceq$ be a z-confidence relation on $(\mathbb{B}, \leq)$ with $\preceq_o \subseteq \preceq$. Since $\preceq \in \mathcal{V}(\mathbb{B}, \leq)$ and $\preceq_o \in \mathcal{M}(\mathbb{B}, \leq)$, it must be that $\preceq_o = \preceq$, whence by definition $\preceq_o \in \mathcal{M}_z(\mathbb{B}, \leq)$. This proves that $\mathcal{V}_z(\mathbb{B}, \leq) \cap \mathcal{M}(\mathbb{B}, \leq) \subseteq \mathcal{M}_z(\mathbb{B}, \leq)$. To prove that $\mathcal{M}_z(\mathbb{B}, \leq) \subseteq \mathfrak{z}(\mathcal{M}(\mathbb{B}, \leq))$, consider $\preceq \in \mathcal{M}_z(\mathbb{B}, \leq)$. There are two possibilities. Either $\preceq \in \mathcal{M}(\mathbb{B}, \leq)$, and therefore $\preceq = \mathfrak{z}(\preceq) \in \mathfrak{z}(\mathcal{M}(\mathbb{B}, \leq))$. Or $\preceq \notin \mathcal{M}(\mathbb{B}, \leq)$, which, by Proposition 5 and Theorem 6 implies that there exists a $\preceq_M$ in $\mathcal{M}(\mathbb{B}, \leq)$ such that $\preceq \subset \preceq_M$ and therefore $\preceq \subseteq \mathfrak{z}(\preceq_M)$, whence $\preceq = \mathfrak{z}(\preceq_M)$, since we assumed that $\preceq \in \mathcal{M}_z(\mathbb{B}, \leq)$. Again we find that $\preceq \in \mathfrak{z}(\mathcal{M}(\mathbb{B}, \leq))$. Next, consider a maximal confidence relation $\preceq$ on $(\mathbb{B}, \leq)$. Assume that $\mathfrak{z}(\preceq)$ is also a maximal confidence relation. Since $\mathfrak{z}(\preceq) \subseteq \preceq$ and $\mathfrak{z}(\preceq)$ is maximal, we find that $\mathfrak{z}(\preceq) = \preceq \in \mathcal{V}_z(\mathbb{B}, \leq)$. Conversely, assume that $\preceq \in \mathcal{V}_z(\mathbb{B}, \leq)$. Then $\preceq = \mathfrak{z}(\preceq) \in \mathcal{M}(\mathbb{B}, \leq)$. That $\preceq_{\mathcal{B}} \in \mathcal{V}_z(\mathbb{B}, \leq)$ is equivalent with (4) follows at once from Proposition 28. Finally, it is a standard result from lattice theory [2] that a filter $\mathcal{B}$ of the Boolean lattice $(\mathbb{B}, \leq)$ satisfies (4) iff it is an ultrafilter. $\square$

This theorem, together with the discussion of the interpretation of maximal confidence relations in Section 3, and particular Eq. (1), tells us that *a maximal confidence relation* $\preceq_{\mathcal{F}}$ *associated with a proper filter* $\mathcal{F}$ *of* $(\mathbb{B}, \leq)$ *describes absolute certainty if and only if it is self-dual.*

For any finite subset Y of $\mathbb{B}^2$ we define $\mathcal{D}_z(Y) = \{ \preceq \in \mathcal{V}_z(\mathbb{B}, \leq) \mid Y \subseteq \preceq \}$, and we call Y *z-confidence-consistent* iff $\mathcal{D}_z(Y) \neq \emptyset$. The set of the finite, z-confidence-consistent subsets of $\mathbb{B}^2$ is denoted by $Con_z(\mathbb{B}, \leq)$. From the following proposition we may easily deduce that Y is z-confidence-consistent iff $Y \cup Y^D$ is confidence-consistent.

Proposition 33 *Let* Y *be a finite subset of* $\mathbb{B}^2$. *Then* $\mathcal{D}_z(Y) = \mathfrak{z}(\mathcal{D}(Y \cup Y^D))$.

PROOF. Let $\preceq \in \mathcal{D}_z(Y)$, then $Y \subseteq \preceq$ and $\preceq$ is self-dual. Therefore $Y^D \subseteq \preceq$, whence $Y \cup Y^D \subseteq \preceq$. This implies that $\preceq \in \mathcal{D}(Y \cup Y^D)$ and $\preceq = \mathfrak{z}(\preceq) \in \mathfrak{z}(\mathcal{D}(Y \cup Y^D))$. We conclude that $\mathcal{D}_z(Y) \subseteq \mathfrak{z}(\mathcal{D}(Y \cup Y^D))$.

Conversely, let $\preceq \in \mathfrak{z}(\mathcal{D}(Y \cup Y^D))$, which means that $\preceq$ is self-dual and that there exists a $\preceq_o \in \mathcal{D}(Y \cup Y^D)$ such that $\preceq = \mathfrak{z}(\preceq_o)$. From $Y \cup Y^D \subseteq \preceq_o$ we deduce that $Y \subseteq \preceq_o$ and $Y \subseteq \preceq_o^D$, and therefore also that $Y \subseteq \mathfrak{z}(\preceq_o) = \preceq$. This implies that $\preceq \in \mathcal{D}_z(Y)$, whence $\mathfrak{z}(\mathcal{D}(Y \cup Y^D)) \subseteq \mathcal{D}_z(Y)$. $\square$

Definition 34 *Let Y be a finite, z-confidence-consistent subset of $\mathbb{B}^2$. We call*

$$\mathfrak{v}_z(Y) = \bigcap \mathcal{D}_z(Y) = \bigcap \{ \preceq \in \mathcal{V}_z(\mathbb{B}, \leq) \mid Y \subseteq \preceq \}$$

the z-confidence relation generated by Y. A z-confidence relation $\preceq$ on $(\mathbb{B}, \leq)$ is called finitely generated iff there exists a $Y \in Con_z(\mathbb{B}, \leq)$ such that $\preceq = \mathfrak{v}_z(Y)$.

From Propositions 31(ii) and 33 we deduce that for a finite, z-confidence-consistent subset Y of $\mathbb{B}^2$, $\mathfrak{v}_z(Y) = \mathfrak{z}(\mathfrak{v}(Y \cup Y^D))$. Using this observation and Proposition 13, the proof of the following proposition becomes trivial.

Proposition 35 *Let Y, Y_1 and Y_2 be finite, z-confidence-consistent subsets of $\mathbb{B}^2$. Then*

(i) $\mathfrak{v}_z(Y) \in \mathcal{V}_z(\mathbb{B}, \leq)$;
(ii) $Y \cup Y^D \subseteq \mathfrak{v}_z(Y)$;
(iii) $\mathfrak{v}_z(Y) = \mathfrak{z}(\mathrm{tc}(\leq \cup Y \cup Y^D))$;
(iv) $Y_1 \subseteq Y_2 \Rightarrow \mathfrak{v}_z(Y_1) \subseteq \mathfrak{v}_z(Y_2)$.

As in the case of confidence relations, we call a subset $\mathcal{D}$ of $\mathcal{V}_z(\mathbb{B}, \leq)$ *directed* if any finite subset $\mathcal{E}$ of $\mathcal{D}$ has an upper bound $D \in \mathcal{D}$, i.e., $(\forall S \in \mathcal{E})(S \subseteq D)$. As before, if $\mathcal{D}$ is a directed set of z-confidence relations on $(\mathbb{B}, \leq)$, and Y is a finite subset of $\bigcup \mathcal{D}$, there exists a $D \in \mathcal{D}$ such that $Y \subseteq D$. Therefore, any finite subset of the union of a directed set of z-confidence relations is z-confidence-consistent. We may use these observations to prove the following result. Theorem 37 follows immediately.

Proposition 36 *Let $\mathcal{D}$ be a directed subset of $\mathcal{V}_z(\mathbb{B}, \leq)$. Then $\bigcup \mathcal{D}$ is a z-confidence relation on $(\mathbb{B}, \leq)$.*

PROOF. $\mathcal{D}$ is in particular also a directed subset of $\mathcal{V}(\mathbb{B}, \leq)$, whence, by Proposition 16, $\bigcup \mathcal{D}$ is a confidence relation on $(\mathbb{B}, \leq)$. $\bigcup \mathcal{D}$ is obviously also self-dual. $\square$

Theorem 37 *$(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$ is an algebraic intersection structure, and therefore $\sup \mathcal{D} = \bigcup \mathcal{D}$ for any directed subset $\mathcal{D}$ of $\mathcal{V}_z(\mathbb{B}, \leq)$, where $\sup$ is the supremum in the complete meet-semilattice $(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$.*

As a corollary $(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$ is also a CPO. As in Section 4, we can invoke the general definition of a *finite element* in a CPO to introduce finite z-confidence relations. We call a z-confidence relation $\preceq$ on $(\mathbb{B}, \leq)$ *finite* iff for every directed subset $\mathcal{D}$ of $\mathcal{V}_z(\mathbb{B}, \leq)$: $\preceq \subseteq \bigcup \mathcal{D} \Rightarrow (\exists D \in \mathcal{D})(\preceq \subseteq D)$. As in the case of confidence relations, we can identify finite and finitely generated z-confidence relations. This is stated in the theorem below. The proof is completely analogous to the proofs of Proposition 20 and Theorem 21, and is therefore omitted. We use the notation

$$FGS_z(\preceq) = \{\, D \in \mathcal{V}_z(\mathbb{B}, \leq) \mid D \text{ is finitely generated and } D \subseteq \preceq \,\}.$$

Theorem 38 *Let* $\preceq\, \in \mathcal{V}_z(\mathbb{B}, \leq)$*. Then* $\preceq\, = \bigcup FGS_z(\preceq)$ *and* $FGS_z(\preceq)$ *is a directed subset of* $\mathcal{V}_z(\mathbb{B}, \leq)$*. Moreover,* $\preceq$ *is finite iff* $\preceq$ *is finitely generated. Finally,* $(\mathcal{V}_z(\mathbb{B}, \leq), \subseteq)$ *is an algebraic semilattice, which implies that every z-confidence relation* $\preceq$ *is the union of the finite (or finitely generated) z-confidence relations it includes:*

$$\preceq\, = \bigcup FGS_z(\preceq) = \bigcup \{\, \mathfrak{v}_z(Y) \mid Y \in Con_z(\mathbb{B}, \leq) \text{ and } Y \subseteq \preceq \,\}.$$

To end this section, we consider the link with information systems. If we let $T_{\mathcal{V}_z(\mathbb{B},\leq)} = \bigcup \mathcal{V}_z(\mathbb{B}, \leq) = \mathbb{B}^2 \setminus \{(1_{\mathbb{B}}, 0_{\mathbb{B}})\}$, $C_{\mathcal{V}_z(\mathbb{B},\leq)} = Con_z(\mathbb{B}, \leq)$ and define the binary relation $\vdash_{\mathcal{V}_z(\mathbb{B},\leq)}$ between $C_{\mathcal{V}_z(\mathbb{B},\leq)}$ and $T_{\mathcal{V}_z(\mathbb{B},\leq)}$ as:

$$(\forall (a, b) \in T_{\mathcal{V}_z(\mathbb{B},\leq)})(\forall Y \in C_{\mathcal{V}_z(\mathbb{B},\leq)})(Y \vdash_{\mathcal{V}_z(\mathbb{B},\leq)} (a, b) \Leftrightarrow (a, b) \in \mathfrak{v}_z(Y)),$$

then we indeed have the following result, the proof of which is completely analogous to that of Theorem 24, and is therefore omitted.

Theorem 39 *The triple* $(T_{\mathcal{V}_z(\mathbb{B},\leq)}, C_{\mathcal{V}_z(\mathbb{B},\leq)}, \vdash_{\mathcal{V}_z(\mathbb{B},\leq)})$ *is an information system.*

8 Conclusion

The reader will have noticed that the course of reasoning in Section 7 is completely similar to that in Sections 4 and 5. Using the techniques given in this paper, it is easily verified that these results can be even further generalised. Indeed, consider any *intersection structure* $\mathcal{C}$ on a set X, i.e., $\mathcal{C}$ is a set of subsets of X that is closed under arbitrary, non-empty intersections. Then obviously $(\mathcal{C}, \subseteq)$ is a complete meet-semilattice with bottom $\bigcap \mathcal{C}$. Call any element of $\mathcal{C}$ a $\mathcal{C}$*-object*. A finite subset Y of X is called $\mathcal{C}$*-consistent* iff $\mathcal{D}_{\mathcal{C}}(Y) = \{\, B \in \mathcal{C} \mid Y \subseteq B \,\} \neq \emptyset$. Whenever Y is $\mathcal{C}$-consistent we call $\mathfrak{v}_{\mathcal{C}}(Y) = \bigcap \mathcal{D}_{\mathcal{C}}(Y)$ the $\mathcal{C}$*-object generated by* Y. The set of the $\mathcal{C}$-consistent finite subsets of X is denoted by $Con_{\mathcal{C}}$. A $\mathcal{C}$-object B is called *finitely generated* iff there exists a $\mathcal{C}$-consistent finite subset Y of X such that $B = \mathfrak{v}_{\mathcal{C}}(Y)$. If for any $\mathcal{C}$-object B we define $FGS_{\mathcal{C}}(B) = \{\, \mathfrak{v}_{\mathcal{C}}(Y) \mid Y \in Con_{\mathcal{C}} \text{ and } Y \subseteq B \,\}$, then $B = \bigcup FGS_{\mathcal{C}}(B)$ and $\bigcup FGS_{\mathcal{C}}(B)$ is a directed subset of $\mathcal{C}$. Call a $\mathcal{C}$-object B *finite* iff for every directed subset $\mathcal{D}$ of $\mathcal{C}$, $B \subseteq \bigcup \mathcal{D} \Rightarrow (\exists D \in \mathcal{D})(B \subseteq D)$. Then a $\mathcal{C}$-object B is finite iff it is finitely generated. Finally, if we define

$T_{\mathcal{C}} = \bigcup \mathcal{C}$, $C_{\mathcal{C}} = Con_{\mathcal{C}}$ and the binary relation $\vdash_{\mathcal{C}}$ between $C_{\mathcal{C}}$ and $T_{\mathcal{C}}$ by $Y \vdash_{\mathcal{C}} a \Leftrightarrow a \in \mathfrak{v}_{\mathcal{C}}(Y)$, then $(T_{\mathcal{C}}, C_{\mathcal{C}}, \vdash_{\mathcal{C}})$ is an information system. Of course, whether $(\mathcal{C}, \subseteq)$ is an algebraic semilattice depends on whether $\mathcal{C}$ is closed under unions of its directed subsets.

We want to point out that these remarks allow us to extend and compare the discussion of confidence relations, given here, to other uncertainty models in the literature. Let us give a few examples.

Walley's classes of *coherent almost-preference relations* (APR) $\geqq$ [13] defined on a linear space of gambles $\mathcal{K}$, constitute an intersection structure. When ordered by set inclusion, they lead to a complete meet-semilattice without top, but with a class of mutually incomparable maximal elements, which are precisely the *complete* coherent almost-preference relations. The bottom of this structure is precisely the pointwise ordering of gambles. Every coherent almost-preference relation is dominated by a complete APR, and is an intersection of complete APR's. Of course, using the results above we see that a number of APR's can be finitely generated by consistent finite subsets of $\mathcal{K}^2$, and that the class of APR's is join-dense in these finitely generated (or finite) APR's. Since a directed union of APR's is not necessarily again an APR, the class of APR's will in general not constitute an algebraic semilattice. Similar results, though admittedly somewhat different in their details, hold for classes of Walley's coherent almost-desirable gambles; coherent really desirable gambles; weak*-compact sets of linear previsions; coherent lower (upper) previsions, . . .

In Giles' so-called pragmatic (or operational) approach to subjective probability, *possibility functions* are mappings from a Boolean algebra to $[0, 1]$, satisfying certain behavioural rationality requirements [9], and representing the betting behaviour of so-called *rational agents* in the face of uncertainty. These mappings can be ordered pointwise, or as Giles put it, 'by caution'. The set of the possibility functions on a Boolean algebra thus ordered constitutes a complete join-semilattice with no bottom, but with a set of mutually incomparable minimal elements, which are precisely the probability measures, describing the behaviour of so-called *Bayesian agents*. The structure has a top, which describes the behaviour of the *cautious agent*. Every possibility function dominates at least one minimal possibility function and is the supremum of the minimal possibility functions that it dominates.

The similarities between the order-theoretic structures underlying all these uncertainty representations are more than striking. We have shown in this paper that the essentials of these structures are already present in very basic uncertainty models, such as our confidence and z-confidence relations. We would feel rewarded if this paper succeeded in drawing attention to this intriguing common order-theoretic framework.

To conclude, we discuss the possible incompleteness of confidence relations. Completeness is often implicitly or explicitly imposed on ordinal models representing uncertainty. It seems natural enough a condition in the context of ordinal models for additive probabilities [8,11], and it has also been required more recently in order-theoretic discussions of possibility measures [6] (but see [3]) and belief functions [15]. It is hardly ever questioned, except in some behavioural accounts of uncertainty [13]. As a matter of fact, potential incomparability seems to be a blind spot in many a scientific tradition, not just the one concerned with modelling uncertainty. Why do we accept and defend incomparability here?

Before we can answer this question, we need at least a sketch of the interpretation we want to give to confidence relations. Indeed, if for events a and b the meaning of $a \preceq b$ is not made sufficiently clear, almost any claim can be made about the relation $\preceq$. We have until now restricted ourselves to a formal mathematical study of a class of relations sharing a number of basic properties, and taking it for granted that the meaning of confidence relations could for the purposes of the discussion be condensed in their defining properties. But if we want to discuss other claims about these relations (and even the origin of their defining properties, see the discussion in the introduction), we *need* an interpretation. First of all, it is a tacit assumption in this paper that information can in some way lead to (partial) beliefs, which can be used to construct preference relations $\preceq$ between events. The interpretation for the preference relations $\preceq$ that we propose for the purpose of the following discussion consists in stating that $a \preceq b$ is not free (or inconsequential). It involves a *commitment* of some kind. For instance, if you should state that $a \preceq b$ you stand to lose something you value (money, respect, utility, $\dots$) if a obtains and b does not, whereas you will win the same something if b obtains and a does not.

Our first reason for defending possible incomparability is fundamental, and we have already stated it in the introduction. Confidence relations should in some way reflect the available information, and it is possible that the available information is not sufficient to conclude that $a \preceq b$ *or* $b \preceq a$. Suppose that the information available allows you to conclude that $a \not\preceq b$. It may be tempting or even natural for you to conclude from this that $b \preceq a$, especially if you are for some reason unaware of, or decided against, the possible existence of incomparability. But you should at least ask yourself if this conclusion is warranted by the information that you have at your disposal. We can see no *a priori* reason why it should be! You should go back to the information and say $b \preceq a$ only if (you believe) this information allows you to do so, not because you assume that it logically or naturally follows from a previous assessment.

It may be argued that a similar line of reasoning will preclude imposing any requirement on relations expressing preference between events on the basis of the available information, in particular the three defining conditions imposed on confidence relations. Our answer to this objection is that the three defining conditions can be defended in a behavioural context as rationality requirements, whereas completeness is not implied by any such requirement [14].

A possible objection to incompleteness is the following. If a confidence relation is incomplete, then it cannot be compatible (in the sense of the introduction) with any real-valued set function. Therefore, the argument goes, it is more general than called for, and has little practical importance, as real-valued set functions seem to constitute a sufficiently vast class of numerical models for uncertainty. Our answer to this objection is two-fold. For a start, it has been argued elsewhere [13] that real-valued *set* functions do not suffice to generate all reasonable numerical models for uncertainty. Secondly, the above course of reasoning can be conceived either as an argument against incompleteness or as an argument against compatibility (as described in the introduction) being the appropriate link in this context between ordinal and numerical models. It will suffice here to say that there exist other links between ordinal and numerical models [13] which do allow for incompleteness [15].

A second, closely related, reason for allowing incompleteness has to do with the representation of complete ignorance. If potential incomparability is accepted, this poses no problem: if you have no information, you will not be willing to state that $a \preceq b$ unless you are forced to by the underlying structure of the event model, that is, unless $a \leq b$. This yields $\leq$ as the appropriate (incomplete) model for complete uncertainty. The underlying idea is that $a \preceq b$ ordinally represents information, and that you need some relevant piece of information before you become willing to commit yourself in stating that $a \preceq b$ (see also Section 5).

All maximal confidence relations are complete, which adds weight to our claim that complete models will only be possible if enough information is available. We therefore fail to see how any complete confidence relation could represent complete ignorance. It is sometimes argued that an appropriate model for ignorance is the maximal (and therefore complete) confidence relation $\preceq_{\mathbb{B}\setminus\{0_{\mathbb{B}}\}} = \mathbb{B} \setminus \{0_{\mathbb{B}}\} \times \mathbb{B} \setminus \{0_{\mathbb{B}}\} \cup \{0_{\mathbb{B}}\} \times \mathbb{B}$, probably because it is the confidence relation compatible with the *vacuous upper probability* $\overline{P}_v$, defined by $\overline{P}_v(0_{\mathbb{B}}) = 0$ and $\overline{P}_v(a) = 1$ for any event $a > 0_{\mathbb{B}}$, which *is* a *numerical* model for

[14] This issue is related to the Dutch book rationality argument, which is presumed to imply the necessity of using additive probability for uncertainty models (whence the completeness of the underlying ordinal models). However, in this kind of argument a number of assumptions are made which are difficult to defend as being implied or forced upon us by rationality (for a good discussion, see [13]). If we do not make them, the Dutch book argument runs out of steam before it can do any harm.

[15] Note by the way that in Walley's approach [13], completeness is also an optional property of his comparative probability orderings.

complete ignorance. Firstly, we do not believe that the *ordinal* model $\preceq_{\mathbb{B}\setminus\{0_{\mathbb{B}}\}}$ represents complete ignorance on our interpretation of confidence relations. It is on the contrary quite informative, since it tells us that there is equal confidence in any pair of non-empty events a and b: we cannot see how anyone without any relevant knowledge at all would be willing to state that $a \preceq b$ (unless $a \leq b$) or $b \preceq a$ (unless $b \leq a$). Secondly, the fact that $\overline{P}_v$ represents complete ignorance whereas on our interpretation the compatible $\preceq_{\mathbb{B}\setminus\{0_{\mathbb{B}}\}}$ does not, adds further weight to our claim that in this context compatibility is not the appropriate link between numerical and ordinal models [16]. Again, it must suffice here to note that there exists another way of linking the two, which does connect $\overline{P}_v$ with $\leq$ (see [13] for more details).

A third argument for allowing incompleteness has its roots in the combination of information. Assume that we have two confidence relations associated with the same experiment, derived from assessments by, say, two different experts. A natural and conservative way to combine this information is by taking the intersection of the two confidence relations: we only use those assessments on which both experts agree. This intersection will again be a confidence relation. But even if we start out with two complete relations, their intersection need not be complete, so we would have to in some way add couples to it in order to get back to a complete confidence relation. Moreover, since the complete confidence relations do not constitute an intersection structure, there need not be a smallest complete confidence relation which includes the intersection, and it seems therefore difficult to give natural procedures to make non-complete confidence relations complete. Combining information from two sources is very natural in the context of confidence relations, but becomes awkward if we restrict ourselves to complete confidence relations. Similar remarks can be made about finding complete confidence relations which are extensions of a finite number of comparisons. In order to make our approach for confidence relations work for complete confidence relations, one would have to find an ordering of relations (other than set inclusion) such that the complete confidence relations constitute at least a complete meet-semilattice with respect to it. We see no obvious candidate for such an ordering, but do not *a priori* exclude its existence.

As a last (more aesthetical) argument in favour of potential incomparability, we again point out the symmetry between the rather simple ordinal models we have studied here, and other numerical and ordinal accounts of uncertainty, which we referred to at the beginning of this section.

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[16] It seems to us that it is the appropriate link in the case of probabilistic models, but that it starts to fail as soon as we turn to non-probabilistic ones.

Acknowledgement

This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction initiated by the Belgian state, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with the author.

The author wishes to express his gratitude to three anonymous referees, who made a number of suggestions to improve the readability of this paper. Their remarks and challenges helped him bring forward important details that otherwise would have been left unmentioned.

He would like to dedicate this paper to his former teacher Vic Versnick, without whom it would never have been written in the first place.

A Basic notions from lattice theory

Consider a non-empty set P. A binary relation $\leq$ on P which is reflexive and transitive is called a *partial preorder*. If it is also antisymmetrical, i.e. if $a \leq b$ and $b \leq a$ imply $a = b$ for any a and b in P, then it is called a *partial order*, and the structure $(P, \leq)$ a *partially ordered set* or *poset*, for short. If the partial order $\leq$ is *complete*, i.e. if $a \leq b$ or $b \leq a$ for any a and b in P, then $(P, \leq)$ is a *chain*. A chain is in particular a poset.

Let $(P, \leq)$ be a partially ordered set. For any a and b in P we say that a dominates b, or that b is dominated by a, if $b \leq a$. Let Q be a subset of P. An element a of P is an *upper bound* of Q if it dominates every element of Q. It is a *lower bound* of Q if it is dominated by every element of Q. The set of the upper bounds of Q is denoted by Q^u, and its set of lower bounds by Q^ℓ.

A subset Q of P is an *up-set* of $(P, \leq)$ if $(\forall a \in Q)(\{a\}^u \subseteq Q)$ and a *down-set* of $(P, \leq)$ if $(\forall a \in Q)(\{a\}^\ell \subseteq Q)$.

If Q has an upper (lower) bound which belongs to Q then this must be unique, and it is called the *greatest* (*smallest* or *least*) element of Q. A *maximal* element of Q is an element of Q that is not dominated by any other element of Q. A *minimal* element of Q is an element of Q that does not dominate any other element of Q. A greatest element is in particular a maximal element, which is furthermore unique, but the converse does not necessarily hold. Similarly for a smallest element.

If the set P itself has a greatest element, this is called the *top* of $(P, \leq)$, and denoted by 1_P. If P has a smallest element, this is called the *bottom* of $(P, \leq)$, and denoted by 0_P. A poset with top and bottom is called *bounded*.

If the set Q^u has a smallest element, this unique element of P is called the *supremum* or least upper bound of Q, and is denoted by $\sup Q$. If Q^ℓ has a greatest element, this unique element of P is called the *infimum* or greatest lower bound of Q, and denoted by $\inf Q$. Note that $\inf \emptyset$ exists iff $\emptyset^\ell = P$ has a top, in which case $\inf \emptyset = 1_P$; similarly for $\sup \emptyset = 0_P$.

A poset $(P, \leq)$ is a *meet-semilattice* (*join-semilattice*) if the infimum (supremum) of every non-empty *finite* subset of P exists. It is a *complete* meet-semilattice (join-semilattice) if the infimum (supremum) of every non-empty subset of P exist. A *lattice* is at the same time a meet-semilattice and a join-semilattice. A *complete* lattice is defined similarly. Note that a complete meet-semilatice with top, and a complete join-semilattice with bottom are automatically also complete lattices.

If $(P, \leq)$ is a meet-semilattice we can define a binary operator $\frown$ on P which maps any two elements a and b to their infimum (or *meet*): $a \frown b = \inf\{a, b\}$. If $(P, \leq)$ is a join-semilattice we can define a binary operator $\smile$ on P which maps any two elements a and b to their supremum (or *join*): $a \smile b = \sup\{a, b\}$.

From now on we consider a lattice $(L, \leq)$. If a non-empty up-set Q of $(L, \leq)$ is meet-closed, i.e. $(\forall (a, b) \in Q^2)(a \frown b \in Q)$, it is called a *filter*. Similarly, a non-empty join-closed down-set is an *ideal*. Up-sets, down-sets, ideals and filters are called *proper* if they do not coincide with $\emptyset$ or L. A proper ideal is called *maximal* if it is not contained in any other proper ideal of $(L, \leq)$, or in other words, if it is a maximal element of the set of all proper ideals of $(L, \leq)$, partially ordered by set inclusion. Similarly for maximal filters. A maximal filter is also called an *ultrafilter*.

The lattice $(L, \leq)$ is called *distributive* if $\frown$ and $\smile$ are mutually distributive binary operators, which is equivalent to $a \frown (b \smile c) = (a \frown b) \smile (a \frown c)$, $(a, b, c) \in L^3$. A unary operator $\neg$ on a bounded lattice $(L, \leq)$ is called a *complement* (operator) if for any a in L: $\neg a \frown a = 0_L$ and $\neg a \smile a = 1_L$. If a bounded distributive lattice has a complement, then this operator is necessarily unique, and the lattice is called a *Boolean lattice*.

Consider a non-empty set X and a mapping $C \colon \wp(X) \to \wp(X)$. C is called a *closure operator* on X if it satisfies the following requirements for all subsets A and B of X: $(C1)$ $A \subseteq C(A)$; $(C2)$ $A \subseteq B \Rightarrow C(A) \subseteq C(B)$; and $(C3)$ $C(C(A)) = C(A)$. A subset A of X for which $C(A) = A$ is called *closed* with respect to C.

Next consider a non-empty collection $\mathcal{C}$ of subsets of X. If $\mathcal{C}$ of closed under arbitrary non-empty intersections, we call $\mathcal{C}$ an *intersection structure*. If we order $\mathcal{C}$ by set inclusion, $(\mathcal{C}, \subseteq)$ is then a complete meet-semilattice. If moreover $X \in \mathcal{C}$, $\mathcal{C}$ is called a *topped* intersection structure, or a *closure system*, and $(\mathcal{C}, \subseteq)$ is a complete lattice. With any closure system $\mathcal{C}$, we can associate a closure operator $C_{\mathcal{C}} \colon \wp(X) \to \wp(X)$ which maps any subset A of X to the smallest element $C_{\mathcal{C}}(A)$ of $\mathcal{C}$ which includes it: $C_{\mathcal{C}}(A) = \bigcap \{\, D \in \mathcal{C} \mid A \subseteq D \,\}$. Conversely, the set of closed sets associated with any closure operator constitutes a closure system.

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