NON-TRUTH-FUNCTIONAL ORDER NORMS

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Abstract: I defend the introduction of triangular (semi)norms and (semi)conorms on bounded partially ordered sets. First, I give a brief survey of the general properties of these binary operators. Then, I work out a number of examples in diverse fields of mathematics, to show that it is indeed useful and natural to generalize the definition of triangular (semi)norms and (semi)conorms from the unit interval towards bounded posets.

Keywords: Order norms, operators, fuzzy integrals, reliability theory.

1 Order Norms on Bounded Posets

If we take a closer look at the definition of triangular norms and conorms [18], we see that none on the defining conditions require the *extra-ordinal* characteristics of the unit interval $[0,1]$. Indeed, it is easily verified that the most general structure for which these defining conditions still make sense, is that of a bounded partially ordered set. This leads to the following definition. In what follows, and unless explicitly stated otherwise, $(L,\leq)$ will denote at least a bounded poset, with bottom 0_L and top 1_L [7].

Definition 1 (Order Norms) *(i) A triangular seminorm P on $(L,\leq)$ is a binary operator on L that is isotonic and satisfies the boundary condition: $(\forall\lambda\in L)(P(1_L,\lambda)=P(\lambda,1_L)=\lambda)$. A triangular norm T on $(L,\leq)$ is a triangular seminorm on $(L,\leq)$ that is furthermore associative and commutative.*

(ii) A triangular semiconorm Q on $(L,\leq)$ is a binary operator on L that is isotonic and satisfies the boundary condition: $(\forall\lambda\in L)(Q(0_L,\lambda)=Q(\lambda,0_L)=\lambda)$. A triangular conorm S on $(L,\leq)$ is a triangular seminorm on $(L,\leq)$ that is furthermore associative and commutative.
Triangular seminorms and semiconorms are collectively called order norms.

It is not surprising that many of the well-known properties of order norms on $([0,1],\leq)$ that can be expressed solely in terms of order, remain valid for their counterparts on $(L,\leq)$, although the potential incomparability of elements of L tends to somewhat complicate their proofs. Let me briefly mention the most important ones. For a detailed account of the properties of order norms on bounded posets, I refer to [13].

First of all, order norms show the following additional boundary behaviour. For arbitrary λ in L: $P(0_L,\lambda)=P(\lambda,0_L)=0_L$ and $Q(1_L,\lambda)=Q(\lambda,1_L)=1_L$. This already implies that a triangular (semi)norm and a triangular (semi)conorm can never completely coincide unless $0_L=1_L$. Obviously, a triangular (semi)norm on $(L,\leq)$ is a triangular (semi)conorm on $(L,\geq)$ and the other way round, which means that triangular (semi)norms and (semi)conorms are *dual notions*. Let us now define the following binary operators Z_L and Z_L^* on L. For arbitrary λ and μ in L:

$$Z_L(\lambda,\mu)=\begin{cases}\lambda & ; \quad \mu=1_L\\ \mu & ; \quad \lambda=1_L\\ 0_L & ; \quad \text{elsewhere}\end{cases} \qquad \text{and} \qquad Z_L^*(\lambda,\mu)=\begin{cases}\lambda & ; \quad \mu=0_L\\ \mu & ; \quad \lambda=0_L\\ 1_L & ; \quad \text{elsewhere.}\end{cases}$$

Z_L is the smallest triangular (semi)norm on $(L,\leq)$ and Z_L^* the greatest triangular (semi)conorm on $(L,\leq)$. Furthermore, if $(L,\leq)$ is a bounded *lattice*, we can define the *meet* or binary infimum operator $\frown$ and the *join* or binary supremum operator $\smile$ on $(L,\leq)$. It turns out that $\frown$ is the greatest triangular (semi)norm and $\smile$ the smallest triangular (semi)conorm on $(L,\leq)$. $\frown$ is the only idempotent triangular (semi)norm and $\smile$ the only idempotent triangular (semi)conorm on $(L,\leq)$. Finally, if a triangular (semi)norm is distributive w.r.t. a triangular (semi)conorm, then this (semi)conorm must be $\smile$. Dually, if a triangular (semi)conorm is distributive w.r.t. a triangular (semi)norm, then this (semi)norm must be $\frown$.

Now, it is one thing to study mathematical notions in their fullest generality, and another thing to show that such a high level of generality is still useful. In the rest of this paper, I intend to show that order norms defined on ordered sets can play an important part in many domains of mathematics.

2 Non-Truth-Functional Fuzzy-Set-Theoretical Operators

Consider a universe X. With an arbitrary subset A of X, we may associate its *characteristic* $X-L$*-mapping* χ_A, defined as follows. For arbitrary x in X: $\chi_A(x) = 1_L \Leftrightarrow x \in A$ and $\chi_A(x) = 0_L \Leftrightarrow x \notin A$. Let us denote by $\Xi_{(L,\leq)}(X)$ the set of the characteristic $X-L$-mappings: $\Xi_{(L,\leq)}(X) = \{\, \chi_A \mid A \subseteq X \,\}$. Following Goguen [17], a $(L,\leq)$-*fuzzy set* in a universe X is defined as a $X-L$-mapping, and is an obvious extension of characteristic $X-L$-mappings. If we consider the set $\mathcal{F}_{(L,\leq)}(X)$ of the $(L,\leq)$-fuzzy sets in X, we can define the following product order $\sqsubseteq$ on $\mathcal{F}_{(L,\leq)}(X)$. For arbitrary $(L,\leq)$-fuzzy sets g and h in X: $g \sqsubseteq h \Leftrightarrow (\forall x \in X)(g(x) \leq h(x))$. Of course, $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a bounded poset with top χ_X and bottom $\chi_\emptyset$. The substructure $(\Xi_{(L,\leq)}(X), \sqsubseteq)$ of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a complete Boolean lattice that is order-isomorphic to the complete Boolean lattice $(\wp(X), \subseteq)$, where $\wp(X)$ is the power set of X and $\subseteq$ is the inclusion relation on $\wp(X)$. The *intersection* $\cap$ of subsets of X is the meet of $(\wp(X), \subseteq)$ and as such corresponds with the meet of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$. The *union* $\cup$ of subsets of X is the join of $(\wp(X), \subseteq)$ and as such corresponds with the join of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$.

What we now want to do is extend the notion of union and intersection from sets (characteristic mappings) to fuzzy sets. That is, we want to define binary operators $\sqcup$ and $\sqcap$ on $\mathcal{F}_{(L,\leq)}(X)$. First of all, these operators must be *extensions* of union respectively intersection of classical sets. Moreover, it is fairly natural to require that these operators should be commutative, associative and isotonic. Finally, we still want $\chi_\emptyset$ ($\emptyset$) to be a neutral element for the fuzzy union $\sqcup$ and χ_X (X) to be a neutral element for the fuzzy intersection $\sqcap$. This leads to the following definition.

Definition 2 *(i) We call intersection operator on $\mathcal{F}_{(L,\leq)}(X)$ any triangular norm on $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$, the restriction to $\Xi_{(L,\leq)}(X)$ of which coincides with the meet of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$.*

(ii) We call union operator on $\mathcal{F}_{(L,\leq)}(X)$ any triangular conorm on $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$, the restriction to $\Xi_{(L,\leq)}(X)$ of which coincides with the join of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$.

As an example, the meet of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ (pointwise infimum) is an intersection operator (the greatest) on $\mathcal{F}_{(L,\leq)}(X)$. Dually, the join of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ (pointwise supremum) is a union operator (the smallest) on $\mathcal{F}_{(L,\leq)}(X)$. On the other hand, if X contains more than one element, the smallest triangular norm $Z_{\mathcal{F}_{(L,\leq)}(X)}$ on $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is not an intersection operator on $\mathcal{F}_{(L,\leq)}(X)$ because its restriction to $\Xi_{(L,\leq)}(X)$ does not coincide with the meet of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$. Indeed, choose subsets A and B of X with $A \cap B \neq \emptyset$, $\emptyset \subset A \subset X$ and $\emptyset \subset B \subset X$. Then $Z_{\mathcal{F}_{(L,\leq)}(X)}(\chi_A, \chi_B) = \chi_\emptyset$ whereas by assumption $A \cap B \neq \emptyset$. Dually, the greatest triangular norm $Z^*_{\mathcal{F}_{(L,\leq)}(X)}$ on $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is not a union operator on $\mathcal{F}_{(L,\leq)}(X)$ because its restriction to $\Xi_{(L,\leq)}(X)$ does not coincide with the join of $(\Xi_{(L,\leq)}(X), \sqsubseteq)$

What is particularly interesting about this way of defining fuzzy union and intersection operators, is that these operators are *not necessarily truth-functional.* Thus, we are led to the study of non-truth-functional fuzzy union and intersection operators, that at the same time retain all the other properties (commutativity, associativity, etc.) of the usual fuzzy unions and intersections. In this light, we have the following interesting proposition [12]. Consider an binary operator Δ on L. Then we call the binary operator θ_Δ on $\mathcal{F}_{(L,\leq)}(X)$, defined by $(\forall (g,h) \in \mathcal{F}_{(L,\leq)}(X)^2)(\theta_\Delta(g,h) = \Delta \circ (g,h))$ the *pointwise extension* of Δ to $\mathcal{F}_{(L,\leq)}(X)$.

Proposition 1 (Truth-Functionality) *Let Δ be a binary operator on L.*

(i) θ_Δ is an intersection operator on $\mathcal{F}_{(L,\leq)}(X)$ if and only if Δ is a triangular norm on $(L,\leq)$.

(ii) θ_Δ is a union operator on $\mathcal{F}_{(L,\leq)}(X)$ if and only if Δ is a triangular conorm on $(L,\leq)$.

This means that the truth-functional fuzzy unions and intersections on $\mathcal{F}_{(L,\leq)}(X)$ are completely characterized by the triangular conorms respectively norms on $(L,\leq)$. Of course, the meet of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ is a truth-functional fuzzy intersection that is the pointwise extension of the meet $\frown$ of $(L,\leq)$. A dual result holds for joins. As an interesting corollary, we find that a union operator is idempotent if and only if it coincides with the join of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ and is therefore truth-functional. Dually, an intersection operator is idempotent if and only if it coincides with the meet of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$ and is therefore truth-functional. Moreover, if a union operator $\sqcup$ is distributive w.r.t. an intersection operator $\sqcap$, then $\sqcap$ must be the meet of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$. Dually, if an intersection operator $\sqcap$ is distributive w.r.t. a union operator $\sqcup$, then $\sqcup$ must be the join of $(\mathcal{F}_{(L,\leq)}(X), \sqsubseteq)$.

In this sense, triangular norms and conorms on bounded posets also appear useful if we only work with $([0,1],\leq)$-fuzzy sets, i.e., the membership functions of Zadeh's fuzzy sets [19].

3 Seminormed and Semiconormed Fuzzy Integrals

A second application of order norms on ordered sets lies in the field of fuzzy integral theory. In integration theory, an integral is usually introduced by first considering functionals on simple mappings, and then extending these towards functionals on more general mappings by some limit transition. In this section, I shall briefly describe how this general procedure can be used to introduce seminormed and semiconormed fuzzy integrals. For more details about seminormed and semiconormed fuzzy integrals, and their relation with the theory of possibility and necessity measures, I refer to [14].

Let us consider a field of subsets $\mathcal{V}$ on the universe X. In this section, $(L,\leq)$ will denote a complete lattice. A $(L,\leq)$-*confidence measure* v on $(X,\mathcal{V})$ is an isotonic $\mathcal{V}-L$-mapping. $(X,\mathcal{V},v)$ is called a $(L,\leq)$-*confidence space*. v is called *normal* iff $v(\emptyset)=0_L$ and $v(X)=1_L$. A $\mathcal{V}$-*simple* $X-L$-mapping s is a $X-L$-mapping with finite range $s(X)=\{s_1,\ldots,s_n\}$, such that $D_k=s^{-1}(\{s_k\})\in\mathcal{V}$, $k\in\{1,\ldots,n\}$. The first step consists in finding *decompositions* of s in terms of the s_k and the D_k. Let ψ, ϕ, ξ and ζ be arbitrary binary operators on L, then it is fairly obvious that the most general candidates for these decompositions are the $X-L$ mappings $\phi_{k=1}^n\psi(s_k,\chi_{D_k}(\cdot))$ and $\xi_{k=1}^n\zeta(s_k,\chi_{\mathrm{co}D_k}(\cdot))$. Of course, ϕ and ξ must be associative and commutative for these expressions to really make sense. Furthermore, it is not difficult to prove that on the one hand $s=\phi_{k=1}^n\psi(s_k,\chi_{D_k}(\cdot))$ if and only if $(\forall\lambda\in L)(\psi(\lambda,1_L)=\lambda)$ and $(\forall(\lambda,\mu)\in L^2)(\phi(\lambda,\psi(\mu,0_L))=\lambda)$. On the other hand $s=\xi_{k=1}^n\zeta(s_k,\chi_{\mathrm{co}D_k}(\cdot))$ if and only if $(\forall\lambda\in L)(\zeta(\lambda,0_L)=\lambda)$ and $(\forall(\lambda,\mu)\in L^2)(\xi(\lambda,\zeta(\mu,1_L))=\lambda)$. Let us therefore assume that these conditions are satisfied.

The next step consists in using the *form* of these decompositions of s to define two functionals on $\mathcal{V}$-simple mappings, that are, in a sense, each other's dual.

Definition 3 *Let A be an arbitrary element of $\mathcal{V}$ and s an arbitrary $\mathcal{V}$-simple $X-L$-mapping. Then we define, with obvious notations, $\alpha^v_{\phi\psi}(A;s)=\phi_{k=1}^n\psi(s_k,v(A\cap D_k))$ and $\beta^v_{\xi\zeta}(A;s)=\xi_{k=1}^n\zeta(s_k,v(A\cap\mathrm{co}D_k))$.*

It is also easily verified that the functionals $\alpha^v_{\phi\psi}(\cdot;\cdot)$ and $\beta^v_{\xi\zeta}(\cdot;\cdot)$ thus defined are only isotonic in both arguments for arbitrary $(X,\mathcal{V},v)$ if ϕ and ψ, respectively ξ and ζ are isotonic. Moreover, it is fairly natural to require that for any $(X,\mathcal{V},v)$: $(\forall A\in\mathcal{V})(\alpha^v_{\phi\psi}(A;\underline{1_L})=v(A))$, where $\underline{1_L}$ is the constant $X-\{1_L\}$-mapping. It is not difficult to show that a necessary condition for this is that $(\forall\lambda\in L)(\psi(1_L,\lambda)=\lambda)$. For $\beta^v_{\xi\zeta}(\cdot;\cdot)$, the dual requirement leads to the necessary condition $(\forall\lambda\in L)(\zeta(0_L,\lambda)=\lambda)$. It therefore turns out that, in order that the functionals $\alpha^v_{\phi\psi}(\cdot;\cdot)$ and $\beta^v_{\xi\zeta}(\cdot;\cdot)$ satisfy some natural properties, ϕ must be a triangular conorm S, ψ a triangular seminorm P, ξ a triangular norm T and ζ a triangular semiconorm Q on $(L,\leq)$. In this way, order norms emerge naturally in this integral-theoretic context.

The last step in this course of reasoning consists in going from functionals on simple mappings towards functionals on arbitrary mappings. If A is an element of $\mathcal{V}$ and h a $X-L$-mapping, we define the $(L,\leq)$-fuzzy SP-integral of h on A (associated with v) as

$$(SP)\,⨍_A hdv=\sup\{\,\alpha^v_{SP}(A;s)\mid s\sqsubseteq h\,\}$$

and the $(L,\leq)$-fuzzy TQ-integral of h on A (associated with v) as

$$(TQ)\,⨍_A hdv=\inf\{\,\beta^v_{TQ}(A;s)\mid h\sqsubseteq s\,\}.$$

For these integrals, the following interesting result may be proven.

Proposition 2 *(i) In order that for an arbitrary $(L,\leq)$-confidence space $(X,\mathcal{V},v)$ with v normal, for an arbitrary triangular seminorm P on $(L,\leq)$ and for arbitrary μ in L, $(SP)⨍_X\underline{\mu}dv=\mu$, it is necessary and sufficient that $S=\smile$.*

(ii) In order that for an arbitrary $(L,\leq)$-confidence space $(X,\mathcal{V},v)$ with v normal, for an arbitrary triangular semiconorm Q on $(L,\leq)$ and for arbitrary μ in L, $(TQ)⨍_X\underline{\mu}dv=\mu$, it is necessary and sufficient that $T=\frown$.

On the one hand, we are thus lead to consider $(L, \leq)$-fuzzy P-integrals $(P)⨍_A h dv = (\smile P)⨍_A h dv$, generally called *seminormed fuzzy integrals*. On the other hand, we are lead to consider $(L, \leq)$-fuzzy Q-integrals $(Q)⨍_A h dv = (\frown Q)⨍_A h dv$, generally called *semiconormed fuzzy integrals*.

In summary, it turns out that (semi)norms and (semi)conorms on complete lattices are very important if we want to consider fuzzy integrals of mappings that assume values in these complete lattices. In my doctoral dissertation, I have shown that such integrals can be made to play a central role in a measure- and integral-theoretic approach to possibility and necessity theory [12] (see also [8, 9, 10, 14, 15]).

4 Extending Barlow-Wu Extensions

We find yet another application of order norms on ordered sets in multi-state reliability theory. This is an extension of the well-known classical, two-state reliability theory. In this theory, a system or a component is assumed to be in one of the two states '*fail*' or '*work*'. In other words, the state set $\mathcal{S}$ is assumed to be $\{fail, work\}$, linearly ordered by the relation $\leq$, defined by $fail < work$. Furthermore, the state of a system is assumed to be a function $\varphi: \mathcal{S}^n \to \mathcal{S}$ of the states of its n components. φ is called the *two-state structure function* of the system, and is in general assumed to be isotonic, and bottom- and top-preserving, i.e., $\varphi(fail, \ldots, fail) = fail$ and $\varphi(work, \ldots, work) = work$.

There are some shortcomings to this model, however. It may be recognized that in some cases the state space $\{fail, work\}$ may not be rich enough to provide a structural description of a system and its components. On the one hand, there is *gradual failure*: a component or system may be working, but only imperfectly. This can be modeled by introducing linearly ordered state sets with more than two elements, such as $(\{1, 2, \ldots, m\}, \leq)$ (see, for instance, the work of El-Neweihi, Proschan and Sethuraman [16]) or the real unit interval $([0, 1], \leq)$ (see, for instance, the work of Baxter et al. [2, 3, 4, 5]). On the other hand, it is possible that a component or system is subject to different, *mutually incomparable types of failure*, i.e., where one type of failure is not a degradation of the other. In this case, the state space will be only partially ordered, and can in general be represented by a bounded poset $(L, \leq)$.

Let us indeed assume that the state space is a bounded poset $(L, \leq)$, and that the state of a system is a function $\varphi: L^n \to L$ of the states of its n components. It is again fairly natural to assume that φ is a *structure function*, i.e., that φ is isotonic, and bottom- and top-preserving. Extending an idea of Barlow and Wu [1] and Cappelle [6], we may now try and extend any two-state structure function φ from $\mathcal{S}$ to L. Using the minimal paths P_r, $r \in \{1, \ldots, n_p\}$, and minimal cuts C_s, $s \in \{1, \ldots, n_c\}$ of the two-state structure function φ, it is well-known result in two-state reliability theory that for arbitrary $(\nu_1, \ldots, \nu_n)$ in $\mathcal{S}^n$

$$\varphi(\nu_1, \ldots, \nu_n) = \bigvee_{1 \leq r \leq n_p} \bigwedge_{i \in P_r} \nu_i = \bigwedge_{1 \leq s \leq n_c} \bigvee_{i \in C_s} \nu_i$$

where, $\vee$ is the join and $\wedge$ the meet of $(\mathcal{S}, \leq)$. Then, if T is a triangular norm and S a triangular conorm on $(L, \leq)$, the $L^n - L$-mappings $\mathcal{P}_{ST}(\varphi)$ and $\mathcal{C}_{TS}(\varphi)$, defined by

$$\mathcal{P}_{ST}(\varphi)(\lambda_1, \ldots, \lambda_n) = S_{1 \leq r \leq n_p} T_{i \in P_r} \lambda_i \quad \text{and} \quad \mathcal{C}_{TS}(\varphi)(\lambda_1, \ldots, \lambda_n) = T_{1 \leq s \leq n_c} S_{i \in C_s} \lambda_i$$

for arbitrary $(\lambda_1, \ldots, \lambda_n)$ in L^n, are structure functions on L. Furthermore, when they are restricted to $\{0_L, 1_L\}$, they behave in exactly the same way as φ. They are therefore called *order norm extensions* of φ from $\mathcal{S}$ to L. In this way, it turns out that we need triangular norms and conorms on ordered sets in order to be able to extend two-state structure functions to state sets that allow for incomparability between states. Moreover, I have recently proven the following interesting result.

Proposition 3 *Let $(L, \leq)$ be a bounded lattice.*

$$(\forall \varphi, \varphi \text{ two-state structure function})(\mathcal{P}_{ST}(\varphi) = \mathcal{C}_{TS}(\varphi)) \Leftrightarrow \begin{cases} (L, \leq) \text{ is distributive} \\ T = \frown \text{ and } S = \smile. \end{cases}$$

For more details about these extensions, and how they play an important part in incorporating possibilistic uncertainty into classical two-state reliability theory, I refer to [11].

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